Lipschitz semigroup approach to two dimensional Navier-Stokes equations

Yoshikazu Kobayashi^a, Naoki Tanaka^{b,*,1}

^a*Department of Mathematics, Faculty of Science and Engineering, Chuo University,
Tokyo 112-8551, Japan*

^b*Department of Mathematics, Faculty of Science, Shizuoka University, Shizuoka
422-8529, Japan*

Abstract

Motivated by the notion of integral solutions in the sense of Benilan, a dissipative condition combined with tangential condition is introduced. A generation problem of semigroups of Lipschitz operators is discussed under such a dissipative condition and the result is applied to the initial-boundary value problem for the Navier-Stokes equation in two dimensional space.

Key words: Semigroup of Lipschitz operators, Mild solution, Integral solution, Metric-like functional, Dissipative condition, Tangential condition, Navier-Stokes equation

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1. Introduction

Let X be a general Banach space with norm $\|\cdot\|$ and D a closed subset of X . By a *semigroup on D* is meant a one-parameter family $\{S(t); t \geq 0\}$ of operators from D into itself satisfying the so-called *semigroup property* and the strong continuity in $t \geq 0$ (see [6]). In order to develop a general theory of nonlinear semigroups, it is necessary to consider the continuity of the operators $S(t)$ in an appropriate way. In this paper, we consider the

*Corresponding author

Email addresses: kobayashi@math.chuo-u.ac.jp (Yoshikazu Kobayashi),
sntanak@ipc.shizuoka.ac.jp (Naoki Tanaka)

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continuity condition of the operators $S(t)$ in such a way that for each $\tau > 0$ there exists $M_\tau > 0$ satisfying

$$\|S(t)x - S(t)y\| \leq M_\tau \|x - y\| \quad \text{for } x, y \in D \text{ and } t \in [0, \tau]. \quad (1.1)$$

A semigroup on D satisfying the above-mentioned continuity condition is called a *semigroup of Lipschitz operators on D* . It was shown ([7]) that a semigroup $\{S(t); t \geq 0\}$ on D satisfies (1.1) if and only if there exist $\omega \geq 0$ and a nonnegative, Lipschitz continuous functional Φ on $X \times X$ satisfying the condition (Φ) below such that $\Phi(S(t)x, S(t)y) \leq e^{\omega t} \Phi(x, y)$ for $x, y \in D$ and $t \geq 0$. In this case, the *infinitesimal generator* A_0 , defined by $A_0x = \lim_{h \downarrow 0} (S(h)x - x)/h$ whenever the limit exists, satisfies the dissipative condition $D_+\Phi(x, y)(A_0x, A_0y) \leq \omega \Phi(x, y)$ for $x, y \in D(A_0)$, where

$$D_+\Phi(x, y)(\xi, \eta) = \limsup_{h \downarrow 0} (\Phi(x + h\xi, y + h\eta) - \Phi(x, y))/h$$

for $(x, y), (\xi, \eta) \in X \times X$. In the special case where A is a continuous operator from D into X , a generation problem under the dissipative condition in terms of the metric-like functional Φ and the so-called subtangential condition was also discussed in [7] and the results due to [8, 9] were improved.

The above-mentioned dissipative condition was introduced in [10, 13] to study the problem of uniqueness of solutions for Cauchy problems of ordinary differential equations in $\mathbb{R}^n$. In the quasi-contractive semigroup theory, the notion of integral solutions in the sense of Bnilan [1] plays an important role in studying the problem of uniqueness of solutions for abstract Cauchy problems.

Motivated by the notion of integral solutions in the sense of Bnilan, a dissipative condition combined with tangential condition is introduced (see (A2) below). A generation problem of semigroups of Lipschitz operators is discussed under such a dissipative condition. The main result is given by Theorem 2.3 and extends the generation theorems given in [4, 5]. Section 3 contains a Lipschitz semigroup approach to the well-posedness for the Navier-Stokes equation in two dimensional space.

2. Generation of semigroups of Lipschitz operators

Let Φ be a nonnegative, Lipschitz continuous functional on $D \times D$ satisfying the condition

(Φ) there exist $M \geq m > 0$ such that $m\|x - y\| \leq \Phi(x, y) \leq M\|x - y\|$ for $(x, y) \in D \times D$.

We denote a Lipschitz constant of Φ by L . Let φ be a proper lower semicontinuous functional from X into $[0, \infty]$ such that

(φ) $D(\varphi) \subset D$ and $D(\varphi)$ is dense in D ,

where $D(\varphi)$ stands for the effective domain $\{x \in X; \varphi(x) < \infty\}$ of φ . Let $\omega \geq 0$ and $a \geq 0$, and let A be a multi-valued operator in X satisfying the following two conditions:

- (A1) $D(A) \subset D(\varphi)$, where $D(A)$ is the domain of the operator A .
- (A2) For each $\varepsilon > 0$ and $x \in D(\varphi)$ there exist $\delta \in (0, \varepsilon]$, $x_\delta \in D(A)$ and $\xi_\delta \in Ax_\delta$ such that the following three conditions are satisfied:
 - (i) $(\Phi(x_\delta, y) - \Phi(x, y))/\delta + (\Phi(x_\delta, y) - \Phi(x_\delta, y_0))/h \leq \omega\Phi(x, y) + \varepsilon + \lambda$ for any $\lambda > 0$, $h \in (0, \lambda]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ such that $\|y - y_0 - h\eta\| \leq h\lambda$.
 - (ii) $\|x_\delta - x\| \leq \varepsilon$ and $\|x_\delta - x - \delta\xi_\delta\| \leq \delta\varepsilon$.
 - (iii) $(\varphi(x_\delta) - \varphi(x))/\delta \leq a\varphi(x) + \varepsilon$.

Lemma 2.1. *Let $\varepsilon > 0$ and $x_0 \in D(\varphi)$. Then there exist $t_i \geq 0$, $x_i \in D(A)$ and $\xi_i \in Ax_i$ for $i = 1, 2, 3, \dots$ such that they satisfy the following conditions:*

- (i) $0 = t_0 < t_1 < \dots < t_i < \dots$
- (ii) $t_i - t_{i-1} \leq \varepsilon$ for $i = 1, 2, 3, \dots$
- (iii) For any $\lambda > 0$, $h \in (0, \lambda]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ such that $\|y - y_0 - h\eta\| \leq h\lambda$,

$$\begin{aligned} & (\Phi(x_i, y) - \Phi(x_{i-1}, y))/(t_i - t_{i-1}) + (\Phi(x_i, y) - \Phi(x_i, y_0))/h \\ & \leq \omega\Phi(x_{i-1}, y) + \varepsilon + \lambda \end{aligned}$$

for $i = 1, 2, 3, \dots$

- (iv) $\|x_i - x_{i-1} - (t_i - t_{i-1})\xi_i\| \leq (t_i - t_{i-1})\varepsilon$ for $i = 1, 2, 3, \dots$
- (v) $\varphi(x_i) \leq e^{a(t_i - t_{i-1})}(\varphi(x_{i-1}) + \varepsilon(t_i - t_{i-1}))$ for $i = 1, 2, 3, \dots$
- (vi) $\lim_{i \rightarrow \infty} t_i = \infty$.

Proof. Let $\varepsilon > 0$. For each $x \in D(\varphi)$ we define $\delta(x)$ by the supremum of all $\delta \in (0, \varepsilon]$ such that there exist $x_\delta \in D(A)$ and $\xi_\delta \in Ax_\delta$ satisfying that $\|x_\delta - x - \delta\xi_\delta\| \leq \delta\varepsilon$, $\varphi(x_\delta) \leq e^{a\delta}(\varphi(x) + \varepsilon\delta)$ and

$$(\Phi(x_\delta, y) - \Phi(x, y))/\delta + (\Phi(x_\delta, y) - \Phi(x_\delta, y_0))/h \leq \omega\Phi(x, y) + \varepsilon + \lambda$$

for any $\lambda > 0$, $h \in (0, \lambda]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ such that $\|y - y_0 - h\eta\| \leq h\lambda$. By condition (A2) we have $\delta(x) > 0$ for any $x \in D(\varphi)$. Since $x_0 \in D(\varphi)$ and $D(A) \subset D(\varphi)$, we inductively choose $h_i \in (0, \varepsilon]$, $x_i \in D(A)$ and $\xi_i \in Ax_i$ for $i = 1, 2, 3, \dots$ such that $\delta(x_{i-1})/2 < h_i$, $\|x_i - x_{i-1} - h_i\xi_i\| \leq h_i\varepsilon$, $\varphi(x_i) \leq e^{ah_i}(\varphi(x_{i-1}) + \varepsilon h_i)$ and such that

$$(\Phi(x_i, y) - \Phi(x_{i-1}, y))/h_i + (\Phi(x_i, y) - \Phi(x_i, y_0))/h \leq \omega\Phi(x_{i-1}, y) + \varepsilon + \lambda$$

for any $\lambda > 0$, $h \in (0, \lambda]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ such that $\|y - y_0 - h\eta\| \leq h\lambda$. Let $t_0 = 0$ and $t_i = \sum_{l=1}^i h_l$ for $i \geq 1$. Then the sequence $\{(t_i, x_i, \xi_i)\}_{i=1}^\infty$ in $[0, \infty) \times D(A) \times X$ satisfies conditions (i) through (v).

To check condition (vi), assume to the contrary that $\lim_{i \rightarrow \infty} t_i = \bar{t} < \infty$. Let $k \geq 1$ be fixed and $\gamma_i = \prod_{l=k+1}^i (1 + \omega h_l)^{-1}$ for $i \geq k$. Then we want to show that

$$\gamma_i \Phi(x_i, x_j) \leq (t_i - t_j) |||Ax_k||| + (t_i - t_k)\varepsilon + (t_j - t_k)\varepsilon \quad (2.1)$$

for $i \geq j \geq k$, where $|||Au||| = \inf\{\|\eta\|; \eta \in Au\}$ for $u \in D(A)$. It is obvious that the inequality (2.1) is true for $i = j$ ($\geq k$). To prove that the inequality (2.1) holds for $i \geq k$ and $j = k$, let $\eta_0 \in Ax_k$ and $\sigma > 0$ be arbitrary, and use condition (iii) with $y = y_0 = x_k$, $\eta = \eta_0$, $\lambda = \|\eta_0\| + \sigma$ and $h = \lambda/2$. Then this yields that $(\Phi(x_i, x_k) - \Phi(x_{i-1}, x_k))/h_i \leq \omega\Phi(x_{i-1}, x_k) + \varepsilon + \|\eta_0\| + \sigma$ for $i \geq k$; hence

$$\gamma_i \Phi(x_i, x_k) \leq \gamma_{i-1} \Phi(x_{i-1}, x_k) + h_i(|||Ax_k||| + \varepsilon)$$

for $i \geq k$. By solving this recursive inequality, the inequality (2.1) is shown to be true for $j = k$. Let $i \geq j \geq k$. By condition (iii) with $\lambda = \varepsilon$, $h = h_j$ and $(y_0, y, \eta) = (x_{j-1}, x_j, \xi_j)$ we find that

$$\gamma_i \Phi(x_i, x_j) \leq \frac{h_j}{h_i + h_j} \gamma_{i-1} \Phi(x_{i-1}, x_j) + \frac{h_i}{h_i + h_j} \gamma_i \Phi(x_i, x_{j-1}) + 2\varepsilon \frac{h_i h_j}{h_i + h_j}.$$

Assume that the inequality (2.1) holds for $(i-1, j)$ and $(i, j-1)$. Then we deduce from this inequality that the inequality (2.1) holds for (i, j) . The desired inequality (2.1) is thus proved by double induction on (i, j) .

Since $m\|x - y\| \leq \Phi(x, y)$ for $(x, y) \in D \times D$, the inequality (2.1) implies that the sequence $\{x_i\}$ is a Cauchy sequence in X , and let $\bar{x} = \lim_{i \rightarrow \infty} x_i$.

By condition (v) we have $\varphi(x_i) \leq e^{a(t_i - t_k)}(\varphi(x_k) + \varepsilon(t_i - t_k))$ for $i \geq k \geq 1$. The lower semicontinuity of φ yields that $\bar{x} \in D(\varphi)$ and

$$\varphi(\bar{x}) \leq e^{a(\bar{t} - t_k)}(\varphi(x_k) + \varepsilon(\bar{t} - t_k)) \quad (2.2)$$

for $k \geq 1$. Since $\bar{x} \in D(\varphi)$, we deduce from condition (A2) that there exist $\delta \in (0, \varepsilon/2]$, $\bar{x}_\delta \in D(A)$ and $\bar{\xi}_\delta \in A\bar{x}_\delta$ satisfying that $\|\bar{x}_\delta - \bar{x} - \delta\bar{\xi}_\delta\| \leq (\varepsilon/2)\delta$, $\varphi(\bar{x}_\delta) \leq e^{a\delta}(\varphi(\bar{x}) + \varepsilon\delta)$ and that

$$(\Phi(\bar{x}_\delta, y) - \Phi(\bar{x}, y))/\delta + (\Phi(\bar{x}_\delta, y) - \Phi(\bar{x}_\delta, y_0))/h \leq \omega\Phi(\bar{x}, y) + \varepsilon/2 + \lambda$$

for any $\lambda > 0$, $h \in (0, \lambda]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ such that $\|y - y_0 - h\eta\| \leq h\lambda$. Now, let $\delta_i = \bar{t} + \delta - t_{i-1}$ for $i = 1, 2, 3, \dots$. Then, by (2.2) we have $\varphi(\bar{x}_\delta) \leq e^{a\delta}(\varphi(\bar{x}) + \varepsilon\delta) \leq e^{a\delta_i}(\varphi(x_{i-1}) + \varepsilon\delta_i)$ for $i \geq 1$. Since $\delta_i \rightarrow \delta$ and $x_i \rightarrow \bar{x}$ as $i \rightarrow \infty$, there exists an integer $i_0 \geq 1$ such that $\delta_i \in (0, \varepsilon]$ and $\|\bar{x}_\delta - x_{i-1} - \delta_i\bar{\xi}_\delta\| \leq \delta_i\varepsilon$ for $i \geq i_0$. Let $\lambda > 0$, $h \in (0, \lambda]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ satisfy $\|y - y_0 - h\eta\| \leq h\lambda$. Then we have

$$\begin{aligned} & (\Phi(\bar{x}_\delta, y) - \Phi(x_{i-1}, y))/\delta_i + (\Phi(\bar{x}_\delta, y) - \Phi(\bar{x}_\delta, y_0))/h - \omega\Phi(x_{i-1}, y) - \lambda \\ & \leq \varepsilon/2 + (\Phi(\bar{x}_\delta, y) - \Phi(x_{i-1}, y))(1/\delta_i - 1/\delta) \\ & \quad + (\Phi(\bar{x}, y) - \Phi(x_{i-1}, y))(1/\delta + \omega) \\ & \leq \varepsilon/2 + L\|\bar{x}_\delta - x_{i-1}\||1/\delta_i - 1/\delta| + L\|\bar{x} - x_{i-1}\|(1/\delta + \omega), \end{aligned}$$

and the right-hand side tends to $\varepsilon/2$ as $i \rightarrow \infty$. Thus, it follows from the definition of $\delta(x_{i-1})$ that $\delta_i \leq \delta(x_{i-1})$ for sufficiently large integer $i \geq 1$. Since $\delta_i \rightarrow \delta$ and $\delta(x_{i-1}) < 2\delta_i \rightarrow 0$ as $i \rightarrow \infty$, we have $\delta \leq 0$. This contradicts the fact that $\delta > 0$. $\square$

Lemma 2.2. *Let $\tau > 0$ and $z_0 \in D(\varphi)$. Assume that for each $\varepsilon \in (0, \varepsilon_0]$ there exist $s_k^\varepsilon \geq 0$, $z_k^\varepsilon \in D(A)$ and $\eta_k^\varepsilon \in Az_k^\varepsilon$ for $k = 1, 2, \dots, K^\varepsilon$ such that they satisfy the following four conditions:*

- (i) $0 = s_0^\varepsilon < s_1^\varepsilon < \dots < s_{K^\varepsilon-1}^\varepsilon < \tau \leq s_{K^\varepsilon}^\varepsilon$.
- (ii) $s_k^\varepsilon - s_{k-1}^\varepsilon \leq \varepsilon$ for $1 \leq k \leq K^\varepsilon$.
- (iii) For any $\nu > 0$, $h \in (0, \nu]$, $v_0 \in D(\varphi)$, $v \in D(A)$ and $\eta \in Av$ such that $\|v - v_0 - h\eta\| \leq h\nu$,

$$\begin{aligned} & (\Phi(z_k^\varepsilon, v) - \Phi(z_{k-1}^\varepsilon, v))/(s_k^\varepsilon - s_{k-1}^\varepsilon) + (\Phi(z_k^\varepsilon, v) - \Phi(z_k^\varepsilon, v_0))/h \\ & \leq \omega\Phi(z_{k-1}^\varepsilon, v) + \varepsilon + \nu \end{aligned}$$

for $1 \leq k \leq K^\varepsilon$, where $z_0^\varepsilon = z_0$.

$$(iv) \quad \|z_k^\varepsilon - z_{k-1}^\varepsilon - (s_k^\varepsilon - s_{k-1}^\varepsilon)\eta_k^\varepsilon\| \leq (s_k^\varepsilon - s_{k-1}^\varepsilon)\varepsilon \text{ for } 1 \leq k \leq K^\varepsilon.$$

For $\varepsilon \in (0, \varepsilon_0]$ let $z^\varepsilon : [0, \tau] \rightarrow X$ be the step function defined by

$$z^\varepsilon(t) = \begin{cases} z_0 & \text{for } t = 0, \\ z_k^\varepsilon & \text{for } t \in (s_{k-1}^\varepsilon, s_k^\varepsilon] \cap [0, \tau] \text{ and } 1 \leq k \leq K^\varepsilon. \end{cases} \quad (2.3)$$

Assume that for each $\lambda \in (0, \lambda_0]$ there exist $x_0^\lambda \in X$, $t_i^\lambda \geq 0$, $x_i^\lambda \in D(A)$ and $f_i^\lambda \in X$ for $1 \leq i \leq N^\lambda$ such that they satisfy the following four conditions:

- (λ1) $0 = t_0^\lambda < t_1^\lambda < \dots < t_{N^\lambda-1}^\lambda < \tau \leq t_{N^\lambda}^\lambda$.
- (λ2) $t_i^\lambda - t_{i-1}^\lambda \leq \lambda$ for $1 \leq i \leq N^\lambda$.
- (λ3) $(x_i^\lambda - x_{i-1}^\lambda)/(t_i^\lambda - t_{i-1}^\lambda) \in Ax_i^\lambda + f_i^\lambda$ for $1 \leq i \leq N^\lambda$.
- (λ4) $\sum_{i=1}^{N^\lambda} (t_i^\lambda - t_{i-1}^\lambda) \|f_i^\lambda\| \leq \lambda$.

For $\lambda \in (0, \lambda_0]$ let $u^\lambda : [0, \tau] \rightarrow X$ be the step function defined by

$$u^\lambda(t) = \begin{cases} x_0^\lambda & \text{for } t = 0, \\ x_i^\lambda & \text{for } t \in (t_{i-1}^\lambda, t_i^\lambda] \cap [0, \tau] \text{ and } 1 \leq i \leq N^\lambda. \end{cases} \quad (2.4)$$

Assume that $\lim_{\lambda \downarrow 0} x_0^\lambda = x$. Then there exists $M_\tau > 0$ depending only on τ such that

$$\begin{aligned} & \limsup_{(\varepsilon, h) \rightarrow (0, 0)} \left(\limsup_{\lambda \downarrow 0} \left(\sup \{ \|z^\varepsilon(t) - u^\lambda(s)\|; t, s \in [0, \tau], |t - s| \leq h \} \right) \right) \\ & \leq M_\tau \|z_0 - x\|. \end{aligned}$$

Proof. Let $h_l^\lambda = t_l^\lambda - t_{l-1}^\lambda$ for $1 \leq l \leq N^\lambda$ and $\hat{h}_l^\varepsilon = s_l^\varepsilon - s_{l-1}^\varepsilon$ for $1 \leq l \leq K^\varepsilon$. Hereafter, superscripts λ and ε will be omitted when no confusion can arise. Consider $\lambda \in (0, \lambda_0]$ such that $\lambda\omega \leq 1/2$ and let $\gamma_{j,i} = \prod_{l=1}^j (1 + \omega \hat{h}_l)^{-1} \prod_{l=1}^i (1 - \omega h_l)$ for $0 \leq i \leq N$ and $0 \leq j \leq K$. Let $\rho > 0$. Since $z_0 \in D(\varphi)$ we deduce from condition (A2) that there exist $\delta \in (0, \rho]$ and $z_\delta \in D(A)$ such that

$$\|z_\delta - z_0\| \leq \rho, \quad (2.5)$$

$$(\Phi(z_\delta, y) - \Phi(z_0, y))/\delta + (\Phi(z_\delta, y) - \Phi(z_\delta, y_0))/h \leq \omega\Phi(z_0, y) + \rho + \gamma \quad (2.6)$$

for any $\gamma > 0$, $h \in (0, \gamma]$, $y_0 \in D(\varphi)$, $y \in D(A)$ and $\eta \in Ay$ such that $\|y - y_0 - h\eta\| \leq h\gamma$. Let $w_0 = z_0$ and $w_i = x_i$ for $1 \leq i \leq N$ and let $g_1 = f_1 + (x_0 - z_0)/(t_1 - t_0)$ and $g_i = f_i$ for $2 \leq i \leq N$. Then we have

$w_0 \in D(\varphi)$, $w_i \in D(A)$ and $g_i \in X$ for $1 \leq i \leq N$, and they satisfy that $(w_i - w_{i-1})/(t_i - t_{i-1}) \in Aw_i + g_i$ for $1 \leq i \leq N$ and that $\sum_{i=1}^N (t_i - t_{i-1}) \|g_i\| \leq \lambda + \|x_0 - z_0\|$. The purpose is to prove that

$$\begin{aligned} \gamma_{j,i} \Phi(z_j, w_i) &\leq 2L(\|u_0 - w_0\| + \rho) + s_j \varepsilon + t_i \lambda + \sum_{l=1}^i h_l \|g_l\| \\ &\quad + ((s_j - t_i)^2 + s_j \varepsilon + t_i \lambda)^{1/2} (\|Au_0\| + (1 + \delta^{-1}L)\rho) \end{aligned} \quad (2.7)$$

for any $u_0 \in D(A)$, $0 \leq i \leq N$ and $0 \leq j \leq K$. Since $\xi_i := (w_i - w_{i-1})/h_i - g_i \in Aw_i$ for $1 \leq i \leq N$, we apply the inequality (2.6) with $\gamma = \lambda + \|g_i\|$, $h = h_i$ and $(y_0, y, \eta) = (w_{i-1}, w_i, \xi_i)$ for $1 \leq i \leq N$. This yields that $(\Phi(z_\delta, w_i) - \Phi(z_\delta, w_{i-1}))/h_i \leq \omega \Phi(z_0, w_i) + \rho + \lambda + \|g_i\| + \delta^{-1}|\Phi(z_\delta, w_i) - \Phi(z_0, w_i)|$ for $1 \leq i \leq N$. By (2.5) and the Lipschitz continuity of Φ we have

$$(1 - \omega h_i) \Phi(z_0, w_i) \leq \Phi(z_0, w_{i-1}) + 2L\rho + h_i(\rho + \lambda + \|g_i\| + \delta^{-1}L\rho)$$

for $1 \leq i \leq N$. By the recursive inequality we can prove that the inequality (2.7) is true for $0 \leq i \leq N$ and $j = 0$. To show that the inequality (2.7) is true for $i = 0$ and $0 \leq j \leq K$, let $u_0 \in D(A)$ and let $\eta_0 \in Au_0$ and $\sigma > 0$ be arbitrary. By condition (iii) with $v = v_0 = u_0$, $\eta = \eta_0$, $\nu = \|\eta_0\| + \sigma$ and $h = \nu/2$ we have $(\Phi(z_j, u_0) - \Phi(z_{j-1}, u_0))/\hat{h}_j \leq \omega \Phi(z_{j-1}, u_0) + \varepsilon + \|\eta_0\| + \sigma$ for $1 \leq j \leq K$; hence $\gamma_{j,0} \Phi(z_j, u_0) \leq \Phi(z_0, u_0) + s_j(\varepsilon + \|Au_0\|)$ for $0 \leq j \leq K$. Since $\gamma_{j,0} \leq 1$ and $|\Phi(z_j, u_0) - \Phi(z_j, w_0)| \leq L\|u_0 - w_0\|$ for $0 \leq j \leq K$, the inequality (2.7) holds for $i = 0$ and $0 \leq j \leq K$. By condition (iii) with $\nu = \lambda + \|g_i\|$, $h = h_i$ and $(v_0, v, \eta) = (w_{i-1}, w_i, \xi_i)$ we find that

$$\begin{aligned} \gamma_{j,i} \Phi(z_j, w_i) &\leq \frac{h_i}{h_i + \hat{h}_j} \gamma_{j-1,i} \Phi(z_{j-1}, w_i) + \frac{\hat{h}_j}{h_i + \hat{h}_j} \gamma_{j,i-1} \Phi(z_j, w_{i-1}) \\ &\quad + \frac{h_i \hat{h}_j}{h_i + \hat{h}_j} \|g_i\| + \frac{h_i \hat{h}_j}{h_i + \hat{h}_j} (\varepsilon + \lambda) \end{aligned}$$

for $1 \leq i \leq N$ and $1 \leq j \leq K$. A double induction argument shows that the inequality (2.7) is true for $0 \leq i \leq N$ and $0 \leq j \leq K$. Notice that $\|x_i - w_i\| \leq \|x_0 - z_0\|$ for $0 \leq i \leq N$. By (2.7) there exists $M_\tau > 0$ such that $\limsup_{\lambda \downarrow 0} (\sup \{ \|z^\varepsilon(t) - u^\lambda(s)\|; t, s \in [0, \tau], |t - s| \leq h \})$ is bounded by $M_\tau(\|z_0 - x\| + \|u_0 - z_0\| + \rho)$ as $(\varepsilon, h) \rightarrow (0, 0)$. Since $\rho > 0$ and $u_0 \in D(A)$ are arbitrary and since $D(A)$ is dense in $D(\varphi)$ by condition (A2), the proof is completed. $\square$

Let $\lambda > 0$. The step function $u^\lambda : [0, \tau] \rightarrow X$ defined by (2.4) is called a λ -approximate solution to the abstract Cauchy problem for A on $[0, \tau]$ if there exist $x_0^\lambda \in X$, $t_i^\lambda \geq 0$, $x_i^\lambda \in D(A)$ and $f_i^\lambda \in X$ for $1 \leq i \leq N^\lambda$ satisfying conditions $(\lambda 1)$ through $(\lambda 4)$. In our setting, a difference between approximate solutions can not be directly estimated. For this reason, the previous lemma is needed and has been proved by the modified arguments due to [2, 3, 5].

A continuous function $u : [0, \tau] \rightarrow X$ is called a *mild solution* to the abstract Cauchy problem for A on $[0, \tau]$, provided that for each $\lambda > 0$ there exists a λ -approximate solution u^λ to the abstract Cauchy problem for A on $[0, \tau]$ such that $\|u^\lambda(t) - u(t)\| \leq \lambda$ for $t \in [0, \tau]$. The main result of this paper is given by

Theorem 2.3. (i) For each $x \in D$ and $\tau > 0$ there exists a mild solution to $(ACP; A, x)$ on $[0, \tau]$. (ii) For each $\tau > 0$, there exists $M_\tau > 0$ such that

$$\|u(t) - \hat{u}(t)\| \leq M_\tau \|u(0) - \hat{u}(0)\| \quad \text{for } t \in [0, \tau], \quad (2.8)$$

where u and $\hat{u}$ are mild solutions to the abstract Cauchy problem for A on $[0, \tau]$. (iii) There exists a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D such that for each $x \in D$ and $\tau > 0$, $S(t)x$ is a unique mild solution to $(ACP; A, x)$ on $[0, \tau]$.

Proof. To verify (i), let $x \in D$ and $\tau > 0$. Since $D(\varphi)$ is dense in D , there exist $x_0^\lambda \in D(\varphi)$ for $\lambda > 0$ such that $\lim_{\lambda \downarrow 0} x_0^\lambda = x$. Let $\lambda > 0$. By Lemma 2.1 there exist $t_i^\lambda \geq 0$, $x_i^\lambda \in D(A)$ and $\xi_i^\lambda \in Ax_i^\lambda$ for $i = 1, 2, 3, \dots$ such that they satisfy the following conditions:

- (i) $0 = t_0^\lambda < t_1^\lambda < \dots < t_i^\lambda < \dots$ and $\lim_{i \rightarrow \infty} t_i^\lambda = \infty$.
- (ii) $t_i^\lambda - t_{i-1}^\lambda \leq \lambda$ for $i = 1, 2, 3, \dots$.
- (iii) $\|x_i^\lambda - x_{i-1}^\lambda - (t_i^\lambda - t_{i-1}^\lambda)\xi_i^\lambda\| \leq (t_i^\lambda - t_{i-1}^\lambda)\lambda$ for $i = 1, 2, 3, \dots$.

Since $\lim_{i \rightarrow \infty} t_i^\lambda = \infty$, there exists an integer $N^\lambda \geq 1$ such that $t_{N^\lambda-1}^\lambda < \tau \leq t_{N^\lambda}^\lambda$. Let $f_i^\lambda = (x_i^\lambda - x_{i-1}^\lambda)/(t_i^\lambda - t_{i-1}^\lambda) - \xi_i^\lambda$ for $1 \leq i \leq N^\lambda$. Then $(x_i^\lambda - x_{i-1}^\lambda)/(t_i^\lambda - t_{i-1}^\lambda) \in Ax_i^\lambda + f_i^\lambda$ for $1 \leq i \leq N^\lambda$ and $\sum_{i=1}^{N^\lambda} (t_i^\lambda - t_{i-1}^\lambda) \|f_i^\lambda\| \leq (\tau + \lambda)\lambda$. This means that for each $\lambda > 0$ there exists a λ -approximate solution u^λ on $[0, \tau]$. Now, let $z_0 \in D(\varphi)$. By Lemma 2.1, for each $\varepsilon > 0$ there exist $s_k^\varepsilon \geq 0$, $z_k^\varepsilon \in D(A)$ and $\eta_k^\varepsilon \in Az_k^\varepsilon$ for $k = 1, 2, \dots, K^\varepsilon$ satisfying conditions (i) through (iv) of Lemma 2.2. Let $z^\varepsilon : [0, \tau] \rightarrow X$ be the step function defined by (2.3). Then we write $u^\lambda(t) - u^\mu(t) =$

$(u^\lambda(t) - z^\varepsilon(t)) + (z^\varepsilon(t) - u^\mu(t))$ for $t \in [0, \tau]$. From Lemma 2.2 we deduce that $\limsup_{\lambda, \mu \downarrow 0} (\sup\{\|u^\lambda(t) - u^\mu(t)\|; t \in [0, \tau]\}) \leq 2M_\tau\|z_0 - x\|$. Since $D(\varphi)$ is dense in D , this inequality implies the existence of a measurable function $u : [0, \tau] \rightarrow X$ such that $\lim_{\lambda \downarrow 0} u^\lambda(t) = u(t)$ uniformly for $t \in [0, \tau]$. The continuity of u on $[0, \tau]$ in X follows by an application of Lemma 2.2. Therefore, the proof of assertion (i) is completed.

To verify (ii), let u^λ and $\hat{u}^\lambda$ be λ -approximate solutions to the abstract Cauchy problem for A on $[0, \tau]$ such that $\lim_{\lambda \downarrow 0} u^\lambda(0) = x$ and $\lim_{\lambda \downarrow 0} \hat{u}^\lambda(0) = \hat{x}$. Let $z_0 \in D(\varphi)$ and let $z^\varepsilon : [0, \tau] \rightarrow X$ be the step function defined by (2.3). The existence of the family $\{z^\varepsilon; \varepsilon > 0\}$ has been already proved in the preceding paragraph. Then, a passage to the limit in the inequality $\|u^\lambda(t) - \hat{u}^\lambda(t)\| \leq \|u^\lambda(t) - z^\varepsilon(t)\| + \|\hat{u}^\lambda(t) - z^\varepsilon(t)\|$ implies that $\|u(t) - \hat{u}(t)\| \leq M_\tau(\|x - z_0\| + \|\hat{x} - z_0\|)$ for $t \in [0, \tau]$, by Lemma 2.2. The desired inequality (2.8) is obtained by letting $z_0 \rightarrow x$. Assertion (iii) is a direct consequence of (i) and (ii). $\square$

3. Application to Navier-Stokes equations

This section is devoted to a Lipschitz semigroup approach to the initial-boundary value problem for the Navier-Stokes equation

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, & \operatorname{div} u = 0 & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (3.1)$$

where the unknown function $u = u(x, t)$ is a vector-valued function from $\Omega \times (0, \tau)$ into $\mathbb{R}^2$, Ω is a domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\nu > 0$. Let X and V be the closures of the set $\{u \in C_0^\infty(\Omega)^2; \operatorname{div} u = 0\}$ in $L^2(\Omega)^2$ and in $H_0^1(\Omega)^2$ respectively. An inner product of X and a bilinear form on $V \times V$ are defined by

$$\langle u, v \rangle = \sum_{i=1}^2 \int_{\Omega} u_i v_i dx \quad \text{and} \quad \langle\langle u, v \rangle\rangle = \sum_{i=1}^2 \langle \nabla u_i, \nabla v_i \rangle,$$

the associated norm of X is denoted by $|\cdot|$ and let $\|u\| = \langle\langle u, u \rangle\rangle^{1/2}$ for $u \in V$. Let b be the trilinear form on $V \times V \times V$ defined by

$$b(u, v, w) = \langle (u \cdot \nabla)v, w \rangle \quad \text{for } u, v, w \in V.$$

It is known ([11]) that

$$b(u, v, w) = -b(u, w, v) \quad \text{for } u, v, w \in V, \quad (3.2)$$

$$|b(u, v, w)| \leq \sqrt{2}|u|^{1/2}||u||^{1/2}||v|||w|^{1/2}||w||^{1/2} \quad \text{for } u, v, w \in V. \quad (3.3)$$

Notice that the function $t \rightarrow b(u(t), u(t), \phi) \in L^1(0, \tau)$ if $u \in L^2(0, \tau; V)$ and $\phi \in V$.

Proposition 3.1. *Let $u \in C([0, \tau]; X) \cap L^2(0, \tau; V)$ and $u(0) = u_0$. Then the function u satisfies the identity*

$$\langle u(t), \phi \rangle - \langle u_0, \phi \rangle + \int_0^t \{ \nu \langle \langle u(s), \phi \rangle \rangle + b(u(s), u(s), \phi) \} ds = 0 \quad (3.4)$$

for any $\phi \in V$ and $t \in [0, \tau]$, if it satisfies the inequality

$$\begin{aligned} \int_0^\tau (|u(t) - \phi|^2 \psi'(t) - 2\{ \nu \langle \langle u(t), u(t) - \phi \rangle \rangle \\ + b(u(t), u(t), u(t) - \phi) \} \psi(t)) dt \geq 0 \end{aligned} \quad (3.5)$$

for any $\phi \in V$ and $\psi \in C_0^\infty(0, \tau)$ with $\psi \geq 0$.

Proof. Let $u \in C([0, \tau]; X) \cap L^2(0, \tau; V)$ and $u(0) = u_0$. Assume that the function u satisfies the inequality (3.5). Let $\kappa > 0$, $\phi_0 \in V$ and let $\psi \in C_0^\infty(0, \tau)$ and $\psi \geq 0$. Setting $\phi = \kappa \phi_0$ in (3.5), dividing the resulting inequality by κ and letting $\kappa \rightarrow \infty$, we use $\int_0^\tau \psi'(t) dt = 0$ to find that

$$\int_0^\tau \left(\langle u(t), \phi_0 \rangle + \int_0^t \{ \nu \langle \langle u(s), \phi_0 \rangle \rangle + b(u(s), u(s), \phi_0) \} ds \right) \psi'(t) dt \leq 0.$$

Since $\phi_0 \in V$ is arbitrary, this inequality implies that u satisfies (3.4). $\square$

Considering Proposition 3.1 we call a *weak solution* to the Navier-Stokes equation (3.1) for a function $u \in C([0, \tau]; X) \cap L^2(0, \tau; V)$ satisfying $u(0) = u_0$ and the inequality (3.5).

Proposition 3.2. *Let u and $\hat{u}$ be weak solutions to the Navier-Stokes equation (3.1) such that $u(0) = u_0$ and $\hat{u}(0) = \hat{u}_0$. Then $|u(t) - \hat{u}(t)|^2 \leq \exp((1/2\nu^2)|u_0|^2)|u_0 - \hat{u}_0|^2$ for $t \in [0, \tau]$.*

Proof. Let u and $\hat{u}$ be weak solutions to the Navier-Stokes equation (3.1) such that $u(0) = u_0$ and $\hat{u}(0) = \hat{u}_0$. Let $f \in C_0^\infty(0, \tau)$ and $f \geq 0$. For $\varepsilon > 0$ let $\psi(t, s) = f((t+s)/2)\rho_\varepsilon(t-s)$, where ρ_ε is the mollifier. Integrating the inequality (3.5) with $\phi = \hat{u}(s)$ over $(0, \tau)$ with respect to s we have

$$\int_0^\tau \left(\int_0^\tau (|u(t) - \hat{u}(s)|^2 \psi_t(t, s) - 2\{\nu \langle u(t), u(t) - \hat{u}(s) \rangle + b(u(t), u(t), u(t) - \hat{u}(s))\} \psi(t, s)) dt \right) ds \geq 0. \quad (3.6)$$

Since $\hat{u}$ is a weak solution to the Navier-Stokes equation (3.1), we have

$$\int_0^\tau (|\hat{u}(s) - u(t)|^2 \psi_s(t, s) - 2\{\nu \langle \hat{u}(s), \hat{u}(s) - u(t) \rangle + b(\hat{u}(s), \hat{u}(s), \hat{u}(s) - u(t))\} \psi(t, s)) ds \geq 0$$

for almost all $t \in (0, \tau)$. Integrating this inequality over $(0, \tau)$ with respect to t , adding the resulting inequality with (3.6) and letting $\varepsilon \downarrow 0$ we find that

$$\int_0^\tau (|u(t) - \hat{u}(t)|^2 f'(t) - 2\{\nu \|u(t) - \hat{u}(t)\|^2 + b(u(t), u(t), u(t) - \hat{u}(t)) - b(\hat{u}(t), \hat{u}(t), u(t) - \hat{u}(t))\} f(t)) dt \geq 0;$$

hence

$$|u(t) - \hat{u}(t)|^2 - |u_0 - \hat{u}_0|^2 + 2 \int_0^t \{\nu \|u(s) - \hat{u}(s)\|^2 + b(u(s), u(s), u(s) - \hat{u}(s)) - b(\hat{u}(s), \hat{u}(s), u(s) - \hat{u}(s))\} ds \leq 0$$

for $t \in [0, \tau]$. By (3.2) we have

$$b(w, z, z - \tilde{z}) - b(\tilde{w}, \tilde{z}, z - \tilde{z}) = b(w - \tilde{w}, z, z - \tilde{z}) \quad (3.7)$$

for $w, \tilde{w}, z, \tilde{z} \in V$. By (3.7) and (3.3) we have $|b(u(s), u(s), u(s) - \hat{u}(s)) - b(\hat{u}(s), \hat{u}(s), u(s) - \hat{u}(s))| \leq \nu \|u(s) - \hat{u}(s)\|^2 + (1/2\nu) \|u(s)\|^2 |u(s) - \hat{u}(s)|^2$ for $s \in (0, \tau)$. It follows that

$$|u(t) - \hat{u}(t)|^2 \leq |u_0 - \hat{u}_0|^2 + (1/\nu) \int_0^t \|u(s)\|^2 |u(s) - \hat{u}(s)|^2 ds \quad (3.8)$$

for $t \in [0, \tau]$. Setting $\phi = 0$ in (3.5) yields that $2\nu \int_0^\tau \|u(s)\|^2 ds \leq |u_0|^2$. Therefore, solving the inequality (3.8) we obtain the desired inequality. $\square$

Lemma 3.3. *Let $u_0 \in V$. Then there exists $\lambda_0 > 0$ such that to each $\lambda \in (0, \lambda_0]$ there corresponds a unique $u_\lambda \in V$ satisfying the equation*

$$\langle (u_\lambda - u_0)/\lambda, \phi \rangle + \nu \langle \langle u_\lambda, \phi \rangle \rangle + b(u_\lambda, u_\lambda, \phi) = 0 \quad \text{for } \phi \in V. \quad (3.9)$$

Moreover, the following assertions hold:

- (i) $|u_\lambda|^2 + 2\lambda\nu||u_\lambda||^2 \leq |u_0|^2$ for $\lambda \in (0, \lambda_0]$.
- (ii) $|u_\lambda - u_0|^2 + \lambda\nu||u_\lambda||^2 \rightarrow 0$ as $\lambda \downarrow 0$.

Proof. Let $u_0 \in V$ and choose $\lambda_0 > 0$ such that

$$k := (\lambda_0/\nu)||u_0||^2 + (2\lambda_0/\nu^7)^{1/2}|u_0|^3||u_0|| < 1.$$

Let $\lambda \in (0, \lambda_0]$ and define a closed subset E of the Banach space V equipped with the norm $\|v\|_V = (|v|^2 + \lambda\nu||v||^2)^{1/2}$ for $v \in V$ by

$$E = \{u \in V; |u|^2 + 2\lambda\nu||u||^2 \leq |u_0|^2\}.$$

Let $u \in E$. Then, the bilinear functional $a : V \times V \rightarrow \mathbb{R}$, defined by $a(v, \phi) = \langle v, \phi \rangle + \lambda\nu \langle \langle v, \phi \rangle \rangle + \lambda b(u, v, \phi)$ for $v, \phi \in V$, is bounded and coercive. Therefore, by the Lax-Milgram theorem we find a unique $v \in V$ such that

$$\langle v - u_0, \phi \rangle + \lambda\nu \langle \langle v, \phi \rangle \rangle + \lambda b(u, v, \phi) = 0 \quad \text{for } \phi \in V. \quad (3.10)$$

Now, consider the mapping $\Psi : E \rightarrow V$ defined in such a way that $\Psi u = v$ if and only if $u \in E$, $v \in V$ and they satisfy the equation (3.10). To show that $\Psi(E) \subset E$, let $u \in E$ and $v = \Psi u$. By (3.10) with $\phi = v$, we have $|v|^2 - |u_0|^2 + 2\lambda\nu||v||^2 \leq 2\langle v - u_0, v \rangle + 2\lambda\nu||v||^2 = 0$. This implies that $v \in E$. To demonstrate that Ψ is strictly contractive, let $\Psi u = v$ and $\Psi \hat{u} = \hat{v}$. By (3.10) and (3.7) we find that

$$|v - \hat{v}|^2 + \lambda\nu||v - \hat{v}||^2 = \lambda b(u - \hat{u}, v, \hat{v} - v).$$

By (3.3) the right-hand side is estimated as follows:

$$\begin{aligned} & \lambda b(u - \hat{u}, v, \hat{v} - v) \\ & \leq |v - \hat{v}|(\lambda\nu)^{1/2}||v - \hat{v}|| + (\lambda/2\nu)||v||^2|u - \hat{u}|(\lambda\nu)^{1/2}||u - \hat{u}|| \\ & \leq (1/2)||v - \hat{v}||_V^2 + (\lambda/4\nu)||v||^2||u - \hat{u}||_V^2. \end{aligned}$$

Hence

$$||v - \hat{v}||_V^2 \leq (\lambda/2\nu)||v||^2||u - \hat{u}||_V^2. \quad (3.11)$$

Applying (3.3) to the equation (3.10) with $\phi = v - u_0$, we have

$$\begin{aligned} & 2|v - u_0|^2 + \lambda\nu(\|v\|^2 - \|u_0\|^2) \\ & \leq (\lambda\nu/2)\|v\|^2 + 4|u|(2\lambda\nu)^{1/2}\|u\| \cdot (\lambda/2\nu)^{1/2}(1/\nu)|u_0|\|u_0\| \\ & \leq (\lambda\nu/2)\|v\|^2 + (\|u\|^2 + 2\lambda\nu\|u\|^2)(2\lambda/\nu)^{1/2}(1/\nu)|u_0|\|u_0\|. \end{aligned}$$

Since $u \in E$ it follows that

$$2|v - u_0|^2 + (\lambda\nu/2)\|v\|^2 \leq \lambda\nu\|u_0\|^2 + (2\lambda/\nu)^{1/2}(1/\nu)|u_0|^3\|u_0\|. \quad (3.12)$$

Combining this inequality and (3.11) we have $\|v - \hat{v}\|_V^2 \leq k\|u - \hat{u}\|_V^2$. This means that the mapping Ψ from E into itself is strictly contractive. By the Banach fixed point theorem, the mapping Ψ has a fixed point u_λ in E . Since $\Psi u_\lambda = u_\lambda$, we have $\lim_{\lambda \downarrow 0}(|u_\lambda - u_0|^2 + \lambda\nu\|u_\lambda\|^2) = 0$ by (3.12). $\square$

Let $\alpha > 0$ and $D = \{u \in X; |u|^2 \leq \alpha\}$. Then, we consider the operator A in X defined in such a way that $Av = \xi$ if and only if $v \in D \cap V$, $\xi \in X$ and they satisfy the equation

$$\langle \xi, \phi \rangle + \nu \langle \langle v, \phi \rangle \rangle + b(v, v, \phi) = 0 \quad \text{for } \phi \in V. \quad (3.13)$$

The domain $D(A)$ is characterized by the set of all elements $v \in D \cap V$ such that there exists $\xi \in X$ satisfying the equation (3.13). Let $\varphi : X \rightarrow [0, \infty]$ be the proper functional defined by

$$\varphi(v) = \begin{cases} \exp(\beta|v|^2)|Av|^2 & \text{for } v \in D(A), \\ \infty & \text{for } v \in X \setminus D(A), \end{cases}$$

where $\beta > 0$ is the constant determined by $2\beta\nu^2 = 1$. Since $D(\varphi) = D(A) \subset D \cap V \subset \overline{D(A)}$ (by Lemma 3.3) and $\overline{D} \cap \overline{V} = D$, condition (φ) is satisfied.

Lemma 3.4. (i) *The functional φ is lower semicontinuous on V .* (ii) *Let $u_0 \in D(\varphi)$. For sufficiently small $\lambda > 0$ let u_λ be the element in V satisfying (3.9). Then, the family $\{\|u_\lambda\|^2\}$ is bounded as $\lambda \downarrow 0$ and*

$$\limsup_{\lambda \downarrow 0} (\varphi(u_\lambda) - \varphi(u_0)) / \lambda \leq 0.$$

Proof. To verify (i), let $\varphi(v_n) \leq r$ for $n \geq 1$ and $v_n \rightarrow v$ in X as $n \rightarrow \infty$. Then, by the definition of A , we have $v_n \in D \cap V$, $\exp(\beta|v_n|^2)|Av_n|^2 \leq r$ and

$$\langle Av_n, \phi \rangle + \nu \langle \langle v_n, \phi \rangle \rangle + b(v_n, v_n, \phi) = 0 \quad \text{for } \phi \in V \quad (3.14)$$

for $n \geq 1$. By (3.14) with $\phi = v_n$ we have $\nu||v_n||^2 = -\langle Av_n, v_n \rangle \leq |Av_n||v_n|$ for $n \geq 1$. Since $\{v_n\}$ and $\{Av_n\}$ are bounded in X , this implies that the sequence $\{v_n\}$ is bounded in V as $n \rightarrow \infty$. Therefore, we have $v \in V$ and there exist $\xi \in X$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that v_{n_k} converges weakly to v in V and Av_{n_k} converges weakly to ξ in X as $k \rightarrow \infty$. Here we have used the fact that $v_n \rightarrow v$ in X as $n \rightarrow \infty$. Since $b(v_n, v_n, \phi) - b(v, v, \phi) = -b(v_n, \phi, v_n) + b(v, \phi, v) = b(v - v_n, \phi, v) + b(v_n, \phi, v - v_n)$ for $\phi \in V$ and $n \geq 1$, we deduce from (3.3) that $\lim_{n \rightarrow \infty} b(v_n, v_n, \phi) = b(v, v, \phi)$ for any $\phi \in V$. A passage to the limit in (3.14) with $n = n_k$ as $k \rightarrow \infty$ implies that $v \in D(A)$ and $Av = \xi$. Since the sequence $\{Av_{n_k}\}$ converges weakly to Av in X and the sequence $\{v_{n_k}\}$ converges to v in X as $k \rightarrow \infty$, the sequence $\{\exp(\beta|v_{n_k}|^2)Av_{n_k}\}$ converges weakly to $\exp(\beta|v|^2)Av$ in X as $k \rightarrow \infty$. This implies that $\varphi(v) \leq \liminf_{k \rightarrow \infty} \exp(\beta|v_{n_k}|^2)|Av_{n_k}|^2 \leq r$. Assertion (i) is thus proved.

To verify (ii), let $u_0 \in D(\varphi)$ ($= D(A)$). Let $\xi_0 = Au_0$ and $\xi_\lambda = Au_\lambda$. Then we have $\xi_\lambda = (u_\lambda - u_0)/\lambda$,

$$\langle \xi_0, \phi \rangle + \nu \langle \langle u_0, \phi \rangle \rangle + b(u_0, u_0, \phi) = 0 \quad \text{for } \phi \in V \quad (3.15)$$

and

$$\langle \xi_\lambda, \phi \rangle + \nu \langle \langle u_\lambda, \phi \rangle \rangle + b(u_\lambda, u_\lambda, \phi) = 0 \quad \text{for } \phi \in V. \quad (3.16)$$

Subtracting (3.15) from (3.16) we have

$$\langle (\xi_\lambda - \xi_0)/\lambda, \phi \rangle + \nu \langle \langle \xi_\lambda, \phi \rangle \rangle + b(\xi_\lambda, u_\lambda, \phi) + b(u_0, \xi_\lambda, \phi) = 0$$

for $\phi \in V$. Setting $\phi = \xi_\lambda$ we deduce from (3.3) that

$$(|\xi_\lambda|^2 - |\xi_0|^2)/\lambda \leq (1/\nu)||u_\lambda||^2|\xi_\lambda|^2. \quad (3.17)$$

By the convexity of the function $t \rightarrow \exp(\beta t)$ we have

$$\exp(\beta t) - \exp(\beta s) \leq \beta \exp(\beta t)(t - s) \quad \text{for } t, s \geq 0. \quad (3.18)$$

Since $2\beta\nu^2 = 1$, the use of (3.18), assertion (i) of Lemma 3.3 and (3.17) yield that

$$\begin{aligned} & (\varphi(u_\lambda) - \varphi(u_0))/\lambda \\ & \leq \beta \exp(\beta|u_\lambda|^2)\lambda^{-1}(|u_\lambda|^2 - |u_0|^2)|\xi_\lambda|^2 + \exp(\beta|u_0|^2)\lambda^{-1}(|\xi_\lambda|^2 - |\xi_0|^2) \\ & \leq 2\beta\nu||u_\lambda||^2|\xi_\lambda|^2(\exp(\beta|u_0|^2) - \exp(\beta|u_\lambda|^2)). \end{aligned}$$

From (ii) of Lemma 3.3 we deduce that $u_\lambda \rightarrow u_0$ in X and $\lambda||u_\lambda||^2 \rightarrow 0$ as $\lambda \downarrow 0$. By (3.17) we have $|\xi_\lambda|^2 \leq (1 - (\lambda/\nu)||u_\lambda||^2)^{-1}|\xi_0|^2 \rightarrow |\xi_0|^2$ as $\lambda \downarrow 0$. By (3.16) with $\phi = u_\lambda$ we have $\nu||u_\lambda||^2 \leq |\xi_\lambda||u_\lambda|$. This implies that the family $\{||u_\lambda||^2\}$ is bounded as $\lambda \downarrow 0$. Thus, it follows that $\limsup_{\lambda \downarrow 0}(\varphi(u_\lambda) - \varphi(u_0))/\lambda \leq 0$. $\square$

Let $\Phi(u, \hat{u}) = c \exp(\gamma|u|^2)|u - \hat{u}|$ for $(u, \hat{u}) \in D \times D$, where $c > 0$ is a constant satisfying $c \exp(\gamma\alpha) \leq 1$ and $\gamma > 0$ is the constant determined by $4\gamma\nu^2 = 1$. Then, the functional Φ is clearly nonnegative and Lipschitz continuous on $D \times D$.

Lemma 3.5. *Let $u_0 \in D(\varphi)$. For sufficiently small $\lambda > 0$ let u_λ be the element in V satisfying (3.9). Then*

$$\limsup_{\lambda \downarrow 0} (\sup \{(\Phi(u_\lambda, v) - \Phi(u_0, v))/\lambda + (\Phi(u_\lambda, v) - \Phi(u_\lambda, v_0))/h - \mu\}) \leq 0,$$

where the supremum is taken over all elements $v_0 \in D(\varphi)$, $v \in D(A)$, $\mu > 0$ and $h \in (0, \mu]$ satisfying $|v - v_0 - hAv| \leq h\mu$.

Proof. Let $v_0 \in D(\varphi)$, $v \in D(A)$, $\mu > 0$ and $h \in (0, \mu]$ satisfy $|v - v_0 - hAv| \leq h\mu$. Let $f = (v - v_0)/h - Av$. Then we have $|f| \leq \mu$ and

$$\langle (v - v_0)/h - f, \phi \rangle + \nu \langle v, \phi \rangle + b(v, v, \phi) = 0 \quad \text{for } \phi \in V.$$

Subtracting this equation from (3.9) and setting $\phi = u_\lambda - v$ we find that

$$\begin{aligned} & |u_\lambda - v| \{ (|u_\lambda - v| - |u_0 - v|)/\lambda + (|u_\lambda - v| - |u_\lambda - v_0|)/h \} + \nu||u_\lambda - v||^2 \\ & \leq |f||u_\lambda - v| + |b(u_\lambda - v, u_\lambda, u_\lambda - v)|. \end{aligned}$$

Here we have used that $|w|(|w| - |z|) \leq \langle w, w - z \rangle$ for $w, z \in X$. By (3.3) the second term on the right-hand side is bounded by $\sqrt{2\nu}||u_\lambda - v|| \cdot (1/\sqrt{\nu})||u_\lambda|||u_\lambda - v|$. An application of the inequality that $ab \leq (a^2 + b^2)/2$ for $a, b \geq 0$ yields that

$$(|u_\lambda - v| - |u_0 - v|)/\lambda + (|u_\lambda - v| - |u_\lambda - v_0|)/h \leq (1/2\nu)||u_\lambda||^2|u_\lambda - v| + \mu.$$

Multiplying this inequality by $c \exp(\gamma|u_\lambda|^2)$ we have, by (3.18) with $\beta = \gamma$,

$$\begin{aligned} & (\Phi(u_\lambda, v) - \Phi(u_0, v))/\lambda + (\Phi(u_\lambda, v) - \Phi(u_\lambda, v_0))/h \\ & \leq c \exp(\gamma|u_\lambda|^2) \{ \gamma|u_0 - v|(|u_\lambda|^2 - |u_0|^2)/\lambda + (1/2\nu)|u_\lambda - v||u_\lambda||^2 + \mu \} \\ & \leq c \exp(\gamma\alpha)(2\nu\gamma||u_\lambda||^2|u_\lambda - u_0| + \mu), \end{aligned}$$

where we have used the fact that $4\gamma\nu^2 = 1$ and assertion (i) of Lemma 3.3 to obtain the last inequality. Since $c \exp(\gamma\alpha) \leq 1$ and since $\|u_\lambda\|^2$ is bounded (by Lemma 3.4) and $u_\lambda \rightarrow u_0$ in X as $\lambda \downarrow 0$, the lemma is shown to be true. $\square$

From Lemmas 3.3 through 3.5 we deduce that the operator A satisfies condition (A2). Therefore, Theorem 2.3 asserts that there exists a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D such that for each $u_0 \in D$ and $\tau > 0$, $u(t) := S(t)u_0$ is a unique mild solution to $(\text{ACP}; A, u_0)$ on $[0, \tau]$. Let $\tau > 0$ and $u_0 \in D$. Then there exists a family $\{u^\lambda\}$ of λ -approximate solutions such that $|u^\lambda(t) - u(t)| \leq \lambda$ for $t \in [0, \tau]$. For each $\lambda > 0$ there exist $x_0^\lambda \in X$, $t_i^\lambda \geq 0$, $x_i^\lambda \in D(A)$ and $f_i^\lambda \in X$ for $1 \leq i \leq N^\lambda$ satisfying conditions $(\lambda 1)$ through $(\lambda 4)$ such that u^λ is defined by (2.4). Since $(x_i^\lambda - x_{i-1}^\lambda)/(t_i^\lambda - t_{i-1}^\lambda) - f_i^\lambda = Ax_i^\lambda$, by the definition of A we have

$$\begin{aligned} & (|x_i^\lambda - \phi|^2 - |x_{i-1}^\lambda - \phi|^2)/(t_i^\lambda - t_{i-1}^\lambda) \\ & + 2\{\nu\langle x_i^\lambda, x_i^\lambda - \phi \rangle + b(x_i^\lambda, x_i^\lambda, x_i^\lambda - \phi)\} \leq 2|f_i^\lambda||x_i^\lambda - \phi| \end{aligned} \quad (3.19)$$

for any $\phi \in V$. Multiplying the inequality (3.19) by $t_i^\lambda - t_{i-1}^\lambda$ and adding the resulting inequality from $i = p + 1$ to $i = q$ we have

$$\begin{aligned} & |u^\lambda(t_q^\lambda) - \phi|^2 - |u^\lambda(t_p^\lambda) - \phi|^2 \\ & + 2 \int_{t_p^\lambda}^{t_q^\lambda} \{\nu\langle u^\lambda(\sigma), u^\lambda(\sigma) - \phi \rangle + b(u^\lambda(\sigma), u^\lambda(\sigma), u^\lambda(\sigma) - \phi)\} d\sigma \\ & \leq 2 \sup\{|u^\lambda(t) - \phi|; t \in [0, \tau]\} \sum_{i=1}^{N^\lambda} |f_i^\lambda|(t_i^\lambda - t_{i-1}^\lambda) \end{aligned} \quad (3.20)$$

for $0 \leq p \leq q \leq N^\lambda$. From (3.20) with $\phi = 0$ we deduce that the family $\{u^\lambda\}$ is bounded in $L^2(0, \tau; V)$ as $\lambda \downarrow 0$. This implies that $u \in L^2(0, \tau; V)$ and there exists a sequence $\{u^{\lambda_n}\}$ in $L^2(0, \tau; V)$ converging weakly to u as $n \rightarrow \infty$. Since $\{u^\lambda(t)\}$ converges in X to $u(t)$ uniformly for $t \in [0, \tau]$ as $\lambda \downarrow 0$, a passage to the limit in the inequality (3.20) with $\lambda = \lambda_n$ shows that the function

$$t \rightarrow |u(t) - \phi|^2 + 2 \int_0^t \{\nu\langle u(\sigma), u(\sigma) - \phi \rangle + b(u(\sigma), u(\sigma), u(\sigma) - \phi)\} d\sigma$$

is nonincreasing. This means that u is a weak solution to the Navier-Stokes equation (3.1). Thus, the following result is proved.

Theorem 3.6. *For each $u_0 \in X$ there exists a weak solution to the Navier-Stokes equation (3.1).*

Remark 3.7. A quasi-contractive semigroup approach to Navier-Stokes equations was discussed in [6]. Their result is different from ours in that a unique global mild solution was obtained for any initial data u_0 belonging only on the space V . In [12] the contractive semigroup theory was applied to the truncated Navier-Stokes operators to construct the solutions.

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