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Statistical physics of intermittency: Phase transitions and fluctuations of scaling indices

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We propose a new statistical-mechanical formalism to extract the multifractal structure from intermittent phenomena. A simple stochastic model reproducing universal properties of intermittency is solved rigorously, and phase transitions associated with the spectrum of the generalized entropies are studied. We also apply the formalism to the logistic map near the critical point for a period-three cycle. The results are in good agreement with numerical results.

I. INTRODUCTION

Intermittent phenomena are found in diverse fields of science. Considerable theoretical effort has been made to understand the mechanism of intermittency and to explore the universality class of intermittent chaos.¹⁻⁶ Although our understanding of intermittency was indeed advanced by these theoretical works, it is now possible to extract more detailed information from intermittent phenomena. For example, fluctuations of dynamical scaling indices must be studied with the aid of multifractal methodology.⁷⁻¹⁷ In this paper we report that statistical mechanics is well suited for this purpose. In our formalism phase transitions associated with the generalized entropies are shown rigorously, and critical exponents are calculated analytically.

We first give a brief review of multifractal theory. Let us consider a symbol sequence $\{s_i\}$ with a probability measure $p(\{s_i\})$, where s_i takes on N values $\{1, 2, \dots, N\}$. In order to describe fluctuations of dynamical scaling indices, we introduce a local information-production rate γ for a subsequence as $p(s_1, s_2, \dots, s_n) \sim e^{-\gamma n}$. The number of subsequences with the rate γ increases exponentially as $e^{h(\gamma)n}$ with an increase of n .^{18,19} If one defines a partition function by

$$Z_n(q) = \sum_{\{s_i\}} \exp\{-q \ln[p(s_1, \dots, s_n)]^{-1}\}, \quad (1.1)$$

$Z_n(q)$ can be assumed to behave as, for large n ,

$$Z_n(q) \sim e^{-f(q)n}, \quad (1.2)$$

where $f(q) = (q-1)K_q$ with K_q the q -order Renyi entropy.²⁰ The function $h(\gamma)$ is obtained from $f(q)$ via the Legendre transform,¹⁰

$$\gamma(q) = \frac{d}{dq} f(q), \quad (1.3a)$$

$$h(q) = g\gamma(q) - f(q). \quad (1.3b)$$

It is obviously seen from Eqs. (1.1)–(1.3) that γ and h correspond to the energy and entropy function, respec-

tively, in statistical mechanics.^{21,22}

Our paper is organized as follows. In Sec. II we construct a general theory of intermittency by extending the theory proposed in Refs. 23 and 24. In Secs. III and IV we apply the formalism to dyadic symbolic dynamics and the logistic map near the critical point for a period-three cycle. Section IV offers a summary with a discussion of the advantage of our method in detecting the multifractal structure.

II. STATISTICAL-MECHANICAL FORMALISM OF INTERMITTENCY

Let us begin to construct a model of intermittent systems with the use of their probability measures. Consider a semi-infinite sequence $\{s_1, s_2, \dots\}$ generated by an intermittent system, where s_j is a symbol that takes on N values; $s_i \in \{1, 2, \dots, N\}$. Using the symbol sequence $\{s_i\}$, we can obtain the probability measure $p(s_1, s_2, \dots, s_n)$ of finding the subsequence $\{s_1, s_2, \dots, s_n\}$. Let $p(s_i, \dots, s_{i+m-1} | s_j, \dots, s_{j+n-1})$ be a conditional probability that a sequence $\{s_j, \dots, s_{j+n-1}\}$ occurs after $\{s_i, \dots, s_{i+m-1}\}$.

Define the “laminar” and “turbulent” sets of symbols by $\mathcal{E}_1 = \{1, 2, \dots, N'\}$ and $\mathcal{E}_2 = \{N'+1, N'+2, \dots, N\}$, respectively. We call symbols $s_i \in \mathcal{E}_1$ the “laminar symbols” and the others the “turbulent symbols.” A series of the identical laminar symbols means successive laminar like states in intermittent signals, while turbulent symbols correspond to bursts. It is not difficult to answer how N' is chosen and how different values of s_i correspond to laminar or turbulent motion in one-dimensional discrete dynamical systems. This will be seen in Sec. IV.

Using the abbreviated expression,

$$\omega_j(s, s') = \{s, s, \dots, s, s'\}, \quad (2.1)$$

where there are j elements in the curly brackets, we model the symbolic dynamics of intermittent systems,

$$p(s | \omega_j(s, s')) = t_{ss'} \exp[-W_s(j)] \quad \text{for } s \in \mathcal{E}_1 \text{ and } s \neq s', \quad (2.2a)$$

$$p(s|s') = t_{ss'} \text{ for } s \in \mathcal{E}_2, \quad (2.2b)$$

where $t_{ss'}$ is a transition probability from a symbol s to s' and $W_s(j)$ is a non-negative function of j satisfying

$$W_s(1)=0 \text{ and } \sum_{j=1}^{\infty} \exp[-W_s(j)] < \infty. \quad (2.3)$$

Normalization requires

$$\sum_{s'=1}^N t_{ss'} \sum_{j=1}^{\infty} e^{-W_s(j)} = 1 \text{ for } s \in \mathcal{E}_1, \quad (2.4a)$$

$$\sum_{s'=1}^N t_{ss'} = 1 \text{ for } s \in \mathcal{E}_2. \quad (2.4b)$$

Note that $\omega_{i+j}(s, s')$ for $s \in \mathcal{E}_1$ cannot be decoupled into $\omega_i(s, s)$ and $\omega_j(s, s')$ since they are strongly correlated to each other. The conditional probability $p(s|\omega_j(s, s'))$ for $s \in \mathcal{E}_1$ decreases more slowly than exponentially with increase of j .

Let us now develop a statistical physics of intermittency on the basis of the model presented in Eqs. (2.2). The key idea is an analogy between intermittent systems and one-dimensional (1D) lattice-gas systems. We replace $\omega_j(s, s')$ for $s \in \mathcal{E}_1$ and $s \neq s'$ by two neighboring gas particles s and s' separated by a distance $j-1$ (the lattice constant being unity), that is, successive identical laminar symbols except for the first correspond to the “empty” sites. The interaction energy $U(s, s', j)$ between neighboring particles separated by a distance j is determined from $t_{ss'}$ and $W_s(j)$ as will be mentioned later. The turbulent symbols $N'+1, N'+2, \dots, N$ are also replaced by occupying gas particles $N'+1, N'+2, \dots, N$, respectively (cf. Fig. 1). The lattice-gas system is thus composed of N kinds of particles. If the conditional probability $p(s|\omega_j(s, s'))$ depends only on the length j and the symbols s and s' , as defined in Eq. (2.2a), this replacement can be safely guaranteed. We will hereafter describe intermittent systems in terms of the 1D lattice-gas system.

Let $Z_{n,m}(q)$ be the partition function for the 1D system, composed of m particles (including N kinds of particles with n lattice sites, that is,

$$Z_n(q) = \sum_{m=0}^{\infty} Z_{n,m}(q). \quad (2.5)$$

Here q is the inverse of “temperature” (but $-\infty \leq q \leq \infty$)

$$\Omega_m(P, q) = \sum_{n=0}^{\infty} e^{-Pn} \sum_{s_1, \dots, s_m=1}^N \sum_{j_1, \dots, j_m=1}^{\infty} e^{-qU(s_1, s_2, j_1)} e^{-qU(s_2, s_3, j_2)} \dots e^{-qU(s_m, s_1, j_m)} \delta_{n, j_1 + \dots + j_m}, \quad (2.8)$$

where δ is the Kronecker δ . Introducing a matrix T defined by

$$\langle s|T|s' \rangle = \begin{cases} (t_{ss'})^q \sum_{j=1}^{\infty} e^{-Pj - qW_s(j)} & \text{for } s \in \mathcal{E}_1 \text{ and } s \neq s', \\ 0 & \text{for } s \in \mathcal{E}_1 \text{ and } s = s', \\ (t_{ss'})^q e^{-P} & \text{for } s \in \mathcal{E}_2, \end{cases} \quad (2.9)$$

FIG. 1. Example of converting dyadic symbols to gas particles for $\mathcal{E}_1 = \{0\}$ and $\mathcal{E}_2 = \{1\}$. There are two kinds of particles denoted by open and solid circles corresponding to “0” and “1,” respectively.

and $Z_{n,m}(q)$ for $n < m$ have been assumed to vanish. The interaction energy $U(s, s', j)$ between neighboring particles s and s' separated by a distance j is given by

$$e^{-U(s, s', j)} = \begin{cases} t_{ss'} e^{-W_s(j)} & \text{for } s \in \mathcal{E}_1 \text{ and } s \neq s', \\ 0 & \text{for } s \in \mathcal{E}_1 \text{ and } s = s', \\ t_{ss'} & \text{for } s \in \mathcal{E}_2. \end{cases} \quad (2.6)$$

Note that U is always positive. In order to calculate $Z_n(q)$, we introduce the generating function,

$$\begin{aligned} \Xi(P, q) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-Pn} Z_{n,m}(q) \\ &= \sum_{m=0}^{\infty} \Omega_m(P, q), \end{aligned} \quad (2.7a)$$

and

$$\Omega_m(P, q) = \sum_{n=0}^{\infty} e^{-Pn} Z_{n,m}(q). \quad (2.7b)$$

P is proportional to the 1D pressure and $\Xi(P, q)$ is the partition function in the isobaric ensemble for the system.

Denote the i th particle by s_i and the distance between the neighboring particles s_i and s_{i+1} by j_i for $i = 1, 2, \dots, m$. j_m is the distance between s_m and s_1 , and n is the sum of j_i ; $n = j_1 + j_2 + \dots + j_m$. We have used the periodic boundary condition for simplicity since effects of boundaries become negligible as n goes to infinity. Then one can write

allows us to rewrite Eq. (2.8) as

$$\begin{aligned} \Omega_m(P, q) &= \sum_{s_1, \dots, s_m} \langle s_1|T|s_2 \rangle \langle s_2|T|s_3 \rangle \dots \langle s_m|T|s_1 \rangle \\ &= \text{tr}(T^m). \end{aligned} \quad (2.10)$$

From Eqs. (2.7a) and (2.10) the generating function results in

$$\Xi(P, q) = \sum_{i=1}^N (1 - \lambda_i)^{-1}, \quad (2.11)$$

where λ_i is an eigenvalue of T with an order of $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$.

Let $\langle Q \rangle$ be the grand canonical ensemble average of a physical quantity Q ,

$$\langle Q \rangle = [\Xi(P, q)]^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-Pn} Q Z_{n,m}(q). \quad (2.12)$$

The ensemble average of n is

$$\begin{aligned} \langle n \rangle &= -\frac{\partial}{\partial P} \ln \Xi(P, q) \\ &= -[\Xi(P, q)]^{-1} \sum_{i=1}^N \frac{\partial \lambda_i}{\partial P} (1 - \lambda_i)^{-2}. \end{aligned} \quad (2.13)$$

To obtain $Z_n(q)$ in the thermodynamic limit, one can follow the familiar method of the Legendre transform. P is inversely solved for a given $\langle n \rangle$ as

$$P(q, \langle n \rangle) = \bar{P}(q) + \langle n \rangle^{-1} + O(\langle n \rangle^{-2}), \quad (2.14)$$

with use of the solution of the implicit equation,

$$\lambda_1(\bar{P}(q), q) = 1. \quad (2.15)$$

Then the ‘‘Helmholtz free energy’’ is obtained as

$$F(q, \langle n \rangle) = -\bar{P}(q) \langle n \rangle - \ln \Xi(P(q, \langle n \rangle), q). \quad (2.16)$$

In the thermodynamic limit, one can identify $f(q)$ in Eq. (1.2) with the Helmholtz free energy per site

$$f(q) = \lim_{\langle n \rangle \rightarrow \infty} \frac{F(q, \langle n \rangle)}{\langle n \rangle}, \quad (2.17)$$

which gives $f(q) = -\bar{P}(q)$. From Eqs. (1.3a) and (1.3b) we have

$$\gamma(q) = -\frac{d}{dq} \bar{P}(q), \quad (2.18a)$$

$$h(q) = q\gamma(q) + \bar{P}(q). \quad (2.18b)$$

It has been now shown that the free energy per site $f(q)$ and the entropy function $h(\gamma)$ can be calculated with the use of the solution $\bar{P}(q)$ of Eq. (2.15).

From general considerations one can immediately derive the following results: the matrix $T(P, q)$ has non-negative elements that are nonincreasing functions of P and q . Therefore the solution $\bar{P}(q)$ is also a nonincreasing function of q . From the normalization condition it can be easily shown that the largest eigenvalue of $T(0, 1)$ is equal to unity (cf. Appendix A), which means $\bar{P}(1) = 0$.

When the interaction energy between neighboring particles logarithmically decreases with increase of j , that is, $W(j) = r \ln j$, one can obtain further important results on phase transitions. Let us consider the case $q > 1$. Since the inequalities $0 < \sum_s \langle s | T(P, q) | s' \rangle < 1$ hold for $P > 0$, it can be seen that positive P cannot be a solution [all eigenvalues of $T(P, q)$ being smaller than unity]. On the other hand, for negative P , $\langle s | T(P, q) | s' \rangle$ diverges to infinity for $s \in \mathcal{E}_1$. Therefore the solution of Eq. (2.15) does not exist for $q > 1$. To overcome this difficulty we cut off the infinite sum in Eq. (2.9) at a finite k , and take $k \rightarrow \infty$ after solving the eigenvalue problem for the finite sum. Then we can get $\bar{P}(q) = 0$ for $q > 1$ in this limit (cf. Appendix

B). The region below $q = 1$ is an ordered phase and smoothness of $\bar{P}(q)$ is lost at $q = 1$, which means that a phase transition occurs at $q = 1$.

In the case where the interaction range between neighboring particles is finite, the singularity disappears because the matrix elements do not diverge even for negative P . This corresponds to the fact that phase transitions do not occur in 1D systems with finite interaction ranges. When the interaction range is sufficiently long, however, the drastic change of $d\bar{P}(q)/dq$ near $q = 1$ may still be persistent.

III. BASIC EXAMPLES

A. The simplest example of symbolic dynamics

We devote this section to studying a model corresponding to a particular case of Eqs. (2.2a) and (2.2b). Let us consider the simplest example that both the laminar and turbulent sets have only one element; $\mathcal{E}_1 = \{0\}$ and $\mathcal{E}_2 = \{1\}$. The conditional probability $p(0|\omega_j(0, 1))$ is assumed to decrease as j^{-r} ($r > 1$). A model for the simplest such case is described as follows:

$$p(0|\omega_j(0, 1)) = ce^{-W(j)} \quad \text{for } j = 1, 2, \dots, \quad (3.1a)$$

$$p(1|0) = a, \quad p(1|1) = b, \quad (3.1b)$$

where $W(j) = r \ln j$ and $\omega_1(0, 1) = \{1\}$. Here $\omega_j(0, 1)$ again cannot be decoupled between successive 0's. Normalization requires that $a + b = 1$ and $c = [\xi(r)]^{-1}$ with Riemann's ξ function $\xi(r)$. This model may be too simple to describe physical processes. Nevertheless, it does not lose its importance with respect to understanding the mechanism of intermittency since universal properties are reproducible from the model and critical exponents of phase transitions can be analytically obtained.

The interaction energy between neighboring particles separated by a distance j is

$$e^{-U(s, s', j)} = \begin{cases} t_{ss'} j^{-qr} & \text{for } s = 0 \text{ and } s' = 1, \\ 0 & \text{for } s = s' = 0, \\ t_{ss'} & \text{for } s = 1, \end{cases} \quad (3.2)$$

where $t_{ss'}$ is a transition probability from a symbol s to s' . The transition probabilities are $t_{00} = 0$, $t_{01} = c$, $t_{10} = a$, $t_{11} = b$, where $t_{00} = 0$ results from the inhibition of decoupling of $\omega_j(0, 1)$. The matrix T is immediately given by

$$T = \begin{bmatrix} 0 & c^q \Phi(P, q) \\ e^{-Pa^q} & e^{-Pb^q} \end{bmatrix}, \quad (3.3)$$

where

$$\Phi(P, q) = \sum_{j=1}^{\infty} \exp(-Pj - qr \ln j). \quad (3.4)$$

The 1D pressure $\bar{P}(q)$ is derived from Eq. (2.15), which is rewritten as

$$e^{-\bar{P}(q)} [b^q + (ac)^q \Phi(\bar{P}(q), q)] = 1. \quad (3.5)$$

From Eqs. (2.18a), (2.18b), and (3.5) the energy and entropy functions are given as

$$\gamma(q) = - \frac{b^q \ln b + (ac)^q \Phi(\bar{P}(q), q) \ln(ac) + (ac)^q \Phi_q(\bar{P}(q), q)}{b^q + (ac)^q \Phi(\bar{P}(q), q) + (ac)^q \Phi_P(\bar{P}(q), q)}, \quad (3.6a)$$

$$h(q) = q\gamma(q) + \bar{P}(q), \quad (3.6b)$$

where $\Phi_P(P, q)$ and $\Phi_q(P, q)$ are partial derivatives of $\Phi(P, q)$ with respect to P and q , respectively.

We also calculate a density of gas particles belonging to $\mathcal{E}_2$ (i.e., an averaged rate of turbulence) as an order parameter as introduced in Ref. 24. Denoting numbers of gas particles “0” and “1” by m_0 and m_1 ($m = m_0 + m_1$), respectively, the order parameter

$$\psi(q) = \lim_{\langle n \rangle \rightarrow \infty} \langle m_1 \rangle / \langle n \rangle \quad (3.7)$$

leads to

$$\psi(q) = e^{\bar{P}} [b^q + (ac)^q \Phi(\bar{P}(q), q) - (ac)^q \Phi_P(\bar{P}(q), q)]^{-1}. \quad (3.8)$$

We give the analytic solution of Eq. (3.5) below, near the transition point $q = 1$, where $\bar{P}(q)$ can be assumed to be sufficiently small. The behavior of the solution is classified into the following three cases.

Case I ($r > 3$). Expansion of the left-hand side of Eq. (3.5) in terms of $\bar{P}$ and $(1-q)$ yields $\bar{P} \simeq A(1-q) + B(1-q)^2$ with $A, B > 0$, which leads immediately to

$$\gamma(q) \simeq A + 2B(1-q) \simeq h(q), \quad (3.9a)$$

$$\psi(q) \simeq [1 + ac\xi(r-1)]^{-1} + W(1-q) \quad (W > 0). \quad (3.9b)$$

Case II ($3 > r > 2$). In this case we cannot expand $\bar{P}$ up to the second order of $(1-q)$ because of the divergence of $\xi(r-2)$. Instead of the above, we find $\bar{P} \simeq A(1-q) + C(1-q)^\eta$ where $\eta = 2/(4-r)$ and $C > 0$. Then

$$\gamma(q) \simeq A + \eta C(1-q)^{\eta-1} \simeq h(q), \quad (3.10a)$$

$$\psi(q) \simeq [1 + ac\xi(r-1)]^{-1} + X(1-q)^{r-2} \quad (X > 0). \quad (3.10b)$$

Cases I and II correspond to the first-order phase transition.

Case III ($2 > r > 1$). Equation (3.5) is rewritten as

$$e^{-\bar{P}} \left[b^q + (ac)^q \xi(qr) + (ac)^q \sum_{j=1}^{\infty} (e^{-\bar{P}j} - 1) j^{-qr} \right] = 1. \quad (3.11)$$

Using the formula, in the limit $q \rightarrow 1$ and $\bar{P} \rightarrow 0$,

$$\sum_{j=1}^{\infty} (e^{-\bar{P}j} - 1) j^{-qr} \rightarrow -\bar{P}^{r-1} \int_0^{\infty} (1 - e^{-x}) x^{-r} dx, \quad (3.12)$$

we obtain $\bar{P} \simeq D(1-q)^{1/(r-1)}$ with $D > 0$, which leads to

$$\begin{aligned} \gamma(q) &\simeq [D/(r-1)](1-q)^\beta \simeq h(q), \\ \beta &= (2-r)/(r-1), \end{aligned} \quad (3.13a)$$

$$\psi(q) \simeq Y(1-q)^\beta \quad (Y > 0). \quad (3.13b)$$

In the marginal case $r = 2$, a logarithmic correction appears as $\bar{P} \simeq -G(1-q)/\ln(1-q)$ with $G > 0$, which gives

$$\gamma(q) \simeq -G/\ln(1-q) \simeq h(q), \quad (3.14a)$$

$$\psi(q) \simeq -Z/\ln(1-q) \quad (Z > 0). \quad (3.14b)$$

These whole functions in Eqs. (3.14a) and (3.14b) with $a = b = \frac{1}{2}$ are shown in Figs. 2(a) and 2(b), respectively. We should emphasize here that the relation $h(\gamma) \simeq \gamma$ holds near the transition point $q \lesssim 1$ in all cases, which characterizes large fluctuations of dynamical scaling indices of intermittency.

FIG. 2. Plots of the entropy function and the order parameter with $a = b = \frac{1}{2}$ and $r = 2$. (a) The entropy function $h(\gamma)$. The solid circles denote the generalized entropies $K_1 = 0$ and $K_\infty = \ln 2 + \frac{1}{3} \ln(\pi^2/6)$. (b) The order parameter $\psi(q)$. The solid circle denotes the transition point $q = 1$.

FIG. 3. Graph of the map in Eq. (3.15).

B. Application to a solvable one-parameter family

It is of much interest to apply the statistical-mechanical formalism to one-parameter families of 1D dynamical systems. Suppose that evolution of states obeys the 1D mapping,

$$x_{i+1} = \begin{cases} x_i + x_i^2 + \epsilon & \text{for } 0 \leq x_i \leq x^*, \\ (1 - x_i)/(1 - x^*) & \text{for } x^* < x_i \leq 1, \end{cases} \quad (3.15)$$

where x^* satisfies $x^* + (x^*)^2 + \epsilon = 1$ and ϵ is a control parameter with $0 \leq \epsilon < 1$ (see Fig. 3). The time series $\{x_i\}$ is encoded dyadically in a way of assigning to $s_i = 1$ (0) if x_i is larger (smaller) than x^* . Then we get $p(1|0) = x^*$ and $p(1|1) = 1 - x^*$. The distribution function $W(j)$ in Eq. (3.1a) is obtained from replacing the recursion relation by a differential equation.² $W(j)$ is thus given as

$$W(j) = A - \ln \left[\frac{\epsilon}{x^*} \left\{ \sec^2 \left[\tan^{-1} \left(\frac{x^*}{\sqrt{\epsilon}} \right) - \epsilon^{1/2} j \right] \right\} \right] \quad \text{for } 1 \leq j \leq k, \quad (3.16)$$

where $k = \epsilon^{-1/2} \tan^{-1}(x^*/\sqrt{\epsilon})$ is the largest length of laminar regions and A is a positive constant introduced from the condition $W(1) = 0$.

In this case the interaction range between neighboring gas particles is finite in contrast to Eq. (3.2). Substitution of Eq. (3.16), $p(1|0) = x^*$, and $p(1|1) = 1 - x^*$ into Eq. (3.5) yields

$$K_{-\infty} = \max_{1 < j \leq k} \{ -\ln(1 - x^*), -[\ln(x^*c) + W(j)]/j \}, \quad (3.18a)$$

$$K_0 = -\ln \xi, \quad (3.18b)$$

$$K_1 = \frac{-(1 - x^*)\ln(1 - x^*) - x^*\ln(x^*c) + x^* \sum_{j=1}^k W(j)e^{-W(j)}}{1 + x^*c \sum_{j=1}^k j e^{-W(j)}}, \quad (3.18c)$$

FIG. 4. Change of $h(\gamma)$ and $\psi(q)$ from $\epsilon = 5 \times 10^{-2}$ to 10^{-4} . (a) The curves of $h(\gamma)$ are indicated by the dotted line ($\epsilon = 5 \times 10^{-2}$), the dashed line ($\epsilon = 10^{-2}$), the dash-dotted line ($\epsilon = 10^{-3}$), and the solid line ($\epsilon = 10^{-4}$). (b) The curves of $\psi(q)$ with the same notation as in (a).

$$e^{-\bar{P}} \left[(1 - x^*)^q + (x^*c)^q \sum_{j=1}^k \exp[-\bar{P}j - qW(j)] \right] = 1, \quad (3.17a)$$

with

$$c = \left[\sum_{j=1}^k e^{-W(j)} \right]^{-1}. \quad (3.17b)$$

The generalized entropies $K_q = -\bar{P}(q)/(q-1)$ for the special values $q = -\infty, 0, 1$, and ∞ are directly estimated from Eq. (3.17a),

$$K_\infty = \min_{1 < j \leq k} \{ -\ln(1-x^*), -[\ln(x^*c) + W(j)]/j \}, \quad (3.18d)$$

where ξ is a real positive solution of $\xi^{k+1} + \xi^k + \dots + \xi = 1$. The Legendre transform of the numerical solution of Eq. (3.17a) gives the functions $h(\gamma)$ and $\psi(q)$ shown in Figs. 4(a) and 4(b). As is mentioned in Sec. II, drastic changes near $q=1$ are observed for sufficiently small ϵ .

IV. APPLICATION TO THE LOGISTIC MAP

We now apply the statistical-mechanical formalism to the logistic map just before the period-three window. This is one of the most familiar examples exhibiting intermittent behavior but less understood from a viewpoint of multifractals.

The logistic map $x_{i+1} = f(x_i)$ is given by

$$x_{i+1} = ax_i(1-x_i) \quad (4.1)$$

on the interval $I=[0,1]$. The critical point bringing on birth of a period-three cycle is $a_c = 1 + \sqrt{8}$. Let I_- and I_+ be the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, and $f^k_{(x)}$ be the k th-iterated map of $f(x)$. We first consider a dyadic symbol sequence $\{s_i\}$ for an orbit $\{x_i\}$ at $a \simeq a_c$ defined by

$$s_i = H(x_i), \quad (4.2a)$$

$$H(x) = \begin{cases} 0 & \text{if } x \in I_- \\ 1 & \text{if } x \in I_+ \end{cases}, \quad (4.2b)$$

which gives a one-to-one relationship between orbits of $f(x)$ and the realization of symbols except for the orbits satisfying $f^k(x) = \frac{1}{2}$ for any integer k . For $a = a_c$ the period-three cycle $\{\dots, x_0, x_1, x_2, x_0, x_1, x_2, \dots\}$ with $x_0 = 0.1598\dots$, $x_1 = 0.5142\dots$, and $x_2 = 0.9563\dots$ exists, which corresponds to the periodic symbol sequence $\{\dots 011011\dots\}$. For $a < a_c$ and sufficiently close to a_c , intermittent bursts appear in the orbit. The laminar states in the intermittent orbit do not correspond to successive 0's or 1's in the corresponding symbol sequence. Hence the dyadic representation of the orbit may not be suitable for our formalism.

To apply the formalism to the problem considered here, let us next consider the third-iterate map $f^3(x)$. The interval $I=[0,1]$ is divided into eight subintervals $I_1, I_2, \dots, I_8$ as shown in Fig. 5. The subinterval I_k corresponds to a triplet of symbols $\{s_1, s_2, s_3\}$ where $s_1 = H(x)$, $s_2 = H(f(x))$, and $s_3 = H(f^2(x))$ for $x \in I_k$. The correspondence between I_k and $\{s_1, s_2, s_3\}$ is also shown in Fig. 5. These eight subintervals can be reduced into only five intervals, $I_3, I_4, \dots, I_7$ because the probability of finding an iterated point x_i of $f^3(x)$ on I_1, I_2 , and I_8 vanishes after a few iterations. Hence we relabel the subintervals I_k ($k=3, \dots, 7$) as $I_3 = V_1$, $I_5 = V_2$, $I_7 = V_3$, $I_4 = V_4$, and $I_6 = V_5$. For an orbit $\{x_i\}$ generated by $f^3(x)$, a new symbol sequence $\{s_i\}$ is defined by

$$s_i = k \quad \text{if } x_i \in V_k \quad (k=1, 2, \dots, 5). \quad (4.3)$$

The laminar and turbulent sets are defined by $\mathcal{E}_1 = \{1, 2, 3\}$ and $\mathcal{E}_2 = \{4, 5\}$, respectively. An iterated point x_i of $f^3(x)$ on the interval V_k for $k \in \mathcal{E}_2$ will lose the memory of its ancestor rapidly after a few iterations. Then we can regard turbulent states as being of a Markovian type. The formalism presented in Sec. II is now applicable.

Let us now consider the matrix T . Suppose that the transition probability $t_{ss'}$ for $s \in \mathcal{E}_1$ is independent of s' if it does not vanish, which gives the fairly simplified matrix T . Let $C \exp[-W_s(j)]$ ($s=1, 2, 3$) be probability distributions of the length j of laminar regions in the intermittent sequence, where C is a normalization constant. $W_1(j)$, $W_2(j)$, and $W_3(j)$ are of the same function of j and denoted by $W(j)$ due to the translational symmetry of laminar states in symbol sequences encoded by Eq. (4.2a), that is,

$$\begin{aligned} \{\dots (011)(011)\dots\} &= \{\dots (110)(110)\dots\} \\ &= \{\dots (101)(101)\dots\}, \end{aligned}$$

where parentheses indicate subintervals. Then $T(P, q)$ is written as

$$T(P, q) = \begin{pmatrix} 0 & C_1\Phi & C_1\Phi & C_1\Phi & C_1\Phi \\ C_1\Phi & 0 & C_1\Phi & C_1\Phi & C_1\Phi \\ 0 & C_2\Phi & 0 & 0 & C_2\Phi \\ 0 & A_2 & A_3 & 0 & A_5 \\ B_1 & B_2 & B_3 & B_4 & B_5 \end{pmatrix}, \quad (4.4a)$$

with

FIG. 5. Third-iterate map of the logistic map at $a = a_c$ on the interval $I=[0,1]$. I is divided into eight subintervals as shown in the figure. See the text for explanation.

$$\Phi = \sum_{j=1}^K e^{-Pj - qW(j)} . \quad (4.4b)$$

Here $C_1 = (C/4)^q$, $C_2 = (C/2)^q$, $A_i = e^{-P(t_{4i})^q}$, $B_i = e^{-P(t_{5i})^q}$, and k is the largest length of laminar regions. The transition probabilities t_{31} , t_{34} , t_{41} , and t_{44} vanish because there do not exist successive 0's in the dyadic sequence of $f(x)$ such as $\{(110)(011)\}$, $\{(110)(010)\}$, $\{(010)(011)\}$, and $\{(010)(010)\}$ in steady states (cf. Fig. 5).

We test the method for the map with the small external noise,

$$x_{i+1} = ax_i(1-x_i) + g\xi_i , \quad (4.5)$$

where ξ_i is the independent random noise whose root-mean-squared average is unity and g is its amplitude. The external noise is introduced to keep the bifurcation structure of the system smooth. The transition probabilities $t_{ss'}$ and the distribution function $W(j)$ are estimated numerically from an orbit (10^6 iterations) generated by the third-iterate map of Eq. (4.5). The function $W(j)$ is shown in Fig. 6. Substituting these quantities into T , the pressure $\bar{P}(q)$ is calculated. For practical calculation we adopt the following technique.¹² Choose a random vector and renormalize the vector operated by T to a unit vector. Again apply T and renormalize the vector. This procedure is continued until the vector converges to a limiting one. Then the renormalization factor converges to the largest eigenvalue $\lambda(P, q)$. The 1D pressure $\bar{P}(q)$ is determined from the condition $\lambda(\bar{P}(q), q) = 1$ [Eq. (2.15)]. Since $\bar{P}(q)$ is a quantity of the third-iterate map, $\bar{P}(q)$ should be replaced by $\bar{P}(q)/3$ in order to get a quantity per iteration. Figure 7 shows the estimated entropy function $h(\gamma)$ obtained via the Legendre transform of $\bar{P}(q)/3$ at $a = 3.83842$ and $g = 10^{-7}$ (cf. $a_c = 3.8284271 \dots$). In order to check the validity of the result, we estimate $h(\gamma)$ in a more direct way. We replace the sum in Eq. (1.1) by ensemble average $\langle\langle \dots \rangle\rangle$;¹⁹

$$Z_n(q) = \langle\langle p(s_1, s_2, \dots, s_n)^{q-1} \rangle\rangle , \quad (4.6)$$

where the ensemble average is taken for random initial

FIG. 6. The function $W(j)$ at $a = 3.82842$. The largest value of j is 143.

FIG. 7. Entropy function of the logistic map for $a = 3.82842$. The solid line was calculated by using the matrix T of Eq. (4.4a). The estimated values with the use of Eq. (4.6) are also shown by the symbols $\times$ and $\circ$. The $\circ$ were calculated for orbits obtained from the inverse mapping (Refs. 19 and 25).

conditions. $p(s_1, \dots, s_n)$ for the orbit $\{x_i\}$ generated by Eq. (4.5) is estimated by using derivatives of $f(x)$, i.e.,

$$p(s_1, \dots, s_n) \propto |f'(x_1)f'(x_2) \cdots f'(x_n)|^{-1} ,$$

where $f'(x) = df(x)/dx$ and $s_i = H(x_i)$. The estimated $h(\gamma)$ is also shown in Fig. 7 for comparison, which are in excellent agreement with the result obtained from the matrix T .

Let us consider the eigenvalue equation $\det[T(P, q) - \lambda E] = 0$, where E is the unit matrix. We remove a trivial solution $\lambda = -C_1\Phi$, not being the largest one, from the eigenvalue equation. Substitution of $\lambda = 1$ into the reduced eigenvalue equation yields

$$B_5 + B_4 A_5 + (C_1 F_1 + C_2 F_2)\Phi + C_1 C_2 F_3 \Phi^2 = 1 , \quad (4.7)$$

where

$$F_1 = 1 + A_2 + B_1 + B_2 - B_5 - B_4 A_5 - A_2 B_5 + A_2 B_4 + A_5 B_1 + A_5 B_2 , \quad (4.8a)$$

$$F_2 = B_3 + A_3 B_4 , \quad (4.8b)$$

$$F_3 = 1 + A_3 + B_1 + B_2 - B_5 + A_3 B_1 + A_3 B_2 - A_3 B_5 - B_4 A_5 + B_4 A_2 + B_3 A_5 - B_3 A_2 . \quad (4.8c)$$

Expanding the left-hand side of Eq. (4.7) around $q = 1$ in terms of $\bar{P}$ and $1 - q$, we have

$$\bar{P}(q) \simeq K_1(1 - q) + O((1 - q)^2) , \quad (4.9)$$

where K_1 is the Kolomogorov-Sinai entropy of $f^3(x)$. For $a < a_c$ and sufficiently close to a_c , K_1 is given as

$$K_1 \simeq \sum_{j=1}^k W(j) e^{-W(j)} / \sum_{j=1}^k j e^{-W(j)} \propto |a - a_c|^{1/2} . \quad (4.10)$$

Equations (4.9) and (4.10) are in good agreement with nu-

merical simulations of the logistic map. For the neutral stability case $a = a_c$, however, T has to be modified since fixed points of $f^3(x)$ appear. A detailed study is planned to be developed elsewhere.

V. SUMMARY

The approach to the multifractal structure of intermittency by means of a statistical-mechanical formalism promotes our understanding through an analogy between intermittency and 1D lattice-gas systems. Phase transitions occur under the existence of long-range interaction between neighboring gas particles. The formalism presented in Sec. II completely grasps this feature and also gives an algebraic equation for the 1D pressure or the free energy per site.

For an intermittent system described by

$$x_{i+1} = x_i + ax_i^z \quad (0 \leq x_i \leq u), \quad (5.1)$$

with random reinjections into the interval $[0, u]$, it is well known that the universality class of the map is determined by the parameter z . The relation between the parameter z and the exponent r in Eq. (3.1) is $z = r/(r-1)$ as seen by the functional renormalization-group method.³ We have shown that the universality class is also determined by z in a simple stochastic model from a point of view of multifractals. For $1 < z < 2$ discontinuous phase transitions occur with varying q , while for $z \geq 2$ continuous ones occur. It is of interest to investigate critical scaling of various thermal quantities such as the “specific heat” and “susceptibility” as had been studied in equilibrium systems. A detailed discussion has been developed in Ref. 24.

We have also studied the logistic map near the critical point for a period-three cycle. With the use of the transfer matrix T derived under some assumptions the entropy function $h(\gamma)$ was obtained, being in excellent agreement with other numerical results.¹⁹

Finally, we mention an advantage of the method presented here in deriving the entropy function from symbol sequences. If one attempts to calculate $f(q)$ or K_q of Eq. (1.2) directly from a sequence composed of symbols taking on N values, accurate estimation of $f(q)$ requires at least N^k subsequences, where k is the largest length of laminar regions. On the other hand, we deal with at most $N^2 k$ terms to calculate the largest eigenvalue of T . Therefore our formalism gives us a shortcut to calculate the free energy $f(q)$ or K_q . Furthermore, one can obtain $f(q)$ without constructing 1D maps since we have used only transition probabilities from states to states and the distribution function $e^{-W(j)}$ of the length j of laminar regions. We have a plan of applying the formalism to the turbulent boundary layer problem.

APPENDIX A

We will show here that the largest eigenvalue of $T(0, 1)$ in Eq. (2.9) is equal to unity. We denote the (i, j) th element of $T(P, q)$ by $T_{ij}(P, q)$ for simplicity. Equations (2.4) are rewritten as

$$\sum_{j=1}^N T_{ij}(0, 1) = 1, \quad (A1)$$

for any i . Note that $T_{ij}(P, q)$ is non-negative for any pair (i, j) .

Let $\lambda(P, q)$ be the largest eigenvalue of $T(P, q)$ and $\mathbf{x} = (x_1, x_2, \dots, x_N)$ be an associated eigenvector. Denote the largest absolute value among $|x_1|, |x_2|, \dots, |x_N|$ by $|x_m|$, where $|x_m|$ never vanishes because $\mathbf{x} \neq \mathbf{0}$. Then we have

$$|\lambda(P, q)| = \left| \sum_{j=1}^N T_{mj}(P, q) x_j / x_m \right| \leq \sum_{j=1}^N T_{mj}(P, q). \quad (A2)$$

From Eqs. (A1) and (A2) we get

$$|\lambda(0, 1)| \leq 1. \quad (A3)$$

On the other hand, with the use of Eq. (A1) it is easy to see that

$$\det(T - E) = 0, \quad (A4)$$

where $\det(\cdot)$ is the determinant of a matrix and E is the unit matrix. It follows from Eqs. (A3) and (A4) that the largest eigenvalue $\lambda(0, 1)$ is equal to unity.

APPENDIX B

The 1D pressure $\bar{P}(q)$ for $W(j) = r \ln j$ ($1 < r$) will be shown to vanish for $q > 1$. Let us define a matrix $T_k(P, q)$ by

$$\langle s | T_k | s' \rangle = \begin{cases} (t_{ss'})^q \Phi_k & \text{for } s \in \mathcal{E}_1, \\ (t_{ss'})^q e^{-P} & \text{for } s \in \mathcal{E}_2, \end{cases} \quad (B1)$$

with

$$\Phi_k = \sum_{j=1}^k e^{-Pj} j^{-qr}. \quad (B2)$$

T_k converges to T of Eq. (2.9) as $k \rightarrow \infty$. We assume $q > 1$ and $P \leq 0$ throughout the remainder of this appendix.

Consider the m th-iterate matrix T_k^m . One can assume without loss of generality that there is a finite positive integer m such that $\langle s | T_k^m | s' \rangle > 0$ for any pairs of s and s' . If such an m does not exist, one can remove s' corresponding to $\langle s | T_k^m | s' \rangle = 0$ from the set of symbols $\{1, 2, \dots, N\}$ since the probability of appearance of such an s' should vanish in steady state. Let $[\lambda_k(P, q)]^m$ be the largest eigenvalue of T_k^m and $\mathbf{x} = (x_1, x_2, \dots, x_N)$ be an associated eigenvector. Note that $\lambda_k(P, q)$ is the largest eigenvalue of T_k , and it is not difficult to see that there exists an eigenvector with $x_i > 0$ for $i = 1, 2, \dots, N$ and $[\lambda_k(P, q)]^m > 0$. The eigenvalue equation of T_k^m yields

$$[\lambda_k(P, q)]^m = \sum_{s'=1}^N \langle s | T_k^m | s' \rangle \frac{x_{s'}}{x_s}, \quad (B3)$$

where x_s is assumed to be positive for any s . Let M be a positive constant defined by

$$M = \min_{1 \leq s, s' \leq N} \{t_{ss'} \text{ with } t_{ss'} > 0\}. \quad (B4)$$

For $s \in \mathcal{E}_1$ we have

$$\langle s | T_k^m | s \rangle \geq M^{qm} e^{-(m-1)P} \Phi_k, \quad (\text{B5})$$

and

$$\Phi_k \geq k^{-qr} \sum_{j=1}^k e^{-Pj} = k^{-qr} e^{-P} \frac{1 - e^{-Pk}}{1 - e^{-P}}. \quad (\text{B6})$$

If one puts $P = -(qr \ln k)/k$, Eqs. (B3), (B5), and (B6) give the inequality

$$[\lambda_k(-(qr \ln k)/k, q)]^m \geq M^{qm} e^{-(mqr \ln k)/k} \frac{1 - k^{-qr}}{e^{(qr \ln k)/k} - 1}. \quad (\text{B7})$$

The right-hand side of Eq. (B7) is clearly larger than uni-

ty for sufficiently large k .

Assuming that $\lambda_k(P, q)$ is a smooth function of P , we have

$$\frac{\partial}{\partial P} \lambda_k(P, q) < 0 \quad (\text{B8})$$

because all elements of T_k^m are decreasing functions of P . Denote the solution of $\lambda_k(P, q) = 1$ by $\bar{P}_k(q)$. From $|\lambda_k(0, q)| < 1$ and Eqs. (B7) and (B8) it follows that

$$0 < |\bar{P}_k(q)| < (qr \ln k)/k, \quad (\text{B9})$$

for sufficiently large k . Therefore we obtain $\bar{P}_k(q) \rightarrow 0$ as $k \rightarrow \infty$.

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