<table>
<thead>
<tr>
<th>著者</th>
<th>Miyazaki Rinko, Naito Toshiki, Shin Jong Son</th>
</tr>
</thead>
<tbody>
<tr>
<td>出版者</td>
<td>SIAM Journal on Mathematical Analysis</td>
</tr>
<tr>
<td>権利</td>
<td>Society for Industrial and Applied Mathematics</td>
</tr>
</tbody>
</table>

doi: 10.1137/090779450
DELYED FEEDBACK CONTROL BY COMMUTATIVE GAIN MATRICES∗

RINKO MIYAZAKI†, TOSHIKI NAITO‡, AND JONG SON SHIN‡

Abstract. The delayed feedback control (DFC) is a control method for stabilizing unstable periodic orbits in nonlinear autonomous differential equations. We give an important relationship between the characteristic multipliers of the linear variational equation around an unstable periodic solution of the equation and those of its delayed feedback equation. The key of our proof is a result about the spectrum of a matrix which is a difference of commutative matrices. The relationship, moreover, allows us to design control gains of the DFC such that the unstable periodic solution is stabilized. In other words, the validity of the DFC is proved mathematically. As an application for the Rössler equation, we determine the best range of $k$ such that the unstable periodic orbit is stabilized by taking a feedback gain $K = kE$.

Key words. delayed feedback control, stability, delay differential equations, Floquet theory

AMS subject classifications. 34K13, 34K20, 34K35, 93C23, 93D15

DOI. 10.1137/090779450

1. Introduction. We consider a nonlinear autonomous differential equation

\[(E) \quad x'(t) = f(x(t)), \]

where $\Omega$ is a domain in the $n$-dimensional real Euclidean space $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ is a continuously differentiable function. Assume that (E) has an unstable periodic orbit $\gamma(\phi(t))$ of solution $\phi(t)$ with period $\omega > 0$.

The delayed feedback control (DFC) is proposed by Pyragas [11] as a method of stabilization of the unstable periodic orbit $\gamma(\phi(t))$ embedded within a chaotic attractor. This method is for stabilizing the unstable periodic orbit $\gamma(\phi(t))$ by using the feedback control $u(t)$ of the form of the difference between the delayed state and the current state, that is, $u(t) = K(y(t - \omega) - y(t))$. Here an $n \times n$ real constant matrix $K$ is the so-called feedback gain. In other words, we intend to find a feedback gain $K$ such that the unstable periodic orbit $\gamma(\phi(t))$ becomes stable in the following delay differential equation:

\[(DF) \quad y'(t) = f(y(t)) + K(y(t - \omega) - y(t)). \]

Notice that the periodic solution $\phi(t)$ of (E) is also a periodic solution of (DF). In general, it is difficult to obtain the unstable periodic solution $\phi(t)$ analytically.

The DFC does not require the concrete form of periodic solution $\phi(t)$; it is possible if the period $\omega$ of the solution is given. This is one of the merits of the DFC. Because of this convenience, by numerical simulations, the DFC has been applied to controlling chaos in a wide variety of systems, for example, laser systems [2], magnetoelastic...
systems [4], and walking robot systems [13] (refer to the survey paper [12]). On the other hand, there are a few attempts to give mathematical analyses of the DFC [5, 6, 8, 9, 10]. Roughly speaking, mathematical analyses of the DFC are reduced to calculations of the characteristic (Floquet) multipliers (or exponents) of the linear variational (linearization) equations

\[
\begin{align*}
  x'(t) &= A(t)x(t), \\
  y'(t) &= A(t)y(t) + K(y(t - \omega) - y(t)),
\end{align*}
\]

of (E) and (DF), respectively. Here \( A(t) = Df(\phi(t)) \) is the Jacobian on \( f(y) \), which is a continuous and \( \omega \)-periodic function. However, it seems that a perfect theory of the DFC has not been established yet mathematically. In building a theory, there are two difficulties:

(i) to give a characteristic equation, that is, an equation which the characteristic multipliers of (1.2) satisfy;

(ii) to determine exactly a feedback gain \( K \) such that the unstable periodic orbit \( \gamma(\phi(t)) \) is stabilized; even if \( K = kE \) (\( E \) : unit matrix), we can find that the range of \( k \) will be very narrow by some numerical simulations for the Rössler equation.

The aim of our work is to give a theory of the DFC which is mathematically perfect. To achieve this aim we assume in this paper that \( A(t) \) and \( K \) are commutative. Then we can obtain a relationship between the characteristic multipliers of (1.1) and (1.2) by using a result about the spectrum of a matrix being a difference of commutative matrices. This means that we can overcome difficulty (i) restrictively. The relationship, moreover, allows us to determine the best range of \( k \) such that the unstable periodic solution is stabilized by taking the feedback gain \( K = kE \), that is, difficulty (ii) is overcome. In other words, the validity of the DFC is proved mathematically.

This paper consists of three parts after summarizing some basic notation and general results on linear periodic differential equations and differential difference equations in section 2.

In the first part (sections 3–6), we intend to overcome difficulty (i). The main results of this part are Theorems 5.2 and 6.2. Theorem 5.2 gives a relationship between the characteristic multipliers of (1.1) and (1.2) under the commutative assumption. Theorem 6.2 gives a sufficient condition under which the nondegenerate property is inheritable from (1.1) to (1.2). Here the nondegenerate property will be necessary to discuss the stability of periodic orbits of nonlinear equations.

In the second part (section 7), we give an answer to difficulty (ii). In fact, Theorem 7.6(ii) gives the range of \( k \) such that the unstable periodic orbit \( \gamma(\phi(t)) \) will be stabilized with a feedback gain \( K = kE \). On the other hand, Theorem 7.5(i) says that if (1.1) has the characteristic multiplier greater than 1, the stabilization with the gain \( K = kE \) is never achieved.

In the last part (section 8), we illustrate our analytical results with the Rössler equation. Theorem 8.1 gives the conditions under which we win success or not in the DEC with the gain \( K = kE \). We also discuss these results numerically. From the bifurcation diagram (Figure 2), we can find that the stability range of \( k \) obtained from Theorem 8.1 might be a necessary condition.

2. Preliminaries. We state some well-known results on linear periodic differential equations and linear periodic differential difference equations. Throughout this
numbers and \( \mathbb{C} \) (2.1) solution of (1.1) period \( 0 < \omega \), and \( K \) is an \( n \times n \) complex matrix.

First, we define some notation and terminologies. Let \( \mathbb{C} \) be the set of all complex numbers and \( \mathbb{C}^n \) the \( n \)-dimensional complex Euclidean space. Let \( X \) be a Banach space and \( T : X \to X \) a bounded linear operator. Denote the null space and the range of the operator \( T : X \to X \) by \( N(T) = \{ x \in X \mid Tx = 0 \} \) and \( \mathcal{R}(T) = \{ Tx \mid x \in X \} \), respectively. We denote by \( \sigma(T) \) and \( P_\sigma(T) \) the spectrum and the point spectrum (the set of eigenvalues) of \( T \), respectively. Corresponding to \( \lambda \in P_\sigma(T) \), the eigenspace \( W_T(\lambda) \) and the generalized eigenspace \( G_T(\lambda) \) are defined as follows:

\[
W_T(\lambda) = N(T - \lambda I), \quad G_T(\lambda) = \bigcup_{i=1}^\infty N((T - \lambda I)^i),
\]

where \( I : X \to X \) is the identity operator. If \( T \) is a compact operator, then the following facts hold true:

1. \( \sigma(T) = P_\sigma(T) \cup \{ 0 \} \).
2. \( P_\sigma(T) \) is an at most countable set of \( \mathbb{C} \) with the only possible accumulation point being zero.
3. For \( \lambda \in P_\sigma(T) \) there is a \( k_0 \geq 1 \) such that

\[
G_T(\lambda) = N((T - \lambda I)^{k_0}), \quad N((T - \lambda I)^{k_0-1}) \neq N((T - \lambda I)^{k_0}),
\]

and hence

\[
X = G_T(\lambda) \oplus \mathcal{R}((T - \lambda I)^{k_0}) \quad \text{and} \quad \dim G_T(\lambda) < \infty.
\]

In particular, if \( X = \mathbb{C}^n \) and \( T = A \) is a matrix, then \( \mathbb{C}^n = \bigoplus_{\lambda \in \sigma(A)} G_A(\lambda) \).

Next, consider the linear periodic differential equation (1.1) with period \( \omega > 0 \). The solution operator of (1.1) is denoted by \( T(t, s) \). Define a periodic operator \( T(t) \), \( t \in \mathbb{R} \), by \( T(t) = T(t + \omega, t) \). Then the relation \( \sigma(T(t)) = \sigma(T(0)) \) holds. The eigenvalue \( \mu \) of \( T(0) \) is called a characteristic multiplier of (1.1). If \( e^{\lambda \omega} \) is a characteristic multiplier of (1.1), then \( \lambda \) is called a characteristic exponent of (1.1). Clearly, \( T(0) \) is a nonsingular matrix, so that all characteristic multipliers of (1.1) are not zero. The following well-known lemmas give the properties of the characteristic multipliers and the characteristic exponents of (1.1).

**Lemma 2.1.** The following statements hold true.

1. \( \lambda \) is a characteristic exponent of (1.1) if and only if there is a nontrivial solution of (1.1) of the form

\[
(2.1) \quad x(t) = e^{\lambda t} p(t), \quad p(t + \omega) = p(t), \quad t \in \mathbb{R}.
\]

2. \( \mu \) is a characteristic multiplier of (1.1) if and only if there is a nontrivial solution of (1.1) of the form

\[
(2.2) \quad x(t + \omega) = \mu x(t), \quad t \in \mathbb{R}.
\]

**Lemma 2.2.** Let \( \mu \in \sigma(T(0)) \) and \( x(t), t \in \mathbb{R} \), be a solution of (1.1) through \( (0, x^0) \in \mathbb{R} \times \mathbb{R}^n \). Then the following statements are equivalent:

1. \( x(t + \omega) = \mu x(t) \) holds for \( t \in \mathbb{R} \).
2. \( x(\omega) = \mu x(0) \).
3. \( x^0 \in W_{T(0)}(\mu) \).
Lemma 2.3. If the characteristic multipliers of (1.1) are $\mu_1, \ldots, \mu_n$, then
\[ \mu_1 \cdots \mu_n = \exp \left( \int_0^\omega \text{tr}A(s)ds \right). \]

Finally, consider the linear periodic differential difference equation (1.2) with period $\omega > 0$. Let $\mathcal{C} := C([-\omega, 0], \mathbb{C}^n)$ be the Banach space of continuous functions mapping the interval $[-\omega, 0]$ into $\mathbb{C}^n$ with the topology of uniform convergence. For an element $\varphi \in \mathcal{C}$, we designate its norm by $|\varphi| = \sup_{t \leq \theta \leq 0} |\varphi(\theta)|$. For a continuous function $x : [\sigma - \omega, \omega] \rightarrow \mathbb{C}^n$ and for any $t \in [\sigma, \infty)$, we let $x_t \in \mathcal{C}$ be defined by $x_t(\theta) = x(t + \theta), -\omega \leq \theta \leq 0$. We denote the solution operator, defined on $\mathcal{C}$, of (1.2) by $U(t, s)$. If $x(t) := x(t, s; \varphi)$ is the solution of (1.2) satisfying $x_s = \varphi$ at $t = s$, then $x_t = U(t, s)\varphi$. The solution operator $U(t, s)$ generates an evolutionary system with period $\omega$, which is given by the following lemma.

Lemma 2.4. The solution operator $U(t, s)$ has the following properties:
\begin{enumerate}
\item $U(t, t) = I$.
\item $U(t, s)U(s, r) = U(t, r)$, $r \leq s \leq t$.
\item $U(t + \omega, s + \omega) = U(t, s)$, $s \leq t$.
\item $U(t + \omega, s) = U(t, s)U(s + \omega, s)$, $s \leq t$.
\end{enumerate}

In particular, if $t \geq \omega$, then $U(t, 0)$ is a compact operator (see [3]). Define a periodic operator $U(t), t \in \mathbb{R}$, by $U(t) := U(t + \omega)$, $t \in \mathcal{C}$. Clearly, $U(t)$ is a compact operator and $P\sigma(U(t)) = P\sigma(U(0))$, $t \in \mathbb{R}$. Any $\nu \in P\sigma(U(0))$ is called a characteristic multiplier of (1.2), and $\lambda$ for which $e^{\lambda \omega} = \nu$ is called a characteristic exponent of (1.2).

Remark 2.5. All the solutions of (1.2) with initial value in $G\omega(0)(\nu)$, $\nu \in P_\sigma(U(0))$, are well defined on $\mathbb{R}$ and are of Floquet type.

Lemma 2.6 (see [3, Lemma 8.1.2]). The following statements hold true:
\begin{enumerate}
\item $\lambda$ is a characteristic exponent of (1.2) if and only if there is a nontrivial solution of (1.2) of the form
\[ x(t) = e^{\lambda p(t)}, \quad p(t + \omega) = p(t), \quad t \in \mathbb{R}. \]
\item $\nu$ is a characteristic multiplier of (1.2) if and only if there is a nontrivial solution of (1.2) of the form
\[ x(t + \omega) = \nu x(t), \quad t \in \mathbb{R}. \]
\end{enumerate}

Lemma 2.7. Let $\nu \in P\sigma(U(0))$ and $x(t)$ be a solution of (1.2) on $J$ through $(0, \varphi) \in \mathbb{R} \times \mathcal{C}$, where $J = [0, \infty)$ or $\mathbb{R}$. Then the following statements are equivalent:
\begin{enumerate}
\item $x(t + \omega) = \nu x(t)$ holds for $t \in J$.
\item $x_{t+\omega} = \nu x_t$ holds for $t \in J$.
\item $\varphi \in W(U(0)(\nu))$.
\end{enumerate}
Proof. We give a proof of the lemma for the case $J = \mathbb{R}$ only. If $x(t)$ is the trivial solution of (1.2), the proof of the lemma is obvious. Assume that $x(t), t \in \mathbb{R}$, is a nontrivial solution of (1.2). The equivalence of (1) and (2) is obvious. Taking $t = 0$ in assertion (2), we have $x_0 = \nu x_0$, i.e., $U(0)\varphi = \nu \varphi$. This is equivalent to assertion (3). If assertion (3) is satisfied, $u(0)\varphi = \nu \varphi$ holds. Multiplying both sides of this relation by $U(t, 0), t \geq 0$, one obtains
\[ U(t + \omega, 0)\varphi = \nu U(t, 0)\varphi, \quad \text{or} \quad x_{t+\omega} = \nu x_t, \]
because $U(t, 0)U(0) = U(t + \omega, \omega)U(0, 0) = U(t + \omega, 0)$. Since $\varphi \in W(U(0)(\nu))$, the relation $x_{t+\omega} = \nu x_t$ holds for $t < 0$ by using Remark 2.5. \qed
3. A reduced feedback equation. We begin with the following observation.

Lemma 3.1. Let $y(t)$ be a function satisfying

\[ y(t + \omega) = \nu y(t), \quad t \in J_{\omega}, \]

where $J_{\omega} = J \cup [-\omega, 0]$ and $J$ is equal to either $[0, \infty)$ or $\mathbb{R}$.

Then $y(t)$ is a solution of (1.2) if and only if it is a solution of the following equation without delay:

\[ y'(t) = A(t)y(t) + \left( \frac{1}{\nu} - 1 \right) Ky(t). \]

Obviously, if $\nu = 1$, (3.2) coincides with (1.1).

Proof. If $y(t)$ is the trivial solution of (1.2) or (3.2), the proof is obvious. Assume that $y(t)$ is a nontrivial solution of (1.2) or (3.2). Then it is obvious that $\nu \neq 0$. Hence it follows from (3.1) that

\[ y(t - \omega) = \nu^{-1} y(t), \quad t \in J. \]

This means that $y(t - \omega) - y(t) = (\nu^{-1} - 1)y(t)$, which proves the lemma.

We call the linear periodic differential equation (3.2) a reduced feedback equation. We investigate further information about the characteristic multiplier $\nu$ of the delayed feedback equation (1.2) in relation to the reduced equation (3.2).

Denote the solution operator and the periodic operator of (3.2) by $V(t, s; \nu^{-1})$ and $V(t; \nu^{-1}) = V(t + \omega, t; \nu^{-1})$, respectively.

The following theorem is easily derived from Lemmas 2.2, 2.7, and 3.1.

Theorem 3.2. $\nu \in P_{\sigma}(U(0))$ if and only if $\nu \in \sigma(V(0; \nu^{-1}))$.

Theorem 3.3. Let $\nu \in P_{\sigma}(U(0))$ and $y: [-\omega, \infty) \to \mathbb{R}^n$. Then $y(t)$ is a solution of (1.2) for $t \geq 0$ such that $\varphi := y_0 \in W_{U(0)}(\nu)$ if and only if $y(t)$ is a solution of (3.2) for $t \geq -\omega$ such that $y(0) \in W_{V(0;\nu^{-1})}(\nu)$.

Proof. Assume that $y(t)$ is a solution of (1.2) for $t \geq 0$ such that $\varphi := y_0 \in W_{U(0)}(\nu)$. By Lemma 2.7, we have that $y(t)$ satisfies

\[ y(t + \omega) = \nu y(t), \quad t \geq 0. \]

Then Lemma 3.1 implies that $y(t)$ is the solution of (3.2) for $t \geq -\omega$. In view of Lemma 2.2 we have that $y(0) = \varphi(0) \in W_{V(0;\nu^{-1})}(\nu)$.

Conversely, assume that $y(t)$ is a solution of (3.2) for $t \geq -\omega$ and that $y(0) \in W_{V(0;\nu^{-1})}(\nu)$. Then it follows from Lemma 2.2 that (3.3) holds, and hence, by Lemma 3.1 we see that $y(t)$ is a solution of (1.2). Lemma 2.7 implies that $y_0 = \varphi \in W_{U(0)}(\nu)$.

Corollary 3.4. Let $\nu \in P_{\sigma}(U(0))$. Then $\varphi \in W_{U(0)}(\nu)$ if and only if $\varphi(0) = V(\theta, 0; \nu^{-1}) \varphi(0) + \varphi(0) \in W_{V(0;\nu^{-1})}(\nu)$.

In the case where $\nu = 1$, $\varphi \in W_{U(0)}(1)$ if and only if $\varphi(0) = T(\theta, 0) \varphi(0), \varphi(0) \in W_{V(0;\nu^{-1})}(\nu), \varphi(0) \in W_{U(0)}(1)$.

Corollary 3.5. Let $\nu \in P_{\sigma}(U(0))$ and $\varphi \in W_{U(0)}(\nu)$. Then $\varphi = 0$ if and only if $\varphi(0) = 0$.

For a $\nu \in P_{\sigma}(U(0))$ we put

\[ W_{U(0)}^0(\nu) = \{ \varphi(0) \mid \varphi \in W_{U(0)}(\nu) \}. \]

Then we have the following corollary.
Corollary 3.6. If $\nu \in P_{\alpha}(U(0))$, then

$$W_{U(0)}^{0}(\nu) = W_{V(0;\nu^{-1})}^{0}(\nu)$$

and

$$\dim W_{U(0)}^{0}(\nu) = \dim W_{V(0;\nu^{-1})}^{0}(\nu).$$

Proof. It is obvious from Corollary 3.4 that $W_{U(0)}^{0}(\nu) = W_{V(0;\nu^{-1})}^{0}(\nu)$. Define a linear mapping $L : W_{U(0)}^{0}(\nu) \ni \phi \rightarrow \phi(0) \in W_{U(0)}^{0}(\nu)$. Corollary 3.5 implies that $\dim N(L) = 0$. Using the dimension theorem in linear algebra, we have that $\dim W_{U(0)}^{0}(\nu) = \dim W_{U(0)}^{0}(\nu)$. This proves the corollary.

4. A representation of solutions for the reduced equation. We give a representation of solution for (3.2). In general, it is difficult to analyze the periodic operators $U(0)$ and $V(0;\nu^{-1})$. Hereafter, throughout this paper, it is assumed that all $A(t)$ commute with $K : A(t)K = KA(t)$, $t \in \mathbb{R}$. Such examples are given in the following:

1. $K = kE$.
2. $A(t) = p(t)A_{1} + p_{2}(t)A_{2}$, $p(t + \omega) = p(t), A_{1}K = KA_{1}$, $(i = 1, 2)$, where $A_{i}$ is a constant matrix and $p(t)$ is a scalar continuous function for each $i = 1, 2$.

We give several conditions equivalent to the commutative condition $A(t)K = KA(t)$, $t \in \mathbb{R}$.

Lemma 4.1. In (1.1), the following statements are equivalent:

1. $A(t)K = KA(t)$, $t \in \mathbb{R}$.
2. $T(t, s)K = KT(t, s)$, $t, s \in \mathbb{R}$.
3. $T(t, 0)K = KT(t, 0)$, $t \in \mathbb{R}$.
4. There is a fundamental matrix $\Phi(t)$ such that $\Phi(t)K = KT(t, 0)$, $t \in \mathbb{R}$.

Proof. Assume that $KA(t) = A(t)K$, $t \in \mathbb{R}$, holds. Then, differentiating the matrix $KT(t, s)$ by $t$, we have

$$\frac{\partial}{\partial t}KT(t, s) = K\frac{\partial}{\partial t}T(t, s) = KA(t)T(t, s) = A(t)KT(t, s).$$

This relation implies that $KT(t, s)$ is a matrix solution of (1.1) and $KT(s, s) = K$. By using the same argument as above, $T(t, s)K$ is also a matrix solution of (1.1) and $T(s, s)K = K$. The uniqueness of solutions for (1.1) implies that $KT(t, s) = T(t, s)K$ holds. Hence statement (1) implies (2).

Obviously, statement (2) implies statement (3). Since $T(t, 0)$ is a fundamental matrix, statement (3) implies (4).

Assume that statement (4) holds. Differentiating both sides of $K\Phi(t) = \Phi(t)K$ by $t$, we have $K\Phi(t) = \Phi(t)K$ and hence

$$KA(t)\Phi(t) = A(t)\Phi(t)K = A(t)K\Phi(t).$$

Since $\Phi(t)$ is invertible, we have $KA(t) = A(t)K$. Therefore statement (4) implies statement (1).

We are now in a position to give a representation of solutions for (3.2) under the condition $A(t)K = KA(t)$, $t \in \mathbb{R}$.

Theorem 4.2. Assume that $A(t)K = KA(t)$, $t \in \mathbb{R}$. Then the solution operator of (3.2) is expressed as

$$(4.1) V(t, s; \nu^{-1}) = e^{(\nu^{-1} - 1)(t-s)K}T(t, s).$$

Moreover, $V(t, s; \nu^{-1})K = KV(t, s; \nu^{-1})$. 

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Proof. We define \( V(t, s) = V(t, s; \nu^{-1}) \) by (4.1). Then, \( V(s, s) = E \). Taking the derivative of both sides of (4.1) by \( t \), we have that
\[
\frac{\partial V}{\partial t}(t, s) = (\nu^{-1} - 1)KV(t, s) + e(t-s)(\nu^{-1} - 1)K A(t)T(t, s).
\]

Since \( A(t)K = KA(t) \), \( e(t-s)(\nu^{-1} - 1)KA(t) \) is nonsingular, and conditions (3) and (4) are equivalent. The equivalence of (4) and (5) is obvious.

We are now in a position to state one of main results; that is, we give a relationship between the characteristic multipliers of (1.1) and (1.2). To do so, we define two

Corollary 4.3. Assume that \( A(t)K = KA(t) \), \( t \in \mathbb{R} \). If \( y(t) \) is a solution of (3.2), then
\[
x(t) := e^{(\nu^{-1} - 1)tK} y(t)
\]
is a solution of (1.1). Conversely, if \( x(t) \) is a solution of (1.1), then
\[
y(t) := e^{(\nu^{-1} - 1)tK} x(t)
\]
is a solution of (3.2).

5. A relationship between characteristic multipliers. We characterize the characteristic multipliers of (1.2) under the commutative condition \( A(t)K = KA(t) \). Moreover, we give a relationship between the characteristic multipliers of (1.1) and (1.2). The idea is based on results about the spectrum of a matrix which is a difference of commutative matrices (see the appendix).

First, Theorem 3.2 is refined in the following form. Define
\[
\Delta(\nu) = \gamma E - e^{(1-\nu^{-1})\omega K} T(0).
\]

Theorem 5.1. Assume that \( A(t)K = KA(t) \), \( t \in \mathbb{R} \). Then the following statements are equivalent:
1. \( \nu \in P_\sigma(U(0)) \).
2. \( \nu \in \sigma(e^{(1-\nu^{-1})\omega K} T(0)) \).
3. \( \det \Delta(\nu) = 0 \).
4. \( \det(\nu e^{(1-\nu^{-1})\omega K} - T(0)) = 0 \).
5. \( 0 \in \sigma(\nu e^{(1-\nu^{-1})\omega K} - T(0)) \).

Proof. By Theorem 4.2, the periodic operator \( V(0; \nu^{-1}) \) of (3.2) is expressed as \( V(0; \nu^{-1}) = e^{(1-\nu^{-1})\omega K} T(0) \). Therefore Theorem 3.2 implies that statements (1) and (2) are equivalent. The equivalence of (2) and (3) is obvious. The matrix \( \Delta(\nu) \) is rewritten as
\[
\Delta(\nu) = e^{(1-\nu^{-1})\omega K} \left( \nu e^{(1-\nu^{-1})\omega K} - T(0) \right).
\]

Since \( e^{(1-\nu^{-1})\omega K} \) is nonsingular, conditions (3) and (4) are equivalent. The equivalence of (4) and (5) is obvious. 

We are now in a position to state one of main results; that is, we give a relationship between the characteristic multipliers of (1.1) and (1.2). To do so, we define two
functions \( f_k(z) \) and \( g_k(z) \) for each complex number \( k \) as follows:

\[
\begin{align*}
 f_k(z) &= z + k(1 - e^{-\omega z}), \quad z \in \mathbb{C}, \\
 g_k(z) &= ze^{(1-z)\omega k}, \quad z \in \mathbb{C} \setminus \{0\}.
\end{align*}
\]

Theorem 5.2. Assume that \( A(t)K = KA(t) \), \( t \in \mathbb{R} \). Then \( \nu \in P_\sigma(U(0)) \) if and only if there exist \( \kappa \in \sigma(K) \) and \( \mu \in \sigma(T(0)) \) such that

\[
(5.1) \quad g_\kappa(\nu) = \mu, \quad W_K(\kappa) \cap W_{T(0)}(\mu) \neq \{0\}.
\]

Proof. Theorem 5.1 asserts that \( \nu \in P_\sigma(U(0)) \) if and only if

\[
(5.2) \quad 0 \in \sigma(\nu e^{(1-\nu^{-1})\omega K} - T(0)).
\]

On the other hand, the spectral mapping theorem implies that

\[
\sigma(\nu e^{(1-\nu^{-1})\omega K}) = \{ g_\kappa(\nu) \mid \kappa \in \sigma(K) \}.
\]

Hence, by Lemma A.1 in the appendix, condition (5.2) shows that \( \nu \in P_\sigma(U(0)) \) if and only if there exist \( \kappa_0 \in \sigma(K) \) and \( \mu \in \sigma(T(0)) \) such that

\[
(5.3) \quad g_{\kappa_0}(\nu) = \mu, \quad G_{\nu e^{(1-\nu^{-1})\omega K}}(g_{\kappa_0}(\nu)) \cap G_{T(0)}(\mu) \neq \{0\}.
\]

For such a \( \kappa_0 \in \sigma(K) \), we denote by \( \{\kappa_0, \kappa_1, \ldots, \kappa_p\} \) the set of \( \kappa \in \sigma(K) \) such that \( g_\kappa(\nu) = g_{\kappa_0}(\nu) \). Using the spectral mapping theorem again, we have

\[
G_{\nu e^{(1-\nu^{-1})\omega K}}(g_{\kappa_0}(\nu)) = \bigoplus_{i=0}^{p} G_K(\kappa_i).
\]

Therefore we see that \( G_{\nu e^{(1-\nu^{-1})\omega K}}(g_{\kappa_0}(\nu)) \cap G_{T(0)}(\mu) \neq \{0\} \) if and only if \( G_{T(0)}(\mu) \cap \bigoplus_{i=0}^{p} G_K(\kappa_i) \neq \{0\} \). Then \( x \in G_{T(0)}(\mu) \cap \bigoplus_{i=0}^{p} G_K(\kappa_i) \), \( x \neq 0 \), is expressed as \( x = \sum_{i=0}^{p} P_i x \), \( P_i x \in G_K(\kappa_i) \), where \( P_i : \mathbb{C}^n \to G_K(\kappa_i) \) is the projection. The condition \( x \in G_{T(0)}(\mu) \) implies that there exists an \( m \geq 1 \) such that \( (T(0) - \mu I)^m x = 0 \). Since \( T(0) \) and \( K \) are commutable, we have

\[
(T(0) - \mu I)^m P_i x = P_i (T(0) - \mu I)^m x = 0, \quad i = 0, \ldots, p.
\]

Since there is at least one \( i \) such that \( P_i x \neq 0 \), there exists an \( i \) such that

\[
(5.4) \quad G_{T(0)}(\mu) \cap G_K(\kappa_i) \neq \{0\}.
\]

Lemma A.2 in the appendix asserts that condition (5.4) is reduced to the condition

\[
W_{T(0)}(\mu) \cap W_K(\kappa_i) \neq \{0\}.
\]

Hence condition (5.3) is replaced by the condition

\[
(5.5) \quad g_{\kappa_0}(\nu) = \mu, \quad W_K(\kappa_i) \cap W_{T(0)}(\mu) \neq \{0\}.
\]

This proves the theorem.

\[\square\]

If \( K = kE \), then \( \sigma(K) = \{k\} \) and \( W_K(k) = \mathbb{C}^n \). Therefore the following corollary holds true.
Corollary 5.3. Let $K = kE$. Then $\nu \in \sigma(U(0))$ if and only if $g_k(\nu) \in \sigma(T(0))$.

Corollary 5.4. Assume that $A(t)K = KA(t)$, $t \in \mathbb{R}$. For any characteristic exponent $\rho$ of (1.2) there is a $\kappa \in \sigma(K)$ such that $\lambda = f_k(\rho)$ is a characteristic exponent of (1.1).

Proof. Let $\rho$ be a characteristic exponent of (1.2). Then $\nu = e^{\rho \omega}$ is a characteristic multiplier of (1.2). Theorem 5.2 shows that there exist $\kappa \in \sigma(K)$ and $\mu \in \sigma(T(0))$ with $g_k(\nu) = \mu$. There exists a characteristic exponent $\lambda$ of (1.1) with $\mu = e^{\lambda \omega}$, so we have that

$$e^{\lambda \omega} = g_k(\nu) = e^{\rho \omega} \cdot \exp \left[(1 - e^{-\rho \omega}) \omega \kappa \right]$$

and that

$$\lambda \omega = \rho \omega + (1 - e^{-\rho \omega}) \omega \kappa + 2\pi i m$$

for some integer $m$. Hence

$$\lambda - i \frac{2\pi}{\omega} m = \rho + (1 - e^{-\rho \omega}) \kappa.$$ 

Since $\exp \left[\left(\lambda - i \frac{2\pi}{\omega} m\right) \omega\right] = e^{\lambda \omega}$, $\lambda - i \frac{2\pi}{\omega} m$ is also a characteristic exponent of (1.1). Thus the proof is complete.

6. Nondegeneracy of delayed feedback equation. We say that a characteristic multiplier is nondegenerate if its eigenspace coincides with its generalized eigenspace. We show that the multiplier of (1.2) inherits the nondegenerate property from the multiplier of (3.2).

Theorem 6.1. Assume that $A(t)K = KA(t)$, $t \in \mathbb{R}$, and that $\nu$ is a characteristic multiplier of (3.2) such that

$$G_{V(0, \nu^{-1})}(\nu) = W_{V(0, \nu^{-1})}(\nu).$$

Then, under the additional condition

$$W_{V(0, \nu^{-1})}(\nu) \cap W_K(-\nu/\omega) = \{0\},$$

the relations

$$G_{U(0)}(\nu) = W_{U(0)}(\nu),$$

$$\dim W_{U(0)}(\nu) = \dim W_{V(0, \nu^{-1})}(\nu)$$

hold for (1.2).

Proof. The relation (6.1) means that

$$N((V(0; \nu^{-1}) - \nu E)^2) = N((V(0; \nu^{-1}) - \nu E)).$$

To prove the relation (6.3), it suffices to show that

$$N((U(0) - \nu I)^2) = N((U(0) - \nu I)).$$

Let $\phi \in N((U(0) - \nu I)^2)$, and set $\psi = (U(0) - \nu I)\phi$. Then we have that $\psi \in N((U(0) - \nu I))$. Let $y(t)$ and $z(t)$ be solutions of (1.2) such that $y_0 = \phi$ and $z_0 = \psi$. 


respectively. Then \( y_t = U(t, 0)\phi \) and \( z_t = U(t, 0)\psi \). Since \( \psi \in W_{U_0}(\nu) \), it follows from Theorem 3.3 that

\[
z(t) = V(t, s; \nu^{-1})z(s)
\]

and

\[
z(0) = \psi(0) \in W_{V(0; \nu^{-1})}(\nu).
\]

Clearly, the solution \( z(t) \) of (1.2) is well defined on \( \mathbb{R} \). Hence, by Lemma 2.2, we see that

\[
z(t + \omega) = \nu z(t), \quad t \in \mathbb{R}.
\]

On the other hand, since

\[
z_t = U(t, 0)\psi = U(t, 0)U(\omega, 0)\phi - \nu U(t, 0)\phi
\]

\[
= U(t + \omega, 0)\phi - \nu U(t, 0)\phi
\]

\[
= y_{t+\omega} - \nu y_t,
\]

we have

\[
z(t) = y(t + \omega) - \nu y(t).
\]

Combining (6.7) with (6.6), we get

\[
y(t - \omega) = \nu^{-1}y(t) - \nu^{-1}z(t - \omega) = \nu^{-1}y(t) - \nu^{-2}z(t).
\]

Hence

\[
y'(t) = A(t)y(t) + K(y(t - \omega) - y(t))
\]

\[
= A(t)y(t) + (\nu^{-1} - 1)Ky(t) - \nu^{-2}Kz(t).
\]

Using the variation of constants formula, Theorem 4.2, and (6.4), we obtain

\[
y(t) = V(t, 0; \nu^{-1})y(0) - \nu^{-2}K \int_0^t V(t, s; \nu^{-1})z(s)\,ds
\]

\[
= V(t, 0; \nu^{-1})y(0) - \nu^{-2}K \int_0^t z(t)\,ds
\]

\[
= V(t, 0; \nu^{-1})\phi(0) - \nu^{-2}tKz(t),
\]

from which it follows that

\[
y(t + \omega) = V(t + \omega, 0; \nu^{-1})\phi(0) - \nu^{-2}(t + \omega)Kz(t + \omega)
\]

\[
= V(t, 0; \nu^{-1})V(0; \nu^{-1})\phi(0) - \nu^{-1}(t + \omega)Kz(t).
\]

Therefore

\[
z(t) = y(t + \omega) - \nu y(t)
\]

\[
= V(t, 0; \nu^{-1})V(0; \nu^{-1})\phi(0) - \nu V(t, 0; \nu^{-1})\phi(0)
\]

\[
+ \nu^{-1}tKz(t) - \nu^{-1}(t + \omega)Kz(t)
\]

\[
= V(t, 0; \nu^{-1})(V(0; \nu^{-1}) - \nu E)\phi(0) - \nu^{-1}\omega Kz(t),
\]
and hence
\[(E + \nu^{-1}\omega K)z(t) = V(t, 0; \nu^{-1}) (V(0; \nu^{-1}) - \nu E) \phi(0)\].

Taking \(t = 0\) in the above relation, we have
\[(E + \nu^{-1}\omega K) \psi(0) = (V(0; \nu^{-1}) - \nu E) \phi(0)\]  
(6.8)

Since
\[(V(0; \nu^{-1}) - \nu E) \psi(0) = 0\]
and \(V(0; \nu^{-1})K = KV(0; \nu^{-1})\), we have
\[(V(0; \nu^{-1}) - \nu E) \left( E + \frac{\omega}{\nu} K \right) \psi(0) = 0\].
Thus, multiplying both sides of (6.8) by \(V(0; \nu^{-1}) - \nu E\), one obtains that
\[(V(0; \nu^{-1}) - \nu E)^2 \phi(0) = 0,\]
which implies that
\[(V(0; \nu^{-1}) - \nu E) \phi(0) = 0.\]
It follows from (6.8) that \((E + \nu^{-1}\omega K) \psi(0) = 0\) holds, that is,
\[
\psi(0) \in W_K(-\nu/\omega).
\]
(6.9)

Summarizing (6.5) and (6.9), we have
\[
\psi(0) \in W_V(0; \nu^{-1}) \cap W_K(-\nu/\omega).
\]
Therefore \(\psi(0) = 0\) from hypothesis (6.2), and, thus, \(z(t) = 0\). Since \(\psi = z_0 = 0\), we have \((U(0) - \nu I)\phi = 0\), that is, \(N((U(0) - I)^2) = N(U(0) - I)\). Corollary 3.6 implies that \(\dim W_{U(0)}(\nu) = \dim W_{U(0; \nu^{-1})}(\nu)\). Therefore the proof is complete. \(\Box\)

We say that \((1.1)\) is nondegenerate if \((1.1)\) has the characteristic multiplier 1, and \(G_{T(0)}(1) = W_{T(0)}(1)\) and \(\dim W_{T(0)}(1) = 1\). Similarly, we say that \((1.2)\) is nondegenerate if \((1.2)\) has the characteristic multiplier 1, and \(G_{U(0)}(1) = W_{U(0)}(1)\) and \(\dim W_{U(0)}(1) = 1\). In these terms we have the following results.

**Theorem 6.2.** If \((1.1)\) is nondegenerate and if
\[
(W_{T(0)}(1) \cap W_K(-1/\omega) = \{0\},
\]
(6.10)
then \((1.2)\) is also nondegenerate.

**Corollary 6.3.** If \((1.1)\) is nondegenerate and if \(-1/\omega \notin \sigma(K)\), then \((1.2)\) is also nondegenerate.

**Corollary 6.4.** Let \(K = kE\). If \((1.1)\) is nondegenerate and if \(\omega k \neq -1\), then \((1.2)\) is also nondegenerate.
7. Property of the characteristic multipliers of (1.2). To determine the stability of the periodic orbit \( \gamma(\phi(t)) \) of (DF), we will evaluate the magnitude of the modulus of characteristic multipliers of (1.2). Before describing a result, we define a new function which is useful in describing our results. Consider a mapping from \( s \in (0, \pi) \) to \( \alpha \in (0, 2) \):

\[
\alpha = \frac{s(1 + \cos s)}{\sin s}, \quad 0 < s < \pi.
\]

This is a one-to-one and onto mapping because \( \frac{d\alpha}{ds} = \frac{(1 + \cos s)(\sin s - s)}{\sin^2 s} < 0 \) for \( 0 < s < \pi \), and

\[
\lim_{s \to 0} \alpha = 2, \quad \lim_{s \to \pi} \alpha = 0.
\]

So, there exists an inverse mapping, and we write it as \( s(\alpha) \). Using this mapping, we define a new function \( \beta(\alpha) \) as follows:

\[
(7.1) \quad \beta(\alpha) = \frac{2s(\alpha)}{\sin s(\alpha)} \quad \text{for} \quad 0 < \alpha < 2.
\]

We will prove lemmas for the characteristic exponents. We define the set of characteristic exponents as follows:

\[
\Lambda(\kappa, \lambda) = \{ z | f_\kappa(z) = \lambda \}.
\]

**Lemma 7.1.** For any \((\kappa, \lambda) \in \mathbb{R} \times \mathbb{C}\), \( \Lambda(\kappa, \lambda) \) is nonempty.

**Proof.** Let \( \omega z = x + iy \), \( \omega \lambda = a + ib \). Then the equation \( f_\kappa(z) = \lambda \) can be rewritten as

\[
(7.2) \quad a = x + \omega \kappa(1 - e^{-x} \cos y),
\]

\[
(7.3) \quad b = y + \omega \kappa e^{-x} \sin y.
\]

On the other hand, the solutions of equations

\[
(7.4) \quad y = b + \sqrt{(\omega \kappa)^2 e^{-2x} - (x + \omega \kappa - a)^2} =: \varphi(x),
\]

\[
(7.5) \quad x = a - \omega \kappa + (b - y) \cot y =: \psi(y)
\]

satisfy (7.2) and (7.3) if

\[
(7.6) \quad (b - y) \sin y > 0.
\]

We can easily see that there exists \( \hat{x} \in \mathbb{R} \) such that \( \varphi(x) \) is monotonically decreasing for \( x \in (-\infty, \hat{x}] \) and \( \lim_{x \to -\infty} \varphi(x) = \infty \). We also find that \( \psi(y) \) is defined on each interval \((j\pi, (j + 1)\pi)\), \( j = 0, \pm 1, \pm 2, \ldots \), and

\[
\lim_{y \to (j+1)\pi-0} \psi(y) = -\infty, \quad \lim_{y \to j\pi+0} \psi(y) = \infty
\]

for any integer \( j > \hat{j} \), where \( \hat{j} \) is the integer satisfying \( b \in [\hat{j}\pi, (\hat{j} + 1)\pi) \). We choose an integer \( j_0 > \hat{j} \) such that \( j_0 \pi > \varphi(\hat{x}) \). Then there exists a solution of (7.4) and
Lemma 7.2. For any \((\kappa, \lambda) \in \mathbb{R} \times \mathbb{C}\) and any \(\sigma \in \mathbb{R}\), the set \(\Lambda_\sigma(\kappa, \lambda) := \{z \in \Lambda(\kappa, \lambda) \mid \Re z \geq \sigma\}\) consists of a finite number of elements. Moreover, for any \((\kappa, \lambda) \in \mathbb{R} \times \mathbb{C}\), there exists \(r > 0\) such that the set \(\Lambda_\sigma(\kappa, \lambda)\) is contained in the disk \(\{z \in \mathbb{C} \mid |z| \leq r\}\).

Proof. For any \((\kappa, \lambda)\) and any \(\sigma \in \mathbb{R}\) we choose \(r > 0\) satisfying \(2(|\kappa|/(1 + e^{-\omega}) + |\lambda|) < r\). If \(|z| > r\) and \(\Re z \geq \sigma\), then
\[
|f_\kappa(z) - \lambda| = |z - \kappa(1 - e^{-\omega}) - \lambda| = |z| - |\kappa(1 - e^{-\omega}) - \lambda| \geq |z| - \{|\kappa|/(1 + e^{-\omega}) + |\lambda|\} > \frac{r}{2} > 0.
\]
Hence \(\Lambda_\sigma(\kappa, \lambda)\) has no elements in the region \(\{z \in \mathbb{C} \mid |z| > r\}\); i.e., \(\Lambda_\sigma(\kappa, \lambda)\) is contained in the disk \(\{z \in \mathbb{C} \mid |z| \leq r\}\). Moreover, since \(f_\kappa(z) - \lambda\) is an analytic function, it can only have a finite number of zeros in the disk which is a closed and bounded set.

From these lemmas for any \((\kappa, \lambda)\) the set \(\Lambda(\kappa, \lambda)\) has an element whose real part is maximal. So we define
\[
m(\kappa, \lambda) = \max\{\Re z | f_\kappa(z) = \lambda\}.
\]
Then the following lemma holds.

Lemma 7.3. \(m(\kappa, \lambda)\) is continuous with respect to \((\kappa, \lambda)\).

Proof. For fixed \((\kappa^*, \lambda^*)\), there exists \(z^* \in \mathbb{C}\) such that \(\Re z^* = m(\kappa^*, \lambda^*)\) and \(f_{\kappa^*}(z^*) = \lambda^*\).

First we prove that for any \(\varepsilon > 0\) there exists \(\delta_1 > 0\) such that
\[
|\kappa - \kappa^*| + |\lambda - \lambda^*| < \delta_1 \implies m(\kappa, \lambda) \geq m(\kappa^*, \lambda^*) - \varepsilon.
\]
Consider a circle \(\gamma_r(z^*) := \{z \in \mathbb{C} \mid |z - z^*| = r\}\) for a positive \(r\). We can choose a radius \(r\) such that \(f_{\kappa^*}(z) - \lambda^*\) has no zero in the interior of the circle \(\gamma_r(z^*)\) except for \(z^*\). For any positive \(\varepsilon < r\), letting \(m = \min\{|f_{\kappa^*}(z) - \lambda^*| \mid z \in \gamma_r(z^*)\}\), it is clear that \(m > 0\). For this \(m\), from the continuity of \(f_{\kappa^*}(z) - \lambda\) with respect to the parameters \((\kappa, \lambda)\), there exists \(\delta_1\) such that
\[
|f_{\kappa^*}(z) - \lambda| < m \leq |f_{\kappa^*}(z) - \lambda^*|.
\]
From Rouché's theorem, \(f_{\kappa^*}(z) - \lambda\) has a zero in the interior of the circle \(\gamma_r(z^*)\). We denote by \(\hat{z}\) the zero. Then
\[
m(\kappa, \lambda) \geq \Re \hat{z} \geq \Re z^* - \varepsilon = m(\kappa^*, \lambda^*) - \varepsilon,
\]
which yields (7.7).

Next we prove that for any \(\varepsilon > 0\) there exists \(\delta_2 > 0\) such that
\[
|\kappa - \kappa^*| + |\lambda - \lambda^*| < \delta_2 \implies m(\kappa, \lambda) \leq m(\kappa^*, \lambda^*) + \varepsilon.
\]
So we assume that this is not true. Then there exist \(\varepsilon > 0\) and \(\{(\kappa_n, \lambda_n)\} (n = 1, 2, \ldots)\) such that
\[
|\kappa_n - \kappa^*| + |\lambda_n - \lambda^*| < \frac{1}{n}, \quad m(\kappa_n, \lambda_n) \geq m(\kappa^*, \lambda^*) + \varepsilon.
\]
Moreover, there exists \( \{z_n\} \ (n = 1, 2, \ldots) \) such that
\[
f_{\kappa_n}(z_n) = \lambda_n, \quad \text{Re } z_n = m(\kappa_n, \lambda_n).
\]
Therefore we have
\[
\text{Re } z_n \geq m(\kappa^*, \lambda^*) + \varepsilon.
\]
From Lemma 7.2, \( \{z_n\} \) is contained in a bounded and closed region. Then there exists a convergent subsequence \( \{z_{n(i)}\} \). Letting \( z_0 = \lim_{i \to \infty} z_{n(i)} \), we have
\[
f_{\kappa^*}(z_0) = \lambda^*, \quad \text{Re } z_0 \geq m(\kappa^*, \lambda^*) + \varepsilon.
\]
This is a contradiction.

**Lemma 7.4.**

(i) If \( \text{Re } \lambda > 0 \) and \( \text{Im } \lambda = 2m\pi/\omega \ (m \in \mathbb{Z}) \), then for any \( \kappa \in \mathbb{R} \) there exists \( \rho \in \Lambda(\kappa, \lambda) \) such that \( \text{Re } \rho > 0 \) and \( \text{Im } \rho = \text{Im } \lambda \).

(ii) If \( \text{Re } \lambda = 0 \) and \( \text{Im } \lambda = 2m\pi/\omega \ (m \in \mathbb{Z}) \), then \( \text{Re } \rho < 0 \) for any \( \kappa > 0 \) and any \( \rho \in \Lambda(\kappa, \lambda) \).

(iii) If \( \text{Re } \lambda = 0 \) and \( \text{Im } \lambda \neq 2m\pi/\omega \ (m \in \mathbb{Z}) \), then \( \text{Re } \rho < 0 \) for any \( \kappa > 0 \) and any \( \rho \in \Lambda(\kappa, \lambda) \).

(iv) If \( \text{Re } \lambda = (0, 2/\omega) \) and \( \text{Im } \lambda = (2m-1)\pi/\omega \ (m \in \mathbb{Z}) \), then \( \text{Re } \rho < 0 \) for any \( \kappa \in (\text{Re } \lambda)/2, \kappa_0) \) and any \( \rho \in \Lambda(\kappa, \lambda) \). Here \( \kappa_0 = \beta(\text{Re } \lambda)/(2\omega) \).

(v) If \( \text{Re } \lambda < 0 \), then \( \text{Re } \rho < 0 \) for any \( \kappa > 0 \) and any \( \rho \in \Lambda(\kappa, \lambda) \).

**Proof.** Suppose that \( \rho \in \Lambda(\kappa, \lambda) \). We set \( \omega \rho = x_0 + iy_0 \). It is clear that \( (x, y) = (x_0, y_0) \) satisfies (7.2) and (7.3).

(i) In this case, we have \( a > 0, b = 2m\pi \ (m \in \mathbb{Z}) \). Since \( \sin b = 0 \), we obtain \( y = b \) from (7.3) and \( a = x + \omega \kappa (1 - e^{-x}) \) from (7.2). It is clear that \( x = 0 \) is no solution because \( a > 0 \). So we can rewrite (7.2) as follows:

\[
(7.9) \quad \omega \kappa = \frac{a - x}{1 - e^{-x}}.
\]

The function of the right-hand side is continuous for \( x > 0 \) and
\[
\lim_{x \to +\infty} \frac{a - x}{1 - e^{-x}} = -\infty, \quad \lim_{x \to 0^+} \frac{a - x}{1 - e^{-x}} = +\infty.
\]
From the intermediate value theorem, (7.9) has at least one solution for \( x > 0 \).

Denoting one of them by \( x_{\kappa}, (x, y) = (x_{\kappa}, b) \) satisfies (7.2) and (7.3). Therefore \( z_{\kappa} = (x_{\kappa} + ib)/\omega \) is an element of \( \Lambda(\kappa, \lambda) \). Furthermore, it is clear that \( \text{Re } z_{\kappa} = x_{\kappa}/\omega > 0 \) and \( \text{Im } z_{\kappa} = b/\omega = \text{Im } \lambda \) hold.

In cases (ii), (iii) one has \( a = 0 \). Assume there exist \( \kappa > 0 \) and \( \rho \in \Lambda(\kappa, \lambda) \). We have

\[
(7.10) \quad 0 = x_0 + \omega \kappa (1 - e^{-x_0} \cos y_0),
\]
\[
(7.11) \quad b = y_0 + \omega \kappa e^{-x_0} \sin y_0
\]
from (7.2) and (7.3). If \( x_0 > 0 \), we have

\[
x_0 = -\omega \kappa (1 - e^{-x_0} \cos y_0) \leq -\omega \kappa (1 - e^{-x_0}) < 0
\]
from (7.10), which is a contradiction. If \( x_0 = 0 \), we have \( \cos y_0 = 1 \) from (7.10), that is, \( y_0 = 2m\pi \ (m \in \mathbb{Z}) \). By substituting this for (7.11), we obtain \( y_0 = b \), which yields
\[
\omega \rho = iy_0 = ib = \omega \lambda = i2m\pi.
\]
In case (ii), this contradicts the assumption that $\rho \neq \lambda$. In case (iii), this contradicts the assumption that $\omega \lambda \neq 2m\pi$. Thus we conclude that $x_0 < 0$, i.e., $\text{Re } \rho < 0$.

(iv) In this case, we have $a \in (0, 2)$ and $b = (2m - 1)\pi$ ($m \in \mathbb{Z}$). We note that $\kappa_0 > 1/\omega$ because

\[
(7.12) \quad \kappa_0 = \frac{\beta(a)}{2\omega} = \frac{1}{\omega \sin s(a)} > 1/\omega.
\]

First we consider the case where $\kappa \in ((\text{Re } \lambda)/2, 1/\omega]$. We assume that there exist $\kappa \in ((\text{Re } \lambda)/2, 1/\omega]$ and $\rho \in \Lambda(\kappa, \lambda)$ such that $\text{Re } \rho \geq 0$, i.e., $x_0 \geq 0$. Because $b = \omega \text{Im } \lambda = (2m - 1)\pi$, we obtain

\[
(7.13) \quad a = x_0 + \omega \kappa \{1 + e^{-x_0} \cos(b - y_0)\},
\]

\[
(7.14) \quad b - y_0 = \omega \kappa e^{-x_0} \sin(b - y_0)
\]

from (7.2) and (7.3). By noting that $\kappa \leq 1/\omega$ and $e^{-x_0} \leq 1$, we can find that (7.14) has only one solution $y_0 = b$. This is shown as follows. An equation $\gamma = \omega \kappa e^{-x_0} \sin \gamma$ has a solution $\gamma = 0$. Assume that this equation has another solution $\gamma_0 \neq 0$. Since $|\sin \gamma_0| < |\gamma_0|$, we have

\[
1 < \left| \frac{\gamma_0}{\sin \gamma_0} \right| = \omega \kappa e^{-x_0} \leq 1.
\]

This is a contradiction. So (7.14) has only one solution $y_0 = b$. By substituting $y_0 = b$ for (7.13), we obtain

\[
\omega \kappa = \frac{a - x_0}{1 + e^{-x_0}}
\]

which yields that $\omega \kappa \leq a/2$. We will prove this in the following. Let

\[
h(s) = \frac{a - s}{1 + e^{-s}}, \quad s \in \mathbb{R}.
\]

Then $\omega \kappa = h(x_0)$. By differentiating $h(s)$, we have

\[
h'(s) = \frac{(a - 1 - s)e^{-s} - 1}{(1 + e^{-s})^2}.
\]

$h'(s) = 0$ is equivalent to $a - 1 - s = e^s$, so that it has a unique solution $\hat{s}$. We can easily see that $\hat{s} < 0$ from $a = \omega \text{Re } \lambda < 2$. So we also find that $h(s)$ is decreasing for $s \geq 0$ and $\omega \kappa = h(x_0) \leq h(0) = a/2$. From $\omega \kappa \leq a/2$, we obtain $\kappa \leq a/(2\omega) = \text{Re } \lambda/2$, which is a contradiction. Therefore, for $\kappa \in (\text{Re } \lambda/2, 1/\omega]$ we have $\text{Re } \rho < 0$.

Next, we consider the case where $\kappa \in (1/\omega, \kappa_0)$. We assume that there exist $\kappa \in (1/\omega, \kappa_0)$ and $\rho \in \Lambda(\kappa, \lambda)$ such that $\text{Re } \rho \geq 0$. From the fact proved in the last paragraph, when $\kappa = 1/\omega$, $m(\kappa, \lambda) := \max\{|\text{Re } z| : z \in \Lambda(\kappa, \lambda)\}$ is negative. Thus, from Lemma 7.3, there exists $\kappa' \in (1/\omega, \kappa_0)$ such that $m(\kappa', \lambda) = 0$. We denote by $\rho^*$ one of the elements in $\Lambda(\kappa', \lambda)$ which give the maximum value. Clearly $\text{Re } \rho^* = 0$. Let $\omega \rho^* = iy^*$, $b = (2m - 1)\pi$. We have

\[
(7.15) \quad a = \omega \kappa^* \{1 + \cos(b - y^*)\},
\]

\[
(7.16) \quad b - y^* = \omega \kappa^* \sin(b - y^*).
\]
If $\sin(b - y^*) = 0$, we get $y^* = b$ from (7.16). By substituting this for (7.15), we obtain $a = 2\omega\kappa^* > 2$. This is a contradiction. So, we have $\sin(b - y^*) \neq 0$ and

$$\omega\kappa^* = \frac{b - y^*}{\sin(b - y^*)} = \frac{a}{1 + \cos(b - y^*)}$$

from (7.15) and (7.16). By putting $b - y^* = s$, we have

$$a = \frac{s(1 + \cos s)}{\sin s}$$

holds. Define a function $h(s) = s(1 + \cos s)/\sin s$. Because $h(s)$ is an even function, it suffices to consider only the case where $s \geq 0$. Since $a \in (0, 2)$, there exists a unique solution of $a = h(s)$ in each interval $(2j\pi, (2j + 1)\pi)$, $j = 0, 1, 2, \ldots$. We write the solution $s_j$, $j = 0, 1, 2, \ldots$. For one of the solutions $\{s_j\}$ we have

$$\omega\kappa^* = \frac{s_j}{\sin s_j}.$$ 

By noting the definition of $s(\alpha)$ given in (7.1), we also have

$$s_0 = s(a).$$

From (7.15) and (7.16), we have

$$(a - \omega\kappa^*)^2 + s_j^2 = (\omega\kappa^*)^2,$$ 

that is,

$$\omega\kappa^* = \frac{a}{2} + \frac{s_j^2}{2a}.$$ 

Moreover, from $a = h(s_0)$, we have

$$\frac{a}{2} + \frac{s_0^2}{2a} = \frac{\{s_0(1 + \cos s_0)/\sin s_0\}^2 + s_0^2}{2s_0(1 + \cos s_0)/\sin s_0} = \frac{s_0}{\sin s_0} = \frac{1}{2}\beta(a) = \omega\kappa_0.$$ 

Since $s_j \geq s_0$, we obtain

$$\omega\kappa^* = \frac{a}{2} + \frac{s_j^2}{2a} \geq \frac{a}{2} + \frac{s_0^2}{2a} = \omega\kappa_0,$$ 

which contradicts $\kappa^* < \kappa_0$.

(v) In this case, we have $a < 0$. We assume that there exist $\kappa > 0$ and $\rho \in \Lambda(\kappa, \lambda)$ such that $\Re \rho \geq 0$. From (7.2), we have

$$a = x_0 + \omega\kappa(1 - e^{-x_0} \cos y_0).$$

Since $\cos y_0 \leq 1$, $x_0 = \Re \rho/\omega \geq 0$, and $\kappa > 0$,

$$a \geq x_0 + \omega\kappa(1 - e^{-x_0}) \geq 0$$

holds. This is a contradiction. 

Remark 7.5. Bélair [1] considered the equation

$$\lambda = -1 + be^{-\lambda \tau},$$
which is induced by stability analysis in a model of a delayed neural network. Bélaïr tried to obtain conditions on $b$ and $\tau$ under which $\text{Re } \lambda < 0$. On the other hand, our equation can be written in the form

$$\rho = -\kappa + \lambda + \kappa e^{-\rho \omega},$$

and we intend to obtain conditions on $\lambda$ and $\kappa$ under which $\text{Re } \rho < 0$. This equation and our aim are very similar to those of Bélaïr. It is hard, however, to obtain our results from Bélaïr’s results or techniques in a straightforward manner.

We classify the characteristic multipliers of (1.1) as follows:

$$\sigma_U = \{ \mu \in \sigma(T(0)) \mid |\mu| > 1 \},$$
$$\sigma_N = \{ \mu \in \sigma(T(0)) \mid |\mu| = 1 \},$$
$$\sigma_S = \{ \mu \in \sigma(T(0)) \mid |\mu| < 1 \}.$$

By using Lemma 7.4 and Corollary 5.3, we obtain the following theorem for the characteristic multipliers of (1.2).

**Theorem 7.6.** Assume $k \in \mathbb{R}$ and $K = kE$.

(i) If there exists $\mu \in \sigma_U$ such that $\mu > 1$, then there exists $\nu \in P_\sigma(U(0))$ such that $\nu > 1$.

(ii) Let $\sigma_U \subset (-e^2, -1)$ and $\alpha_0 = \max_{\mu \in \sigma_U} \text{log } |\mu|$. For any $k$, if

$$\frac{\alpha_0}{2\omega} < k < \frac{\beta(\alpha_0)}{2\omega}$$

holds, then $|\nu| < 1$ or $\nu = 1$ for any $\nu \in P_\sigma(U(0))$.

**Proof.** (i) For $\lambda = (\text{log } \mu)/\omega$, we consider $z$ such that $f_k(z) = \lambda$. Since $\mu > 1$, we have that $\text{Re } \lambda > 0$ and $\text{Im } \lambda = 0$ hold. By Lemma 7.4(i), there exists $z = \rho$ in $\Lambda(k, \lambda)$ such that $\text{Re } \rho > 0$ and $\text{Im } \rho = \text{Im } \lambda = 0$. Let $\nu = e^{\omega \rho}$; then $\nu > 1$ and

$$g_\kappa(\nu) = e^{\omega \rho} \exp\{1 - e^{-\omega \rho}\} = e^{\omega \lambda} = \mu,$$

which yields $\nu \in P_\sigma(U(0))$ from Corollary 5.3.

(ii) We assume that there exists $\nu \in P_\sigma(U(0))$ such that $|\nu| \geq 1$ and $\nu \neq 1$. Let $\mu = g_\kappa(\nu)$; then we find $\mu \in \sigma(T(0))$ by Corollary 5.3. We put $\nu = e^{\omega \rho}$ and $\mu = e^{\omega \rho}$ and take the principal value of log in the equality $g_\kappa(\nu) = \mu$. Then $\lambda = f_k(\rho)$ holds, and the hypotheses of proof by refutation become

$$\text{Re } \rho = \frac{\text{log } |\nu|}{\omega} \geq 0 \quad \text{and} \quad \rho \neq \frac{2m\pi}{\omega} i \quad (m \in \mathbb{Z}).$$

First, we consider the case where $|\mu| < 1$, that is, $\text{Re } \lambda < 0$. Since $k > 0$ from (7.17), we can use Lemma 7.4(v) and have $\text{Re } \rho < 0$. This contradicts (7.18).

Next, we consider the case where $|\mu| = 1$. If $\mu = 1$, then $\text{Re } \lambda = 0$ and $\text{Im } \lambda = 2m\pi/\omega$ ($n \in \mathbb{Z}$). From the condition $k > 0$ and Lemma 7.4(ii), we have $\text{Re } \rho < 0$ or $\rho = \lambda = 2m\pi i/\omega$. This contradicts (7.18). If $\mu \neq 1$, then $\text{Re } \lambda = 0$ and $\text{Im } \lambda \neq 2m\pi/\omega$ ($m \in \mathbb{Z}$). From the condition $k > 0$ and Lemma 7.4(iii), we have $\text{Re } \rho < 0$, which contradicts (7.18).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Finally, we consider $|\mu| > 1$. From the condition of the theorem, we have $-\varepsilon^2 < \mu < -1$, that is, $0 < \Re \lambda < 2/\omega$ and $\Im \lambda = (2m - 1)\pi/\omega$ ($m \in \mathbb{Z}$). By the definition of $\alpha_0$, we find that

$$2 > \alpha_0 \geq \log |\mu| = \omega \Re \lambda > 0.$$ 

On the other hand, $\beta(\alpha)$ is a decreasing function of $\alpha$, because for $0 < \alpha < 2$,

$$\frac{d\beta(\alpha)}{d\alpha} = \frac{2(\sin s - s \cos s)}{\sin^2 s} \frac{ds}{d\alpha} < 0.$$ 

Hence we have $\beta(\omega \Re \lambda) \geq \beta(\alpha_0)$. From (7.18) we obtain

$$\frac{\Re \lambda}{2} \leq \frac{\alpha_0}{2\omega} < k < \frac{\beta(\alpha_0)}{2\omega} \leq \frac{\beta(\omega \Re \lambda)}{2\omega}.$$ 

Since $\beta(\omega \Re \lambda)/(2\omega)$ in the right-hand side of the above inequalities is equal to $\kappa_0$ in the statement of Lemma 7.4(iv), $\Re \rho < 0$ is obtained from Lemma 7.4(iv). This contradicts (7.18). \[\square\]

8. Application to the Rössler system. We consider the following Rössler system with a DFC input:

$$
\begin{cases}
    x'(t) = -y(t) - z(t) + k(x(t - \omega) - x(t)), \\
    y'(t) = x(t) + 0.2y(t) + k(y(t - \omega) - y(t)), \\
    z'(t) = 0.2 + z(t)(x(t) - 5.7) + k(z(t - \omega) - z(t)).
\end{cases}
$$

(8.1)

Here $k$ is a real constant. When $k = 0$, (8.1) is free from the control, and the dynamics is known to be chaotic. In this case, there are many unstable periodic solutions, and we intend to stabilize one of them. Let $(x^*(t), y^*(t), z^*(t))$ be the desired unstable periodic solution whose period is $\omega$. It is clear that this is also a solution of (8.1). The linear variational equation around the periodic solution is given by (1.2), where

$$A(t) := \begin{pmatrix}
    0 & -1 & -1 \\
    1 & 0.2 & 0 \\
    z^*(t) & 0 & x^*(t) - 5.7
\end{pmatrix}, \quad K = kE.$$ 

We note that for this $K$ the commutative condition $A(t)K = KA(t)$, $t \in \mathbb{R}$, is satisfied.

Then there exist two characteristic multipliers of (1.1) other than 1. We put them with $\mu_1, \mu_2$ ($|\mu_1| < |\mu_2|$). The following results give a criteria for DFC being successful.

**Theorem 8.1.** The following statements about the periodic solution $(x^*(t), y^*(t), z^*(t))$ of (8.1) hold:

(i) If $\mu_2 > 1$, then it is unstable for any $k \neq 1/\omega$.

(ii) If $-\varepsilon^2 < \mu_2 < -1$, then it is stable for $\alpha_0/(2\omega) < k < \beta(\alpha_0)/(2\omega)$, where $\alpha_0 = \log |\mu_2|$.

**Remark 8.2.** In Theorem 8.1 we define the stability of a periodic solution by using characteristic multipliers of the linear variational equation around the periodic solution as follows. Assume that the linear variational equation is nondegenerate. If there exists a characteristic multiplier such that the absolute value is greater than 1, then it is unstable. If the absolute value of any characteristic multipliers other than 1 is less than 1, then the solution is stable.
Proof of Theorem 8.1. (i) From Theorem 7.6(i), the linear variational equation (1.2) has a multiplier whose absolute value is greater than 1. Moreover, from \( k \neq -1/\omega \) and Theorem 6.2, the linear variational equation (1.2) is nondegenerate. Therefore the periodic solution \((x^*(t), y^*(t), z^*(t))\) of (8.1) is unstable.

(ii) From Lemma 2.3,
\[
\mu_1 \mu_2 = \exp \left( \int_0^\omega \text{tr} A(s)ds \right) > 0
\]
holds, so that \(-e^2 < \mu_2 \leq \mu_1 < 0\) because of \(|\mu_2| \geq |\mu_1|\). If \(\mu_1 > -1\), then \(\sigma_N = \{1\}\) and \(\sigma_U = \{\mu_2\}\). If \(\mu_1 = -1\), then \(\sigma_N = \{1, -1\}\) and \(\sigma_U = \{\mu_2\}\). If \(\mu_1 < -1\), then \(\sigma_N = \{1\}\) and \(\sigma_U = \{\mu_1, \mu_2\}\). Hence, the assumptions in Theorem 7.6(ii) are satisfied. Moreover, by noting that \(\max_{\mu \in \sigma_N} \log |\mu| = \log |\mu_2|\), the condition (7.17) of Theorem 7.6(ii) holds. Therefore the absolute value of any characteristic multipliers other than 1 of the linear variational equation (1.2) is less than 1. We also find that \(k \neq -1/\omega\) and the linear variational equation (1.2) is nondegenerate from Theorem 6.2. Therefore the periodic solution \((x^*(t), y^*(t), z^*(t))\) of (8.1) is stable.

Recently Minamoto and Nakao [7] considered the Rössler system with the DFC method successfully stabilizes a periodic solution. We will give some numerical results. Figure 1 displays attractors of (8.1) for \(k = 6.2\) therefore the periodic solution \((x^*(t), y^*(t), z^*(t))\) of (8.1) is unstable. We note that this periodic orbit approximates one of the unstable periodic orbits of the original Rössler system. Therefore, by using the numerical method, we can obtain a stable periodic solution of (8.2) numerically. We also find that \(k = 6.2\) and \(\omega = 5.88109\times 10^{-7}\) as the characteristic multipliers. It can be considered that 0.999986 is 1, \(\mu_1 = -8.40037 \times 10^{-7}\), and \(\mu_2 = -2.40399\). Hence, from Theorem 8.1, if
\[
0.074572 < k < 0.658948,
\]
the DFC method successfully stabilizes a periodic solution. We will give some numerical results. Figure 1 displays attractors of (8.1) for \(k = 6.2\) near both ends of the interval (8.3). Figure 2 shows a bifurcation diagram of (8.1) for \(\omega = 5.88109\), which is given by plotting the local extremum of \(x(t)\) for large \(t\) to each value \(k\).

Appendix. Let \(A\) and \(B\) be \(n \times n\) complex matrices. In this appendix, we consider the structure of the set \(\sigma(A-B)\) in the case where \(A\) and \(B\) are commutative.

Lemma A.1. If two matrices \(A\) and \(B\) are commutative, then
\[
\sigma(A-B) = \{\alpha - \beta \mid \alpha \in \sigma(A), \beta \in \sigma(B), G_A(\alpha) \cap G_B(\beta) \neq \{0\}\}.
\]

Proof. Let
\[
\sigma(A) = \{\alpha_1, \ldots, \alpha_r\}, \quad \sigma(B) = \{\beta_1, \ldots, \beta_s\}.
\]
Fig. 1. Attractor of (8.1) for $\omega = 5.88109$.

Fig. 2. Bifurcation diagram of (8.1) for $\omega = 5.88109$. 
By the spectral decomposition theorem for the matrices A and B there are projective matrices $P_i : \mathbb{C}^n \to G_A(\alpha_i)$ and $Q_j : \mathbb{C}^n \to G_B(\beta_j)$ and nilpotent matrices $M$ and $N$ such that

$$A = \sum_{i=1}^{r} \alpha_i P_i + M, \quad B = \sum_{j=1}^{s} \beta_j Q_j + N.$$ 

$$E = P_1 + \cdots + P_r, \quad E = Q_1 + \cdots + Q_s.$$ 

Moreover $P_i, M$ are expressed by polynomials of $A$; $Q_j, N$ are expressed by polynomials of $B$. As a result, we have

$$P_i Q_j = Q_j P_i, \quad MP_i = P_i M, \quad MQ_j = Q_j M,$$

$$NP_i = P_i N, \quad NQ_j = Q_j N, \quad MN = NM$$

from the condition $AB = BA$. Therefore we obtain

$$A - B = \sum_{i=1}^{r} \alpha_i P_i \sum_{j=1}^{s} Q_j - \sum_{j=1}^{s} \beta_j Q_j \sum_{i=1}^{r} P_i + M - N$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \alpha_i P_i Q_j - \sum_{j=1}^{s} \beta_j \sum_{i=1}^{r} P_i Q_j + M - N$$

$$\sum_{i=1}^{r} \sum_{j=1}^{s} (\alpha_i - \beta_j) P_i Q_j + M - N. \quad (A.1)$$

Setting $R_{ij} = P_i Q_j$, we see that

$$E = \sum_{i=1}^{r} \sum_{j=1}^{s} R_{ij}, \quad R_{ij} R_{kl} = \delta_{ik} \delta_{jl} R_{ij},$$

where $\delta_{ij} = 1 (i = j)$, $\delta_{ij} = 1 (i \neq j)$. Since for $x \in \mathbb{C}^n$

$$R_{ij} x = P_i Q_j x \in G_A(\alpha_i), \quad R_{ij} x = P_i Q_j x = Q_j P_i x \in G_B(\beta_j),$$

we have $R_{ij} x \in G_A(\alpha_i) \cap G_B(\beta_j)$. Conversely, if $x \in G_A(\alpha_i) \cap G_B(\beta_j)$, then $x = P_i x = Q_j x$. Hence we have

$$x = P_i x = P_i Q_j x = R_{ij} x.$$

Consequently, we arrive at the relation

$$R_{ij}(\mathbb{C}^n) = G_A(\alpha_i) \cap G_B(\beta_j).$$

Set

$$S = \sum_{(i,j) \in I} (\alpha_i - \beta_j) R_{ij} \quad (I = \{(i,j) \mid G_A(\alpha_i) \cap G_B(\beta_j) \neq \{0\})}.$$ 

Then it follows from (A.1) that

$$A - B = S + M - N. \quad (A.2)$$
and
\[ \sum_{(i,j) \in I} R_{ij} = E, \quad R_{ij}R_{kl} = \delta_{ij}\delta_{kl}R_{ij}. \]

Since \( M \) and \( N \) are commutative, \( M - N \) is a nilpotent matrix; since \( M \) and \( N \) are
commutative with \( P \) and \( Q \), \( M - N \) is commutative with \( S \). Thus the above relation
(A.2) gives a spectral decomposition of \( A - B \). This proves the lemma. \( \square \)

Lemma A.2. Let two matrices \( A \) and \( B \) be commutative, and let \( \alpha \in \sigma(A) \), \( \beta \in \sigma(B) \). Then \( G_A(\alpha) \cap G_B(\beta) \neq \{0\} \) if and only if \( W_A(\alpha) \cap W_B(\beta) \neq \{0\} \).

Proof. We need only prove that if \( G_A(\alpha) \cap G_B(\beta) \neq \{0\} \), then \( W_A(\alpha) \cap W_B(\beta) \neq \{0\} \). Let \( x \in G_A(\alpha) \cap G_B(\beta) \), \( x \neq 0 \). Then there are \( i, j \geq 1 \) such that
\[(A - \alpha E)^{i-1}x \neq 0, \quad (A - \alpha E)^j x = 0, \quad (B - \beta E)^{j-1}x \neq 0, \quad (B - \beta E)^j x = 0.
\]

Setting \( y = (A - \alpha E)^{i-1}x \), we have that \( y \in W_A(\alpha) \), \( y \neq 0 \), and that
\[(B - \beta E)^j y = (A - \alpha E)^{i-1}(B - \beta E)^j x = 0.
\]
Hence there is a \( k \geq 1 \) such that
\[(B - \beta E)^{k-1}y \neq 0, \quad (B - \beta E)^k y = 0.
\]
Setting \( z = (B - \beta E)^{k-1}y \), we have that \( z \neq 0 \) and \( (B - \beta E)z = 0 \). On the
other hand, we have \( (A - \alpha E)z = (B - \beta E)^{k-1}(A - \alpha E)y = 0 \). This implies that
\( z \in W_A(\alpha) \cap W_B(\beta) \), \( z \neq 0 \). Therefore the proof is complete. \( \square \)

Summarizing Lemmas A.1 and A.2, the following result holds true.

Theorem A.3. If two matrices \( A \) and \( B \) are commutative, then
\[ \sigma(A - B) = \{\alpha - \beta \mid \alpha \in \sigma(A), \beta \in \sigma(B), \quad W_A(\alpha) \cap W_B(\beta) \neq \{0\}\}. \]

Acknowledgment. We appreciate the many useful comments and suggestions
offered by the referees.

REFERENCES

control, in Proceedings of the Conference on PDEs and Phenomena in Miyazaki 2005,

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

