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Kowalevski Top in Quantum Mechanics

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Abstract

The quantum mechanical Kowalevski top is studied by the direct diagonalization of the Hamiltonian. The spectra show different behaviors depending on the region divided by the bifurcation sets of the classical invariant torus. Some of these spectra are nearly degenerate due to the multiplicity of the invariant torus. The Kowalevski top has several symmetries and symmetry quantum numbers can be assigned to the eigenstates. We have also carried out the semiclassical quantization of the Kowalevski top by the EBK formulation. It is found that the semiclassical spectra are close to the exact values, thus the eigenstates can be also labeled by the integer quantum numbers. The symmetries of the system are shown to have close relations with the semiclassical quantum numbers and the near-degeneracy of the spectra.

Keywords:

Kowalevski top, quantum spectrum, action variable, Maslov index, semiclassical quantization

1 Introduction

The Kowalevski top [1] is one of the most popular integrable systems, and it has been investigated extensively more than one century. It has an exceptional constant of motion in addition to the energy and the angular momentum, and classical solutions are given in terms of hyperelliptic integrals. Detailed reviews of its special properties can be found in Ref.[2] for example.

The Kowalevski top is a typical rigid body under the gravitational force. The quantum mechanical formulation of the rigid body was given in Ref.[3]. Nowadays, it is well described in standard textbooks [4, 5], since it is a basic starting point to obtain quantum states of extended objects like molecules and deformed nuclei. The wave function of the rigid body is given by a linear combination of D -function $D_{MK}^J(\Omega)$, where $\Omega = (\alpha, \beta, \gamma)$ are the Euler angles, J is the angular momentum (in units of $\hbar$), M (K) is the z -component of the angular momentum in the space-fixed (body-fixed) frame. For the symmetric Euler top, (J, M, K) and D -function are exactly the quantum numbers and the eigenfunction. In the case of asymmetric Euler top, the wave function is given by the superposition of D -functions with different K -values. A symmetric top molecule in an electric field is an analogue of the Lagrange top. The Stark energy levels of symmetric top molecules were calculated by Shirley [6], and have been well studied in chemical physics. In the case of Lagrange top, M and K are conserved quantities and the eigenfunction is expressed by the superposition of D -functions with different J -values. For the Kowalevski top, Komarov and Kuznetsov [7] have succeeded in obtaining the approximate quantum spectra in the semiclassical EBK formulation, which is applicable only for integrable systems. However, the calculation of the exact quantum spectra of the Kowalevski top has not been carried out so far.

The purpose of this paper is to explore the quantum mechanical properties of the Kowalevski top. We will briefly review the Hamiltonian and the conserved quantities in Section 2. Classical mechanics and action variables are described in Section 3 and 4, which are basic ingredients for the semiclassical EBK quantization in Section 5. Numerical methods of calculating the quantum spectra of the Kowalevski top are described in Section 6, and the results are presented and discussed in detail in Section 7. Section 8 will be devoted to a summary.

2 Hamiltonian and conserved quantities

2.1 Euler equations

The motion of a rigid body with a fixed point is described by the Euler equations. The space-fixed frame is named (x, y, z) -frame and unit vectors along the axes are denoted by $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. Similarly, the body-fixed frame and its unit vectors are named $(1, 2, 3)$ -frame and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The space-fixed frame is placed with the z -axis vertically upward in

the gravitational field, and the body-fixed (1,2,3)-axes are chosen to coincide with the principal axes of inertia of the rigid body. The dynamical variables are the direction-cosines $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, which are (1,2,3)-components of the unit vector of the space-fixed z -axis $\mathbf{e}_z = (0, 0, 1)$. Then the Euler equations in the body-fixed frame are

$$\frac{d\mathbf{J}}{dt} = \mathbf{J} \times \boldsymbol{\omega} + \mathbf{N}, \quad (2.1)$$

where $\boldsymbol{\omega}$ is the angular velocity vector, and $\boldsymbol{\omega} = (p, q, r)$ are its components in customary notation. $\mathbf{J}$ is the angular momentum of the rigid body and its components are $\mathbf{J} = (Ap, Bq, Cr)$ with the principal moments of inertia (A, B, C) . The torque $\mathbf{N}$ of the gravitational force Mg is given by

$$\mathbf{N} = -Mg\mathbf{r}_G \times \mathbf{e}_z, \quad (2.2)$$

where $\mathbf{r}_G$ is the coordinate of the center of mass (C.M.).

The characteristic features of the Kowalevski top are

(1) $A = B = 2C$,

(2) the C.M. is on the (1,2)-plane of the body-fixed frame.

The C.M. vector can be set as $\mathbf{r}_G = (x_0, 0, 0)$ $x_0 > 0$. Then the resulting Euler equations become

$$2\dot{p} = qr, \quad 2\dot{q} = c_0\gamma_3 - rp, \quad \dot{r} = -c_0\gamma_2 \quad (c_0 = Mg x_0 / C). \quad (2.3)$$

Time derivative $\frac{df}{dt}$ will be abbreviated by $\dot{f}$ hereafter. The time evolution of $\boldsymbol{\gamma}$ is given by $\dot{\mathbf{e}}_z + \boldsymbol{\omega} \times \mathbf{e}_z = \mathbf{0}$. In the body-fixed frame, they are

$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2. \quad (2.4)$$

These six equations completely specify the time development of the variables $(p, q, r, \gamma_1, \gamma_2, \gamma_3)$, and determine the motion of the Kowalevski top.

2.2 Hamiltonian and canonical equations

The Euler equations are useful for the numerical calculation of orbits since they are simple autonomous equations. However, those variables $(\boldsymbol{\omega}, \boldsymbol{\gamma})$ are not canonical, thus not suitable for the discussion of action-angle variables and Hamilton equations.

In order to write canonical equations, we will choose the Euler angles $\Omega = (\alpha, \beta, \gamma)$, $0 \leq \alpha, \gamma < 2\pi$, $0 \leq \beta \leq \pi$, instead of the direction-cosines to specify the relative position of the body-fixed frame to the space-fixed frame. We will adopt the convention of the Euler angles given in Ref.[8]. Then the velocity vector $\boldsymbol{\omega} = (p, q, r)$ and the direction-cosines $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ are written in terms of the Euler angles $\Omega = (\alpha, \beta, \gamma)$ as

$$p = \dot{\beta} \sin \gamma - \dot{\alpha} \sin \beta \cos \gamma, \quad q = \dot{\beta} \cos \gamma + \dot{\alpha} \sin \beta \sin \gamma, \quad r = \dot{\gamma} + \dot{\alpha} \cos \beta, \quad (2.5)$$

and

$$\gamma_1 = -\sin \beta \cos \gamma, \quad \gamma_2 = \sin \beta \sin \gamma, \quad \gamma_3 = \cos \beta, \quad (2.6)$$

and the z -component z_G of the C.M. vector $\mathbf{r}_G$ is

$$z_G = \mathbf{r}_G \cdot \mathbf{e}_z = x_0 \gamma_1 = -x_0 \sin \beta \cos \gamma. \quad (2.7)$$

The Lagrangian L is

$$L = \dot{\beta}^2 + \dot{\alpha}^2 \sin^2 \beta + \frac{1}{2}(\dot{\gamma} + \dot{\alpha} \cos \beta)^2 + c_0 \sin \beta \cos \gamma, \quad (2.8)$$

where the moments of inertia are set $A = B = 2$, $C = 1$ for simplicity. The canonical angular momentum components $\mathbf{p}_\Omega = (p_\alpha, p_\beta, p_\gamma)$ are

$$\begin{aligned} p_\alpha &= \frac{\partial L}{\partial \dot{\alpha}} = 2\dot{\alpha} \sin^2 \beta + (\dot{\gamma} + \dot{\alpha} \cos \beta) \cos \beta, \\ p_\beta &= \frac{\partial L}{\partial \dot{\beta}} = 2\dot{\beta}, \\ p_\gamma &= \frac{\partial L}{\partial \dot{\gamma}} = \dot{\gamma} + \dot{\alpha} \cos \beta. \end{aligned} \quad (2.9)$$

Then the angular momentum components (J_1, J_2, J_3) are written by

$$\begin{aligned} J_1 &= 2p = p_\beta \sin \gamma - \frac{\cos \gamma}{\sin \beta} (p_\alpha - p_\gamma \cos \beta), \\ J_2 &= 2q = p_\beta \cos \gamma + \frac{\sin \gamma}{\sin \beta} (p_\alpha - p_\gamma \cos \beta), \\ J_3 &= r = p_\gamma. \end{aligned} \quad (2.10)$$

Let us define $J_\pm$ for the later convenience as

$$J_\pm = J_1 \pm iJ_2 = e^{\mp i\gamma} \left[\pm ip_\beta - \frac{1}{\sin \beta} (p_\alpha - p_\gamma \cos \beta) \right]. \quad (2.11)$$

The Hamiltonian H is expressed in terms of the Euler angles as

$$\begin{aligned} H &= \frac{1}{4}(J_1^2 + J_2^2 + 2J_3^2) + c_0 \gamma_1 \\ &= \frac{1}{4 \sin^2 \beta} (p_\alpha^2 + p_\gamma^2 - 2p_\alpha p_\gamma \cos \beta) + \frac{1}{4} (p_\beta^2 + p_\gamma^2) - c_0 \sin \beta \cos \gamma. \end{aligned} \quad (2.12)$$

Then the canonical equations of motion are given by

$$\frac{d\Omega_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial \Omega_i}, \quad i = \alpha, \beta, \gamma. \quad (2.13)$$

Since the Hamiltonian does not depend on the variable α , the conjugate momentum p_α is a constant of motion. If the components of the angular momentum $\mathbf{J}$ in the space-fixed frame are denoted by $(L_x, L_y, L_z) = (\mathbf{J} \cdot \mathbf{e}_x, \mathbf{J} \cdot \mathbf{e}_y, \mathbf{J} \cdot \mathbf{e}_z)$, then $p_\alpha = L_z = J_1 \gamma_1 + J_2 \gamma_2 + J_3 \gamma_3$. L_z is a constant of motion since the gravitational field is parallel to the z -axis.

2.3 Kowalevski constant

It was shown by Kowalevski [1] that there exists an integral of motion in this dynamical system, which will be called the Kowalevski constant K hereafter. The constant K is given by

$$K = k_+ k_- = (p^2 - q^2 - c_0 \gamma_1)^2 + (2pq - c_0 \gamma_2)^2, \quad (2.14)$$

where

$$k_{\pm} = \omega_{\pm}^2 - c_0 \gamma_{\pm} = \frac{1}{4} J_{\pm}^2 - c_0 \gamma_{\pm} \quad (2.15)$$

and

$$\omega_{\pm} = p \pm iq = (\pm i \dot{\beta} - \dot{\alpha} \sin \beta) e^{\mp i \gamma}, \quad \gamma_{\pm} = \gamma_1 \pm i \gamma_2 = -\sin \beta e^{\mp i \gamma}. \quad (2.16)$$

In the Euler angle representation, $k_{\pm}$ is expressed as

$$k_{\pm} = \frac{1}{4} e^{\mp 2i \gamma} \left[\pm i p_{\beta} - \frac{1}{\sin \beta} (p_{\alpha} - p_{\gamma} \cos \beta) \right]^2 + c_0 \sin \beta e^{\mp i \gamma}, \quad (2.17)$$

and K is written by

$$K = \frac{1}{16} \left[p_{\beta}^2 + \frac{1}{\sin^2 \beta} (p_{\alpha} - p_{\gamma} \cos \beta)^2 \right]^2 + \frac{c_0}{2} \sin \beta \cos \gamma \left[\frac{1}{\sin^2 \beta} (p_{\alpha} - p_{\gamma} \cos \beta)^2 - p_{\beta}^2 \right] - c_0 \sin \gamma p_{\beta} (p_{\alpha} - p_{\gamma} \cos \beta) + c_0^2 \sin^2 \beta. \quad (2.18)$$

2.4 Symmetric properties

As is shown in Section 2.2, L_z is a constant of motion since the Hamiltonian is invariant under the rotation around the z -axis. In Ref.[9], Dullin, Juhnke and Richter discussed another symmetries of the system, which are summarized as follows in a slightly different manner.

$$(a) R : (\alpha, \beta, \gamma) \rightarrow (\alpha + \pi, \pi - \beta, -\gamma)$$

$$(J_1, J_2, J_3) \rightarrow (J_1, -J_2, -J_3), \quad (\gamma_1, \gamma_2, \gamma_3) \rightarrow (\gamma_1, -\gamma_2, -\gamma_3). \quad (2.19)$$

$$(b) S_1 : (\alpha, \beta, \gamma) \rightarrow (-\alpha, \beta, -\gamma)$$

$$(J_1, J_2, J_3) \rightarrow (-J_1, J_2, -J_3), \quad (\gamma_1, \gamma_2, \gamma_3) \rightarrow (\gamma_1, -\gamma_2, \gamma_3). \quad (2.20)$$

$$(c) S_2 : (\alpha, \beta, \gamma) \rightarrow (\pi - \alpha, \pi - \beta, \gamma)$$

$$(J_1, J_2, J_3) \rightarrow (-J_1, -J_2, J_3), \quad (\gamma_1, \gamma_2, \gamma_3) \rightarrow (\gamma_1, \gamma_2, -\gamma_3). \quad (2.21)$$

Since the arguments α and γ are periodic with 2π -period, these arguments should be shifted by $\pm 2\pi$ if they are out of the defined interval.

The R -operator is physically the π -rotation of the rigid body around the 1-axis in the body-fixed frame, which leaves the Hamiltonian invariant since we choose the 1-axis along the C.M. vector $\mathbf{r}_G$. These three operators and the identity operator E , i.e., (E, R, S_1, S_2)

form the dihedral group D_2 . The Hamiltonian H and the Kowalevski constant K are invariant under these operators. Although the operator R also leaves L_z unchanged, the operators S_1 and S_2 change the sign, $L_z \rightarrow -L_z$. Thus, it is sufficient to assume $L_z \geq 0$, since those states with $L_z < 0$ are obtained by the time reversal transformation. Therefore, if we constrain the system with a conserved $L_z \geq 0$ value, only the R -operator is a genuine symmetric transformation. The operators S_1 and S_2 change the system to that of time reversed state. These symmetric properties will be used for the classification of the quantum states in Section 6 and 7.

3 Classical mechanics

The Kowalevski top is an integrable system and the classical orbits are given by hyperelliptic integrals on the Riemann surface with genus $g = 2$. Detailed accounts of the way of integration are summarized in Ref.[2] for example. In this section, we will discuss the structure of the invariant tori, which will be necessary for the action integrals and the semiclassical quantization in later sections.

3.1 Classification of the invariant tori

The Euler equations are given by Eqs.(2.3),(2.4). The conserved quantities are

$$h = p^2 + q^2 + \frac{1}{2}r^2 + c_0\gamma_1, \quad : \text{energy} \quad (3.1)$$

$$-\ell = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3, \quad : \text{angular momentum} \quad (3.2)$$

$$1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \quad : \text{length of direction cosine} \quad (3.3)$$

$$k = (p^2 - q^2 - c_0\gamma_1)^2 + (2pq - c_0\gamma_2)^2, \quad : \text{Kowalevski constant} \quad (3.4)$$

The angular momentum in the space-fixed system is set $L_z = \mathbf{J} \cdot \mathbf{e}_z = -\ell$ in accordance with the customary convention where the z -axis is oriented vertically downward. In order to match the original notations [1], one should replace $c_0 \rightarrow -c_0$ and $\ell \rightarrow -2\ell$. Hereafter, the parameter c_0 will be set $c_0 = 1$ by a suitable choice of units.

Kharlamov [10] investigated the invariant tori and the bifurcation patterns between them. One should note that he used the original notations. The method is to find the critical bifurcation set Σ in the (ℓ, h, k) -space and classify them into five regions which have different structure of the invariant tori. The bifurcation set Σ had been analyzed by Appel'rot [11] who divided them into four classes. Let us briefly summarize their way of classifications since the structures of the invariant tori are very important in order to calculate the action integrals.

Let us define (x_1, x_2) as $x_1 = p + iq$, $x_2 = p - iq$, and following functions for fixed values (ℓ, h, k) .

$$R(x) = -x^4 + 2hx^2 + 2\ell x + 1 - k, \quad (3.5)$$

$$R(x_1, x_2) = -x_1^2 x_2^2 + 2hx_1 x_2 + \ell(x_1 + x_2) + 1 - k, \quad (3.6)$$

$$R_1(x_1, x_2) = -2hx_1^2 x_2^2 - 2\ell(x_1 + x_2)x_1 x_2 - (1 - k)(x_1 + x_2)^2 + 2h(1 - k) - \ell^2. \quad (3.7)$$

There is an identity

$$R(x_1)R(x_2) - R(x_1, x_2)^2 = (x_1 - x_2)^2 R_1(x_1, x_2). \quad (3.8)$$

We define variables w_1 and w_2 as the roots of the quadratic equation

$$(x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2) = 0. \quad (3.9)$$

Then the solutions w_1, w_2 are given by

$$w_{1,2} = [R(x_1, x_2) \mp \sqrt{R(x_1)R(x_2)}] / (x_1 - x_2)^2. \quad (3.10)$$

Further, according to Kowalevski [1], the shifted variables s_1, s_2 are introduced as

$$s_1 = w_1 + h, \quad s_2 = w_2 + h. \quad (3.11)$$

The remarkable feature is that the variables s_1 and s_2 are commutable in Poisson bracket, i.e., $\{s_1, s_2\} = 0$, which was shown by Dullin, Richter and Veselov [12], and Komarov [13]. This fact is in contrast to the variables (w_1, w_2) , which are not commutable. Therefore the variables (s_1, s_2) can serve as canonical coordinates. The equations of motion can be written

$$\dot{s}_1 = \frac{\sqrt{R_5(s_1)}}{s_2 - s_1}, \quad \dot{s}_2 = \frac{\sqrt{R_5(s_2)}}{s_1 - s_2}, \quad (3.12)$$

where

$$R_5(s) = -2R_2(s)R_3(s), \quad (3.13)$$

and

$$R_2(s) = (s - h)^2 - k, \quad (3.14)$$

$$R_3(s) = s(R_2(s) + 1) - \frac{1}{2}\ell^2 = s^3 - 2hs^2 + (h^2 - k + 1)s - \frac{1}{2}\ell^2. \quad (3.15)$$

Note that the polynomial $R_3(s)$ is the resolvent cubic of the 4th order polynomial $R(x)$ [10]. Thus $R(x) = 0$ has a multiple root if $R_3(s) = 0$ has a multiple root, and vice versa. The solutions of Eq.(3.12) are expressed in terms of hyperelliptic integrals of genus $g = 2$ since $R_5(s)$ is the 5th order polynomial.

Appel'rot [11] classified the critical bifurcation set Σ by analyzing whether the polynomial $R_5(s)$ has a multiple root. The classes are

$$\text{class 1 : } k = 0, \quad (3.16)$$

$$\text{class 2 : } h = -\sqrt{k} + \frac{1}{2}\ell^2, \quad (3.17)$$

$$\text{class 3 : } h = \sqrt{k} + \frac{1}{2}\ell^2, \quad (3.18)$$

$$\text{class 4 : } (1 - k)(h^2 + 1 - k)^2 - \frac{1}{2}h\ell^2[9(1 - k) + h^2] + \frac{27}{16}\ell^4 = 0. \quad (3.19)$$

The curve (h, k) of the class 4 is parameterized in terms the multiple root x of the equation $R(x) = 0$ as [10]

$$h = x^2 - \ell/(2x), \quad k = x^4 + \ell x + 1. \quad (3.20)$$

$R_2(s)$ has a double root for the class 1, $R_2(s)$ and $R_3(s)$ have a common root for the class 2 and 3, and $R_3(s)$ has a multiple root for the class 4, respectively.

Kharlamov [10] classified the (ℓ, h, k) -space into five regions which are divided by the bifurcation set Σ . In Fig.1, typical five regions are shown in (h, k) -diagram for $\ell = 0$, $0 < \ell^2 < 2$, $2 < \ell^2$, which will be named the case 1, 2 and 3 respectively. More detailed patterns are found in Ref.[10]. The region 5 exists only for $k \geq 0$ area in the case 3.

Figure 1: (h, k) -diagram. Regions are labeled by bold letters. Classically forbidden region is shown by shaded area.

Since the variables (s_1, s_2) move only in the allowed region defined by $R_5(s) \geq 0$, the sequence of the roots of $R_5(s)$ is crucial. Let the roots of $R_2(s)$ be (a_1, a_2) , i.e.,

$$a_1 = h - \sqrt{k}, \quad a_2 = h + \sqrt{k}. \quad (3.21)$$

$R_3(s) = 0$ has either three real roots b_1, b_2, b_3 ($b_1 \leq b_2 \leq b_3$), or one real root b_1 and complex conjugate roots b, b^* . In Table 1, the sequences of the roots of $R_5(s)$ are shown for five regions, together with the range of those values (s_1, s_2) for $R_5(s) \geq 0$. There are six patterns of sequence depending on the (ℓ, h, k) -values.

In Ref.[10], Kharlamov has shown that the boundary between the region 3 and 3', which is shown by the dashed line in Fig.1, is not included in the bifurcation set Σ , i.e., the structures of the invariant tori of the region 3 and 3' are the same. Thus the region 3 and 3', where the sequence of (a_1, b_2) is reversed, are not distinguished hereafter and named simply the region 3. Kharlamov also counted the number of invariant tori in each region by analyzing the projection of the invariant tori on the (p, q) -plane. The physical

region	sequence of $R_5(s) = 0$ roots	possible (s_1, s_2) values	number of tori
1	(a_1, b_1, a_2)	$-\infty \leq s_2 \leq a_1, b_1 \leq s_1 \leq a_2$	1
2	(b_1, a_1, a_2)	$-\infty \leq s_2 \leq b_1, a_1 \leq s_1 \leq a_2$	2
3	$(b_1, a_1, b_2, b_3, a_2)$	$-\infty \leq s_2 \leq b_1, b_3 \leq s_1 \leq a_2$	2
3'	$(b_1, b_2, a_1, b_3, a_2)$	$-\infty \leq s_2 \leq b_1, b_3 \leq s_1 \leq a_2$	2
4	$(a_1, b_1, b_2, b_3, a_2)$	$-\infty \leq s_2 \leq a_1, b_3 \leq s_1 \leq a_2$	2
5	$(b_1, b_2, b_3, a_1, a_2)$	$-\infty \leq s_2 \leq b_1, a_1 \leq s_1 \leq a_2$	4

Table 1: Roots of $R_5(s)$, possible (s_1, s_2) values and number of tori

real motions exist in the real (p, q) -domain for which $R_5(s) \geq 0$. In practice, the possible (s_1, s_2) values are given by the condition $R_5(s) \geq 0$, thus allowed (w_1, w_2) values are also limited. Since (w_1, w_2) are the solutions of the quadratic equation Eq.(3.9), the possible $x_{1,2} = p \pm iq$ values give us the real (p, q) -domain. The number of invariant tori is also listed in Table 1. The projected (p, q) -domains for each region are given in Ref.[10, 11]. For the later reference, schematic pictures of the (p, q) -domains are shown in Fig.2 for each region. An orbit with s_1 -varying (solid line) is like a "radial oscillation mode", while that with s_2 -varying (dashed line) is like an "angular rotational mode". The dashed lines always pass through the p -axis ($q = 0$), which corresponds to $s_2 = -\infty$, and go around in a limited angular range in the region 1 and 4. The shaded domains are always symmetric about the p -axis since Eq.(3.9) is an even function of q . They are also symmetric about the q -axis for the case 1 ($\ell = 0$), which are shown in Fig.2.

3.2 Allowed configuration space

The real angular velocity (p, q) -domains can be given by solving the 4th order algebraic equation of (p, q) for given (s_1, s_2) values. Further, one would like to know the allowed domain in the configuration space, i.e., direction-cosines $(\gamma_1, \gamma_2, \gamma_3)$ or the Euler angles $\Omega = (\alpha, \beta, \gamma)$. It is very important to know classically available configuration space since we will calculate quantum mechanical wave functions in terms of the Euler angles in Section 7. In other words, we would know the projection of the invariant tori on the configuration space.

According to Ref.[1, 2], one can obtain two equations by eliminating the variables r and γ_3 from Eqs.(3.1)-(3.4)

$$R(x_2)\xi_1 + R(x_1)\xi_2 + R_1(x_1, x_2) + k(x_1 - x_2)^2 = 0, \quad (3.22)$$

$$\xi_1\xi_2 = k, \quad (3.23)$$

where

$$\xi_1 = p^2 - q^2 - \gamma_1 + i(2pq - \gamma_2), \quad \xi_2 = p^2 - q^2 - \gamma_1 - i(2pq - \gamma_2). \quad (3.24)$$

Figure 2: Schematic (p, q) -domains for real motions. Solid (dashed) lines show the orbits when only s_1 (s_2)-variable moves. Dots represent the roots $R(x) = 0$, ($x = p + iq$).

For given (p, q) values, Eqs.(3.22),(3.23) are algebraic equations of the variables (γ_1, γ_2) . They are a linear and a quadratic type and are solved easily. Finally, the eliminated variables (r, γ_3) are recovered by Eqs.(3.1)-(3.3). In this way, the invariant tori are parameterized by the variables (s_1, s_2) . By changing the parameters in possible ranges given in Table 1, one can construct invariant tori in the phase space, and project them on the configuration space.

A more practical way of looking for the allowed configuration space is to solve directly the original equations of motion Eqs.(2.3),(2.4) for a long time enough until orbits fill densely an available phase space. Then one can construct an envelope curve of the orbits to obtain the allowed configuration space.

Here, we will adopt another way of fixing the boundary of the allowed region. We will derive the condition for which real (p, q) solutions exist. At the boundary of the allowed region, the solution (p, q) of Eqs.(3.22),(3.23) should have a multiple root. From

Eqs.(3.1)-(3.3), one can obtain, with setting $c_0 = 1$,

$$(p^2 + q^2 + \gamma_1 - h)(1 - \gamma_1^2 - \gamma_2^2) + \frac{1}{2}[\ell + 2(p\gamma_1 + q\gamma_2)]^2 = 0, \quad (3.25)$$

by eliminating the variables r and γ_3 . Changing variables as $p = \rho \cos \theta$, $q = \rho \sin \theta$, and substituting the Euler angles, Eq.(3.25) becomes

$$\rho^2 + \rho^2 \sin^2 \beta \cos 2(\theta + \gamma) - 2\ell \rho \sin \beta \cos(\theta + \gamma) - a = 0, \quad (3.26)$$

where $a = \cos^2 \beta (\sin \beta \cos \gamma + h) - \ell^2/2$. This Eq.(3.26) is also obtained by Eqs.(3.22),(3.23). Setting $u = \rho^2 \cos 2(\theta + \gamma)$, $v = \rho^2 \sin 2(\theta + \gamma)$, it becomes

$$\sqrt{u^2 + v^2} + u \sin^2 \beta \pm \sqrt{2}\ell \sin \beta (\sqrt{u^2 + v^2} + u)^{1/2} - a = 0. \quad (3.27)$$

By squaring Eq.(3.27) twice, one can obtain

$$[2\ell^2 \sin^2 \beta + 2(a - u \sin^2 \beta)]^2 (u^2 + v^2) = [(a - u \sin^2 \beta)^2 + u^2 + v^2 - 2\ell^2 u \sin^2 \beta]^2. \quad (3.28)$$

Eq.(3.23) is written as

$$u^2 + v^2 = k - 2u \sin \beta \cos \gamma - 2v \sin \beta \sin \gamma - \sin^2 \beta. \quad (3.29)$$

Thus, Eq.(3.28) becomes by substituting Eq.(3.29)

$$\begin{aligned} & [2\ell^2 \sin^2 \beta + 2(a - u \sin^2 \beta)]^2 (k - 2u \sin \beta \cos \gamma - 2v \sin \beta \sin \gamma - \sin^2 \beta) \\ & = [(a - u \sin^2 \beta)^2 + k - 2u \sin \beta \cos \gamma - 2v \sin \beta \sin \gamma - \sin^2 \beta - 2\ell^2 u \sin^2 \beta]^2. \end{aligned} \quad (3.30)$$

Now the condition that the solution (p, q) should have a multiple root is reduced to that of (u, v) variables. Both Eqs.(3.29),(3.30) are quadratic equations of the variable v . They have a common solution when the resultant of two polynomials vanishes. This resultant is the 8th order polynomial of u , which will be denoted by $g(u)$. For the solution (u, v) has a multiple root, the resultant $g(u)$ should have a multiple root, and this condition is again expressed that the resultant Δ of $g(u)$ and its derivative $g'(u)$ vanishes. Since $g(u)$ is the 8th order polynomial, Δ is expressed by the determinant of 15×15 matrix, where the matrix elements are given by the coefficients of $g(u)$ and $g'(u)$. The resultant Δ becomes a very involved function of (β, γ) and is not written here explicitly. However, for fixed values (ℓ, h, k) , it is not so difficult to solve numerically the equation $\Delta(\beta, \gamma) = 0$.

Let us make one comment for the case 1 ($\ell = 0$). In this case, Eq.(3.30) becomes a complete square,

$$[k - 2u \sin \beta \cos \gamma - 2v \sin \beta \sin \gamma - (a - u \sin^2 \beta)^2 - \sin^2 \beta]^2 = 0. \quad (3.31)$$

So the resultant becomes $\Delta(\beta, \gamma) = 0$ since (u, v) always have a multiple root. Thus, instead of Eq.(3.30), one should use

$$k - 2u \sin \beta \cos \gamma - 2v \sin \beta \sin \gamma - (a - u \sin^2 \beta)^2 - \sin^2 \beta = 0, \quad (3.32)$$

Figure 3: (β, γ) -domains for real motions. (a)-(d) and (e1),(e2) correspond to the region 1-4, and two domains of the region 5, respectively. Angular momentum is $\ell = 0$ for the region 1-4, while $\ell = 1.8$ for the region 5.

and Eq.(3.29), which becomes

$$u^2 + v^2 - (a - u \sin^2 \beta)^2 = 0, \quad (3.33)$$

and the resultant of the polynomials of Eqs.(3.32),(3.33).

In Fig.3, we show the allowed (β, γ) -domains for typical (ℓ, h, k) -values of each region. Since the equation $\Delta(\beta, \gamma) = 0$ has several solutions, we choose proper boundaries by comparing them with those given by the envelopes of the orbits. The ranges are $0 \leq \beta \leq \pi$, $-\pi \leq \gamma \leq \pi$. The lowest potential energy is at $\beta = \pi/2$ and $\gamma = 0$. The classical orbits going through $\gamma_3 = \pm 1$ ($\beta = 0, \pi$) points are possible only for the $h = \sqrt{k} + \ell^2/2$ boundary. The angular momentum is $\ell = 0$ for the region 1-4, thus the projections of the two tori are the same for the region 2, 3 and 4. In the region 5, (e1) and (e2) correspond to the outer and inner domains of Fig.2-(e). Since $\ell = 1.8$, there exist another (β, γ) -domains

which are not shown. They are given by $\beta \rightarrow \pi - \beta$, $(\gamma_3 \rightarrow -\gamma_3)$ of the figures (e1) and (e2), i.e., reflections with respect to the line $\beta = \pi/2$.

4 Action variables

Since the Kowalevski top is an integrable system, dynamical variables are canonically transformed to action-angle variables by Liouville's theorem. Dullin, Juhnke and Richter [9] have carried out the calculation of the action variables and constructed energy surfaces in the space of action variables. They used the algorithm of Dullin and Wittek [14], which made it possible to find numerically two independent closed paths generated by the flows of the Hamiltonian and the Kowalevski constant. In this section, we will calculate the action integrals given in the analytic forms, and derive the connection formulas of the action integrals on the boundary of different regions.

4.1 Action-angle variables

The canonical conjugate momenta of the Kowalevski top were first given by Veselov and Novikov in Ref.[15]. Komarov and Kuznetsov [7] derived the same formula by the method of separation of variables which requires that the action S has a truncated form, i.e., $S = S_1(s_1) + S_2(s_2)$ with commuting variables (s_1, s_2) .

The canonical momentum $p(s_i)$ conjugate to s_i is given by

$$p(s) = \frac{1}{2\sqrt{-2s}} \log \left[\sqrt{-s}R_2(s) - \frac{(2s - \ell^2)}{4\sqrt{-s}} + \sqrt{-R_2(s)R_3(s)} \right], \quad (4.1)$$

and the action integrals I_i are expressed by

$$I_i = \frac{1}{2\pi} \oint_{C_i} p(s)ds, \quad i = 1, 2. \quad (4.2)$$

The closed contour C_i encircles each interval of the real s -axis of Table 1 where $R_5(s) \geq 0$ and there exist real motions. (C_1, C_2) correspond to (s_1, s_2) variables respectively. In Ref.[12], Dullin, Richter and Veselov have reported that the numerical calculations of Ref.[9] agreed with those given by Eq.(4.2). From Eq.(4.2), (h, k) are given by the functions of (I_1, I_2) . The function $S_i(s_i)$ is given by

$$S_i(s_i, I_1, I_2) = \int^{s_i} p(s, I_1, I_2)ds, \quad (4.3)$$

and the angle variables φ_i are

$$\varphi_i = \frac{\partial S}{\partial I_i}. \quad (4.4)$$

By integrating by parts, Eq.(4.3) becomes [12],

$$2\pi I_i = [\Phi]_{C_i} - \oint_{C_i} (\mu(s) + \nu(s))ds, \quad (4.5)$$

where

$$\Phi = -\frac{1}{2}\sqrt{-2s} \log \left[\sqrt{-s}R_2(s) - \frac{(2s - \ell^2)}{4\sqrt{-s}} + \sqrt{-R_2(s)R_3(s)} \right], \quad (4.6)$$

and

$$\begin{aligned} \mu(s) &= \frac{2s(s - h)(2s - \ell^2) - \ell^2 R_2(s)}{\sqrt{R_5(s)}(2s - \ell^2)} \\ &= \frac{4s^3 - (4h + 3\ell^2)s + 4h\ell^2 - \ell^2(h^2 - k)}{\sqrt{R_5(s)}(2s - \ell^2)}, \end{aligned} \quad (4.7)$$

$$\nu(s) = \frac{1}{2\sqrt{-2s}} \frac{2s + \ell^2}{2s - \ell^2}. \quad (4.8)$$

$R_5(s)$ is the 5th order polynomial, thus the contours C_i are on the Riemann surface with genus $g = 2$. However, due to the another cut $\sqrt{-s}$, the integral should be calculated on its double covering surface with genus $g = 4$ [12]. The cuts coming from $\sqrt{R_5(s)}$ and $\sqrt{-s}$ are different, and both cuts have the same branch point at $s = \infty$. Therefore, the integration parameter s_i generally goes around twice the contour C_i to make a physical orbit closed in the phase space. The exception is C_1 in the region 2 and 5, where s_1 goes around once the cut $[a_1, a_2]$. This situation is also understood by the fact that the value $\sqrt{-R_2(s)R_3(s)}$ in Eq.(4.1) is the same after one cycle since a_1 and a_2 are the solutions of $R_2(s) = 0$. In Table 2, the multiplicities (N_1, N_2) of the contours (C_1, C_2) are summarized for each region.

region	1	2	3	4	5
N_1	2	1	2	2	1
N_2	2	2	2	2	2

Table 2: Multiplicity of contours

In the region 1, s_1 goes around the contour $C_1 = (a_2 \rightarrow b_1 \rightarrow a_2)$ of the first $\sqrt{-s}$ sheet, and then goes on the second sheet. This situation is the same in the region 3 and 4 where $C_1 = (a_2 \rightarrow b_3 \rightarrow a_2)$. The s_2 variable always goes around the contour C_2 twice since the $s = \infty$ is the common branch point for $\sqrt{-s}$ and $\sqrt{R_5(s)}$. These behavior also can be understood by the trajectory in the (p, q) -plane of Fig.2.

4.2 Numerical calculations of action integrals

Let us carry out numerical calculations of Eq.(4.5). In fact,

$$[\Phi]_{\alpha_i} = 0, \quad \oint_{\alpha_i} \nu(s) ds = 0, \quad (4.9)$$

where the closed loops α_i are understood to include the multiplicity number of Table 2. For $\ell \neq 0$, the functions $\mu(s)$ and $\nu(s)$ have a pole at $s_p = \ell^2/2$. It turns out that the

location s_p is always out of the physically allowed region. For simplifying notations, the action integrals are redefined as follows

$$2\pi I_r = \oint_{\alpha_1} \mu(s_1) ds_1, \quad 2\pi J_r = \oint_{\alpha_2} \mu(s_2) ds_2, \quad (4.10)$$

where $r = 1, \dots, 5$ specifies different regions. We choose the way of making cuts and the directions of the loops α_i such that the action integrals (I_r, J_r) are real positive values and also numerical calculations can be carried out as easily as possible. The phase of $\sqrt{R_5(s)}$ is defined $\arg[\sqrt{R_5(s)}] = 0$ on the integration interval of s_1 along the real axis. The phases of $\sqrt{R_5(s)}$ in the integrands I_r and J_r are related in the same region, and may have an opposite sign. In that case, the direction of α_i should be reversed simultaneously. The schematic illustrations of cuts, integration loops and the location of the pole are shown in Fig.4.

Figure 4: Integration contours for each region. Cuts are shown by thick lines and poles are shown by $\times$ -symbol.

At the boundary of the integral on the real axis, except for $s_2 = -\infty$, the integrand diverges as $1/\sqrt{\varepsilon} \rightarrow \infty$ ($\varepsilon \rightarrow 0$). These singularities of $1/\sqrt{x}$ type can be integrated and harmless. The integrand of J_r at $s_2 = -\infty$ goes like

$$\mu(s_2) \simeq 2/\sqrt{-2s_2}, \quad (4.11)$$

and the integral J_r along the real axis diverges. However, this divergence is canceled by the contribution encircling the $s_2 = -\infty$ point, and the action integral J_r remains finite. We will employ the subtraction method to calculate the integral J_r . The details of practical calculations are rather technical, and will be explained in Appendix A. Numerical results of the action integrals will be shown in the next section where the semiclassical quantization is described.

4.3 Boundary values and connection formulas of action integrals

At the Appel'rot's critical set Σ where $R_5(s) = 0$ has a multiple root, one of the cuts is closed and the genus becomes $g = 1$. Thus the calculations of the action integrals become much simpler. The set Σ corresponds to either (a) boundaries of possible real motion, or (b) boundaries of different regions.

The type (a) boundaries consist of four parts for the case 1 ($\ell = 0$), and three parts for the case 2 and 3 ($\ell > 0$), as is seen in Fig.1. The boundary AX for the case 1 is the line A-C-X. For this special case, AC and CX are the boundaries of the region 1 and 4 respectively. For the case 2 and 3, the boundary AX is a smooth curve. The boundary values are shown in Table 3.

boundary	PY	PA	AX
case 1	$I_2 = 0$	$I_1 = 0$	$I_1 = J_1, I_4 = J_4$
case 2 and 3	$I_2 = 0$	$I_1 = \ell$	$I_1 = J_1 + \ell$

Table 3: Boundary values

There are many patterns of type (b) boundaries depending on the ℓ -value and the neighboring regions. For several cases, the action integrals have discontinuities across these boundaries. It is because the structure of the invariant torus changes across the boundary due to the bifurcation. In addition, the pole singularity of the integrand at $s_p = \ell^2/2$ gives rise to another discontinuities. However, one can relate the action integrals of the neighboring regions by smoothly deforming the integration contour on the Riemann surface at the boundary. These connection formulas of the actions of neighboring regions are summarized in Table 4. Table 4-(a) is common for three cases of Fig.1, while 4-(b) is for the cases 1, 2, and 4-(c) is for the case 3. The practical way of calculations is described in Appendix B.

5 Semiclassical quantization

The semiclassical quantization can be carried out through the EBK (Einstein-Brillouin-Keller) formulation. If a classical Hamiltonian is expressed in terms of action integrals I_i , the semiclassical quantum energy is given by the substitutions

$$I_i = (n_i + \beta_i/4)\hbar, \quad (5.1)$$

where $n_i \geq 0$ is an integer quantum number and β_i is the Maslov index. In this section, we will calculate the Maslov indices in each region, and assign quantum numbers to the semiclassical spectra.

boundary	1-2	2-3	1-3
connection	$I_1 = 2I_2 + \ell$	$I_2 = I_3 - J_3/2 - \ell/2$	$I_1 = I_3$
formula	$J_1 = J_2 + \ell$	$J_2 = J_3$	$J_1 = 2J_3 - I_3 + \ell$

(a) boundaries 1-2-3

boundary	BC	CD	BD
connection	$I_1 = 2I_4 - J_4 + \ell$	$I_1 = I_4$	$I_3 = I_4$
formula	$J_1 = J_4$	$J_1 = J_4$	$J_3 = J_4 - \ell$

(b) case 1 and 2

boundary	BC	CD	BD
connection	$I_2 = I_5$	$I_2 = I_5$	$I_3 = 2I_5 + \ell$
formula	$J_2 = 2J_5 - 2I_5 - \ell$	$J_2 = J_5$	$J_3 = J_5$

(c) case 3

Table 4: Connection formulas

5.1 Maslov indices

The Maslov index is the number of points which are intersections of a curve on the invariant torus and a caustic. Here, caustics are defined by the set on which the projection from the invariant torus to the configuration space is singular. For example, $\beta = 2$ for the one-dimensional oscillation since it has two turning points. In the case of the Kowalevski top, the invariant torus is 2-dimensional manifold in 4-dimensional phase space. Thus the projection to the 2-dimensional configuration space is not easy to imagine. It will also depend on the region, since the structure of the invariant torus is different.

In order to calculate the Maslov indices systematically, we will employ the winding number formula given by Robbins [16]. In the case of integrable systems, they are given as follows. Let $(\mathbf{p}, \mathbf{q})$ denote canonical conjugate coordinates and momenta, and $(\mathbf{I}, \boldsymbol{\varphi})$ action-angle variables. We will choose an invariant torus with constant action integrals, and j -th closed curve on the torus with only φ_j -varying. Let us define the matrices A and C with matrix elements as

$$A_{ij} = \frac{\partial q_i}{\partial \varphi_j}, \quad C_{ij} = \frac{\partial p_i}{\partial \varphi_j}. \quad (5.2)$$

The determinant $\det(A - iC)$ is a periodic function of φ_j . Then, the Maslov index of the j -th angle contour is given by the winding number of $\det(A - iC)$ for the j -th closed curve ($0 \leq \varphi_j \leq 2\pi$),

$$\beta_j = 2 \text{ wn}[\det(A - iC)], \quad (5.3)$$

where the winding number $\text{wn}[z]$ is defined by the amount of phase-change of a complex number z moving $z_1 \rightarrow z_2$ as

$$\text{wn}[z] = (\arg[z_2] - \arg[z_1])/2\pi. \quad (5.4)$$

The formula (5.3) is very simple and useful, since all we need is to trace the argument of $\det(A - iC)$ for a closed contour. In between, it is also useful to check the value $\det A$, because the caustics occur precisely at $\det A = 0$. At the caustics, Jacobian is $\det\left(\frac{\partial p_i}{\partial q_j}\right) = \infty$ and the projection to the configuration space is singular.

In the numerical calculations, we have used the Euler angles and their conjugate momenta, i.e., $\mathbf{q} = (\beta, \gamma)$, $\mathbf{p} = (p_\beta, p_\gamma)$ of Eq.(2.9). Note that $p_\alpha = L_z$ is a constant of motion. Instead of the angle variables (φ_1, φ_2) , we adopt the parameters (s_1, s_2) which move on the given invariant torus. For given (s_1, s_2) , (ω, γ) variables are obtained by the same way as Section 3.2. Since (ω, γ) variables are not canonical, we need to transform them further to (β, γ) by Eq.(2.6). (p_β, p_γ) become by using Eqs.(2.5),(2.9),

$$p_\beta = 2\dot{\beta} = \frac{2(p\gamma_2 - q\gamma_1)}{\sin \beta}, \quad (5.5)$$

$$p_\gamma = \dot{\gamma} + \dot{\alpha} \cos \beta = r. \quad (5.6)$$

Now the variables $(\beta, \gamma, p_\beta, p_\gamma)$ are obtained as the functions of (s_1, s_2) , the matrix elements of A and C can be numerically calculated. Then it is easy to extract the winding numbers and the caustics. Table 5 shows the Maslov indices (β_1, β_2) in each region, where β_1 and β_2 correspond to the loops α_1 and α_2 of Fig.4 respectively. In Fig.5, we will show typical orbits in the configuration space where (ℓ, h, k) -values are $(0, 0.3, 0.36)$ in the region 1. (a) is the α_1 -loop with $s_2 = -37/90$, while (b) is the α_2 -loop with $s_1 = 0.45$. Thick lines are orbits and thin lines are the projections of caustics on the configuration space. Small dots show the crossing points of the orbit and the caustic, the number of which is the Maslov index.

region	1	2	3	4	5
β_1	2	2	2	2	2
β_2	4	2	2	4	2

Table 5: Maslov indices

5.2 Semiclassical quantum numbers

We will assign integer quantum numbers (m_r, n_r) for each region $r = 1, \dots, 5$. They are shown in Table 6. The quantum numbers are deliberately chosen by considering the boundary values and the Maslov indices. The classical angular momentum ℓ is replaced by the integer quantum number M as $\ell = M\hbar$.

The numbers with a prime (m'_2, m'_5) denote that these quantum numbers correspond to the loops with the multiplicity 1 in Table 2. The connection formulas of Table 4 give us the connection formulas of the quantum numbers if they are assumed continuous. For example, at the 1-2 boundary,

$$m_1 = 2m'_2 + 1/2, \quad n_1 = n_2 + M. \quad (5.7)$$

Figure 5: Typical orbits in the configuration space

region	I	J
1	$M + m_1 + 1/2$	$m_1 + n_1 + 1$
2	$m'_2 + 1/2$	$(2m'_2 + 1) + n_2 + 1/2$
3	$M + m_3 + 1/2$	$m_3 + n_3 + 1/2$
4	$M + m_4 + 1/2$	$m_4 + n_4 + 1$
5	$m'_5 + 1/2$	$(2m'_5 + 1) + n_5 + 1/2$

Table 6: Semiclassical quantum numbers with Maslov indices

5.3 Numerical results

The numerical values of the action integrals are given by Eq.(4.10), i.e., $I_r = I_r(h, k)$, $J_r = J_r(h, k)$ for a fixed ℓ -value. Conversely, for given quantum numbers (m_r, n_r) and $\hbar$ -value, conserved quantities (h, k) can be calculated. In Fig.6, we will show curves with constant m_r -value and n_r -value on the (h, k) -diagram. Crossing points of $m_r = \text{const}$ and $n_r = \text{const}$ lines correspond to the semiclassical quantum states. The $\hbar$ -value is chosen in such a way that there exist sufficient number of semiclassical quantum states in each region. For the region 5, $m'_5 = 0$, $n_5 = 35$ and there exists one state. In practice, we have employed the bisection method or the simplex method to search (h, k) solutions for given (m_r, n_r) . From Fig.6, one can understand that the assignments of the quantum numbers are reasonable. It is clearly seen how the action variables (I_r, J_r) depend on the conserved quantities (h, k) or vice versa. They are very regular in the middle of each region

and show some discontinuities on the boundaries. Since the Kowalevski top has multiple tori, semiclassical quantum states have some degeneracies. For example, there exist two tori in the region 2, which give rise to doubly degenerate states. Eq.(5.7) indicates that $m_1 = 2m'_2 + 1/2$ is an average of $m_1 = 2m'_2$ and $2m'_2 + 1$ at the boundary, while n_1 and n_2 are discontinuous by M . Similar relations such as Eq.(5.7) are obtained by the connection formula (Table 4), and they are certainly consistent with Fig.6.

6 Quantum Mechanics

In this section, we will show the quantum operators of the Hamiltonian and the Kowalevski constant, and orthogonal bases which diagonalize these operators.

6.1 Hamiltonian and Kowalevski operator

The Hamiltonian $\hat{H}$ is given simply by the replacement of the classical one Eq.(2.12),

$$\hat{H} = \frac{\hbar^2}{4C}(J_1^2 + J_2^2 + 2J_3^2) + \mu_0\gamma_1, \quad \mu_0 = Mgx_0. \quad (6.1)$$

We resume writing the moment of inertia C and the gravitational potential strength $\mu_0 = Mgx_0$ explicitly. The angular momenta J_i are measured in units of $\hbar$, and they are given by

$$J_1 = -i \left(-\frac{\cos \gamma}{\sin \beta} \frac{\partial}{\partial \alpha} + \sin \gamma \frac{\partial}{\partial \beta} + \cot \beta \cos \gamma \frac{\partial}{\partial \gamma} \right), \quad (6.2)$$

$$J_2 = -i \left(\frac{\sin \gamma}{\sin \beta} \frac{\partial}{\partial \alpha} + \cos \gamma \frac{\partial}{\partial \beta} - \cot \beta \sin \gamma \frac{\partial}{\partial \gamma} \right), \quad (6.3)$$

$$J_3 = -i \frac{\partial}{\partial \gamma}. \quad (6.4)$$

For the Kowalevski operator, let us define operators $k_{\pm}$ as the same forms of their classical ones Eq.(2.15),

$$k_{\pm} = \frac{\hbar^2}{4C^2} J_{\pm}^2 - c_0 \gamma_{\pm} = \frac{1}{4C^2} (\hbar^2 J_{\pm} - 4\mu_0 C \gamma_{\pm}), \quad J_{\pm} = J_1 \pm iJ_2. \quad (6.5)$$

Laporte [17] showed that the Hamiltonian $\hat{H}$ commutes with

$$\hat{K} = \frac{1}{2} \{k_+, k_-\} + \frac{2\hbar^4}{(4C^2)^2} \{J_+, J_-\}, \quad (6.6)$$

where $\{A, B\} = AB + BA$ is the anticommutator. The quantum operator $\hat{K}$ differs from the classical form by the symmetrization of the first term, and the quantum correction of the second term proportional to $\hbar^4$

(a) $\ell = 0$, $\hbar = 0.2$

(b) $\ell = 0.2$, $\hbar = 0.2$

(c) $\ell = 0.4$, $\hbar = 0.2$

(d) $\ell = 1.8$, $\hbar = 0.05$

Figure 6: Semiclassical spectra labeled by integer quantum numbers. Thick lines show the boundaries.

Let us measure the Hamiltonian in units of μ_0 and the Kowalevski operator in units of $(\mu_0/C)^2$, and simply rewrite $\hat{H}/\mu_0 \rightarrow \hat{H}$, $\hat{K}/(\mu_0/C)^2 \rightarrow \hat{K}$. Then

$$\hat{H} = \frac{\hbar^2}{4}(J_1^2 + J_2^2 + 2J_3^2) + \gamma_1, \quad (6.7)$$

$$16\hat{K} = \frac{1}{2}\{k'_+, k'_-\} + 2\hbar^4\{J_+, J_-\}, \quad k'_\pm = \hbar^2 J_\pm - 4\gamma_\pm, \quad (6.8)$$

where we further set $\hbar/\sqrt{\mu_0 C} \rightarrow \hbar$. Since $\sqrt{\mu_0 C}$ has the same dimension as $\hbar$, the scaled $\hbar$ is a dimensionless parameter. The parameter $\hbar$ represents the magnitude of the gravitational field since $\hbar \propto 1/\sqrt{\mu_0}$. $\hbar \rightarrow 0$ (∞) corresponds to the large (small) gravitational field limit. For the practical calculations of the matrix elements, it is better to rewrite Eqs.(6.7),(6.8) as

$$\hat{H} = \frac{\hbar^2}{4}(\mathbf{J}^2 + J_3^2) + \frac{1}{2}(\gamma_+ + \gamma_-), \quad (6.9)$$

$$16\hat{K} = \hbar^4 \left[\frac{1}{4}\{J_+, J_-\}^2 + \{J_+, J_-\} + 3J_3^2 \right] - 2\hbar^2 \left[\{J_+^2, \gamma_-\} + \{J_-^2, \gamma_+\} \right] + 8\{\gamma_+, \gamma_-\}. \quad (6.10)$$

6.2 Orthogonal bases and matrix elements

If the gravitational field is switched off, the Hamiltonian reduces to that of the symmetric Euler top ($A = B \neq C$). The eigenfunction is given by the D -function which is the representation matrix element of the rotational group. The definition of the rotation operator is

$$\hat{R}(\Omega) = \hat{R}(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}, \quad (6.11)$$

where angular momentum operators satisfy ordinary commutation relations

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y, \quad (6.12)$$

and the D -function is defined as

$$D_{mm'}^j(\Omega) = \langle jm | \hat{R}(\Omega) | jm' \rangle^* = \langle jm' | e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z} | jm \rangle = e^{im\alpha} d_{mm'}^j(\beta) e^{im'\gamma} \quad (6.13)$$

with

$$d_{mm'}^j(\beta) = \langle jm | e^{-i\beta J_y} | jm' \rangle^* = \langle jm' | e^{i\beta J_y} | jm \rangle. \quad (6.14)$$

The $d_{mm'}^j(\beta)$ function is real and its explicit form is given in terms of Jacobi polynomial or hypergeometric function [18].

The orthogonal bases are given by

$$\varphi_{MK}^J(\Omega) = \left(\frac{2J+1}{8\pi^2} \right)^{1/2} D_{MK}^J(\Omega) \quad (6.15)$$

with the normalization

$$\int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma \varphi_{M_1 K_1}^{J_1}(\Omega)^* \varphi_{M_2 K_2}^{J_2}(\Omega) = \delta_{J_1, J_2} \delta_{M_1, M_2} \delta_{K_1, K_2}. \quad (6.16)$$

$\varphi_{MK}^J(\Omega)$ is the simultaneous eigenfunction of the operators $(\mathbf{J}^2, L_z, J_3)$ and the eigenvalues are

$$\mathbf{J}^2 \varphi_{MK}^J(\Omega) = J(J+1) \varphi_{MK}^J(\Omega), \quad (6.17)$$

$$L_z \varphi_{MK}^J(\Omega) = M \varphi_{MK}^J(\Omega), \quad (6.18)$$

$$J_3 \varphi_{MK}^J(\Omega) = K \varphi_{MK}^J(\Omega), \quad (6.19)$$

where $L_z = -i \frac{\partial}{\partial \alpha}$ is the space-fixed z -component of the angular momentum.

When the gravitational field is included, the operators $(\mathbf{J}^2, J_3)$ do not commute with the Hamiltonian. However, the operator L_z is still commutable with the Hamiltonian and the eigenvalue M is a constant of motion. This constant of motion corresponds to the classical value ℓ as $\ell = M\hbar$. We will diagonalize the Hamiltonian $\hat{H}$ and the Kowalevski operator $\hat{K}$ in terms of the orthonormal bases $\varphi_{MK}^J(\Omega)$. For a fixed value $M \geq 0$, $J = M, M+1, M+2, \dots$, and $|K| = 0, 1, 2, \dots, J$.

The transformations by the operators $\gamma_{\pm}$ can be calculated by the functional relations of Jacobi polynomials. The results are

$$\begin{aligned} \gamma_+ \varphi_{MK}^J(\Omega) &= -\frac{1}{J} \left[\frac{(J^2 - M^2)(J+K-1)(J+K)}{(2J-1)(2J+1)} \right]^{1/2} \varphi_{M,K-1}^{J-1}(\Omega) \\ &\quad + \frac{M[(J-K+1)(J+K)]^{1/2}}{J(J+1)} \varphi_{M,K-1}^J(\Omega) \\ &\quad + \frac{1}{J+1} \left[\frac{\{(J+1)^2 - M^2\}(J-K+1)(J-K+2)}{(2J+1)(2J+3)} \right]^{1/2} \varphi_{M,K-1}^{J+1}(\Omega), \end{aligned} \quad (6.20)$$

$$\begin{aligned} \gamma_- \varphi_{M,K}^J(\Omega) &= \frac{1}{J} \left[\frac{(J^2 - M^2)(J-K-1)(J-K)}{(2J-1)(2J+1)} \right]^{1/2} \varphi_{M,K+1}^{J-1}(\Omega) \\ &\quad + \frac{M[(J+K+1)(J-K)]^{1/2}}{J(J+1)} \varphi_{M,K+1}^J(\Omega) \\ &\quad - \frac{1}{J+1} \left[\frac{\{(J+1)^2 - M^2\}(J+K+1)(J+K+2)}{(2J+1)(2J+3)} \right]^{1/2} \varphi_{M,K+1}^{J+1}(\Omega). \end{aligned} \quad (6.21)$$

Since the Hamiltonian and the Kowalevski operator are invariant under the R -operator, which is the π -rotation about the 1-axis in the body-fixed frame, the eigenstates are classified by the symmetry quantum number $r = \pm 1$. Accordingly, it is better to classify the bases with $r = \pm 1$. The rotation operator R changes the Euler angles $\Omega = (\alpha, \beta, \gamma)$ as Eq.(2.19)

$$R: (\alpha, \beta, \gamma) \rightarrow (\alpha + \pi, \pi - \beta, -\gamma), \quad (6.22)$$

then the D -function is transformed as

$$RD_{MK}^J(\alpha, \beta, \gamma) = D_{MK}^J(\alpha + \pi, \pi - \beta, -\gamma) = (-1)^J D_{M,-K}^J(\alpha, \beta, \gamma). \quad (6.23)$$

Hereafter we will suppress the quantum number M , since M is a trivial constant of motion. The argument Ω will be also suppressed for simplicity. Then

$$R\varphi_K^J = (-1)^J \varphi_{-K}^J. \quad (6.24)$$

Let us define the bases with definite eigenvalue r as, for $K > 0$,

$$\varphi_{JK}^+ = \frac{1}{\sqrt{2}}(\varphi_K^J + (-1)^J \varphi_{-K}^J), \quad r = 1, \quad (6.25)$$

$$\varphi_{JK}^- = \frac{1}{\sqrt{2}}(\varphi_K^J - (-1)^J \varphi_{-K}^J), \quad r = -1, \quad (6.26)$$

and for $K = 0$,

$$\varphi_{J0}^+ = \varphi_0^J, \quad r = 1, \quad J = \text{even}, \quad (6.27)$$

$$\varphi_{J0}^- = \varphi_0^J, \quad r = -1, \quad J = \text{odd}. \quad (6.28)$$

The matrix elements of $\hat{H}$ and $\hat{K}$ in terms of the bases $\varphi_{JK}^\pm$ are easily calculated for those terms which consist of only angular momenta, since those terms do not change the J -value. However, the matrix elements of the terms with $\gamma_\pm$ are rather complicated although they are given by recursively applying Eqs.(6.20),(6.21). The matrix elements of the Hamiltonian and the Kowalevski operator are summarized in Appendix C. Once the Hamiltonian is diagonalized and the eigenfunctions are obtained, it is straightforward to calculate the eigenvalues of the Kowalevski operator.

Let us comment on the special case of $M = 0$, where the system has also the symmetries generated by the operators S_1 and S_2 of Eqs.(2.20),(2.21). The operations of S_1 and S_2 on the D -function are

$$\begin{aligned} S_1 D_{MK}^J(\alpha, \beta, \gamma) &= D_{MK}^J(-\alpha, \beta, -\gamma) \\ &= (-1)^{M-K} D_{-M, -K}^J(\alpha, \beta, \gamma) = D_{MK}^J(\alpha, \beta, \gamma)^*, \end{aligned} \quad (6.29)$$

$$\begin{aligned} S_2 D_{MK}^J(\alpha, \beta, \gamma) &= D_{MK}^J(\pi - \alpha, \pi - \beta, \gamma) \\ &= (-1)^{J+M-K} D_{-M, K}^J(\alpha, \beta, \gamma) = (-1)^J D_{M, -K}^J(\alpha, \beta, \gamma)^*. \end{aligned} \quad (6.30)$$

Therefore, they transform the wave function to its complex conjugate which represents the time-reversed state. The bases do not depend on α and are transformed as

$$S_1 \varphi_{JK}^\pm = \varphi_{JK}^{\pm*} = (-1)^{J-K} (\pm 1) \varphi_{JK}^\pm, \quad (6.31)$$

$$S_2 \varphi_{JK}^\pm = R S_1 \varphi_{JK}^\pm = (-1)^{J-K} \varphi_{JK}^\pm. \quad (6.32)$$

Let the symmetry quantum numbers of the operators (S_1, S_2) be (s_1, s_2) . Then those two bases with $J - K = \text{even (odd)}$ are decoupled and quantum states are further classified with the symmetry number $s_2 = (-1)^{J-K} = \pm 1$. This further symmetry can be seen also in Eqs.(C.10)-(C.25) in Appendix C. For the non-vanishing transitions $(J, K) \rightarrow (J', K')$, the relation $(-1)^{J-K} = (-1)^{J'-K'}$ always holds since $M = 0$. Therefore, those bases with $J - K = \text{even/odd}$ are decoupled.

7 Results and discussion

We have diagonalized the Hamiltonian and obtained eigenvalues and eigenfunctions. Then the eigenvalues of the Kowalevski operator are calculated by using these wave functions. In this section, we will discuss quantum spectra (h, k) which show near-degeneracies except for the region 1. The quantum states are labeled by the semiclassical quantum numbers of Section 5. The relations between the symmetry and quantum numbers are clarified by making use of the semiclassical wave functions. These relations are analogous to those in the case of the quantum Toda lattice [19, 20]. We will plot typical wave functions in the configuration space of the Euler angles. The limiting cases of the large or small gravitational force and the size of quantum correction will be discussed in some detail.

7.1 Quantum spectra

In Fig.7, we show the quantum spectra (h, k) and compare them with the semiclassical values. Since the region 5 is very tiny, we choose deliberately the angular momentum $\ell = 1.8$ ($M = 36$, $\hbar = 0.05$) in order to obtain aimed quantum states. For graphs (a)-(c) ($M = 0, 1, 2$), we take the bases $\varphi_{JK}^\pm$ with $M \leq J \leq J_{max} = 40$ and $0 \leq K \leq J$ for each J -value, and the number of bases is about 840 for each $r = \pm 1$. For the graph (d) ($M = 36$), $J_{max} = 90$ and the number of bases is about 3500.

The quantum spectra are generally very close to their semiclassical values. Thus, it is possible to label quantum states by the semiclassical quantum numbers. The spectra are very regular in the middle of the region, and show some irregular behavior near the boundaries.

One can see that the quantum spectra are nearly degenerate except for the region 1, although it may not easy to distinguish two overlapped same symbols in the region 4. This is because there exist several invariant tori except for the region 1. However, these quantum spectra are not exactly degenerate due to the tunnel effects between different tori. Especially, these near-degeneracies are lifted around the boundary.

It is interesting that these near degenerate states have different symmetry quantum numbers $r = \pm 1$ in the region 2 and 3, while they are the same in the region 4. In the region 5, four states are nearly degenerate, and two of them have $r = 1$ and the other two have $r = -1$. In the region 1 and 4, there is a relation between the symmetry quantum number $r = \pm 1$ and quantum numbers (m_r, n_r) ($r = 1, 4$) as

$$r = (-1)^{M+m_r+n_r}. \quad (7.1)$$

This simple relation will be explained by making use of the semiclassical wave functions.

(a) $M = 0$, $\hbar = 0.2$

(b) $M = 1$, $\hbar = 0.2$

(c) $M = 2$, $\hbar = 0.2$

(d) $M = 36$, $\hbar = 0.05$

Figure 7: Quantum spectra compared with the semiclassical values. The symbol $\circ$ ($+$) indicates $r = 1$ (-1) state.

7.2 Semiclassical wave functions

Since the quantum spectra are close to the semiclassical values, it is expected that the wave functions can be also approximated by those of the semiclassical quantization. To the lowest order in $\hbar$, the semiclassical wave function which is consistent with the EBK quantization is given by

$$\phi(\mathbf{q}) = A(\mathbf{q}, \mathbf{I}) \exp[iS(\mathbf{q}, \mathbf{I})/\hbar], \quad (7.2)$$

where $\mathbf{I} = (I_r, J_r)$ are action variables. The phase function $S(\mathbf{q}, \mathbf{I})$ is a solution of the Hamilton-Jacobi equation and also a generating function of the canonical transformation for $(\mathbf{p}, \mathbf{q}) \rightarrow (\mathbf{I}, \boldsymbol{\varphi})$,

$$\mathbf{p} \cdot d\mathbf{q} + \boldsymbol{\varphi} \cdot d\mathbf{I} = dS, \quad (7.3)$$

and can be given by the integral

$$S(\mathbf{q}, \mathbf{I}) = \int_C \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x}, \quad (7.4)$$

where C is a path from a fixed point $O(\mathbf{p}_0, \mathbf{q}_0)$ to $P(\mathbf{p}, \mathbf{q})$ on the invariant torus $M_{\mathbf{I}}$ defined by $\mathbf{I}(\mathbf{p}, \mathbf{q}) = \text{const}$. The amplitude $A(\mathbf{q}, \mathbf{I})$ is given by

$$A^2(\mathbf{q}, \mathbf{I}) = \left| \det \left(\frac{\partial^2 S}{\partial q_i \partial I_j} \right) \right|. \quad (7.5)$$

The function $S(\mathbf{q}, \mathbf{I})$ is a multivalued function since there are several points $\mathbf{p}_k$ satisfying $\mathbf{I}(\mathbf{p}_k, \mathbf{q}) = \text{const}$ for a fixed $\mathbf{q}$. Thus the wave function is given by a superposition,

$$\phi(\mathbf{q}) = \sum_k A_k(\mathbf{q}, \mathbf{I}) \exp[iS_k(\mathbf{q}, \mathbf{I})/\hbar], \quad (7.6)$$

where

$$S_k(\mathbf{q}, \mathbf{I}) = S(\mathbf{q}, \mathbf{I}; \mathbf{p}_k) = \int_{C_k} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x}, \quad (7.7)$$

and the integration path C_k connects $O(\mathbf{p}_0, \mathbf{q}_0)$ to $P_k(\mathbf{p}_k, \mathbf{q})$, and

$$A_k^2(\mathbf{q}, \mathbf{I}) = \left| \det \left(\frac{\partial^2 S_k}{\partial q_i \partial I_j} \right) \right|. \quad (7.8)$$

The function $S_k(\mathbf{q}, \mathbf{I})$ is defined on each piece of the invariant torus divided by the caustics, and has a phase loss of $\pi/2$ at the points of intersection with the caustics.

In the case of the Kowalevski top, $\mathbf{q} = \Omega = (\alpha, \beta, \gamma)$ are the Euler angles. In Section 2.4, we have shown that there are three symmetry operators (R, S_1, S_2) under which the Hamiltonian and the Kowalevski constant are invariant, and thus the action variables are also invariant. Therefore symmetry operators move a point $P(\mathbf{p}, \mathbf{q})$ to another point $P'(\mathbf{p}', \mathbf{q}')$ on the invariant torus. However, the points P and P' may be on the different torus since there exist several tori. Let us denote the two tori as (T_1, T_2) for the region 2, 3 and 4, and the four tori as $(T_{11}, T_{12}, T_{21}, T_{22})$ for the region 5. In the region 5, (T_{11}, T_{12})

have the same projection on the (p, q) -plane, and (T_{21}, T_{22}) have the other, as is seen in Fig.2-(e).

The numerical calculations of classical orbits give us the following important observations. Let the point P be transformed to the point P' by the symmetry operator.

(1) S_1 -operator: P and P' are on the same torus.

(2) R and S_2 -operators:

(2-a) region 2 and 3: P and P' are on the different torus.

(2-b) region 4: P and P' are on the same torus.

(2-c) region 5: P and P' are transformed on the torus as $T_{i1} \rightarrow T_{i2}, T_{i2} \rightarrow T_{i1}$.

In the region 1 and 4, the symmetry operators transform the point P to P' on the same torus. Let us apply the operator R to $\phi(\mathbf{q})$,

$$R\phi(\mathbf{q}) = A(R\mathbf{q}, \mathbf{I}) \exp[iS(R\mathbf{q}, \mathbf{I}; R\mathbf{p})/\hbar], \quad (7.9)$$

where the index k is suppressed since the transformations of each component of Eq.(7.6) are the same. The phase is

$$S(R\mathbf{q}, \mathbf{I}; R\mathbf{p}) = \int_{C'} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x}, \quad (7.10)$$

where C' is a path from a fixed point $O(\mathbf{p}_0, \mathbf{q}_0)$ to $P'(R\mathbf{p}, R\mathbf{q})$. By separating the path $C' = C + C_1$ as $C : (\mathbf{p}_0, \mathbf{q}_0) \rightarrow (\mathbf{p}, \mathbf{q})$ and $C_1 : (\mathbf{p}, \mathbf{q}) \rightarrow (R\mathbf{p}, R\mathbf{q})$, Eq.(7.10) becomes

$$\begin{aligned} \int_{C'} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} &= \int_C \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} + \int_{C_1} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} + \oint \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} \\ &= S(\mathbf{q}, \mathbf{I}) + \int_{C_1} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} + \oint \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x}. \end{aligned} \quad (7.11)$$

The last term is necessary since an integration path is not specified, and it becomes

$$\oint \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} = \oint \mu(s_1)ds_1 + \oint \mu(s_2)ds_2 = 2\pi(\ell_1 I_r + \ell_2 J_r) \quad (\ell_i \text{ integer}), \quad (7.12)$$

since the mapping is a canonical transformation, and the action integrals are defined by Eq.(4.10). When the EBK quantization is applied, the term $2\pi \times \beta_i/4 = \beta_i\pi/2$ coming from the Maslov index is canceled by the phase loss at the intersection points with the caustics. Thus the last term of Eq.(7.11) is a multiple of $2\pi\hbar$ and can be neglected.

The second term of Eq.(7.11) is the integration along the path $C_1 : (\mathbf{p}, \mathbf{q}) \rightarrow (R\mathbf{p}, R\mathbf{q})$. This R -operation corresponds to the transformation of the angle variable as $\varphi_2 \rightarrow \varphi_2 + \pi$ with φ_1 fixed. This situation can be understood by the transformation of the angular velocity Eq.(2.19) as $(p, q) \rightarrow (p, -q)$, or the orbit of Fig.5-(b). Therefore,

$$\int_{C_1} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} = \frac{1}{2} \oint_{\alpha_2} \mu(s_2)ds_2 = \frac{1}{2} \times 2\pi J_r = \pi(m_r + n_r + 1)\hbar, \quad (7.13)$$

since the EBK quantization is $J_r = (m_r + n_r + 1)\hbar$ in the region 1 and 4. The phase π coming from the Maslov index is again canceled by the phase loss at the caustics and

can be dropped. For the case of $M \neq 0$, there is another phase $(-1)^M$, since the R -transformation changes $\alpha \rightarrow \alpha + \pi$ and $e^{i\alpha M} \rightarrow e^{i(\alpha+\pi)M} = (-1)^M e^{i\alpha M}$. The amplitude $A(\mathbf{q}, \mathbf{I})$ is the same as Eq.(7.8), since $S(R\mathbf{q}, \mathbf{I}; R\mathbf{p}) = S(\mathbf{q}, \mathbf{I}; \mathbf{p}) + \text{const.}$ Finally the symmetry quantum number Eq.(7.1) $r = (-1)^{M+m_r+n_r}$ is obtained. In the region 4, the quantum numbers (m_4, n_4) are the same for near degenerate states, thus the symmetry quantum numbers are the same.

In the region 2 and 3, two states with $r = \pm 1$ are nearly degenerate. In these regions, the point P is transformed to the point P' on the other torus. This interesting situation has been already observed in Ref.[9]. Let the semiclassical wave function be $\phi(\mathbf{q})$, $P(\mathbf{p}, \mathbf{q}) \in T_1$ for example. The wave function $\phi(R\mathbf{q})$ is an independent function since $P'(R\mathbf{p}, R\mathbf{q}) \in T_2$. Let us make a function which is a superposition of the different tori as

$$\phi^\pm(\mathbf{q}) = \phi(\mathbf{q}) \pm \phi(R\mathbf{q}). \quad (7.14)$$

Then

$$R\phi^\pm(\mathbf{q}) = \phi^\pm(R\mathbf{q}) = \pm\phi^\pm(\mathbf{q}), \quad (7.15)$$

since $R^2 = 1$. Therefore near degenerate states have different symmetry quantum number $r = \pm 1$. In the region 5, this type of degeneracy occurs for the pair of tori (T_{11}, T_{12}) and (T_{21}, T_{22}) , and there are two states with $r = 1$ and two states with $r = -1$, i.e., there exist four near degenerate states.

In quantum mechanical calculations, the only difference of matrix elements for $r = \pm 1$ bases comes from those connected to $\varphi_{j_0}^\pm$ bases of Eqs.(6.27),(6.28) (See Appendix C). In other words, if the $\varphi_{j_0}^\pm$ bases are removed, the spectra of $r = \pm 1$ states are exactly the same. Conversely, the degenerate spectra are lifted due to the perturbation coupled to the $\varphi_{j_0}^\pm$ bases. However, Eqs.(6.20)(6.21) show that the interaction $\gamma_1 = -\sin\beta \cos\gamma$ can change the K -value by $\Delta K = \pm 1$. Therefore, for the unperturbed state with a large K -value, the effect of the perturbation is very small due to many steps necessary to reach the $K = 0$ base.

This situation may be understood by the simple model of the potential $V(x) = V_0 \cos x$ ($V_0 \geq 0$) in the one-dimensional space ($0 \leq x \leq 2\pi$) with a periodic boundary condition. The wave functions are given by the Mathieu function, and the eigenvalues E are almost degenerate for $E \gg V_0$. If $V_0 = 0$, the energy is doubly degenerate except for the ground state. This degeneracies are lifted when the potential term is turned on, and the eigenvalue of $\text{ce}_n(x)$ Mathieu function becomes larger than that of $\text{se}_n(x)$ function.

7.3 Symmetries and wave functions

In Section 2.4, we take only $L_z \geq 0$ states and the R -operator. Since the (S_1, S_2) operators transform the wave function to its complex conjugate (time-reversed) state, the state with $L_z = M$ becomes that with $L_z = -M$. For the discussion of the symmetries of

the wave function, it may be better to take into account (S_1, S_2) symmetries and $L_z = \pm M$ degenerate states at the same time.

The bases of Eqs.(6.25)-(6.28) are transformed as

$$R\varphi_{JK}^{\pm} = \pm\varphi_{JK}^{\pm}, \quad (7.16)$$

$$S_1\varphi_{JK}^{\pm} = \varphi_{JK}^{\pm*}, \quad (7.17)$$

$$S_2\varphi_{JK}^{\pm} = \pm\varphi_{JK}^{\pm*}. \quad (7.18)$$

The eigenfunction $\psi^{\pm}$ is given by the linear combination of $\varphi_{JK}^{\pm}$ with real coefficients $c_{JK}^{\pm}$,

$$\psi^{\pm} = \sum_{J,K} c_{JK}^{\pm} \varphi_{JK}^{\pm}. \quad (7.19)$$

If the real (imaginary) part of the wave function $\psi_r^{\pm}$ ($\psi_i^{\pm}$) is defined as

$$\psi_r^{\pm} = \frac{1}{2}(\psi^{\pm} + \psi^{\pm*}), \quad \psi_i^{\pm} = \frac{1}{2i}(\psi^{\pm} - \psi^{\pm*}), \quad (7.20)$$

then $\psi_r^{\pm}$, $\psi_i^{\pm}$ are the eigenfunctions of the (S_1, S_2) operators,

$$S_1\psi_r^{\pm} = \psi_r^{\pm}, \quad S_1\psi_i^{\pm} = -\psi_i^{\pm}, \quad (7.21)$$

$$S_2\psi_r^{\pm} = \pm\psi_r^{\pm}, \quad S_2\psi_i^{\pm} = \mp\psi_i^{\pm}. \quad (7.22)$$

However, $\psi_r^{\pm}$, $\psi_i^{\pm}$ are not the eigenfunctions of the L_z -operator, and superpositions of $L_z = \pm M$ components.

Since the R -operator is expressed as $R = S_1S_2$, there is a relation $r = s_1s_2$ of the symmetry numbers. Therefore, the states with $r = 1$ are made of 2 degenerate states (ψ_r^+, ψ_i^+) , where ψ_r^+ has $(s_1, s_2) = (1, 1)$ and ψ_i^+ has $(s_1, s_2) = (-1, -1)$. The states with $r = -1$ are made of (ψ_r^-, ψ_i^-) , where ψ_r^- has $(s_1, s_2) = (1, -1)$ and ψ_i^- has $(s_1, s_2) = (-1, 1)$.

In the special case of $M = 0$, there is no degeneracy due to time-reversal, and the wave function has (S_1, S_2) symmetries. It was shown in Section 6.2 that the bases are further classified by $s_2 = (-1)^{J-K} = \pm 1$. In this case, the wave function $\psi^{\pm}$ is either real or pure imaginary. Therefore, for $r = 1$,

$$\begin{aligned} \psi^+ &= \psi_r^+, & \text{for } s_2 = 1, \\ \psi^+ &= \psi_i^+, & \text{for } s_2 = -1, \end{aligned} \quad (7.23)$$

and for $r = -1$,

$$\begin{aligned} \psi^- &= \psi_i^-, & \text{for } s_2 = 1, \\ \psi^- &= \psi_r^-, & \text{for } s_2 = -1. \end{aligned} \quad (7.24)$$

In the region 1 of $M = 0$ case, there exist following simple relations between the symmetry numbers (s_1, s_2) and the integer quantum numbers (m_1, n_1) ,

$$s_1 = (-1)^{m_1}, \quad s_2 = (-1)^{n_1}. \quad (7.25)$$

This relation can be understood by using the semiclassical wave function. The S_1 -operator corresponds to the transformation of the angle variable as $\varphi_1 \rightarrow \varphi_1 + \pi$ with φ_2 fixed. Repeating the same argument for the R -operator in Section 7.2, one can show $s_1 = (-1)^{m_1}$. Then $s_2 = (-1)^{n_1}$ is concluded since $r = s_1 s_2 = (-1)^{m_1+n_1}$.

Figure 8: Wave functions of $M = 0$ states. Thick lines show the classical boundaries.

In Fig.8 and 9, we show contour plots of typical wave functions of low energy states for $M = 0$ and $M = 1$. The coordinates of the configuration space are $(\gamma, \cos \beta)$ ($-\pi \leq \gamma \leq \pi$, $-1 \leq \cos \beta \leq 1$). The states are denoted by $E^\pm(M, m_1, n_1)$, where $\pm$ means $r = \pm 1$. The ground states are $E^+(0, 0, 0)$ and $E^-(1, 0, 0)$. The dark (light) region indicates positive (negative) value, except for $E^+(0, 0, 0)$ and $E^-(1, 0, 0)$ (Real part), where the wave functions are localized around the bottom of the potential.

In Fig.8 ($M = 0$), the wave function does not depend on the variable α , and the real and imaginary parts of the wave function have definite symmetries with $S_1 : \gamma \rightarrow -\gamma$ and $S_2 : \beta \rightarrow \pi - \beta$ ($\cos \beta \rightarrow -\cos \beta$), in accordance with Eqs.(7.23),(7.24). In Fig.9 ($M = 1$), the wave functions with $\alpha = 0$ are shown. The transformations of the wave function are

$$S_1 \psi(0, \beta, \gamma) = \psi(0, \beta, -\gamma) \quad (7.26)$$

Figure 9: Wave functions of $M = 1$ states.

$$S_2\psi(0, \beta, \gamma) = \psi(\pi, \pi - \beta, \gamma) = -\psi(0, \pi - \beta, \gamma). \quad (7.27)$$

One can see that the symmetry relations Eqs.(7.21),(7.22) are realized in Fig.9. For $\alpha \neq 0$ case, we have confirmed that those relations are certainly satisfied.

In Fig.8, we also show the boundaries of the classically allowed region given in Section 3. The wave functions extend out of the classically allowed region due to the tunnel effect. However, they do not decrease rapidly beyond the boundary. This is because the configuration space (β, γ) is compact, and the periodic boundary condition should be satisfied. This condition requires an increasing component outside the boundary, which makes the wave function decrease slowly. This situation is again similar to the simple model of $V(x) = V_0 \cos x$ explained in Section 7.2. The Mathieu functions for $E < V_0$ do not decrease rapidly in the classically forbidden region.

7.4 Limiting cases

As is explained in Section 6.1, the parameter $\hbar$ represents the magnitude of the gravitational force. The $\hbar \rightarrow 0$ (∞) limit corresponds to the large (small) gravitational force.

In these limiting cases, the Hamiltonian is simplified and one can obtain the eigenvalues easily.

In the $\hbar \rightarrow 0$ limit, the Kowalevski top oscillates in the bottom of the potential. The bottom of the potential is $(\beta_0, \gamma_0) = (\pi/2, 0)$. Let us expand the potential term around the equilibrium as

$$V(\beta = \frac{\pi}{2} + \beta', \gamma = \gamma') = -\sin(\frac{\pi}{2} + \beta') \cos \gamma' \simeq -1 + \frac{1}{2}(\beta'^2 + \gamma'^2), \quad (7.28)$$

where (β', γ') are small deviations. The kinetic energy term becomes

$$\frac{\hbar^2}{4}(J_1^2 + J_2^2 + 2J_3^2) \simeq -\frac{\hbar^2}{4} \left(\frac{\partial^2}{\partial \beta'^2} + 2 \frac{\partial^2}{\partial \gamma'^2} \right), \quad (7.29)$$

where the constant term $M^2 \hbar^2 / 4$ is dropped, since the energy is $O(\hbar)$. By comparing Eqs.(7.28),(7.29) with the harmonic oscillator case, the energy is approximately given by

$$h = -1 + (m_1 + \frac{1}{2})\hbar + \frac{1}{\sqrt{2}}(n_1 + \frac{1}{2})\hbar, \quad (7.30)$$

where the quantum numbers (m_1, n_1) in the region 1 correspond to the oscillation of (γ', β') variables respectively. Here we have deliberately assigned quantum numbers (m_1, n_1) to be consistent with the semiclassical quantum numbers. The Kowalevski operator becomes

$$\hat{K} \simeq 1 - 2 \left(-\frac{\hbar^2}{4} \frac{\partial^2}{\partial \beta'^2} + \frac{1}{2} \beta'^2 \right), \quad (7.31)$$

and its eigenvalue is

$$k = 1 - \sqrt{2}(n_1 + \frac{1}{2})\hbar. \quad (7.32)$$

On the other hand, one can calculate the action integrals Eq.(4.10), and the semiclassical quantization with $\ell = 0$ becomes, by setting $h = -1 + h'$, $k = 1 - k'$ with small deviations (h', k') ,

$$I_1 = h' - \frac{1}{2}k' = (m_1 + \frac{1}{2})\hbar, \quad (7.33)$$

$$J_1 = h' - \frac{1}{2}k' + \frac{1}{\sqrt{2}}k' = (m_1 + n_1 + 1)\hbar. \quad (7.34)$$

Therefore, one can understand that the semiclassical quantum numbers (m_1, n_1) represent the small oscillations of the (γ, β) variables around the bottom of the potential.

In the $\hbar \rightarrow \infty$ limit, the potential term can be neglected, and the leading terms of the Hamiltonian and the Kowalevski operator are

$$\hat{H} \simeq \frac{\hbar^2}{4}(J_1^2 + J_2^2 + 2J_3^2), \quad (7.35)$$

$$\hat{K} \simeq \frac{\hbar^4}{16} \left[\frac{1}{4} \{J_+, J_-\}^2 + \{J_+, J_-\} + 3J_3^2 \right]. \quad (7.36)$$

Therefore the bases $\varphi_{JK}^{\pm}$ are eigenfunctions with eigenvalues

$$h = \frac{\hbar^2}{4} [J(J+1) + K^2], \quad (7.37)$$

$$k = \frac{\hbar^4}{16} \left[\left\{ J(J+1) - K^2 \right\}^2 + 2 \left\{ J(J+1) - K^2 \right\} + 3K^2 \right]. \quad (7.38)$$

The spectra do not depend on the $L_z = M$ value, and they have $(2J+1)$ -fold degeneracy.

In the semiclassical quantization, the $\hbar \rightarrow \infty$ limit corresponds to the $c_0 = \mu_0/C \rightarrow 0$ limit. The Kowalevski top becomes the Euler top which is free of the gravitational force. However, the $\hbar \rightarrow \infty$ limit is not compatible with the semiclassical quantization, and we must reconsider the formulation in another way.

So far, we have used the scaled $\hbar$, Hamiltonian $\hat{H}$ and Kowalevski operator $\hat{K}$ as

$$\hbar/\sqrt{\mu_0 C} \rightarrow \hbar, \quad \hat{H}/\mu_0 \rightarrow \hat{H}, \quad \hat{K}/(\mu_0/C)^2 \rightarrow \hat{K}.$$

In the case of $c_0 \rightarrow 0$ limit, we will use the original Hamiltonian $\hat{H}$, and the scaled $\hbar$, $\hat{K}$ as

$$\hbar/\sqrt{C} \rightarrow \hbar, \quad \hat{K}/(1/C)^2 \rightarrow \hat{K},$$

and assume the $\mu_0 \rightarrow 0$ limit. For a small $c_0^2 = \varepsilon$, we should use the function $R_3(s)$ of Eq.(3.15) in its original form as

$$R_3(s) = s(R_2(s) + c_0^2) - \frac{c_0^2}{2}\ell^2 = s^3 - 2hs^2 + (h^2 - k + c_0^2)s - \frac{c_0^2}{2}\ell^2, \quad (7.39)$$

while the function $R_2(s)$ is the same. The solutions (a_1, a_2) of $R_2(s) = 0$ are the same, and the solutions (b_1, b_2, b_3) of $R_3(s) = 0$ become

$$b_1 = \frac{\ell^2 \varepsilon}{2(h^2 - k)}, \quad b_2 = a_1 + \frac{[2(h - \sqrt{k}) - \ell^2]\varepsilon}{4\sqrt{k}(h - \sqrt{k})}, \quad b_3 = a_2 - \frac{[2(h + \sqrt{k}) - \ell^2]\varepsilon}{4\sqrt{k}(h + \sqrt{k})}. \quad (7.40)$$

Then the action integrals can be calculated and the EBK quantization conditions are

$$I_3 = \sqrt{2(h + \sqrt{k})} = \left(M + m_3 + \frac{1}{2}\right) \hbar, \quad (7.41)$$

$$J_3 = \sqrt{2(h + \sqrt{k})} + \sqrt{2(h - \sqrt{k})} - \ell = \left(m_3 + n_3 + \frac{1}{2}\right) \hbar, \quad (7.42)$$

since the spectra are mainly in the region 3. From Eqs.(7.41),(7.42), the semiclassical spectra are given by

$$h = \frac{\hbar^2}{4} \left[\left(M + m_3 + \frac{1}{2}\right)^2 + n_3^2 \right], \quad (7.43)$$

$$k = \frac{\hbar^4}{16} \left[\left(M + m_3 + \frac{1}{2}\right)^2 - n_3^2 \right]^2. \quad (7.44)$$

These formulas are similar to those given by Komarov and Kuznetsov in Ref.[7], although their formulas are obtained by the multiplicity $N_1 = N_2 = 1$ in Table 2 and M -dependence

Figure 10: Spectra of the Euler top compared with the semiclassical spectra. Parameters are $\hbar = 1$, $M = 2$. Legends are the same as Fig.7.

is not taken into account. By comparing with the leading terms of Eqs.(7.37),(7.38), there are relations

$$J = M + m_3, \quad K = n_3. \quad (7.45)$$

In Fig.10, we will compare the semiclassical spectra Eqs.(7.43),(7.44) with the quantum spectra for the case of $\hbar = 1$ ($M = 2$) for example. One can see that the EBK quantization conditions give good approximate values. Most of the states are in the region 3, and the degeneracy corresponds to that of $\pm K$ states of Eq.(6.15), since the bases are related by the linear transformation Eqs.(6.25),(6.26).

It is interesting to see how the spectra (h, k) change when the parameter $\hbar$ is varied. In Fig.11, we show the trajectories of the spectra of the low energy states in the (h, k) -diagram. For the $M = 0$ states, the range of $\hbar$ is $0.1 \leq \hbar \leq 1.5$ for the 1st and 2nd levels, while $0.1 \leq \hbar \leq 1$ for the 3rd level. Dots show the position of $\hbar = 0.1, 0.2, 0.5, 1.0, 1.5$. For the $M = 1$ states, $0.1 \leq \hbar \leq 1$ for all levels. Dots show the position of $\hbar = 0.1, 0.2, 0.5, 1.0$. For a small $\hbar$, the spectra are given by Eqs.(7.30),(7.32), i.e., the trajectory is a straight line, while for a large $\hbar$, Eqs.(7.43),(7.44) give $h \simeq \sqrt{k}$ behavior for small n_3 .

For the $M = 0$ states, the low energy states move from the region 1 to the region 3. However, the a' -line with $(m_1, n_1) = (0, 1)$ and the c -line with $(m_1, n_1) = (1, 1)$ go into the classically forbidden region. Although it seems certainly strange behavior, this phenomenon happens in the case of $K = 0$, i.e., Eqs.(7.37),(7.38) become

$$h = \frac{\hbar^2}{4} J(J+1), \quad k = \frac{\hbar^4}{16} \left[\{J(J+1)\}^2 + 2J(J+1) \right], \quad (7.46)$$

Figure 11: Trajectories of the low energy spectra. a - c (a' - c') represent the 1st (ground state), 2nd and 3rd levels of $r = 1$ (-1) states.

and $h < \sqrt{k}$, which is in the forbidden region. Since we are discussing the classically allowed region, it seems more reasonable to use the Kowalevski operator without the quantum correction term. Then, its spectrum k_c is, instead of Eq.(7.38),

$$k_c = \frac{\hbar^4}{16} \left[\left\{ J(J+1) - K^2 \right\}^2 - 2 \left\{ J(J+1) - K^2 \right\} + 3K^2 \right], \quad (7.47)$$

i.e., the second term has an opposite sign. In this case, one can show that $h > \sqrt{k_c}$ and the spectra stay in the classically allowed region. This effect of the quantum correction term will be discussed in the next subsection.

For the $M = 1$ states, the spectra of the low energy states move also from the region 1 to the region 3. In this case, the angular momentum $\ell = M\hbar$ also varies. For examples shown in Fig.11, the trajectories cross the 1-3 boundary for $\hbar = 1.5 \sim 3.5$, which corresponds to the case 3 in Fig.1.

7.5 Size of quantum correction term

As is shown in Section 6.1, the Kowalevski operator should be corrected with a term of order $\hbar^4$ in order to commute with the Hamiltonian. Let $\hat{K}_c$ be the 'classical' Kowalevski operator without the quantum correction term,

$$16\hat{K}_c = \frac{1}{2}\{k'_+, k'_-\}, \quad k'_\pm = \hbar^2 J_\pm - 4\gamma_\pm, \quad (7.48)$$

and its expectation value be k_c in the eigenstate of $\hat{H}$. The size δ of the correction term is defined by $\delta = (k - k_c)/k$, where k and k_c are the exact values given by the numerical calculation, not the approximated values of the limiting cases in Section 7.4.

$\hbar$	$r = 1$			$r = -1$		
	E_1	E_2	E_3	E_1	E_2	E_3
0.05	2.2×10^{-5}	1.3×10^{-4}	7.5×10^{-5}	7.3×10^{-5}	2.3×10^{-5}	2.0×10^{-4}
0.1	1.8×10^{-4}	1.3×10^{-3}	6.8×10^{-4}	6.5×10^{-4}	1.9×10^{-4}	2.1×10^{-3}
0.2	1.5×10^{-3}	1.5×10^{-2}	6.8×10^{-3}	6.5×10^{-3}	1.6×10^{-3}	3.0×10^{-2}
0.5	2.3×10^{-2}	0.44	0.14	0.18	2.9×10^{-2}	0.50
1.0	0.14	0.43	0.39	0.85	0.24	0.49
2.0	0.31	0.61	0.49	0.99	0.60	0.52
5.0	0.33	0.66	0.50	0.9998	0.66	0.53

(a) $M = 0$

$\hbar$	$r = 1$			$r = -1$		
	E_1	E_2	E_3	E_1	E_2	E_3
0.05	7.5×10^{-5}	2.5×10^{-5}	2.0×10^{-4}	2.4×10^{-5}	1.3×10^{-4}	7.7×10^{-5}
0.1	6.7×10^{-4}	2.2×10^{-4}	2.1×10^{-3}	2.1×10^{-4}	1.3×10^{-3}	7.1×10^{-4}
0.2	6.7×10^{-3}	2.3×10^{-3}	3.0×10^{-2}	1.9×10^{-3}	1.5×10^{-2}	7.0×10^{-3}
0.5	0.14	6.4×10^{-2}	0.50	4.1×10^{-2}	0.37	9.8×10^{-2}
1.0	0.51	0.34	0.56	0.33	0.62	0.44
2.0	0.65	0.48	0.54	0.83	0.75	0.52
5.0	0.67	0.50	0.53	0.99	0.67	0.53

(b) $M = 1$

Table 7: Size of correction term

We will show the size δ in Table 7 for the cases of $M = 0, 1$ and the lowest three energy states. The correction term is non-negative. Its contributions are very small for $\hbar \leq 0.2$, while become sizable for $\hbar \geq 0.5$. For a large $\hbar$, the size of the correction can be readily seen by comparing Eqs.(7.38) and (7.47). For $(J, K) = (0, 0)$ and $(1, 0)$, $k_c = 0$ in Eq.(7.47) and $O(\hbar^4)$ contribution is entirely due to the correction term. The state $(1, 0)$ corresponds to the $r = -1$, E_1 -state, which is clearly seen in Table 7. The state $(0, 0)$ corresponds to the $M = 0$, $r = 1$, E_1 -state. In this case, however, the leading order $O(\hbar^2)$ of the energy h also vanishes. The careful calculation shows that $(h, k) \rightarrow (0, 4/9)$ and $(h, k_c) \rightarrow (0, 8/27)$ for $\hbar \rightarrow \infty$. Therefore, $\delta = 1/3$ is concluded.

For a large $\hbar$, the quantum correction term becomes also large and the semiclassical spectra deviate from the exact values. Therefore, the semiclassical quantization is valid when the contribution of the correction term is negligible. For example, in the case of $\hbar = 0.2$, the semiclassical spectra of the ground states are $(h, k) = (-0.8302, 0.8712)$, $(-0.8197, 0.8918)$, $(-0.7882, 0.9550)$ for $M = 0, 1, 2$, while the exact values are $(h, k) = (-0.8363, 0.8636)$, $(-0.8257, 0.8844)$, $(-0.7941, 0.9483)$, respectively.

8 Summary

We have calculated the quantum spectra of the Kowalevski top by diagonalizing the Hamiltonian in terms of D -function bases. They are very regular, which can be expected since the classical Kowalevski top is integrable. However, they show different behaviors depending on the region classified by the bifurcation sets of the classical invariant torus. They become irregular at the boundaries of the different regions. Furthermore, some of the spectra tend to nearly degenerate.

In order to understand these phenomena, the semiclassical EBK quantization is carried out. We have calculated the action integrals and the Maslov indices. It turns out that semiclassical spectra are close to the exact values as far as the quantum correction term is small, and the eigenstates can be labeled by the semiclassical quantum numbers. Since the Kowalevski top has the symmetry of the dihedral group D_2 , symmetry quantum numbers can be also assigned to the spectra. It is found that there are simple relations between the symmetry quantum numbers and the semiclassical integer quantum numbers. There exist several invariant tori, which make the spectra degenerate. However, the degeneracy is not exact, and the small splitting comes from the tunnel effect between different tori. The eigenfunctions of near-degenerate states are classified by the symmetry quantum numbers.

By increasing the gravitational force, the spectra change from those of the symmetric Euler top to those of the harmonic vibration. It is found that the EBK quantization conditions can also give good approximate values for the Euler top limit. The semiclassical quantum numbers correspond to the angular momenta.

We have estimated the size of the quantum correction term in the Kowalevski operator. Its contributions are very small for $\hbar \leq 0.2$, while become sizable for $\hbar \geq 0.5$. For some special states, the correction term entirely contributes to the eigenvalue of the Kowalevski operator in the large $\hbar$ limit.

Appendix A

Numerical calculation of the action integrals

In this appendix, we will describe the practical way of the numerical calculations of the action integrals Eq.(4.10). The explicit forms of the action integrals are

$$2\pi I_r = N_1 \times 2 \int_{s_{10}}^{a_2} \mu(s_1) ds_1, \quad (\text{A.1})$$

$$2\pi J_r = N_2 \times \left[2 \int_{s_{20}}^{-\infty} \mu(s_2) ds_2 + J_r(\infty) \right], \quad (\text{A.2})$$

where (N_1, N_2) are multiplicities of the cycle shown in Table 2. s_{10} and s_{20} are the lower bounds of the integral, which depend on the region shown in Table 1.

The integrand of I_r diverges as $1/\sqrt{\varepsilon}$ ($\varepsilon \rightarrow 0$) at the boundary. This divergence is explicitly removed by changing the variable $s_1 = s_{10} + t^2$ and $s_1 = a_2 - t^2$ for the lower and upper bounds respectively.

In order to calculate the J_r integral, let us divide the integral J_r into two parts, i.e., one is the contribution along the real axis, and the other is $J_r(\infty)$ which is the contribution encircling the $s_2 = \infty$ point shown in Fig.12. Note that $s_2 = \pm\infty$ points are identical since s -plane is treated as a compactified Riemann sphere. By setting the integral in the finite interval $(s_{20}, -R)$ with large $R > 0$, it is shown

$$2 \int_{s_{20}}^{-R} \mu(s_2) ds_2 \simeq -4\sqrt{2R}, \quad J_r(R) \simeq 4\sqrt{2R}, \quad (\text{A.3})$$

where $J_r(R)$ is the contribution of the circle C and can be calculated by changing the variable $s_2 = 1/t$ with $t \rightarrow 0$. Thus at $R \rightarrow \infty$, these two infinite contributions cancel and a finite J_r value is obtained.

Figure 12: Integration contour of α_2

The practical calculation of J_r can be carried out by the subtraction method. The key observation is

$$J_{r0} = \oint_{\alpha_2} \mu_0(s_2) ds_2 = 0, \quad \mu_0(s_2) = \frac{2}{\sqrt{-2(s_2 - s_{20})}}, \quad (\text{A.4})$$

since $\mu_0(s_2)$ is regular inside the contour α_2 . By separating the contribution in the same way as J_r , J_{r0} is written

$$J_{r0} = N_2 \times \left[2 \int_{s_{20}}^{-\infty} \mu_0(s_2) ds_2 + J_{r0}(\infty) \right]. \quad (\text{A.5})$$

Here, $J_{r0}(\infty)$ diverges as the same way as $J_r(\infty)$, i.e., $J_{r0}(R) \simeq 4\sqrt{2R}$ for $R \rightarrow \infty$. Therefore, J_r is expressed as the subtracted form

$$2\pi J_r = N_2 \times 2 \left[\int_{s_{20}}^{-\infty} \{\mu(s_2) - \mu_0(s_2)\} ds_2 \right]. \quad (\text{A.6})$$

In this form, at $s_2 \rightarrow -\infty$,

$$\mu(s_2) - \mu_0(s_2) \simeq \frac{4h + \ell^2}{2\sqrt{-2s_2^3}}, \quad (\text{A.7})$$

and the integral Eq.(A.6) is free from the divergence at $s_2 = -\infty$. The infinite $(s_{20}, -\infty)$ interval is mapped to a finite interval. The mapping function is

$$s_2 = s_{20} - u(t)^2, \quad u(t) = u_0 \left(\frac{1+t}{1-t} \right)^\alpha, \quad (\text{A.8})$$

where $u_0 > 0$, $\alpha \geq 1$ and $-1 \leq t \leq 1$. With this change of the integration variable, both divergences at $s_2 = s_{20}$ and $s_2 = -\infty$ are removed from the integrand. The standard mapping parameters are $u_0 = 1$ and $\alpha = 1$, which give good convergent results. Finally the numerical integration is carried out by using the Gauss-Legendre quadrature.

Appendix B

Action integrals on the boundaries and connection formulas

In this appendix, we will describe the calculation of the action integrals on the boundaries and the connection formulas for several cases.

The boundary value along PY ($k = 0$) of Fig.1 is $I_2 = 0$, since the integration interval ($a_1 = a_2 = h$) vanishes and the integrand has no singularity. On the boundary PA ($h = -\sqrt{k} + \ell^2/2$), $b_1 = a_2 = \ell^2/2$, and the integration interval also vanishes. However, the denominator becomes $s - \ell^2/2 \simeq 0$ and the integrand diverges at the same time. The only finite contribution comes from the term

$$2\pi I_1 = -4\ell^2 \int_{b_1}^{a_2} ds \frac{(\ell^2/2 - h)^2 - k}{\sqrt{R_5(s)}(2s - \ell^2)}. \quad (\text{B.1})$$

Let us choose a point close to PA as

$$h = -\sqrt{k} + \ell^2/2 + \varepsilon, \quad \varepsilon > 0, \quad (\text{B.2})$$

$$a_2 - \ell^2/2 = h + \sqrt{k} - \ell^2/2 = \varepsilon. \quad (\text{B.3})$$

The root of $R_3(b_1) = 0$ is also close to $\ell^2/2$ and expressed as

$$b_1 = \ell^2/2 + \varepsilon'. \quad (\text{B.4})$$

Then, one can find

$$\varepsilon' = \frac{\ell^2 \sqrt{k}}{1 + \ell^2 \sqrt{k}} \varepsilon. \quad (\text{B.5})$$

$R_5(s)$ is approximated near $s \simeq \ell^2/2$ as

$$R_5(s) \simeq 4\sqrt{k}(1 + \ell^2 \sqrt{k})(a_2 - s)(s - b_1). \quad (\text{B.6})$$

By employing the formula

$$\int_{b_1}^{a_2} ds \frac{1}{\sqrt{(a_2 - s)(s - b_1)(2s - \ell^2)}} = \frac{\pi}{2\sqrt{(a_2 - \ell^2/2)(b_1 - \ell^2/2)}} = \frac{\pi}{2\sqrt{\varepsilon\varepsilon'}}, \quad (\text{B.7})$$

Eq.(B.1) becomes

$$\begin{aligned} 2\pi I_1 &= -4\ell^2(-2\sqrt{k}\varepsilon) \frac{1}{2[\sqrt{k}(1 + \ell^2 \sqrt{k})]^{1/2}} \frac{\pi}{2\sqrt{\varepsilon\varepsilon'}} \\ &= 2\ell^2 \pi \left(\frac{\sqrt{k}}{1 + \ell^2 \sqrt{k}} \right)^{1/2} \left(\frac{\varepsilon}{\varepsilon'} \right)^{1/2} \\ &= 2\pi\ell. \end{aligned} \quad (\text{B.8})$$

Thus $I_1 = \ell$ is obtained on the boundary PA.

On the boundary AX, $R_3(s) = 0$ has another multiple real solution $s = b'$ besides $s = b_1$. Then $1/\sqrt{R_5(s)}$ gives rise to a pole at $s = b'$. However, the residue at this pole turns out zero. Therefore,

$$\oint_{C_1} \mu(s_1) ds_1 = \oint_{C_2} \mu(s_2) ds_2 + 2\pi i \text{Res}[\mu(\ell^2/2)] \quad (\text{B.9})$$

by the residue theorem. The residue at the pole $s_p = \ell^2/2$ is $\text{Res}[\mu(\ell^2/2)] = -i\ell/2$. The multiplicities are $N_1 = N_2 = 2$ in the region 1 and 4, and finally one can obtain

$$I_1 = J_1 + \ell. \quad (\text{B.10})$$

In the case 1 ($\ell = 0$), they are $I_1 = J_1$ and $I_4 = J_4$

Next, let us explain the connection formulas on the boundary 1-2 ($h = \sqrt{k} + \ell^2/2$). The location of the roots a_1 and b_1 is exchanged between the region 1 and 2. However, there is a pole in between, and the action integrals have discontinuities. Let us fix a point $h_1 = \sqrt{k} + \ell^2/2 - \varepsilon$, $\varepsilon > 0$ in the region 1. Then $a_1 = h_1 - \sqrt{k} = \ell^2/2 - \varepsilon$. Setting $b_1 = \ell^2/2 + \varepsilon'$, ε' is given by $R_3(b_1) = 0$ and turns out

$$\varepsilon' = \frac{\ell^2 \sqrt{k}}{1 - \ell^2 \sqrt{k}} \varepsilon. \quad (\text{B.11})$$

In the region 2, at a point $h_2 = \sqrt{k} + \ell^2/2 + \varepsilon$, boundaries are $a'_1 = \ell^2/2 + \varepsilon$, $b'_1 = \ell^2/2 - \varepsilon'$. The difference of the actions I_1 and $2I_2$ is

$$2\pi(I_1 - 2I_2) = 4 \left[\int_{b_1}^{a_2} \mu(s) ds - \int_{a'_1}^{a'_2} \mu(s) ds \right]. \quad (\text{B.12})$$

The finite contribution comes from the difference of the lower bounds,

$$2\pi(I_1 - 2I_2) = 4 \int_{b_1}^{a'_1} ds \frac{-\ell^2 R_2(s)}{\sqrt{R_5(s)}(2s - \ell^2)}. \quad (\text{B.13})$$

The RHS turns out $2\pi\ell$ by the similar calculation of Eq.(B.8). Finally $I_1 = 2I_2 + \ell$ is obtained. The connection formula $J_1 = J_2 + \ell$ is also derived in the same way.

Figure 13: Change of the cut structure between the region 1 and 4.

The connection formula on the boundary 1-4 (BC) is somewhat complicated since the action integrals (I_r, J_r) are mixed. Fig.13 shows how the cut structure and the integration paths change at the boundary. We choose the cuts in such a way that the transition is clearly displayed. The cut $[b, b^*]$ in the region 1 disappears at the boundary, and comes out as $[b_2, b_3]$ on the real axis in the region 4. On the boundary,

$$\pi I_1 = \oint_{C'_1} \mu(s_1) ds_1 + \oint_{C'_3} \mu(s_1) ds_1, \quad \pi I_4 = \oint_{C'_1} \mu(s_1) ds_1, \quad (\text{B.14})$$

$$\pi J_1 = \pi J_4 = \oint_{C'_2} \mu(s_2) ds_2, \quad (\text{B.15})$$

where the loops denote one cycle. On the other hand, the following relation is obtained by the residue theorem in the region 4,

$$\oint_{C'_1} \mu(s_1) ds_1 - \oint_{C'_3} \mu(s_1) ds_1 = \oint_{C'_2} \mu(s_2) ds_2 + 2\pi i \text{Res}[\mu(\ell^2/2)]. \quad (\text{B.16})$$

$\text{Res}[\mu(\ell^2/2)] = i\ell/2$ since we define the phase $\arg[\sqrt{R_5(s)}] = 0$ at the $[b_3, a_2]$ interval in the region 4. Therefore,

$$\oint_{C'_3} \mu(s_1) ds_1 = \pi(I_4 - J_4) + \pi\ell. \quad (\text{B.17})$$

With Eqs.(B.14),(B.15), one can derive the connection formulas

$$I_1 = 2I_4 - J_4 + \ell, \quad J_1 = J_4. \quad (\text{B.18})$$

Other connection formulas listed in Table 4 are verified in a similar way. They have been confirmed also by the numerical calculations.

Appendix C

Matrix elements of the Hamiltonian and the Kowalevski operator

The Hamiltonian $\hat{H}$ and the Kowalevski operator $\hat{K}$ are given by Eqs.(6.9),(6.10),

$$\hat{H} = \frac{\hbar^2}{4}(\mathbf{J}^2 + J_3^2) + \frac{1}{2}(\gamma_+ + \gamma_-), \quad (\text{C.1})$$

$$\begin{aligned} 16\hat{K} = & \hbar^4 \left[\frac{1}{4}\{J_+, J_-\}^2 + \{J_+, J_-\} + 3J_3^2 \right] \\ & - 2\hbar^2 \left[\{J_+^2, \gamma_-\} + \{J_-^2, \gamma_+\} \right] + 8\{\gamma_+, \gamma_-\}, \end{aligned} \quad (\text{C.2})$$

where (J_1, J_2, J_3) are the components of the angular momentum in the body-fixed frame, and $J_{\pm} = J_1 \pm iJ_2$, $\gamma_{\pm} = \gamma_1 \pm i\gamma_2$. The commutation relations of the J_i are

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = -iJ_2. \quad (\text{C.3})$$

Note that the commutators involve a change of sign with respect to those Eq.(6.12) in the space-fixed frame.

The angular momentum operators transform the bases Eq.(6.15) as follows,

$$\mathbf{J}^2 \varphi_K^J = J(J+1) \varphi_K^J, \quad (\text{C.4})$$

$$J_3 \varphi_K^J = K \varphi_K^J, \quad (\text{C.5})$$

$$\{J_+, J_-\} \varphi_K^J = 2(\mathbf{J}^2 - J_3^2) \varphi_K^J = 2[J(J+1) - K^2] \varphi_K^J, \quad (\text{C.6})$$

$$J_+ \varphi_K^J = \sqrt{(J+K)(J-K+1)} \varphi_{K-1}^J, \quad (\text{C.7})$$

$$J_- \varphi_K^J = \sqrt{(J-K)(J+K+1)} \varphi_{K+1}^J, \quad (\text{C.8})$$

thus the matrix elements of those terms which consist of angular momenta are calculated without difficulty.

For the terms with $\gamma_{\pm}$, let us define V_1, V_2 and V_3 as

$$V_1 = \gamma_+ + \gamma_-, \quad V_2 = \{J_+^2, \gamma_-\} + \{J_-^2, \gamma_+\}, \quad V_3 = \frac{1}{2}\{\gamma_+, \gamma_-\} = \sin^2 \beta. \quad (\text{C.9})$$

Then the matrix elements are summarized in the followings.

$$(1-1) \langle \varphi_{J'K'}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle \text{ for } J \geq 1$$

(a) $K \geq 2$

$$\langle \varphi_{J-1, K-1}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle = -\frac{1}{J} \left[\frac{(J^2 - M^2)(J + K - 1)(J + K)}{(2J - 1)(2J + 1)} \right]^{1/2}, \quad (\text{C.10})$$

$$\langle \varphi_{J, K-1}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle = \frac{M[(J - K + 1)(J + K)]^{1/2}}{J(J + 1)}, \quad (\text{C.11})$$

$$\langle \varphi_{J+1, K-1}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle = \frac{1}{J + 1} \left[\frac{\{(J + 1)^2 - M^2\}(J - K + 1)(J - K + 2)}{(2J + 1)(2J + 3)} \right]^{1/2}, \quad (\text{C.12})$$

$$\langle \varphi_{J-1, K+1}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle = \frac{1}{J} \left[\frac{(J^2 - M^2)(J - K - 1)(J - K)}{(2J - 1)(2J + 1)} \right]^{1/2}, \quad (\text{C.13})$$

$$\langle \varphi_{J, K+1}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle = \frac{M[(J + K + 1)(J - K)]^{1/2}}{J(J + 1)}, \quad (\text{C.14})$$

$$\langle \varphi_{J+1, K+1}^{\pm} | V_1 | \varphi_{JK}^{\pm} \rangle = -\frac{1}{J + 1} \left[\frac{\{(J + 1)^2 - M^2\}(J + K + 1)(J + K + 2)}{(2J + 1)(2J + 3)} \right]^{1/2}. \quad (\text{C.15})$$

Matrix elements are the same for the symmetry quantum number $r = \pm 1$.

(b-1) $K = 1 \rightarrow K' = 2$

$$\langle \varphi_{J-1, 2}^{\pm} | V_1 | \varphi_{J, 1}^{\pm} \rangle = \frac{1}{J} \left[\frac{(J^2 - M^2)(J - 2)(J - 1)}{(2J - 1)(2J + 1)} \right]^{1/2}, \quad (\text{C.16})$$

$$\langle \varphi_{J, 2}^{\pm} | V_1 | \varphi_{J, 1}^{\pm} \rangle = \frac{M[(J + 2)(J - 1)]^{1/2}}{J(J + 1)}, \quad (\text{C.17})$$

$$\langle \varphi_{J+1, 2}^{\pm} | V_1 | \varphi_{J, 1}^{\pm} \rangle = -\frac{1}{J + 1} \left[\frac{\{(J + 1)^2 - M^2\}(J + 2)(J + 3)}{(2J + 1)(2J + 3)} \right]^{1/2}. \quad (\text{C.18})$$

Matrix elements are the same for the symmetry quantum number $r = \pm 1$.

(b-2) $K = 1 \rightarrow K' = 0$, ($J=\text{even}$, $r = +1$) or ($J=\text{odd}$, $r = -1$)

$$\langle \varphi_{J, 0}^{\pm} | V_1 | \varphi_{J, 1}^{\pm} \rangle = \frac{\sqrt{2}M}{\sqrt{J(J + 1)}}. \quad (\text{C.19})$$

(b-3) $K = 1 \rightarrow K' = 0$, ($J=\text{even}$, $r = -1$) or ($J=\text{odd}$, $r = +1$)

$$\langle \varphi_{J-1, 0}^{\pm} | V_1 | \varphi_{J, 1}^{\pm} \rangle = -\left[\frac{2(J^2 - M^2)(J + 1)}{J(2J - 1)(2J + 1)} \right]^{1/2}, \quad (\text{C.20})$$

$$\langle \varphi_{J+1, 0}^{\pm} | V_1 | \varphi_{J, 1}^{\pm} \rangle = \left[\frac{2\{(J + 1)^2 - M^2\}J}{(J + 1)(2J + 1)(2J + 3)} \right]^{1/2}. \quad (\text{C.21})$$

(c) $K = 0 \rightarrow K' = 1$, ($J=\text{even}$, $r = +1$) or ($J=\text{odd}$, $r = -1$)

$$\langle \varphi_{J-1, 1}^{\pm} | V_1 | \varphi_{J, 0}^{\pm} \rangle = \left[\frac{2(J^2 - M^2)(J - 1)}{J(2J - 1)(2J + 1)} \right]^{1/2}, \quad (\text{C.22})$$

$$\langle \varphi_{J,1}^{\pm} | V_1 | \varphi_{J,0}^{\pm} \rangle = \frac{\sqrt{2}M}{\sqrt{J(J+1)}}, \quad (\text{C.23})$$

$$\langle \varphi_{J+1,1}^{\pm} | V_1 | \varphi_{J,0}^{\pm} \rangle = - \left[\frac{2\{(J+1)^2 - M^2\}(J+2)}{(J+1)(2J+1)(2J+3)} \right]^{1/2}. \quad (\text{C.24})$$

(1-2) $\langle \varphi_{J',K'}^{\pm} | V_1 | \varphi_{J,K}^{\pm} \rangle$ for $J = 0$

$$\langle \varphi_{1,1}^{+} | V_1 | \varphi_{0,0}^{+} \rangle = -\frac{2}{\sqrt{3}}. \quad (\text{C.25})$$

This is the only possible non-zero matrix element.

(2-1) $\langle \varphi_{J',K'}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle$ for $J \geq 1$

(a) $K \geq 2$

$$\begin{aligned} \langle \varphi_{J-1,K-1}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle &= \frac{2[(J-K)^2 + J - K + 1]}{J} \\ &\times \left[\frac{(J^2 - M^2)(J+K-1)(J+K)}{(2J-1)(2J+1)} \right]^{1/2}, \end{aligned} \quad (\text{C.26})$$

$$\langle \varphi_{J,K-1}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle = \frac{2M(J^2 - K^2 + J + K - 1)\sqrt{(J-K+1)(J+K)}}{J(J+1)}, \quad (\text{C.27})$$

$$\begin{aligned} \langle \varphi_{J+1,K-1}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle &= -\frac{2[(J+K)^2 + J + K + 1]}{J+1} \\ &\times \left[\frac{\{(J+1)^2 - M^2\}(J-K+1)(J-K+2)}{(2J+1)(2J+3)} \right]^{1/2}, \end{aligned} \quad (\text{C.28})$$

$$\begin{aligned} \langle \varphi_{J-1,K+1}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle &= -\frac{2[(J+K)^2 + J + K + 1]}{J} \\ &\times \left[\frac{(J^2 - M^2)(J-K-1)(J-K)}{(2J-1)(2J+1)} \right]^{1/2}, \end{aligned} \quad (\text{C.29})$$

$$\langle \varphi_{J,K+1}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle = \frac{2M(J^2 - K^2 + J - K - 1)\sqrt{(J+K+1)(J-K)}}{J(J+1)}, \quad (\text{C.30})$$

$$\begin{aligned} \langle \varphi_{J+1,K+1}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle &= \frac{2[(J-K)^2 + J - K + 1]}{J+1} \\ &\times \left[\frac{\{(J+1)^2 - M^2\}(J+K+1)(J+K+2)}{(2J+1)(2J+3)} \right]^{1/2}. \end{aligned} \quad (\text{C.31})$$

(b-1) $K = 1 \rightarrow K' = 2$

$$\langle \varphi_{J-1,2}^{\pm} | V_2 | \varphi_{J,1}^{\pm} \rangle = -\frac{2(J^2 + 3J + 3)}{J} \left[\frac{(J^2 - M^2)(J-2)(J-1)}{(2J-1)(2J+1)} \right]^{1/2}, \quad (\text{C.32})$$

$$\langle \varphi_{J,2}^{\pm} | V_2 | \varphi_{J,1}^{\pm} \rangle = \frac{2M(J^2 + J - 3)}{J(J+1)} \sqrt{(J+2)(J-1)}, \quad (\text{C.33})$$

$$\langle \varphi_{J+1,2}^{\pm} | V_2 | \varphi_{J,1}^{\pm} \rangle = \frac{2(J^2 - J + 1)}{J+1} \left[\frac{\{(J+1)^2 - M^2\}(J+2)(J+3)}{(2J+1)(2J+3)} \right]^{1/2}. \quad (\text{C.34})$$

Matrix elements are the same for the symmetry quantum number $r = \pm 1$.

(b-2) $K = 1 \rightarrow K' = 0$, (J =even, $r = +1$) or (J =odd, $r = -1$)

$$\langle \varphi_{J,0}^{\pm} | V_2 | \varphi_{J,1}^{\pm} \rangle = \frac{2\sqrt{2}M(J^2 + J - 1)}{\sqrt{J(J+1)}}. \quad (\text{C.35})$$

(b-3) $K = 1 \rightarrow K' = 0$, (J =even, $r = -1$) or (J =odd, $r = +1$)

$$\langle \varphi_{J-1,0}^{\pm} | V_2 | \varphi_{J,1}^{\pm} \rangle = 2(J^2 - J + 1) \left[\frac{2(J^2 - M^2)(J+1)}{J(2J-1)(2J+1)} \right]^{1/2}, \quad (\text{C.36})$$

$$\langle \varphi_{J+1,0}^{\pm} | V_2 | \varphi_{J,1}^{\pm} \rangle = -2(J^2 + 3J + 3) \left[\frac{2\{(J+1)^2 - M^2\}J}{(J+1)(2J+1)(2J+3)} \right]^{1/2}. \quad (\text{C.37})$$

(c) $K = 0 \rightarrow K' = 1$, (J =even, $r = +1$) or (J =odd, $r = -1$)

$$\langle \varphi_{J-1,1}^{\pm} | V_2 | \varphi_{J,0}^{\pm} \rangle = -2(J^2 + J + 1) \left[\frac{2(J^2 - M^2)(J-1)}{J(2J-1)(2J+1)} \right]^{1/2}, \quad (\text{C.38})$$

$$\langle \varphi_{J,1}^{\pm} | V_2 | \varphi_{J,0}^{\pm} \rangle = \frac{2\sqrt{2}M(J^2 + J - 1)}{\sqrt{J(J+1)}}, \quad (\text{C.39})$$

$$\langle \varphi_{J+1,1}^{\pm} | V_2 | \varphi_{J,0}^{\pm} \rangle = 2(J^2 + J + 1) \left[\frac{2\{(J+1)^2 - M^2\}(J+2)}{(J+1)(2J+1)(2J+3)} \right]^{1/2}. \quad (\text{C.40})$$

(2-2) $\langle \varphi_{J,K'}^{\pm} | V_2 | \varphi_{J,K}^{\pm} \rangle$ for $J = 0$

$$\langle \varphi_{1,1}^{+} | V_2 | \varphi_{0,0}^{+} \rangle = \frac{4}{\sqrt{3}}. \quad (\text{C.41})$$

This is the only possible non-zero matrix element.

(3) $\langle \varphi_{J',K'}^{\pm} | V_3 | \varphi_{J,K}^{\pm} \rangle$

$$\begin{aligned} \langle \varphi_{J-2,K}^{\pm} | V_3 | \varphi_{J,K}^{\pm} \rangle &= -\frac{1}{J(J-1)(2J-1)} \\ &\times \left[\frac{(J^2 - M^2)(J^2 - K^2)\{(J-1)^2 - M^2\}\{(J-1)^2 - K^2\}}{(2J-3)(2J+1)} \right]^{1/2}, \end{aligned} \quad (\text{C.42})$$

$$\langle \varphi_{J-1,K}^{\pm} | V_3 | \varphi_{J,K}^{\pm} \rangle = -\frac{2MK}{(J-1)J(J+1)} \left[\frac{(J^2 - M^2)(J^2 - K^2)}{(2J-1)(2J+1)} \right]^{1/2}, \quad (\text{C.43})$$

$$\begin{aligned} \langle \varphi_{J,K}^{\pm} | V_3 | \varphi_{J,K}^{\pm} \rangle &= 1 - \frac{(J^2 - M^2)(J^2 - K^2)}{J^2(2J-1)(2J+1)} - \frac{M^2K^2}{J^2(J+1)^2} \\ &- \frac{[(J+1)^2 - M^2][(J+1)^2 - K^2]}{(J+1)^2(2J+1)(2J+3)}, \end{aligned} \quad (\text{C.44})$$

$$\langle \varphi_{J+1,K}^{\pm} | V_3 | \varphi_{J,K}^{\pm} \rangle = -\frac{2MK}{J(J+1)(J+2)} \left[\frac{\{(J+1)^2 - M^2\}\{(J+1)^2 - K^2\}}{(2J+1)(2J+3)} \right]^{1/2}, \quad (\text{C.45})$$

$$\begin{aligned}
\langle \varphi_{J+2,K}^{\pm} | V_3 | \varphi_{JK}^{\pm} \rangle &= - \frac{1}{(J+1)(J+2)(2J+3)} \\
&\times \left[\frac{\{(J+1)^2 - M^2\} \{(J+1)^2 - K^2\} \{(J+2)^2 - M^2\} \{(J+2)^2 - K^2\}}{(2J+1)(2J+5)} \right]^{1/2}. \quad (\text{C.46})
\end{aligned}$$

The operator V_3 does not change the K -value and those matrix elements with $J' < K'$ are automatically zero.

References

- [1] S. Kowalevski, *Acta Math.* **12** (1889), 177.
- [2] V.V. Golubev, "Lectures on Integration of the Equations of Motion of a Rigid Body about a Fixed point",
State Publishing House of Theoretical Technical Literature, Moscow, 1953.
- [3] H.B.G. Casimir, "Rotation of a Rigid Body in Quantum Mechanics", Wolters, Groningen, 1931.
- [4] L. Pauling and E.B. Wilson, "Introduction to Quantum Mechanics" McGraw-Hill, New York, 1935.
- [5] L.D. Landau and E.M. Lifschitz, "Quantum Mechanics, Nonrelativistic Theory", Pergamon Press, London and Paris, 1958.
- [6] J.H. Shirley, *J.Chem.Phys.* **38** (1963), 2896.
- [7] I.V. Komarov and V.B. Kuznetsov, *Theor. and Math. Physics* **73** (1987), 335.
- [8] E.P. Wigner, "Group Theory and its Application to the Quantum Mechanics of Atomic Spectra", Academic Press, New York and London, 1959.
- [9] H.R. Dullin, M. Juhnke and P.H. Richter, *Int.J. Bifurcation and Chaos* **4** (1994), 1535.
- [10] M.P. Kharlamov, *Prikl.Matem.Mekhan.* USSR **47** (1983), 737.
- [11] G.G. Appel'rot, "Not entirely symmetrical heavy gyroscopes" in *Rigid Body Motion About a Fixed Point*, Acad.Sci. USSR, Moscow, p.61-155
- [12] H.R. Dullin, P.H. Richter and A.P. Veselov, *Regular and Chaotic Dynamics* **3** (1998), 18.
- [13] I.V. Komarov, *J.Phys.A. Math.Gen.* **34** (2001), 2111.
- [14] H.R. Dullin and A. Wittek, *J.Phys.A. Math.Gen.* **27** (1994), 7461.
- [15] A.P. Veselov and S.P. Novikov, "Poisson brackets and complex tori" Proc.Steklov Inst. of Math. **165** (1985), p.53-65.
- [16] J. Robbins, *Chaos* **2** (1992), 145.
- [17] O. Laporte, *Phys.Rev.* **43** (1933), 548.
- [18] A.R. Edmonds, "Angular momentum in Quantum Mechanics", Princeton University Press, Princeton, 1957.

- [19] A. Matsuyama, *Ann. Phys.* **220** (1992), 300.
- [20] A. Matsuyama, *Phys.Lett.A* **177** (1993), 415.