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Semantics for Intuitionistic Epistemic Logics of Shallow Depths for Game Theory

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Abstract Epistemic logic of shallow depths has been developed by Kaneko-Suzuki using classical logic. In this paper, we adopt intuitionistic logic as the base logic to limit each player's reasoning to constructive one. The resulting logical system is denoted by IG_{EF} , and we introduce epistemic possible world semantics for IG_{EF} . We show the soundness and completeness of IG_{EF} with respect to this semantics for it. We also show some applications to game-theoretical decision making.

Keywords logic and game theory · epistemic logic · intuitionistic logic · Kripke-type semantics

JEL Classification Numbers: C70

1 Introduction

In order to study players' limited interpersonal inferences for decision making in games, Kaneko-Suzuki (2002b, 2003) developed epistemic logic GL_{EF} of shallow depths and its semantics. In Kaneko-Suzuki (2005), they introduced also intuitionistic epistemic logic IG_{EF} to capture some constructive nature of decision making. Logic IG_{EF} differs from GL_{EF} in their base logics ; the base logic of the former is intuitionistic logic¹, while that of the latter is classical logic. Technically, IG_{EF} is obtained from GL_{EF} by adding one constraint on expressions of statements in GL_{EF} . However, the corresponding semantic framework is not so straightforward. In this paper, we develop the semantical framework for IG_{EF} by extending the semantics for

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¹ For the concise introduction to mathematical intuitionism, see Dummett (2000). For the fundamentals in studying intuitionistic logic, we refer the reader to Drágalin's textbook Drágalin (1988).

GL_{EF} . We will show the *soundness* and *completeness theorems* for IG_{EF} with respect to our semantics.

The semantics for GL_{EF} , developed in Kaneko-Suzuki (2002a, 2003), is a modification of Kripke-semantics incorporating *epistemic structures* E and F . For the semantics for IG_{EF} , we should consider two types of Kripke semantics; one for epistemic logic and the other for intuitionistic logic. In our framework, these two different structures coexist and are connected, so as to capture the interpersonal and intuitionistic features together. For this extended semantics, we obtain the soundness and completeness theorems for IG_{EF} .

Epistemic logic GL_{EF} was developed to capture boundedly rational players in game situations. It employs “shallow depths” of reasoning and “**KD**-type” logic, to free us from common knowledge and the other epistemic axioms adopted in **S5**-type approaches. IG_{EF} takes one step further toward modeling boundedly rational players by adopting constructive intuitionistic reasoning as its base.

We now give a small game example to show that the shallow reasoning is inherited by IG_{EF} from GL_{EF} , and that IG_{EF} is able to discuss new questions that cannot be in GL_{EF} . This example will be elaborated on more in Section 4.

Example 1.1 [Large Store vs. Small Store] Consider the situation where a large store (a supermarket, player 1) and a small store (a mini-mart, player 2) are competing. Now, we consider decision making by the small store. Player 2 has the subjective understanding of the situation: Player 1 is large enough to ignore the small store. but 2’s profits are influenced by 1’s choice, 2’s understanding is described by the following tables:

		$s_{2,1}$	$s_{2,2}$
$s_{1,1}$	600	0	2
$s_{1,2}$	300	10	−1

Player 1 chooses his action simply based on the left table and his payoff maximization. However, player 2 may have various potential decision criteria, even if payoff maximization is assumed.

Here, we consider two decision criteria for player 2.

- *Max-Min Criterion*: Player 2 evaluates each of his actions by its worst payoff; and he should choose an action to maximize this evaluation.

The worst payoff to player 2 for $s_{2,1}$ is 0, and that for $s_{2,2}$ is −1. Thus the Max-Min action for 2 is $s_{2,1}$.

- *Prediction-Decision Criterion*: Player 2 predicts that player 1 would choose the best action. Then, player 2 should choose the best action toward his prediction.

In the above example, player 2 predicts that 1 would choose $s_{1,1}$, and then 2 chooses the best action $s_{2,2}$ toward $s_{1,1}$.

When player 2 adopts the Max-Min Criterion, he does not consider player 1’s decision; so no interpersonal inferences are required. On the other hand, when player 2 adopts the Prediction-Decision Criterion, interpersonal inferences are involved. The

required interpersonal structure differs with the choice of a decision criterion. In both cases, common knowledge is not involved, and the required interpersonal structures are shallow, not infinite. Also, these inferences do not require the epistemic axioms of positive introspection, negative introspection, or truth, which are typically adopted in the standard epistemic logic approach (cf. Fagin et al. (1995)).

The above argument can be executed in epistemic logic GL_{EF} . To show the new feature captured by IG_{EF} , however, we consider a different decision criterion as follows.

- *Oracular Decision Criterion*²: If player 1's choice is $s_{1,1}$, then player 2 chooses the best action toward $s_{1,1}$; and otherwise use the Max-Min Criterion.

This criterion recommends one action in each of the two alternative cases. However, the action recommended differs in the two cases; the recommendation is not decisive. Nevertheless, it gives the existence of a recommendation. This assertion of the mere existence of a recommendation is provable in GL_{EF} . In this sense, GL_{EF} allows non-constructive reasoning. On the other hand, it is shown in Section 4 (Theorem 4.3), that if the players follow intuitionistic logic, player 2 does not reach the conclusion of existence of a decision. The constructive nature of intuitionistic logic built in IG_{EF} allows us to distinguish clearly between the existence of a recommended action and the ability to obtain a specific recommendation.

In sum, IG_{EF} inherits the important features of GL_{EF} . In addition, IG_{EF} overcomes some non-constructive nature of GL_{EF} . Using Example 1.1, this distinction will be shown by our semantics.

This paper is organized as follows. In Section 2, we introduce a proof-theoretical formulation of intuitionistic epistemic logic IG_{EF} . Section 3 is devoted to formulating our semantics. There, we state the soundness and completeness theorems. Since the proof of the completeness theorem is long and technical, it is postponed to Section 6. Example 1.1 is examined in Section 4 as an application of our IG_{EF} -formalism and semantics. In Section 5, we will close the paper with concluding remarks.

2 Intuitionistic Epistemic Logic IG_{EF} of Shallow Depths

In this section, we introduce logical system IG_{EF} . In IG_{EF} , the subscripts E and F in IG_{EF} are restrictions on ‘nesting’ of belief operators in formulas and on interpersonal inferences. They will be defined presently.

2.1 Epistemic Depths and Epistemic Structures

We fix the finite non-empty set $N = \{1, 2, \dots, n\}$ of players. Our propositional epistemic language $\mathcal{L}$ consists of logical connectives³ $\vee$ (disjunction), $\wedge$ (conjunction),

² The term “oracular” suggests some ability to foretell the other's choice as an oracle. But the ability itself is not described in the criterion. In this sense, this criterion looks oracular as well.

³ In Kaneko-Suzuki (2002a, 2002b, 2003, 2005), we treated $\vee$ and $\wedge$ as operators acting on finite sets. Namely, for any finite set Φ , $\vee \Phi$ ($\wedge \Phi$) is the disjunction (conjunction, resp.) of formulas in Φ . Such defi-

$\supset$ (implication), $\neg$ (negation), *belief operators* B_i for each $i \in N$, and a countable non-empty set PV of propositional variables. *Formulas* are defined inductively in the standard manner. The expression $B_i(A)$ means that player i believes formula A . $A \equiv B$ is the abbreviation of $(A \supset B) \wedge (B \supset A)$.

Next, we define $N^{<\omega} := \{(i_1, \dots, i_m) \mid i_j \in N \text{ and } m \geq 0\}$. A sequence $(i_1, \dots, i_m) \in N^{<\omega}$ is called an *epistemic status*. When $m = 0$, the sequence is called the *empty status*, and is denoted by ε . For $e = (i_1, \dots, i_m)$, $B_{i_1} \cdots B_{i_m}$ is denoted by B_e . Similarly, the formula $B_{i_1} \cdots B_{i_m}(A)$ is denoted by $B_e(A)$. If e is the empty status ε , B_e is the empty sequence of symbols, and $B_e(A)$ is the formula A itself.

We have the standard concatenation operation $\circ$, namely $(i_1, \dots, i_m) \circ (j_1, \dots, j_n) := (i_1, \dots, i_m, j_1, \dots, j_n)$. For each subset S of $N^{<\omega}$ and each $e \in N^{<\omega}$, $e \circ S$ is the set $\{e \circ e' \mid e' \in S\}$. For simplicity, we sometimes write the length-one epistemic states (i) by i , e.g., $e \circ (i) \circ (j) \circ e'$ is simply written as $e \circ i \circ j \circ e'$.

Definition 2.1 We define the *epistemic depth* $\delta(A)$ of a formula A inductively as follows:

- ($\delta 0$) $\delta(A) = \{\varepsilon\}$ if A is a propositional variable in PV ,
- ($\delta \neg$) $\delta(\neg C) = \delta(C)$,
- ($\delta \wedge$) $\delta(C \wedge D) = \delta(C) \cup \delta(D)$,
- ($\delta \vee$) $\delta(C \vee D) = \delta(C) \cup \delta(D)$,
- ($\delta \supset$) $\delta(C \supset D) = \delta(C) \cup \delta(D)$,
- (δB) $\delta(B_i(C)) = \{i \circ e \mid e \in \delta(C)\}$.

For example, $\delta(p \supset B_1 B_2(q)) = \{\varepsilon, (1, 2)\}$, where p and q are propositional variables. We define the set $\delta(\Gamma) = \{\delta(C) \mid C \text{ occurs in } \Gamma\}$ for a finite sequence Γ of formulas.

A non-empty subset E of $N^{<\omega}$ is said to be an *epistemic structure* if for $m \geq 1$;

$$(i_1, \dots, i_m) \in E \text{ implies } (i_1, \dots, i_{m-1}) \in E. \quad (1)$$

The set $N^{<\omega}$ of all epistemic statuses is the largest epistemic structure, and the smallest one is $\{\varepsilon\}$. Every epistemic structure contains the empty status ε .

We use epistemic structures in two ways⁴. One is to control ‘nesting’ of epistemic operations in formulas (*the descriptive epistemic structure*), the other is to control interpersonal epistemic inferences in a formal system (*the inferential epistemic structure*). In this paper, inferential epistemic structures play a central role for establishing logical systems and their semantics, while descriptive epistemic structure is used in describing beliefs of players in game-theoretic applications (cf. Section 4).

Consider the following examples: $F^1 := \{\varepsilon, (1), (2)\}$ and $F^2 := \{\varepsilon, (1), (2), (2, 1)\}$, which are illustrated in Figure 2.1. Each epistemic structure has a form of a tree by (1). Since each player thinks about only his own payoffs, F^1 suffices for the Max-Min criterion in Example 1.1. The black circles named as (1) and (2) express 1’s and 2’s minds, respectively. The symbol ε expresses the outside investigator’s mind.

notations are more convenient for describing game theoretical applications. In this paper, $\vee$ and $\wedge$ are treated as binary connectives in accordance with the literature of mathematical logic, especially with Gentzen’s LJ (Gentzen (1934-35)). This treatment makes proofs in Section 6 simpler.

⁴ See Kaneko-Suzuki (2002b) for extensive discussions on the use of these two epistemic structures.

Figure 2.1

In this situation, neither player thinks about the other's mind. Epistemic structure F^2 is required for the Prediction-Decision criterion. The black circle named by (2,1) expresses 1's hypothetical mind in 2's mind.

Definition 2.2 A formula A is said to be *e-admissible* in E , if $e \circ \delta(A) \subseteq E$. The set of all *e-admissible* formulas in E is denoted by $\mathcal{P}_E(e) = \{A \mid e \circ \delta(A) \subseteq E\}$.

The set $\mathcal{P}_E(e)$ is also described as $\{A \mid B_e(A) \in \mathcal{P}_E(\varepsilon)\}$. Formulas in $\mathcal{P}_E(e)$ are admissible within the nested minds $e = (i_1, \dots, i_n)$ in E .

2.2 Formal System IG_{EF}

In the following, we fix the epistemic structures E and F with $F \subseteq E$. We introduce one additional symbol $\rightarrow$ to define the concept of a sequent. For a finite (possibly empty) sequence Γ of formulas and a formula A , the two expressions $\Gamma \rightarrow A$ and $\Gamma \rightarrow$ are called (*intuitionistic*) *sequents*.

The meaning of the expression $\Gamma \rightarrow A$ is “we have a (constructive) proof of A from Γ ”, and that of $\Gamma \rightarrow$ is “we have a (constructive) proof of a contradiction from Γ ”. We write a sequent simply by $\Gamma \rightarrow \Theta$, with assuming that Θ contains at most one formula.

We add a pair of auxiliary symbols $[$ and $]$, and introduce the expression: $B_e[\Gamma \rightarrow \Theta]$, where $e = (i_1, \dots, i_m) (\in N^{<\omega})$ is an epistemic status and $\Gamma \rightarrow \Theta$ is a sequent. We call this expression a *thought-sequent*. The thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is an abbreviation of $B_{i_1} \dots B_{i_m}[\Gamma \rightarrow \Theta]$. $B_{(1,2)}[\Gamma \rightarrow \Theta] = B_1 B_2[\Gamma \rightarrow \Theta]$ means “player 1 believes that player 2 believes that $\Gamma \rightarrow \Theta$ ”. The thought-sequent $B_e[\Gamma \rightarrow \Theta]$ with ε is often written as $\Gamma \rightarrow \Theta$.

A thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is said to be *admissible* in IG_{EF} , if $e \in F$ and $e \circ \delta(\Gamma, \Theta) \subseteq E$, i.e., $e \in F$ and every formula in Γ and Θ is *e-admissible* in E .

The basic principle for IG_{EF} is: Non-epistemic thinking of hypothetical player i_m in $e = (i_1, \dots, i_m)$ is governed by Gentzen's LJ; it is formulated by 15 non-epistemic inference rules with one axiom schema (initial sequent). This part has the same outer $B_e[\cdot]$. The thinking of player i_m described in this part is connected with that of player i_{m-1} by one epistemic inference rule. The epistemic structure F constrains the use of this rule.

In the list below, Γ, Δ, Θ and Λ are finite possibly empty sequences of formulas; A and B are arbitrary formulas. Moreover, Θ and Λ must contain at most one formula.

Initial Sequent: $B_e[A \rightarrow A]$

Non-Epistemic Inference Rules:

Structural Rules

$$\frac{B_e[\Gamma \rightarrow \Theta]}{B_e[A, \Gamma \rightarrow \Theta]} (w \rightarrow)$$

$$\frac{B_e[\Gamma \rightarrow]}{B_e[\Gamma \rightarrow A]} (\rightarrow w)$$

$$\frac{B_e[A, A, \Gamma \rightarrow \Theta]}{B_e[A, \Gamma \rightarrow \Theta]} (c \rightarrow)$$

$$\frac{B_e[\Delta, A, B, \Gamma \rightarrow \Theta]}{B_e[\Delta, B, A, \Gamma \rightarrow \Theta]} (e \rightarrow)$$

$$\frac{B_e[\Gamma \rightarrow A] \quad B_e[A, \Delta \rightarrow \Lambda]}{B_e[\Gamma, \Delta \rightarrow \Lambda]} (cut)$$

Rules for logical connectives

$$\frac{B_e[A, \Gamma \rightarrow \Theta]}{B_e[A \wedge B, \Gamma \rightarrow \Theta]} (\wedge \rightarrow 1) \quad \frac{B_e[\Gamma \rightarrow A] \quad B_e[\Gamma \rightarrow B]}{B_e[\Gamma \rightarrow A \wedge B]} (\rightarrow \wedge)$$

$$\frac{B_e[B, \Gamma \rightarrow \Theta]}{B_e[A \wedge B, \Gamma \rightarrow \Theta]} (\wedge \rightarrow 2)$$

$$\frac{B_e[A, \Gamma \rightarrow \Theta] \quad B_e[B, \Gamma \rightarrow \Theta]}{B_e[A \vee B, \Gamma \rightarrow \Theta]} (\vee \rightarrow) \quad \frac{B_e[\Gamma \rightarrow A]}{B_e[\Gamma \rightarrow A \vee B]} (\rightarrow \vee 1)$$

$$\frac{B_e[\Gamma \rightarrow B]}{B_e[\Gamma \rightarrow A \vee B]} (\rightarrow \vee 2)$$

$$\frac{B_e[\Gamma \rightarrow A]}{B_e[\neg A, \Gamma \rightarrow]} (\neg \rightarrow)$$

$$\frac{B_e[A, \Gamma \rightarrow]}{B_e[\Gamma \rightarrow \neg A]} (\rightarrow \neg)$$

$$\frac{B_e[\Gamma \rightarrow A] \quad B_e[B, \Delta \rightarrow \Lambda]}{B_e[A \supset B, \Gamma, \Delta \rightarrow \Lambda]} (\supset \rightarrow) \quad \frac{B_e[A, \Gamma \rightarrow B]}{B_e[\Gamma \rightarrow A \supset B]} (\rightarrow \supset)$$

Epistemic Inference Rule: (distribution rule)

$$\frac{B_{e \circ i}[\Gamma \rightarrow \Theta]}{B_e[B_i(\Gamma) \rightarrow B_i(\Theta)]} (B \rightarrow B) \text{ where } e \circ i \in F.$$

Here $B_i(\Gamma)$ is the sequence $B_i(A_1), B_i(A_2), \dots, B_i(A_n)$, when Γ is $A_1, A_2, \dots, A_n$.

All thought-sequents are controlled by E , and all epistemic inferences are controlled by F . Thus, E and F are called *descriptive* and *inferential* epistemic struc-

tures, respectively. As mentioned above, all nonepistemic rules are exactly the same as Gentzen's LJ (Gentzen (1934-35)), except that each sequent has the outer $B_e[\cdot]$.

A (formal) proof in IG_{EF} is a finite rooted tree satisfying:

- (i) Each node is associated with a thought-sequent admissible in IG_{EF} ;
- (ii) Each leaf of the tree is associated with an initial sequent;
- (iii) Each adjoining node together with associated thought-sequents forms an instance of some inference rule.

A sequent $B_e[\Gamma \rightarrow \Theta]$ is said to be *provable in IG_{EF}* if there exists a proof in IG_{EF} such that $B_e[\Gamma \rightarrow \Theta]$ is associated with its root. A formula A is said to be *provable in IG_{EF}* if the sequent $B_e[\rightarrow A]$ is provable in IG_{EF} .

IG_{EF} is based on the same idea as classical epsitemic logic GL_{EF} in Kaneko-Suzuki (2002b) (cf. also Kaneko-Suzuki (2002a, 2003)), except for taking intuitionistic sequent calculus LJ as the base logic for IG_{EF} . Logic GL_{EF} is obtained from IG_{EF} by allowing the right-side of each thought-sequent to have more than one formula. The reader may notice some differences to the treatment of GL_{EF} in Kaneko-Suzuki (2002b). Though one difference is the use of sequences rather than finite sets in this paper, the main difference is that the right introduction rules are restricted here not to allow more than one formula in the right sequence. This corresponds to restricting Θ to have at most one formula in Kaneko-Suzuki (2002b). This difference has an important implication. In GL_{EF} , $B_i(A) \vee \neg B_i(A)$ is always provable for any formula A , while not always in IG_{EF} . We will show this fact in Example 3.7.

3 Epistemic Possible World Semantics

We introduce our Kripke-type semantics, i.e., epistemic possible world semantics.

3.1 Soundness and Completeness for a formula

The semantics of IG_{EF} must be generalization of semantics both of intuitionistic logic and modal logic **KD**. Our semantics has two kinds of accessibility relations: intuitionistic⁵ and modal accessibility relations.

Definition 3.1 An $(n+2)$ -tuple $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ is said to be an *Int_F-frame*, if the following conditions are satisfied.

- F1 (*F*-classification): W is an arbitrary non-empty set and is partitioned to a family $\{W_e : e \in F\}$ of non-empty sets indexed by F . When $w \in W_e$, we write $\lambda_{\mathcal{F}}(w) = e$, which is called the *epistemic status* of w .
- F2 (Intuitionistic mind): $\preceq$ is a partial order on W satisfying: for every $w, w' \in W$, if $w \preceq w'$, then $\lambda_{\mathcal{F}}(w) = \lambda_{\mathcal{F}}(w')$.
- F3 $R_1, \dots, R_n$ are binary relations on W satisfying:

⁵ For a concise and heuristic explanation of Kripke models for intuitionistic logic, see van Dalen (2004).

- F3.1 (Accessibility of epistemic statuses): for every $w, w' \in W$ and every $i \in N$, if wR_iw' , then $\lambda_{\mathcal{F}}(w') = \lambda_{\mathcal{F}}(w) \circ i$.
- F3.2 (F -seriality): each R_i is F -serial, i.e., for every $w \in W$ and every $i \in N$, if $\lambda_{\mathcal{F}}(w) \circ i \in F$, there exists a $w' \in W$ with wR_iw' .
- F4 (Connections between intuitionistic and epistemic accessibilities): for every $w_1, w_2, w_3 \in W$ and every $i \in N$,
- F4.1: $w_1 \preceq w_2$ and $w_2R_iw_3$ imply $w_1R_iw_3$.
- F4.2: $w_1R_iw_2$ and $w_2 \preceq w_3$ imply $w_1R_iw_3$.

The Int_F -frame $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ is regarded as a family of intuitionistic Kripke models connected by the inferential epistemic structure F . Each $W_e = W_{(i_1, \dots, i_m)}$ together with $\preceq$, restricted to W_e , turns out to be an intuitionistic frame (see e.g. van Dalen (2004)). This is a model of (imagined) player i_m in the mind of i_{m-1} of $\dots$ of i_1 . Each $w \in W_{(i_1, \dots, i_m)}$ expresses a stage of i_m 's intellectual activity in the mind of i_{m-1} of $\dots$ of i_1 . Condition F3 describes accessibilities between possible stages in the mind. Condition F3.1 requires a legitimate connection between epistemic statuses and accessibility. Condition F3.2 corresponds to our choice of **KD**-type system: It requires that if a possible world w has a next epistemic status $\lambda_{\mathcal{F}}(w) \circ i \in F$, then there exists another w' having the status $\lambda_{\mathcal{F}}(w) \circ i$. Condition F4 requires relations between intrapersonal and interpersonal accessibilities. The relation $\preceq$ describes the progression of stages of one mind in the intuitionistic way. The relations $R_1, \dots, R_n$ connect the minds of imagined players. Conditions F4.1 and F4.2 connect those two types of relations.

Kaneko-Suzuki (2002b, 2003) introduced semantics for epistemic logic GL_{EF} with classical logic as its base logic. In the Kripke-type semantics for GL_{EF} , every W_e is a set of *classical* models. If $\preceq$ is the equality $=$ on W_e (i.e., no intuitionistic accessibility relation between distinct worlds) in our present semantics, then $(W_e, =)$ is essentially a set of classical models, i.e., our semantics coincides with that of Kaneko-Suzuki (2002b, 2003) when $\preceq$ is the equality on the frame (subject to $E = F$ in Kaneko-Suzuki (2003)). We give a very simple example of such an Int_F -frame which will be used in Section 4. Let F be an arbitrary epistemic structure. We regard F itself as the set of possible worlds $\{W_e\}_{e \in F} = \{\{e\}\}_{e \in F}$. We define $\preceq$ and R_i ($i \in N$) by:

- $e \preceq e'$ if and only if $e = e'$;
- eR_ie' if and only if $e' = e \circ i$.

This is called the *degenerate* Int_F -frame, and denoted by $\mathcal{D}(F)$. In the case of $F = F^2$, the $\mathcal{D}(F^2)$ is depicted as Fig. 3.1. As mentioned above, this $\mathcal{D}(F^2)$ is a frame for GL_{EF} . An example of Int_F -frame which is not a frame for GL_{EF} will be shown after defining other concepts of our semantics.

For simplicity, we introduce the product $R \circ S$ of two relations R and S : $xR \circ Sy$ if and only if xRz and zSy for some z . We write the product of m relations $R_{i_1}, \dots, R_{i_m}$ by: $R_e = R_{i_1} \circ \dots \circ R_{i_m}$, and R_e is understood as the identity relation $=$.

Figure 3.1: $\mathcal{D}(F^2)$

The following lemma gives basic properties of Int_F -frames corresponding to F4.1 and F4.2, and will be used throughout this paper without referring to it.

Lemma 3.2 *Let $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ be an Int_F -frame. For every $w_1, w_2, w_3 \in W$ and every epistemic status $e \neq \varepsilon$,*

- (1) $w_1 \preceq w_2$ and $w_2 R_e w_3$ imply $w_1 R_e w_3$;
- (2) $w_1 R_e w_2$ and $w_2 \preceq w_3$ imply $w_1 R_e w_3$.

We first introduce w -atomic formulas. Let $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ be an Int_F -frame. For each $w \in W$ with $\lambda_{\mathcal{F}}(w) = e$, formulas in $PV \cup \{B_i(A) \in \mathcal{P}_E(e) : e \circ i \notin F\}$ are called w -atomic⁶. The set of w -atomic formulas is denoted by $\mathcal{A}_{EF}(w)$. Note that when $\lambda_{\mathcal{F}}(w) \circ i = e \circ i \in F$, $\{B_i(A) \in \mathcal{P}_E(e) : e \circ i \notin F\}$ is the empty set. Then, we define the crucial requirement, hereditariness, for intuitionistic semantics (cf. van Dalen (2004)).

Definition 3.3 [Hereditariness Condition] A relation $\models$ between each $w \in W$ and each $A \in \mathcal{A}_{EF}(w)$ is said to be an Int_{EF} -valuation on $\mathcal{F}$, if it satisfies:

(HC): for every $w, w' \in W$ and every $A \in \mathcal{A}_{EF}(w)$, if $w \models A$ and $w \preceq w'$, then $w' \models A$.

A pair $(\mathcal{F}, \models)$ satisfying (HC) is said to be an Int_{EF} -model.

We will extend this relation $\models$ to the relation between w ($\in W_e \subseteq W$) and C ($\in \mathcal{P}_E(e)$). For each of $A \wedge B, A \vee B, A \supset B, \neg A, B_i(A) \in \mathcal{P}_E(e)$ ($e \circ i \in F$), we define by induction on the length of a formula:

- (V- $\wedge$) $w \models A \wedge B$ if and only if $w \models A$ and $w \models B$,
- (V- $\vee$) $w \models A \vee B$ if and only if $w \models A$ or $w \models B$,
- (V- $\supset$) $w \models A \supset B$ if and only if for every w' with $w \preceq w', w' \not\models A$ or $w' \models B$,
- (V- $\neg$) $w \models \neg A$ if and only if for every w' with $w \preceq w', w' \not\models A$,
- (V- B_i) $w \models B_i(A)$ if and only if for every w' with $w R_i w', w' \models A$.

If $e \circ i \notin F$, then $B_i(A) \in \mathcal{A}_{EF}(w)$, which implies either $w \models B_i(A)$ or $w \not\models B_i(A)$. Note that w' in (V- B_i) has the epistemic status $\lambda_{\mathcal{F}}(w') = e \circ i$, i.e., $w' \in W_{e \circ i}$, and hence $e \circ i \in F$.

⁶ A formula $B_i(A)$ is called w -atomic, since in world w it is treated as an atomic formula.

The next lemma states that hereditariness extends to all of A . We can show this lemma by induction on A .

Lemma 3.4 (Hereditariness) *For every formula A , the set $\{w \mid w \models A\}$ is $\preceq$ -hereditary. That is, $w \models A$ and $w \preceq w'$ imply $w' \models A$.*

Definition 3.5 A formula A is said to be *true* at $w \in W$, if $A \in \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$ and $w \models A$. A formula A is said to be *true* in an Int_{EF} -model $(\mathcal{F}, \models)$, if $A \in \mathcal{P}_E(\mathcal{E})$ and $w \models A$ for every $w \in W_{\mathcal{E}}$. A formula A is said to be Int_{EF} -*valid* (or, simply *valid*) in an Int_F -frame $\mathcal{F}$, if for every Int_{EF} -valuation $\models$ on $\mathcal{F}$, A is true in $(\mathcal{F}, \models)$ (i.e., for every $w \in W_{\mathcal{E}}$, $w \models A$).

The main theorem of the paper is the equivalence between the provability in IG_{EF} and the validity in semantics.

Theorem 3.6 (Completeness Theorem for IG_{EF}) *A formula A is provable in IG_{EF} if and only if A is Int_{EF} -valid in every Int_F -frame.*

The “only-if” part is called the *soundness*. The “if”-part is called the *completeness*. Their proofs are given in Section 6. Theorem 3.6 is an extension of various modal completeness theorems (cf. Hughes-Cresswell (1985)). It is also an extension of the completeness theorem for intuitionistic logic (cf. van Dalen (2004)). Those completeness theorems are using different types of Kripke models. Our logical system has two structures corresponding to those different Kripke models. In our proof, we execute two different completeness theorems in a consistent manner.

Theorem 3.6 has another aspect described by two epistemic structures E and F in IG_{EF} . In Kaneko-Suzuki (2003), this aspect was already considered in epistemic logic GL_{EF} , but with classical logic as the base logic. Our proof will treat this and the aspect stated above in a consistent manner; modal, intuitionistic, and shallow epistemic depths.

We show a simple but important application of our soundness theorem.

Example 3.7 [A violation of the law of excluded middle in IG_{EF}] Let E and F be epistemic structures with $F = \{\mathcal{E}, (1)\} \subseteq E$. We show that $B_1(p) \vee \neg B_1(p)$ is not provable in IG_{EF} , where p is a propositional variable. We define an Int_F -frame $\mathcal{F}_1$ as follows:

$$W = W_{\mathcal{E}} \cup W_{(1)}; \quad W_{\mathcal{E}} = \{a, b\}, \quad W_{(1)} = \{v, w\},$$

$$a \preceq a, \quad b \preceq b, \quad a \preceq b; \quad w \preceq w, \quad v \preceq v$$

$$aR_1w, \quad bR_1v, \quad aR_1v$$

See Fig. 3.2. Here, solid arrows ($\rightarrow$) mean R_1 , and dashed arrows ($\dashrightarrow$) mean $\preceq$. Dashed arrows from worlds to themselves (i.e., arrows showing reflexivity) are omitted for the sake of simplicity.

Figure 3.2: $\mathcal{F}_1$

We define an Int_{EF} -valuation $\models$ by

$$x \models p \text{ if and only if } x = v,$$

for every $x \in W$. Then, $b \models B_1(p)$. Since $a \preceq b$, we have $a \not\models \neg B_1(p)$ by $(V-\neg)$. Next, since $w \not\models p$ and aR_1w , we have $a \not\models B_1(p)$. Therefore, we have $a \not\models B_1(p) \vee \neg B_1(p)$. By the soundness part of Theorem 3.6, we have that $B_1(p) \vee \neg B_1(p)$ is not provable in IG_{EF} . In classical logic, this type of example cannot arise. Classical logic requires the intuitionistic relation to be the equality relation, so the clause $(V-\neg)$ reduces to: $w \models \neg A$ if and only if $w \not\models A$.

Note that $\rightarrow$ from a to v (i.e., aR_1v) is not used in the process of checking $a \not\models B_1(p) \vee \neg B_1(p)$. This arrow is indispensable for making $\mathcal{F}_1$ to be an Int_F -frame. This observation suggests that this violation of the law of excluded middle is caused not by introducing modalities but by adopting intuitionistic logic.

3.2 Soundness and Completeness for a thought-sequent

Since our system IG_{EF} is formulated as a sequent calculus, we define truthfulness and validity of thought-sequents in models and frames. In this subsection, we introduce these definitions, and state the completeness Theorem 3.9, from which Theorem 3.6 is derived.

Definition 3.8 (cf. Definition 3.5) Let $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ be an Int_F -frame and $\models$ an Int_{EF} -valuation on $\mathcal{F}$. A(n intuitionistic) sequent $\Gamma \rightarrow \Theta$ is said to be *true* at $w \in W$, if $\Gamma \cup \Theta \subseteq \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$ and the following implication holds: If for every A in Γ is true at w , then there exists a C in Θ such that C is true at w . A thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is said to be *true* in $(\mathcal{F}, \models)$, if $\Gamma \rightarrow \Theta$ is true at every $w \in W_e$. (Note that e in the subscripts of $B_e[\Gamma \rightarrow \Theta]$ and W_e are the same.) A thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is said to be *Int_{EF} -valid* (or simply, *valid*) in $\mathcal{F}$, if it is true in $(\mathcal{F}, \models)$ for every Int_{EF} -valuation $\models$ on $\mathcal{F}$.

Then, the completeness theorem for thought-sequents is stated as follows.

Theorem 3.9 (The Second Form of Completeness Theorem for IG_{EF}) *A thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is provable in IG_{EF} if and only if it is Int_{EF} -valid in every Int_F -frame.*

Recall that a formula A is provable in IG_{EF} if and only if $B_e[\rightarrow A]$ is provable in IG_{EF} . Therefore, by Definitions 3.5 and 3.8, Theorem 3.6 follows from Theorem 3.9. In Section 6, we show the completeness in this second form i.e., Theorem 3.9.

Note that the concept, “truth at $w \in W$ ”, is defined only for formulas not for thought-sequents, while “truth in an Int_{EF} -model” is defined both for formulas and thought-sequents. The outer part $B_e[\cdot]$ of thought-sequent refers the set W_e in an Int_{EF} -model (cf. Definition 3.8). This difference explains the difference between formulas and thought-sequents from the viewpoint of semantics. Recall the meaning of thought-sequents explained in Section 2.2. Then, the following definition of a translation of thought-sequents to formulas is plausible.

Definition 3.10 Let $B_e[\Gamma \rightarrow \Theta]$ be a thought-sequent. We write by $B_e[\Gamma \rightarrow \Theta]^*$ the following formula:

$$\begin{cases} B_e(\bigwedge \Gamma \supset A) & \text{if } \Theta \text{ is } A, \\ B_e(\neg \bigwedge \Gamma) & \text{if } \Theta \text{ is empty,} \end{cases}$$

where $\bigwedge \Gamma$ is the formula obtained from formulas in Γ by taking $\wedge$ when Γ is non-empty, and when Γ is empty it is stipulated to be the formula $p \supset p$ with a propositional variable p .

Then, the next proposition easily follows from the definition.

Proposition 3.11 (1) *If a thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is true in an Int_{EF} -model, then so is $B_e[\Gamma \rightarrow \Theta]^*$.*

(2) *If $B_e[\Gamma \rightarrow \Theta]$ is Int_{EF} -valid in an Int_F -frame, then so is $B_e[\Gamma \rightarrow \Theta]^*$.*

Although this translation is natural, the converse implications of (1) and (2) fail to hold. That is, truth in an Int_{EF} -model and validity in an Int_F -frame do not coincide in $B_e[\Gamma \rightarrow \Theta]$ and $B_e[\Gamma \rightarrow \Theta]^*$. Let us see this fact. Let $\mathcal{F}_2$ be an Int_F -frame defined by Fig. 3.3 where $F = \{\varepsilon, (1)\}$ and $\mathcal{F}_2 = (W : \preceq, R_1)$ with $W_\varepsilon = \{w\}$, $W_{(1)} = \{a, b\}$ and $W = W_\varepsilon \cup W_{(1)}$. Here, the solid arrow ($\rightarrow$) means R_1 , and the dashed arrow ($--\rightarrow$) means $\preceq$.

Figure 3.3: $\mathcal{F}_2$

Then, it is easy to check that for every p and every Int_{EF} -valuation $\models$ on $\mathcal{F}_2$, it holds that $a \models p \vee \neg p$. Therefore, $B_1[\rightarrow p \vee \neg p]^*$, i.e., $B_1(p \vee \neg p)$, is valid in $\mathcal{F}_2$. Next we define an Int_{EF} -valuation $\models$ on $\mathcal{F}_2$ by: $x \models p$ if and only if $x = a$. By this Int_{EF} -valuation, we have $a \models p$ and $b \not\models p$. We shall check that $b \not\models \neg p$. Suppose otherwise, i.e., $b \models \neg p$. By (HC) and $b \preceq a$, it follows that $a \models \neg p$. This contradicts $a \models p$. Hence, we have both of $b \not\models p$ and $b \not\models \neg p$. That is, $b \not\models p \vee \neg p$. Therefore, $B_1[\rightarrow p \vee \neg p]$ is not true in $(\mathcal{F}_2, \models)$.

The culprit in this model is b in $W_{(1)}$, which is not reachable from $w \in W_\varepsilon$ by the solid arrow. At the end of Section 6, we will take care of this point by introducing the concept *R-reachability* and will show a variant of completeness theorem with the *R-reachability* condition (Theorem 6.19). Then we show that “validity in every Int_F -frame” coincides in both of thought-sequents and their translations by the virtue of this completeness (cf. Theorem 6.22).

4 Applications

We analyze the example with large and small stores given in Section 1, in terms of IG_{EF} with Theorem 3.6. Specifically, we will formulate the two decision criteria given in Section 1. We show that these two criteria need different inferential epistemic structures for decision making. Then, it is argued introducing the third criterion that our logic IG_{EF} differentiates between a specific recommendation and the existence of a recommendation.

Let $N = \{1, 2\}$. Recall that player 1 is the large store and player 2 the small store. We fix the epistemic structures E and F with $F \subseteq E$ for IG_{EF} . Let $s_{i,k}$ ($i, k = 1, 2$) be the symbol indicating the k -th strategy of player i . Denote $S_i = \{s_{i,1}, s_{i,2}\}$ ($i = 1, 2$) and $S = S_1 \times S_2$. Throughout this section we assume $(2) \in F$.

To describe payoff functions g_1 and g_2 in terms of preferences, we introduce 16 propositional variables $P_i(u, s : v, t)$ ($(u, s), (v, t) \in S$) for each $i = 1, 2$. Propositional variable $P_i(u, s : v, t)$ describes $g_i(u, s) \geq g_i(v, t)$, meaning that player i (weakly) prefers (u, s) to (v, t) . Define:

$$\Gamma_i = \{P_i(u, s : v, t) \mid g_i(u, s) \geq g_i(v, t)\} \cup \{\neg P_i(u, s : v, t) \mid g_i(u, s) < g_i(v, t)\}.$$

The payoff function g_i in the non-formalized language is expressed by this set Γ_i in our formalized language. For a later purpose, the first set in the above union is denoted as Γ_i^+ and the second is Γ_i^- .

Max-Min Decision Criterion: Let us consider the Max-Min decision criterion for player 2 discussed in Section 1. It is the first step that player 2 evaluates each s of his strategies against a worst case u . In our formalization, it is expressed as⁷:

$$\min_2(u, s) := \bigwedge_{v \in S_1} P_2(v, s : u, s) . \quad (2)$$

⁷ The expressions $\bigwedge_{v \in S_i} A(v)$ and $\bigvee_{v \in S_i} A(v)$ are understood as $A(s_{i,1}) \wedge A(s_{i,2})$ and $A(s_{i,1}) \vee A(s_{i,2})$, respectively, since $S_i = \{s_{i,1}, s_{i,2}\}$. We use this abbreviation to make the meaning of formulas intuitively clear. Similarly, for any finite set $X = \{x_1, \dots, x_n\}$, $\bigwedge_{x \in X} A(x)$ is understood as $(\dots (A(x_1) \wedge A(x_2)) \wedge \dots) \wedge A(x_n)$.

The second step is to take the strategy s as a maximizer of this evaluation, which is expressed by:

$$\text{maxmin}_2(u, s) := \min_2(u, s) \wedge \bigwedge_{(v, t) \in S} \left(\min_2(v, t) \supset P_2(u, s : v, t) \right). \quad (3)$$

The Max-Min decision criterion is formulated as:

$$\text{MaxMin}_2(s) := \bigvee_{u \in S_1} \text{maxmin}_2(u, s). \quad (4)$$

This states that for some u , the pair (u, s) maximizes player 2's guaranteed payoff level.

We can give a formal-proof of the thought-sequent $B_2[I_2 \rightarrow \text{MaxMin}_2(s_{2,1})]$ in IG_{EF} without using $(B \rightarrow B)$. Since neither I_2 nor $\text{MaxMin}_2(s_{2,1})$ involves epistemic operators, F can be taken as $F = \{\varepsilon, (2)\}$. Thus, in player 2's mind, $\text{MaxMin}_2(s_{2,1})$ is inferred from I_2 without thinking about the other player's mind.

Prediction-Decision Criterion: Consider the Prediction-Decision criterion for player 2 discussed in Section 1. To describe this criterion, we first introduce formulas $Dm_1(u, v)$ and $Br_2(s; u)$ expressing “ u dominates v for player 1” and “ s is the best response of player 2 to u ”, respectively as follows:

$$Dm_1(u, v) := \bigwedge_{s \in S_2} P_1(u, s : v, s),$$

$$Br_2(s; u) := \bigwedge_{t \in S_2} P_2(u, s : u, t) .$$

We use the first formula to describe player 2's prediction about player 1's decision. The formula $B_1(Dm_1(s_{1,1}, s_{1,2}))$ means that player 1 believes $s_{1,1}$ dominates $s_{1,2}$. When $B_1(Dm_1(s_{1,1}, s_{1,2}))$ occurs in $B_2[\cdot]$, it describes player 2's prediction about player 1's decision. We denote this formula $B_1(Dm_1(s_{1,1}, s_{1,2}))$ by $Pd_2(s_{1,1})$.

The *Prediction-Decision criterion* for strategy s of player 2 is formulated as:

$$\bigvee_{u \in S_1} Pd_2(u) \wedge \bigwedge_{u \in S_1} \left(Pd_2(u) \supset Br_2(s; u) \right) . \quad (5)$$

The first conjunct means the existence of a prediction for player 2 on 1's decision. The second states that s is the best response to each predicted strategy u . We write this formula by $PD_2(s)$. Similar to the discussion of the Max-Min decision criterion, we can show that if $(2, 1) \in F$, then $B_2[B_1(I_1), I_2 \rightarrow PD_2(s_{2,2})]$ is provable in IG_{EF} . Moreover, we can show the converse of this using our semantics:

Theorem 4.1 $B_2[B_1(I_1), I_2 \rightarrow PD_2(s_{2,2})]$ is provable in IG_{EF} if and only if $(2, 1) \in F$.

Proof. We show here the “only if”-part. Suppose $(2, 1) \notin F \subseteq E$. Take the degenerate Int_F -frame $\mathfrak{D}(F)$. Note that for each $e \in F$, W_e consists of a unique element e : $W_e = \{e\}$. We will show that there is an Int_{EF} -valuation on $\mathfrak{D}(F)$ such that $(2) \not\models B_2[B_1(\Gamma_1), \Gamma_2 \rightarrow PD_2(s_{2,2})]$.

The set $\mathcal{A}_{EF}(e)$ of e -atomic formulas is given by:

$$\mathcal{A}_{EF}(e) = \begin{cases} PV & \text{if } e \neq (2), \\ PV \cup \{B_1(A) \mid A \text{ is a formula}\} & \text{otherwise.} \end{cases}$$

Let $\models$ be an Int_{EF} -valuation so that for $e \in \mathfrak{D}(F)$ with $e \neq (2)$,

$$e \models A \text{ if and only if } A \in \Gamma_1^+ \cup \Gamma_2^+$$

and for $e = (2)$,

$$(2) \models A \text{ if and only if } \begin{cases} A \in \Gamma_1^+ \cup \Gamma_2^+ \\ \text{or} \\ A \text{ is of the form } B_1(C) \text{ with } C \in \Gamma_1^- \end{cases}.$$

Then, every formula in $B_1(\Gamma_1)$ and Γ_2^+ is true at (2) . Note that for every $f \in F$, $(2) \preceq f$ if and only if $f = (2)$. Since $(2) \not\models A$ for every formula A with $\neg A \in \Gamma_2^-$, we have $(2) \models A$ for every $A \in \Gamma_2^-$. Therefore, $(2) \models A$ for every $A \in B_1(\Gamma_1) \cup \Gamma_2$.

Let us show $(2) \not\models PD_2(s_{2,2})$. This is shown by checking $(2) \not\models Pd_2(u)$ for every $u \in S_2$, which directly follows from the definition of $\models$. $\square$

The crucial point of Theorem 4.1 is the shallow depths of epistemic inferences. This is not a problem of the choice of the base logic: In fact, Theorem 4.1 holds in GL_{EF} , i.e., classical epistemic logic of shallow depths (Kaneko-Suzuki (2003)). The next example needs the intuitionistic base of our epistemic logic IG_{EF} .

Oracular-Decision Criterion: The third feature is related to constructive reasoning. Consider the Oracular-Decision Criterion for player 2 in Example 1.1. As seen in Section 1, this criterion implies the existence of a recommendation, but it does not imply that player 2 can play with this criterion.

In fact, a similar difficulty was discussed in the literature of constructive mathematics. It is known (cf. Dummett (2000) p. 16): There is an irrational number a with $a^{\sqrt{2}}$ being rational. Indeed, if $\sqrt{2}^{\sqrt{2}}$ is rational, then $a = \sqrt{2}$, and if $\sqrt{2}^{\sqrt{2}}$ is irrational, then, putting $a = \sqrt{2}^{\sqrt{2}}$, we have $a^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$. This proof is acceptable in classical mathematics, but not in constructive mathematics. Even though we have the existence of such an irrational number a , the above proof does not tell which of $\sqrt{2}$ or $\sqrt{2}^{\sqrt{2}}$ a is, i.e., it does not provide a concrete instance. The culprit here is the law of excluded middle $A \vee \neg A$, which is provable in classical logic. The law of excluded middle is central in distinguishing constructive reasoning from non-constructive reasoning. Thus, intuitionistic logic is regarded as the logic of constructive reasoning (cf. van Dalen (2004)).

In game situations, constructive reasoning is regarded as a requirement for decision making; mere existence without specifying a concrete instance may not provide

a decision. To discuss this kind of problem, we have developed logic IG_{EF} and its semantics.

Now, let us consider the oracular-decision criterion in IG_{EF} . To describe this criterion, we extend our language by adding new propositional variables $D_i(s)$ ($i = 1, 2$ and $s \in S_i$). Using them, the idea “player i adopts decision criterion A for strategy s ” is expressed by $D_i(s) \equiv A$.

We formulate the oracular-decision criterion for player 2. Note that $B_1(D_1(s_{1,1}))$ means that player 1 believes that his decision is $s_{1,1}$. We denote by $Expd$ and $PD^*(s)$ ($s \in S_2$), respectively:

$$B_1(D_1(s_{1,1})) \vee \neg B_1(D_1(s_{1,1})),$$

$$\left(B_1(D_1(s_{1,1})) \supset Br_2(s; s_{1,1}) \right) \wedge \left(\neg B_1(D_1(s_{1,1})) \supset MaxMin_2(s) \right).$$

Then, $Expd$ means the existence of prediction, and $PD^*(s)$ is the main body of the oracular-decision criterion. Thus, the formula $D_2(s) \equiv Expd \wedge PD^*(s)$, states that player 2 has a prediction about what player 1 would choose by referring the information $B_1(D_1(s_{1,1}))$ or $\neg B_1(D_1(s_{1,1}))$; if the prediction is $s_{1,1}$, then player 2’s decision is the best response to $s_{1,1}$, and if not, then player 2’s decision is given by a Max-Min strategy. Let $Ax(ODC)$ be the set $\{D_2(s) \equiv Expd \wedge PD^*(s) \mid s \in S_2\}$. This set $Ax(ODC)$ is called the *decision criterion axiom set*⁸. Then, the thought-sequent $B_i[\Gamma, Ax(ODC) \rightarrow D_2(s)]$ means that 2’s decision $D_2(s)$ is derived from assumptions in Γ by using the oracular-decision criterion. We have the following.

Proposition 4.2 *In classical epistemic logic GL_{EF} with $(2) \in F$,*

$$B_2[\Gamma_2, Ax(ODC) \rightarrow D_2(s_{2,1}) \vee D_2(s_{2,2})] \quad (6)$$

is provable. However, neither $B_2[\Gamma_2, Ax(ODC) \rightarrow D_2(s_{2,1})]$ nor $B_2[\Gamma_2, Ax(ODC) \rightarrow D_2(s_{2,2})]$ is provable in GL_{EF} .

In epistemic logic based on classical logic, the decision criterion ODC enjoys the existence of player 2’s decision: $D_2(s_{2,1}) \vee D_2(s_{2,2})$. However, neither $D_2(s_{2,1})$ nor $D_2(s_{2,2})$ follows from $\Gamma_2 \cup Ax(ODC)$.

In classical logic, an existential claim has constructive and non-constructive sources, while intuitionistic logic avoids the non-constructive one. Proposition 4.2 gives an example of a purely non-constructive assertion in GL_{EF} : Specifically, existence does not recommend a specific action. From the game theoretical point of view, the existence of a recommendation should be equivalent to finding a specific decision. In intuitionistic epistemic logic IG_{EF} , this difficulty is “eliminated” as follows.

Theorem 4.3 *In intuitionistic epistemic logic IG_{EF} with $(2) \in F$, the thought-sequent (6) is not provable.*

⁸ Here we discuss a simplified version of decision criteria. See Kaneko-Suzuki (2002b) for details. The set $ODC = \{Expd \wedge PD^*(s) \mid s \in S_2\}$ is called the Oracular-decision criterion, while $D_2(s) \equiv Expd \wedge PD^*(s)$ is an instance of the criterion. In the case of the Max-Min Criterion, it is $MaxMin_2(s) \equiv D_2(s)$ ($s \in S_2$), which means that s is a player 2’s decision if and only if $MaxMin_2(s)$. In the case of the Prediction-Decision criterion, the set PDC consisting of the above defined formulas (5) is the decision criterion set for the small store introduced in Section 1. Then, we can show that $B_2[B_1(\Gamma_1), \Gamma_2, Ax(PDC) \rightarrow D_2(s_{2,2})]$ is provable in IG_{EF} if and only if $(2, 1) \in F$.

In fact, the following stronger form holds: In IG_{EF} with $(2) \in F$,

$$\text{B}_2[\text{B}_1(\Gamma_1), \text{B}_1(\Gamma_2), \Gamma_1, \Gamma_2, \text{Ax}(\text{ODC}) \rightarrow \text{D}_2(s_{2,1}) \vee \text{D}_2(s_{2,2})] \quad (7)$$

is not provable.

Proof. We give the proof of the above strong form. For the sake of simplicity, we assume that $F = \{\varepsilon, (2), (2, 1)\}$. Let us define an Int_F -frame $\mathcal{F}_3$ by Fig. 4.1. To make the figure simple, some arrows corresponding to R_1 and R_2 are omitted. For example, by (F4.1), there must be arrows corresponding to R_1 from c to w and v .

We define $\models$ on $\mathcal{F}_3$ such that

$$c \not\models \text{B}_1(\Gamma_1), \text{B}_1(\Gamma_2), \Gamma_1, \Gamma_2, \text{Ax}(\text{ODC}) \rightarrow \text{D}_2(s_{2,1}) \vee \text{D}_2(s_{2,2}) .$$

Put $c \uparrow := \{a, b, c, v, w\}$. It suffices to define $\models$ between each $x \in c \uparrow$ and each $A \in \mathcal{A}_{EF}(x)$:

$$x \models A \text{ if and only if } \begin{cases} A \in \Gamma_1^+ \cup \Gamma_2^+, \text{ or} \\ x = w \text{ and } A \text{ is } \text{D}_1(s_{1,1}), \text{ or} \\ x = a \text{ and } A \text{ is } \text{D}_2(s_{2,2}), \text{ or} \\ x = b \text{ and } A \text{ is } \text{D}_2(s_{2,1}) \end{cases}$$

It is easily seen that every formula in $\Gamma_1^+ \cup \Gamma_2^+$ is true at every $x \in c \uparrow$. Hence, $\text{Br}_2(s_{2,2}; s_{1,1})$ and $\text{MaxMin}_2(s_{2,1})$ are true at every $x \in c \uparrow$, while $\text{Br}_2(s_{2,1}; s_{1,1})$ and $\text{MaxMin}_2(s_{2,2})$ are not true at every $x \in c \uparrow$.

Since aR_1x implies $x = w$, we have $a \models \text{B}_1(\text{D}_1(s_{1,1}))$ and $a \not\models \neg \text{B}_1(\text{D}_1(s_{1,1}))$. Since $a \preceq x$ implies $x = a$ and $a \models \text{Br}_2(s_{2,2}; s_{1,1})$, we have $a \models \text{PD}^*(s_{2,2})$. Therefore, $a \models \text{ExpD} \wedge \text{PD}^*(s_{2,2})$. On the other hand, since $a \not\models \text{Br}_2(s_{2,1}; s_{1,1})$, we have $a \not\models \text{PD}^*(s_{2,1})$. By Lemma 3.4, $c \not\models \text{PD}^*(s_{2,1})$. Similarly, we have $b \models \text{PD}^*(s_{2,1})$ and $b \not\models \text{PD}^*(s_{2,2})$. By Lemma 3.4, $c \not\models \text{PD}^*(s_{2,2})$. Therefore, $c \not\models \text{ExpD} \wedge \text{PD}^*(s_{2,2})$.

Figure 4.1

Then, for every x with $c \preceq x$, $x \models \text{Expd} \wedge \text{PD}^*(s_{2,2})$ if and only if $x = a$ if and only if $x \models D_2(s_{2,2})$. Namely, $c \models D_2(s_{2,2}) \equiv \text{Expd} \wedge \text{PD}^*(s_{2,2})$. Similarly, we have $c \models D_2(s_{2,1}) \equiv \text{Expd} \wedge \text{PD}^*(s_{2,1})$. Thus, both of formulas in $\text{Ax}(\text{ODC})$ are true at c .

It is clear from the definition of $\models$ that $c \not\models D_2(s_{2,1}) \vee D_2(s_{2,2})$. Therefore, we have that the thought-sequent $B_1(\Gamma_1), B_1(\Gamma_2), \Gamma_1, \Gamma_2, \text{Ax}(\text{ODC}) \rightarrow D_2(s_{2,1}) \vee D_2(s_{2,2})$ is not true at c . $\square$

In the previous discussion, we simply adopt the formulation of decision criteria in Kaneko-Suzuki (2002b) which was introduced for analysis of shallow-depths aspect of decision making in GL_{EF} . The aim of the previous discussion is to see a sharp contrast between GL_{EF} and IG_{EF} . To demonstrate constructive aspect of IG_{EF} more generally, we have the following Fact which can be shown by extensive applications of the completeness theorem.

Fact 4.4 *Suppose that every formula in a sequence Γ satisfies the following condition:*

(H) *Every occurrence of $\vee$ is in the left-scope of $\supset$ or in the scope of $\neg$.*

Then, $B_2[\Gamma \rightarrow \bigvee_{t \in S} D_2(t)]$ is provable in IG_{EF} if and only if for some $t \in S$, $B_2[\Gamma \rightarrow D_2(t)]$ is provable in IG_{EF} .

In $\text{Ax}(\text{ODC})$, we have subformulas out of the left-scope of $\supset$ (e.g., $\text{Expd} = B_1(D_1(s_{1,1})) \vee \neg B_1(D_1(s_{1,1}))$). However, we can use the following weak version instead of $\text{Ax}(\text{ODC})$:

$$\text{Ax}^w(\text{ODC}) := \{\text{Expd} \wedge \text{PD}^*(s) \supset D_2(s) \mid s \in S_2\}.$$

As for the recommendation of decision, $\text{Ax}^w(\text{ODC})$ has the same effect as $\text{Ax}(\text{ODC})$. This suggests that there are at least two choices of formulation for a decision criteria. Therefore, we should reconsider the formulation of a decision criteria suitable for the IG_{EF} -setting. We leave this as a future research problem.

5 Concluding Remarks

We have introduced the semantical framework for intuitionistic epistemic logic IG_{EF} . We have shown the soundness and completeness theorems for IG_{EF} . Intuitionistic logic and epistemic logic are integrated systematically in syntax and semantics. A complete system of intuitionistic epistemic logic of shallow depths is established.

System IG_{EF} and its semantics have merits also from the viewpoints of game theory. IG_{EF} enables us to discuss limitations on interpersonal inferences, as in GL_{EF} (Kaneko-Suzuki (2002b)). The main point of IG_{EF} is, however, the capability of discussing the distinction between the existence of a decision and a recommendation of a specific action. These merits are discussed in the three decision criteria in the game in Sections 1 and 4.

This paper provides the first step of our interdisciplinary research project. A continuation of the present paper is “horticulture” of our models, as in Kaneko-Suzuki (2003) for $\text{GL}_E (= \text{GL}_{EE}$ in our present notation). By such techniques, we can discuss

more game theoretical problems. Another direction is to investigate game theoretical situations which will provide a new problems to intuitionistic epistemic logic. For example, finding games and decision criteria requiring constructive reasoning and formulating them suitable for IG_{EF} -setting seem to be interesting and significant for future interdisciplinary research project.

We also have the following technical remarks:

1. In Section 4, we expressed a payoff function by the set Γ_i of (weak) preferences. This set Γ_i is a *complete* set in the sense that non-epistemic statements about the payoff functions are completely determined by Γ_i . They are not only determined but inferred from Γ_i even in intuitionistic logic. This way of expressing payoff function is the “strongest” one relative to non-epistemic facts on payoffs. More precisely, we have the following.

Proposition 5.1 *Let A be a non-epistemic formula containing only propositional variables of the form $P_i(s, t : s', t')$. Then,*

- (1) $\Gamma_1, \Gamma_2 \rightarrow A$ is provable in classical logic if and only if so is in intuitionistic logic;
- (2) either $\Gamma_1, \Gamma_2 \rightarrow A$ or $\Gamma_1, \Gamma_2 \rightarrow \neg A$ is provable.

2. Finite model property and Decidability.

IG_{EF} is decidable. One natural way of proving decidability is to show its *finite model property* first. Then finite model property automatically implies decidability. Here, a logical system is said to have the finite model property, if every formula not provable in the logic is not valid in some frame the size of which is finite (see e.g., Hughes-Cresswell (1985)). However, it is not the case in our semantics. As we stated above, our semantics reflects F . If F is infinite, then the size of the model must be infinite. To get rid of this difficulty, we have to prove one theorem on the finite approximability of the size of F .

Theorem 5.2 *Let A be a formula with $\delta^*(A) \subseteq E$. Then, A is provable in IG_{EF} if and only if there exists a finite epistemic structure $G \subseteq F$ such that A is provable in IG_{EG} .*

Then we can show the decidability by establishing the finite model property with the assumption that F is finite. The finite model property is proved with some technical consideration. In this paper, we do not go into such technical details.

6 Proofs of the Completeness Theorems

We show the Completeness Theorems 3.6 and 3.9. As we stated in Section 3.2, it suffices to show Theorem 3.9.

6.1 Soundness

In this subsection, we show the soundness of IG_{EF} with respect to the epistemic possible world semantics, i.e., the “only-if” part of Theorem 3.9.

Lemma 6.1 (Soundness: the “only-if” part of Theorem 3.9) *Let $B_e[\Gamma \rightarrow \Theta]$ be provable in IG_{EF} . Then, $B_e[\Gamma \rightarrow \Theta]$ is Int_{EF} -valid in every Int_F -frame.*

Proof. We show this lemma by the induction on the height of the proof of $B_e[\Gamma \rightarrow \Theta]$ in IG_{EF} . As the base case, we show that $B_e[A \rightarrow A]$ ($e \in F$) is Int_{EF} -valid in every Int_F -frame. Suppose otherwise. There exist an Int_F -model $(\mathcal{F}, \models)$ with $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ and $w \in W_e$ such that $A \rightarrow A$ is not true at w . It is clear by the definition of proof-figures in IG_{EF} that $A \in \mathcal{P}_E(e) = \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$. Then, A is true at w and A is not true at w . This is a contradiction.

Suppose that $B_e[\Gamma \rightarrow \Theta]$ has the proof with the height being n . As the induction hypothesis, we assume that every thought-sequent having its proof with the height less than n is Int_{EF} -valid in every Int_F -frame. We have some cases depending on the lower most application of rules in its proof.

I. the lower most application is of some non-epistemic rules.

We have several subcases. We show here the case that the lower most application is of $(\rightarrow \supset)$. The lower most part of the proof is as follows:

$$\frac{\begin{array}{c} \vdots \\ B_e[A, \Gamma \rightarrow B] \end{array}}{B_e[\Gamma \rightarrow A \supset B]} .$$

Suppose that $B_e[\Gamma \rightarrow A \supset B]$ is not true in an Int_{EF} -model $(\mathcal{F}, \models)$ with $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$. Then there exists a $w \in W_e$ such that $\Gamma \rightarrow A \supset B$ is not true at w , i.e., every C in Γ is true at w and $A \supset B$ is not. Then there exists a $v \in W_e$ such that $w \preceq v$ and A is true at v and B is not true at v . By Lemma 3.4, every C in Γ is true at v , as well. Hence, we have that $A, \Gamma \rightarrow B$ is not true at $v \in W_e$. That is, $B_e[A, \Gamma \rightarrow B]$ is not true in $(\mathcal{F}, \models)$. This contradicts the induction hypothesis.

II. the lower most application is of the epistemic rule.

The lower most part of the proof is as follows:

$$\frac{\begin{array}{c} \vdots \\ B_{eoi}[\Gamma \rightarrow \Theta] \end{array}}{B_e[B_i(\Gamma) \rightarrow B_i(\Theta)]} .$$

Recall that $e \circ i \in F$. Suppose that there exist an Int_{EF} -model $(\mathcal{F}, \models)$ with $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ and $w \in W_e$ such that $B_i(\Gamma) \rightarrow B_i(\Theta)$ is not true at w . Then, $w \models B_e(C)$ for every $B_e(C)$ in $B_e(\Gamma)$ and there is no $B_e(D)$ in $B_e(\Theta)$ such that $w \models B_e(D)$. Note we have $W_{eoi} \neq \emptyset$ by F3.2(F -seriality). We have two subcases: II-i. Θ is a single formula A ; II-ii. Θ is empty.

II-i. We have $w \not\models B_i(A)$. Hence, there exists a $v \in W_{eoi}$ such that wR_iv and $v \not\models A$. Since $w \models B_e(C)$ for every $B_e(C)$ in $B_e(\Gamma)$, $v \models C$ for every C in Γ . Therefore, $\Gamma \rightarrow \Theta$ is not true at $v \in W_{eoi}$. This contradicts the induction hypothesis.

II-ii. Take any $v \in W_{eoi} \neq \emptyset$. Since $w \models B_e(C)$ for every $B_e(C)$ in $B_e(\Gamma)$, $v \models C$ for every C in Γ . Therefore, $\Gamma \rightarrow$ is not true at $v \in W_{eoi}$. This contradicts the induction hypothesis.

This completes the proof. $\square$

6.2 Completeness

In this subsection, we show the completeness theorem for IG_{EF} , i.e., the “if” part of Theorem 3.9. More precisely we show the so-called *strong* completeness of IG_{EF} . To state the strong completeness, we need some definitions.

Definition 6.2 For any finite sequence Θ of formulas, define $\Theta^\vee$ by:

$$\Theta^\vee = \begin{cases} A_1 \vee \cdots \vee A_n & \text{if } \Theta \text{ is } A_1, \dots, A_n \\ & \text{if } \Theta \text{ is empty.} \end{cases}$$

Note that if Θ is empty, the $\Theta^\vee$ is the empty expression.

Definition 6.3 A triple (S_1, S_2, e) of sets S_1, S_2 of e -admissible formulas in E and an epistemic status $e \in F$ is said to be *EF-inconsistent* if there exist a finite subset Γ of S_1 and a finite subset Θ of S_2 such that $B_e[\Gamma \rightarrow \Theta^\vee]$ is provable in IG_{EF} . A triple (S_1, S_2, e) is said to be *EF-consistent* if (1) $e \in F$, (2) $S_1 \cup S_2 \subseteq \mathcal{P}_E(e)$, and (3) it is not *EF-inconsistent*.

In the rest of this section, we show the following theorem.

Theorem 6.4 (The Strong Completeness Theorem) *Let (S_1, S_2, e_0) be EF-consistent. Then there exist an Int_{EF} -model $(\mathcal{F}, \models)$ with $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$, and $w \in W_{e_0}$ such that $w \models A$ for every $A \in S_1$ and $w \not\models C$ for every $C \in S_2$.*

Let us check that the “if” part of Theorem 3.9 follows from Theorem 6.4. Suppose that a thought-sequent $B_e[\Gamma \rightarrow \Theta]$ admissible in IG_{EF} is not provable in IG_{EF} . Then by putting $\Gamma^* = \{A \mid A \text{ is in } \Gamma\}$ and $\Theta^* = \{A \mid A \text{ is in } \Theta\}$, we have Γ^* and Θ^* are sets of e -admissible formulas in E . By Definition 6.3, we have

Proposition 6.5 *A thought-sequent $B_e[\Gamma \rightarrow \Theta]$ admissible in IG_{EF} is provable in IG_{EF} if and only if (Γ^*, Θ^*, e) is EF-inconsistent.*

Therefore, by putting $S_1 = \Gamma^*$ and $S_2 = \Theta^*$ in Theorem 6.4, we have

Corollary 6.6 (Completeness: “If” part of Theorem 3.9) *A thought-sequent $B_e[\Gamma \rightarrow \Theta]$ is provable in IG_{EF} if and only if it is Int_{EF} -valid in every Int_F -frame.*

Definition 6.7 A subformula C of a formula A is said to be *epistemically direct* subformula (simply *d*-subformula), if C does not occur in the scope of any B_i ($i \in N$) in A . The set of all *d*-subformula of A is denoted by $\mathcal{D}(A)$. A set S of formulas is said to be *ds-closed*, if S is closed under taking *d*-subformulas, i.e., $\mathcal{D}(A) \subseteq S$ for every $A \in S$.

The next lemma follows directly from the definition.

Lemma 6.8 *Let A be a formula.*

- (1) *If $A \in \mathcal{P}_E(e)$, then $\mathcal{D}(A) \subseteq \mathcal{P}_E(e)$.*
- (2) *If $B_i(A) \in \mathcal{P}_E(e)$, then $A \in \mathcal{P}_E(e \circ i)$.*

Definition 6.9 A triple (S_1, S_2, e) of sets S_1, S_2 of e -admissible formulas in E is said to be *saturated*, if it is *EF-consistent* and $S_1 \cup S_2 = \mathcal{P}_E(e)$.

Lemma 6.10 *Let (S_1, S_2, e_0) be EF -consistent triple. Then there exists (S_1^*, S_2^*, e_0) such that (S_1^*, S_2^*, e_0) is saturated.*

Proof. Let $C_1, C_2, \dots, C_n, \dots$ be a fixed enumeration of formulas in $\mathcal{P}_E(e)$. We define two sequences $\{U_m\}_{m < \omega}$ and $\{V_m\}_{m < \omega}$ of formulas of $\mathcal{P}_E(e)$ inductively as follows;

$$\begin{aligned} U_0 &= S_1, & V_0 &= S_2, \\ U_m &= \begin{cases} U_{m-1} \cup \{C_m\} & \text{if } (U_{m-1} \cup \{C_m\}, V_{m-1}, e_0) \text{ is } EF\text{-consistent,} \\ U_{m-1} & \text{otherwise,} \end{cases} \\ V_m &= \begin{cases} V_{m-1} & \text{if } (U_{m-1} \cup \{C_m\}, V_{m-1}, e_0) \text{ is } EF\text{-consistent,} \\ V_{m-1} \cup \{C_m\} & \text{otherwise.} \end{cases} \end{aligned}$$

And put $S_1^* = \bigcup_{m < \omega} U_m$ and $S_2^* = \bigcup_{m < \omega} V_m$. By induction on m , we can show that each (U_m, V_m, e_0) is EF -consistent. Hence, (S_1^*, S_2^*, e_0) is EF -consistent. From the construction of (S_1^*, S_2^*, e_0) , it follows that $\mathcal{P}_E(e) = S_1^* \cup S_2^*$. Therefore, (S_1^*, S_2^*, e_0) is saturated. $\square$

Definition 6.11 For each set S of formulas, we denote by $S^{\setminus i}$ the set $\{A \mid B_i(A) \in S\}$.

Lemma 6.12 *Let (S_1, S_2, e_0) be saturated. If $e_0 \circ i \in F$, then there exists a $(S'_1, S'_2, e_0 \circ i)$ such that $(S'_1, S'_2, e_0 \circ i)$ is saturated and $(S_1)^{\setminus i} \subseteq S'_1$.*

Proof. We first show that $((S_1)^{\setminus i}, \emptyset, e_0 \circ i)$ is EF -consistent. Suppose otherwise. Then there exists a finite subset Γ of $(S_1)^{\setminus i}$ such that $B_{e_0 \circ i}[\Gamma \rightarrow \]$ is provable in IG_{EF} . Note that we have $(S_1)^{\setminus i} \subseteq \mathcal{P}_E(e_0 \circ i)$, since $S_1 \subseteq \mathcal{P}_E(e_0)$. By applying $(B_i \rightarrow B_i)$, $B_{e_0}[B_i(\Gamma) \rightarrow \]$ is provable in IG_{EF} . Since $B_i(\Gamma)$ is a finite subset of S_1 , this contradicts the assumption that (S_1, S_2, e_0) is EF -consistent. Therefore $((S_1)^{\setminus i}, \emptyset, e_0 \circ i)$ is EF -consistent. By Lemma 6.10, there exists a saturated triple $(S'_1, S'_2, e_0 \circ i)$ such that $(S_1)^{\setminus i} \subseteq S'_1$. $\square$

Lemma 6.13 *Let (S_1, S_2, e_0) be saturated.*

- (1) *For every $A \wedge C \in \mathcal{P}_E(e_0)$, $A \wedge C \in S_1$ if and only if $A \in S_1$ and $C \in S_1$.*
- (2) *For every $A \vee C \in \mathcal{P}_E(e_0)$, $A \vee C \in S_1$ if and only if $A \in S_1$ or $C \in S_1$.*
- (3) *For every $\neg A \in \mathcal{P}_E(e_0)$, $\neg A \in S_1$ if and only if for every saturated triple (S'_1, S'_2, e_0) , $S_1 \subseteq S'_1$ implies $A \notin S'_1$.*
- (4) *For every $A \supset C \in \mathcal{P}_E(e_0)$, $A \supset C \in S_1$ if and only if for every saturated triple (S'_1, S'_2, e_0) , $S_1 \subseteq S'_1$ implies $A \notin S'_1$ or $C \in S'_1$.*

Proof. We show here (1) and (4). Others are proved similarly.

(1): Take any $A \wedge C \in \mathcal{P}_E(e_0)$. Since $\mathcal{P}_E(e_0)$ is ds-closed, both of A and C belong to $\mathcal{P}_E(e_0)$. Suppose $A \wedge C \in S_1$. Recall that $B_{e_0}[A \wedge C \rightarrow A]$ and $B_{e_0}[A \wedge C \rightarrow C]$ are provable in IG_{EF} . If $A \notin S_1$, then $A \in S_2$ and hence (S_1, S_2, e_0) is EF -inconsistent. This contradicts the assumption. Therefore $A \in S_1$. Similarly, we have $C \in S_1$. Next suppose that $A \in S_1$ and $C \in S_1$. Recall that $B_{e_0}[A, C \rightarrow A \wedge C]$ is provable in IG_{EF} . If $A \wedge C \notin S_1$, then $A \wedge C \in S_2$ and hence (S_1, S_2, e_0) is EF -inconsistent. Therefore $A \wedge C \in S_1$.

(4): Take any $A \supset C \in \mathcal{P}_E(e_0)$. Since $\mathcal{P}_E(e_0)$ is ds-closed, both of A and C belong to $\mathcal{P}_E(e_0)$. Suppose $A \supset C \in S_1$. Let (S'_1, S'_2, e_0) be a saturated triple with $S_1 \subseteq S'_1$. Assume that $A \in S'_1$. Then $A, A \supset C \in S'_1$. If $C \notin S'_1$, then $C \in S'_2$. Recall that $B_{e_0}[A, A \supset C \rightarrow C]$ is provable in IG_{EF} . It follows that (S'_1, S'_2, e_0) is EF -inconsistent. This is a contradiction. Therefore $C \in S'_1$. Next suppose $A \supset C \notin S_1$. Then $A \supset C \in S_2$. We claim that $(S_1 \cup \{A\}, \{C\}, e_0)$ is EF -consistent. Otherwise, there exists a finite $\Gamma \subseteq S_1$ such that $B_{e_0}[\Gamma, A \rightarrow C]$ is provable in IG_{EF} . Hence $B_{e_0}[\Gamma \rightarrow A \supset C]$ is provable, and (S_1, S_2, e_0) turns to be EF -inconsistent. This contradicts the assumption. By Lemma 6.10, There is a saturated triple (S'_1, S'_2, e_0) such that $S_1 \cup \{A\} \subseteq S'_1$ and $\{C\} \subseteq S'_2$. That is, (S'_1, S'_2, e_0) is a saturated triple such that $S_1 \subseteq S'_1$, $A \in S'_1$, and $C \notin S'_1$. $\square$

Lemma 6.14 *Let (S_1, S_2, e_0) be saturated. Suppose $e_0 \circ i \in F$. For every $B_i(A) \in \mathcal{P}_E(e_0)$, $B_i(A) \in S_1$ if and only if for every saturated triple $(S'_1, S'_2, e_0 \circ i)$, $(S_1)^{\setminus i} \subseteq S'_1$ implies $A \in S'_1$.*

Proof. Take an arbitrary $B_i(A) \in \mathcal{P}_E(e_0)$. Suppose $B_i(A) \in S_1$. Then $A \in (S_1)^{\setminus i}$. Hence, $(S_1)^{\setminus i} \subseteq S'_1$ implies $A \in S'_1$ for every saturated triple $(S'_1, S'_2, e_0 \circ i)$. Next suppose $B_i(A) \notin S_1$. Then $B_i(A) \in S_2$. We claim that $((S_1)^{\setminus i}, \{A\}, e_0 \circ i)$ is EF -consistent. Otherwise, there exists a finite $\Gamma \subseteq (S_1)^{\setminus i}$ such that $B_{e_0 \circ i}[\Gamma \rightarrow A]$ is provable in IG_{EF} . By applying $(B_i \rightarrow B_i)$, we have that $B_{e_0}[B_i(\Gamma) \rightarrow B_i(A)]$ is provable. That is, (S_1, S_2, e_0) is EF -inconsistent, since $B_i(\Gamma)$ is a finite subset of S_1 . This contradicts the assumption. By Lemma 6.10, there exists a saturated triple $(S'_1, S'_2, e_0 \circ i)$ such that $(S_1)^{\setminus i} \subseteq S'_1$ and $\{A\} \subseteq S'_2$. Since $(S'_1, S'_2, e_0 \circ i)$ is EF -consistent, we have $A \notin S'_1$. $\square$

Now let us prove Theorem 6.4. Suppose that (S_1, S_2, e) is EF -consistent. For each $f \in F$, we define:

$$W_f = \{(U_1, U_2, f) \mid (U_1, U_2, f) \text{ is saturated}\},$$

and put $W = \bigcup_{f \in F} W_f$. Then $\{W_f \mid f \in F\}$ is a partition of W . On W we define relations $\preceq$ and $R_1, \dots, R_n$ by:

$$\begin{aligned} (U_1, U_2, f) \preceq (V_1, V_2, g) & \text{ if and only if } f = g \text{ and } U_1 \subseteq V_1, \\ (U_1, U_2, f) R_i (V_1, V_2, g) & \text{ if and only if } g = f \circ i \text{ and } (U_1)^{\setminus i} \subseteq V_1. \end{aligned}$$

Lemma 6.15 $\mathfrak{F} = (W : \preceq, R_1, \dots, R_n)$ is an Int_F -frame.

Proof. We have only to check the conditions. Note that F1, F2 and F3.1 are automatically satisfied. We show F3.2. Let (U_1, U_2, f) be saturated. Assume that $f \circ i \in F$. By Lemma 6.12, there exists a saturated triple $(U'_1, U'_2, f \circ i)$ such that $(U_1)^{\setminus i} \subseteq U'_1$. That is, $(U_1, U_2, f) R_i (U'_1, U'_2, f \circ i)$.

Next we show F4. F4.1: Let (U_1, U_2, f) , (U'_1, U'_2, g) and (U''_1, U''_2, h) be saturated with $(U_1, U_2, f) \preceq (U'_1, U'_2, g) R_i (U''_1, U''_2, h)$. Then, we have $f = g$, $h = g \circ i$, $U_1 \subseteq U'_1$ and $(U'_1)^{\setminus i} \subseteq U''_1$. Hence $h = f \circ i$ and $(U_1)^{\setminus i} \subseteq U''_1$. That is, $(U_1, U_2, f) R_i (U''_1, U''_2, h)$. F4.2: Let (U_1, U_2, f) , (U'_1, U'_2, g) and (U''_1, U''_2, h) be saturated with $(U_1, U_2, f) R_i (U'_1, U'_2, g) \preceq (U''_1, U''_2, h)$. Then, we have $g = f \circ i$, $h = g$, $(U_1)^{\setminus i} \subseteq U'_1$ and $U'_1 \subseteq U''_1$. Hence $h = f \circ i$ and $(U_1)^{\setminus i} \subseteq U''_1$. That is, $(U_1, U_2, f) R_i (U''_1, U''_2, h)$. $\square$

Let us define $\models$ by:

(*) For each $(U_1, U_2, f) \in W$ and $C \in PV \cup \{B_i(A) \in \mathcal{P}_E(f) : f \circ i \notin F\}$,

$$(U_1, U_2, f) \models C \text{ if and only if } C \in U_1.$$

Then we have

Lemma 6.16 *This $\models$ is an EF-valuation on $\mathfrak{F}$.*

Proof. By the definition, we can easily check that the condition (HC) holds. $\square$

Lemma 6.17 *Let A be a formula. Suppose $e \in F$ and $A \in \mathcal{P}_E(e_0)$. Then, for every (U_1, U_2, e) , $(U_1, U_2, e) \models A$ if and only if $A \in U_1$.*

Proof. We show this by induction on the length of A . If A is a propositional variable, then the statement is just the definition of $\models$. If A is of the form $C \wedge D$, $C \vee D$, $\neg C$ or $C \supset D$, then we have the lemma by Lemma 6.13. If A is of the form $B_i(C)$, we have two cases.

I. $e \circ i \notin F$: Then $A \in \{B_i(C) \in \mathcal{P}_E(e) : e \circ i \notin F\}$. The statement is just the definition of $\models$.

II. $e \circ i \in F$: Then $A \notin \{B_i(C) \in \mathcal{P}_E(e) : e \circ i \notin F\}$. Suppose $(U_1, U_2, e) \models B_i(C)$. Then, for every $(V_1, V_2, e \circ i)$ with $(U_1)^i \subseteq V_1$, it holds $(V_1, V_2, e \circ i) \models C$. By the induction hypothesis, we have $C \in V_1$ for every $(V_1, V_2, e \circ i)$ with $(U_1)^i \subseteq V_1$. By Lemma 6.14, we have $B_i(C) \in U_1$. Next suppose $(U_1, U_2, e) \not\models B_i(C)$. Then, there exists a triple $(V_1, V_2, e \circ i)$ with $(U_1)^i \subseteq V_1$ such that $(V_1, V_2, e \circ i) \not\models C$. By the induction hypothesis, we have $C \notin V_1$. By Lemma 6.14 again, we have $B_i(C) \notin U_1$. $\square$

Now we show that this model $(\mathfrak{F}, \models)$ is ‘the’ model we wanted. We have an EF-consistent triple (S_1, S_2, e) . By Lemma 6.10, there exists a saturated extension (S_1^*, S_2^*, e) of (S_1, S_2, e) . Clearly $(S_1^*, S_2^*, e) \in W_e$ in $\mathfrak{F}$. By Lemma 6.17, $(S_1^*, S_2^*, e) \models A$ for every $A \in S_1^*$ and $(S_1^*, S_2^*, e) \not\models C$ for every $C \in S_2^*$. Since $S_1 \subseteq S_1^*$ and $S_2 \subseteq S_2^*$, $(S_1^*, S_2^*, e) \models A$ for every $A \in S_1$ and $(S_1^*, S_2^*, e) \not\models C$ for every $C \in S_2$. This completes the proof of Theorem 6.4. $\square$

As stated in Section 3.2, we will introduce the following definition.

Definition 6.18 Let $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$ be an Int_F -frame. An element $w_1 \in W$ is said to be *R-reachable from* $w_2 \in W$, if there exists an epistemic status e such that $w_2 R_e w_1$.

Then we can show the following version of the strong completeness theorem.

Theorem 6.19 (The Strong Completeness with Reachability) *Let (S_1, S_2, e_0) be EF-consistent. Then there exist an Int_{EF} -model $(\mathcal{F}, \models)$ with $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$, and $w \in W_{e_0}$ such that*

- (1) $w \models A$ for every $A \in S_1$ and $w \not\models C$ for every $C \in S_2$,
- (2) w is R-reachable from some $w_\varepsilon \in W_\varepsilon$.

We have only to manage (S_1^*, S_2^*, e) in the Int_{EF} -model $(\mathfrak{F}, \models)$ to be R-reachable from some w whose epistemic status $\lambda_{\mathcal{F}}(w)$ is ε . If $e = \varepsilon$, then just take (S_1^*, S_2^*, e) as w . If $e \neq \varepsilon$, we work on $(\mathfrak{F}, \models)$ by an easy model theoretic technique defined below.

Definition 6.20 Define a binary relation $\sqsubseteq$ on the set $N^{<\omega}$ of all epistemic statuses by: $e \sqsubseteq f$ if and only if there exists an e' such that $f = e \circ e'$. This $\sqsubseteq$ is a partial order on $N^{<\omega}$, and $N^{<\omega}$ forms an n -branching infinite tree with the root ε . We use the notation $e \sqsubset f$ to denote that $e \sqsubseteq f$ and $e \neq f$.

Let $(\mathcal{F}, \models)$ be an Int_{EF} -model with $\mathcal{F} = (W : \preceq, R_1, \dots, R_n)$, f a fixed epistemic status with $f \in F$. We define other binary relations $R_1^f, R_2^f, \dots, R_n^f$ by

$$w_1(R_i^f)w_2 \text{ if and only if } \begin{cases} w_1 R_i w_2 & \text{if } f \sqsubseteq \lambda_{\mathcal{F}}(w_1), \\ \lambda_{\mathcal{F}}(w_1) \circ i = \lambda_{\mathcal{F}}(w_2) & \text{otherwise.} \end{cases}$$

Then, $\mathcal{F}^f = (W : \preceq, R_1^f, \dots, R_n^f)$ is an Int_F -frame. Moreover, it is clear that every $w \in W_f$ is R^f -reachable from some $u \in W_\varepsilon$. By the definition, $\lambda_{\mathcal{F}}(w) = \lambda_{\mathcal{F}^f}(w)$ for every $w \in W$. Take the same Int_{EF} -valuation $\models$ and consider $(\mathcal{F}^f, \models)$. Then, for every $w \in W$ with $f \sqsubseteq \lambda_{\mathcal{F}}(w)$, each step in deciding $w \models A$ is the same in $(\mathcal{F}, \models)$ and $(\mathcal{F}^f, \models)$. We have the following.

Lemma 6.21 *In $(\mathcal{F}^f, \models)$, it holds that for every $w \in W$ with $f \sqsubseteq \lambda_{\mathcal{F}}(w)$ and every $A \in \mathcal{P}_E(\mathcal{F}^f(w))$, $w \models A$ in $(\mathcal{F}, \models)$ if and only if $w \models A$ in $(\mathcal{F}^f, \models)$*

Now, take $(\mathcal{F}^e, \models)$. Then, (S_1^*, S_2^*, e) is reachable from ε in W_e . This completes the proof of Theorem 6.19. $\square$

Then we have

Theorem 6.22 (Faithfull Translation) *Let $B_e[\Gamma \rightarrow \Theta]$ be a thought-sequent admissible in IG_{EF} . Then, $B_e[\Gamma \rightarrow \Theta]$ is valid in every Int_F -frame if and only if so is $B_e[\Gamma \rightarrow \Theta]^*$.*

Proof. Suppose that $B_e[\Gamma \rightarrow \Theta]$ is valid in every Int_F -frame. Then, it is provable in IG_{EF} . By applying $(B \rightarrow B)$ finitely many times, we have that $B_e[\Gamma \rightarrow \Theta]^*$ is provable in IG_{EF} . Hence, $B_e[\Gamma \rightarrow \Theta]^*$ is valid in every Int_F -frame.

Next suppose that $B_e[\Gamma \rightarrow \Theta]$ is not valid. Then, it is not provable in IG_{EF} . Take an Int_{EF} -model ensured by Theorem 6.19. In this model, $B_e[\Gamma \rightarrow \Theta]^*$ is not true. $\square$

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