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Semiclassical wave function of the periodic Toda lattice

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Abstract:

The semiclassical wave function of the periodic Toda lattice is constructed by the Keller's method. It is shown that the integer quantum numbers have a close relationship to a structure of the invariant torus of the classical motion.

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The Toda lattice is an integrable nonlinear dynamical system and has been studied extensively over two decades [1]. For the N -particle periodic Toda lattice, there are N constants of motion and they define the N -dimensional invariant torus in the $2N$ -dimensional phase space. Once the invariant torus is known, it is possible to construct the semiclassical wave function on the torus according to the Keller's method [2]. However, no explicit integrating variables have been found to separate coordinates and, thus, it is very difficult to visualize the explicit structure of the invariant torus. The quantum version of the periodic Toda lattice has been studied [3-9], and in Ref.[9], we have carried out systematic investigation of the periodic Toda lattice and numerical calculation was done for $N=3,4,5$ and 6 particle Toda lattices. We have found that the semiclassical EBK quantization is a good approximation and there exists a close relationship between the integer quantum numbers and the symmetry of the eigenfunction. The purpose of this Letter is to explore this relationship in more detail and point out that it is based on the structure of the invariant torus.

Let us briefly review the N -particle periodic Toda lattice. The Hamiltonian is given in a dimensionless form as

$$H = \frac{1}{2} \sum_i p_i^2 + \sum_i \exp(q_i - q_{i+1}) \quad (1)$$

where we set $q_{N+1} = q_1$. The classical equations of motion can be written in a Lax form

$$\frac{dL}{dt} = [B, L] \quad (2)$$

where

$$L = \begin{pmatrix} b_1 & a_1 & & & a_N \\ a_1 & b_2 & a_2 & & \\ & \cdots & \cdots & \mathbf{0} & \\ & & \cdots & \cdots & \\ \mathbf{0} & a_{N-2} & b_{N-1} & a_{N-1} & \\ a_N & & a_{N-1} & b_N & \end{pmatrix}, B = \begin{pmatrix} 0 & -a_1 & & & a_N \\ a_1 & 0 & -a_2 & & \\ & \cdots & \cdots & \mathbf{0} & \\ & & \cdots & \cdots & \\ \mathbf{0} & a_{N-2} & 0 & -a_{N-1} & \\ -a_N & & a_{N-1} & 0 & \end{pmatrix} \quad (3)$$

and $a_n = \frac{1}{2} \exp((q_n - q_{n+1})/2)$, $b_n = \frac{1}{2} p_n$. Therefore the eigenvalues of the matrix L are constants of motion and, thus, the coefficients of the characteristic polynomial of the

matrix L are also constants of motion, which are defined as follows (I stands for the $N \times N$ identity matrix.)

$$\det(2\lambda I - 2L) = \sum_i A_i (2\lambda)^{N-i} - 2 \quad (4)$$

($A_0 = 1$) The conserved quantities A_i are in involution with the Hamiltonian, $[H, A_i] = 0$, and also mutually in involution $[A_i, A_j] = 0$. (note that $H = \frac{1}{2}A_1^2 - A_2$)

In quantum mechanics, the Hamiltonian and the conserved quantities are operators and are given by the canonical quantization, i.e. the momentum p_i is replaced by the operator $-i\hbar \frac{\partial}{\partial q_i}$. Since we have used three physical parameters (the mass of each particle, the strength and the range of the potential) as units, Plank's constant $\hbar$ cannot be scaled out. Although $\hbar$ is basically arbitrary, we set $\hbar = 1$ in the following arguments. We are interested in symmetric properties of the semiclassical wave function and they do not depend on the size of $\hbar$.

In Ref.[9], we have shown that the Hamiltonian has a symmetry of the dihedral group D_N and the simultaneous eigenfunctions of the Hamiltonian H and the conserved quantities A_i are classified according to the irreducible representation of the D_N group. The dihedral group D_N consists of two generators s and t satisfying $s^N = 1$, $t^2 = 1$, $ts t^{-1} = s^{-1}$. These operators are defined to shift the coordinates as

$$s(q_1, q_2, \dots, q_N) = (q_2, q_3, \dots, q_N, q_1) \quad (5)$$

$$t(q_1, q_2, \dots, q_N) = (-q_N, -q_{N-1}, \dots, -q_2, -q_1) \quad (6)$$

In a matrix representation, s and t are orthogonal matrices, i.e. ${}^T s s = {}^T t t = 1$ where T denotes a transposition. For $N=\text{odd}$ ($N=2m+1$), the D_N group has two 1-dimensional representations (A_1, A_2) and m 2-dimensional representations E_k ($1 \leq k \leq m$), while it has another two 1-dimensional representations (B_1, B_2) for $N=\text{even}$ ($N=2m+2$). We will call A_1/A_2 , B_1/B_2 and E_k representations as A -, B - and E -symmetries respectively, since they come out in a pair. The eigenfunctions and the matrix elements of the conserved quantities A_i are summarized in Table 1. One should note that the eigenstates generally

degenerate in two-fold. These eigenfunctions are transformed as follows

$$s\phi(\mathbf{q}) = \phi(s\mathbf{q}) = \zeta^l \phi(\mathbf{q}) \quad (7)$$

$$t\phi(\mathbf{q}) = \phi(t\mathbf{q}) = \phi(\mathbf{q})^* \quad (8)$$

where $\zeta = \exp(2\pi i/N)$ and $l = 0, k$ for the A -, E_k -symmetries respectively. In the case of N =even, $l = N/2$ for the B -symmetry should be added.

The semiclassical quantization can be carried out by the EBK formulation. Since the Toda lattice is integrable, the Hamiltonian can be expressed in terms of the actions I_i by a canonical transformation, and the semiclassical energy is given by the substitutions $I_i \rightarrow (n_i + \frac{1}{2})h$, where n_i is an integer quantum number and the Maslov index is 2. In order to obtain action-angle variables (I_i, φ_i) , it is known that another canonical conjugate variables (ν_i, μ_i) are useful since the action I_i is expressed as an integral over one period of μ_i variable

$$I_i = \oint \nu_i(\mu_i) d\mu_i \quad (9)$$

The auxiliary spectra μ_i are defined by the eigenvalues of the reduced matrix L^* which is the $(N-1) \times (N-1)$ matrix given by removing the first row and the first column from the matrix L . They are confined in $N-1$ intervals satisfying $|\Delta(\mu)| \geq 2$ (they are labeled as $\mu_1 < \mu_2 < \dots < \mu_{N-1}$). The function $\Delta(\mu)$ is an N -th order polynomial with coefficients A_i as $\Delta(\mu) = \sum_i A_i (2\mu)^{N-i}$. Thus the integral of eq.(9) is carried out in the bounded region where $|\Delta(\mu)| \geq 2$. If we consider complex μ -plane, the Riemann sheet has a hyperelliptic structure with N slits along the real axis and $N-1$ μ_i variables move between these slits. Whether μ_i is on the upper or lower sheet can be determined by the following formula [10].

$$a_1^2 = -2^{-N} \sum_{i=1}^{N-1} \frac{y^\pm(\mu_i)}{P'(\mu_i)} \quad (10)$$

where

$$P(\lambda) = \prod_{i=1}^{N-1} (\lambda - \mu_i), \quad y^\pm(\mu_i) = \frac{1}{2} \left(\Delta(\mu_i) \pm \left(\Delta(\mu_i)^2 - 4 \right)^{\frac{1}{2}} \right) \quad (11)$$

and the sign $+/-$ is for the upper/lower Riemann sheets respectively. The momentum

variable ν_i conjugate to μ_i is

$$\nu_i = 2(-1)^{N-i} \log \left| \frac{1}{2} \left(\Delta(\mu_i) + \left(\Delta(\mu_i)^2 - 4 \right)^{\frac{1}{2}} \right) \right| \quad (12)$$

where the positive branch of $\sqrt{\Delta(\mu_i)^2 - 4}$ is on the upper Riemann sheet. For the later convenience, we will change the indices of the quantum numbers n_i as follows

$$(n_1, n_2, \dots, n_{N-1}) \rightarrow (n_m, n_{m-1}, \dots, n_{-m+1}, n_{-m})$$

where $N = 2m + 1(2m + 2)$ for $N=\text{odd}(\text{even})$. For $N=\text{odd}$, the quantum number n_0 is missing. The variables $(\mu_1, \mu_2, \dots, \mu_{N-1})$ are also relabeled as $(\mu_m, \mu_{m-1}, \dots, \mu_{-m+1}, \mu_{-m})$.

In Ref.[9], we have noticed that the quantum numbers n_i and the symmetry of the eigenfunction have a close relationship as follows. When we define the integer N_S as

$$N_S = \begin{cases} (n_m - n_{-m}) + 2(n_{m-1} - n_{-m+1}) + 3(n_{m-2} - n_{-m+2}) + \\ \dots + m(n_1 - n_{-1}) \pmod{N} & \text{for } N = 2m + 1 \\ (n_m - n_{-m}) + 2(n_{m-1} - n_{-m+1}) + 3(n_{m-2} - n_{-m+2}) + \\ \dots + m(n_1 - n_{-1}) + (m+1)n_0 \pmod{N} & \text{for } N = 2m + 2 \end{cases} \quad (13)$$

then

$$\begin{aligned} N_S &= 0 && \text{for } A\text{-symmetry} \\ N_S &= N/2 && \text{for } B\text{-symmetry} \\ N_S &= \pm k \quad (1 \leq k \leq m) && \text{for } E_k\text{-symmetry} \end{aligned}$$

Of course, the B -symmetry is missing in the case of $N=\text{odd}$.

The semiclassical wave function which is consistent with the EBK quantization is given by (setting $\hbar = 1$) [11]

$$\phi(\mathbf{q}) = A(\mathbf{q}, \mathbf{I}) \exp(iS(\mathbf{q}, \mathbf{I})) \quad (14)$$

where $\mathbf{I} = (I_1, I_2, \dots, I_{N-1})$ are action variables. The phase function $S(\mathbf{q}, \mathbf{I})$ is a solution of the Hamilton -Jacobi equation and also a generating function of the canonical transformation for $(\mathbf{p}, \mathbf{q}) \rightarrow (\mathbf{I}, \boldsymbol{\varphi})$, i.e.

$$\mathbf{p} \cdot d\mathbf{q} + \boldsymbol{\varphi} \cdot d\mathbf{I} = dS \quad (15)$$

and can be given by the integral

$$S(\mathbf{q}, \mathbf{I}) = \int_C \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} \quad (16)$$

where C is a path from $O(\mathbf{p}_0, \mathbf{q}_0)$ to $P(\mathbf{p}, \mathbf{q})$ in the invariant torus $M_{\mathbf{I}}$ defined by $\mathbf{I}(\mathbf{p}, \mathbf{q}) = \text{const.}$. The amplitude $A(\mathbf{q}, \mathbf{I})$ is given by

$$A^2(\mathbf{q}, \mathbf{I}) = \left| \det \frac{\partial^2 S}{\partial q_i \partial I_j} \right| \quad (17)$$

The function $S(\mathbf{q}, \mathbf{I})$ is a multivalued function of $\mathbf{q}$ since there are several points $\mathbf{p}_k$ satisfying $\mathbf{I}(\mathbf{p}_k, \mathbf{q}) = \text{const.}$ for a fixed $\mathbf{q}$. Thus the wave function is given by a superposition, i.e.

$$\phi(\mathbf{q}) = \sum_k A_k(\mathbf{q}, \mathbf{I}) \exp(iS_k(\mathbf{q}, \mathbf{I})) \quad (18)$$

where

$$S_k(\mathbf{q}, \mathbf{I}) = S(\mathbf{q}, \mathbf{I}; \mathbf{p}_k) = \int_{C_k} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} \quad (19)$$

and the integration path C_k is connecting $O(\mathbf{p}_0, \mathbf{q}_0)$ with $P_k(\mathbf{p}_k, \mathbf{q})$ and $A_k^2(\mathbf{q}, \mathbf{I}) = \left| \det \frac{\partial^2 S_k}{\partial q_i \partial I_j} \right|$. The function $S_k(\mathbf{q}, \mathbf{I})$ is defined on the each piece of the phase space and it is continuous and finite. However, there is a phase loss $\pi/2$ at each transition from one piece to another [2].

The conserved quantities A_i defined by eq.(4) are functions of $(\mathbf{p}, \mathbf{q})$ and they have the properties $A_i(s\mathbf{p}, s\mathbf{q}) = A_i(\mathbf{p}, \mathbf{q})$, $A_i(-t\mathbf{p}, t\mathbf{q}) = A_i(\mathbf{p}, \mathbf{q})$. Therefore the action variables I_i also satisfy $I_i(s\mathbf{p}, s\mathbf{q}) = I_i(\mathbf{p}, \mathbf{q})$, $I_i(-t\mathbf{p}, t\mathbf{q}) = I_i(\mathbf{p}, \mathbf{q})$ and thus the shifted coordinates $(s\mathbf{p}, s\mathbf{q}), (-t\mathbf{p}, t\mathbf{q})$ stay on the invariant torus $M_{\mathbf{I}}$. When the shift operators s and t are applied to the wave function $\phi(\mathbf{q})$, transformations of the each component of eq.(18) are the same and it is sufficient to consider one of the components. Thus, a suffix k of eqs.(18,19) is dropped hereafter.

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Let us firstly apply the s operator to $\phi(\mathbf{q})$, i.e.

$$s\phi(\mathbf{q}) = A(s\mathbf{q}, \mathbf{I}) \exp(iS(s\mathbf{q}, \mathbf{I}; s\mathbf{p})) \quad (20)$$

and the phase is

$$\begin{aligned}
S(s\mathbf{q}, \mathbf{I}; s\mathbf{p}) &= \int_{(\mathbf{p}_0, \mathbf{q}_0)}^{(s\mathbf{p}, s\mathbf{q})} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} \\
&= \int_{(\mathbf{p}_0, \mathbf{q}_0)}^{(\mathbf{p}, \mathbf{q})} \mathbf{p} \cdot d\mathbf{x} + \int_{(\mathbf{p}, \mathbf{q})}^{(s\mathbf{p}, s\mathbf{q})} \mathbf{p} \cdot d\mathbf{x} + \oint \mathbf{p} \cdot d\mathbf{x} \\
&= S(\mathbf{q}, \mathbf{I}) + \int_{(\mathbf{p}, \mathbf{q})}^{(s\mathbf{p}, s\mathbf{q})} \mathbf{p} \cdot d\mathbf{x} + \sum_i l_i I_i
\end{aligned} \tag{21}$$

The term $\oint \mathbf{p} \cdot d\mathbf{x}$ is necessary since we do not specify an integration path and it becomes

$$\oint \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} = \oint \boldsymbol{\nu}(\boldsymbol{\mu}, \mathbf{I}) \cdot d\boldsymbol{\mu} = \sum_i l_i I_i \quad (l_i; \text{integer}) \tag{22}$$

since the phase space can be mapped to $(\boldsymbol{\nu}, \boldsymbol{\mu})$ -space by the canonical transformation. When the path $(\mathbf{p}, \mathbf{q}) \rightarrow (s\mathbf{p}, s\mathbf{q})$ is mapped to $(\boldsymbol{\nu}, \boldsymbol{\mu})$ -space, it winds the μ_i -interval several times. So we will define the minimum path in $(\boldsymbol{\nu}, \boldsymbol{\mu})$ -space such that the variation of the angle variable φ_i is $0 \leq \Delta\varphi_i < 2\pi$. If we denote the corresponding minimum path $(\mathbf{p}, \mathbf{q}) \rightarrow (s\mathbf{p}, s\mathbf{q})$ in $(\mathbf{p}, \mathbf{q})$ -space as C_1 , then

$$\int_{(\mathbf{p}, \mathbf{q})}^{(s\mathbf{p}, s\mathbf{q})} \mathbf{p} \cdot d\mathbf{x} = \int_{C_1} \mathbf{p} \cdot d\mathbf{x} + \sum_i l'_i I_i \quad (l'_i; \text{integer}) \tag{23}$$

Now the problem is how to calculate $\int_{C_1} \mathbf{p} \cdot d\mathbf{x}$. Let us define the path C_2 by applying the s operator $C_2 = sC_1$, i.e.

$$(\mathbf{p}, \mathbf{q}) \in C_1, \quad (\mathbf{p}', \mathbf{q}') \in C_2, \quad \mathbf{p}'(\mathbf{q}', \mathbf{I}) = \mathbf{p}'(s\mathbf{q}, \mathbf{I}) = s\mathbf{p}(\mathbf{q}, \mathbf{I}) \tag{24}$$

Then

$$\begin{aligned}
\int_{C_2} \mathbf{p}'(\mathbf{x}', \mathbf{I}) \cdot d\mathbf{x}' &= \int_{C_1} \mathbf{p}'(s\mathbf{x}, \mathbf{I}) \cdot d(s\mathbf{x}) \\
&= \int_{C_1} s\mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d(s\mathbf{x}) \\
&= \int_{C_1} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x}
\end{aligned} \tag{25}$$

since $T_{ss} = 1$ in a matrix representation. Therefore

$$S(s^2\mathbf{q}, \mathbf{I}) - S(s\mathbf{q}, \mathbf{I}) = S(s\mathbf{q}, \mathbf{I}) - S(\mathbf{q}, \mathbf{I}) \tag{26}$$

Making use of $\varphi_i = \frac{\partial S}{\partial I_i}$, one can get the variation of the angle φ_i as

$$\varphi_i(s^2\mathbf{q}, \mathbf{I}) - \varphi_i(s\mathbf{q}, \mathbf{I}) = \varphi_i(s\mathbf{q}, \mathbf{I}) - \varphi_i(\mathbf{q}, \mathbf{I}) \tag{27}$$

Thus

$$\Delta\varphi_i(C_2) = \Delta\varphi_i(C_1) \quad (28)$$

Since C_1 is a minimum path ($0 \leq \Delta\varphi_i(C_1) < 2\pi$), C_2 is also a minimum path ($0 \leq \Delta\varphi_i(C_2) < 2\pi$) connecting $(s\mathbf{p}, s\mathbf{q}) \rightarrow (s^2\mathbf{p}, s^2\mathbf{q})$. In the same way

$$\int_{C_1} \mathbf{p} \cdot d\mathbf{x} = \int_{sC_1} \mathbf{p} \cdot d\mathbf{x} = \int_{s^2C_1} \mathbf{p} \cdot d\mathbf{x} = \cdots = \int_{s^{N-1}C_1} \mathbf{p} \cdot d\mathbf{x} \quad (29)$$

$$\int_{C_1} \mathbf{p} \cdot d\mathbf{x} = \frac{1}{N} \int_{C_1 + sC_1 + \cdots + s^{N-1}C_1} \mathbf{p} \cdot d\mathbf{x} = \frac{1}{N} \oint_C \mathbf{p} \cdot d\mathbf{x} \quad (30)$$

where $C = C_1 + sC_1 + \cdots + s^{N-1}C_1$ and C is a closed path.

Let us evaluate $\oint_C \mathbf{p} \cdot d\mathbf{x}$ by mapping C to $(\boldsymbol{\nu}, \boldsymbol{\mu})$ -space, i.e.

$$\oint_C \mathbf{p} \cdot d\mathbf{x} = \oint \boldsymbol{\nu} \cdot d\boldsymbol{\mu} = \sum_{i=-m}^m k_i I_i \quad (31)$$

where k_i is a winding number of the variable μ_i in the i -th interval. In order to obtain k_i , we have carried out numerical calculation. Firstly, we take an arbitrary point $P_1 = (\mathbf{p}, \mathbf{q})$ is the phase space. This point P_1 determines the conserved quantities I_i and, thus, the invariant torus. The closed path consists of $P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_N \rightarrow P_{N+1} = P_1$ where $P_k = s^{k-1}P_1 = (s^{k-1}\mathbf{p}, s^{k-1}\mathbf{q})$. Secondly, the point P_k is mapped to $(\boldsymbol{\nu}, \boldsymbol{\mu})$ -space and a number of winding is counted for each μ_i . Since the path $P_k \rightarrow P_{k+1}$ is a minimum one, the winding number is uniquely determined. We have done the calculation by taking thousands of random points P_1 for $N=3,4,5$ and 6. The typical cases for $N=5$ and 6 are shown in Figs.1 and 2. In any case, the winding numbers which are obtained by the numerical calculation are $(1, 1)$, $(-1, 2, 1)$, $(1, -2, -2, 1)$ and $(-1, 2, 3, -2, 1)$ for $N=3,4,5$ and 6 respectively. (the increasing direction of the upper Riemann sheet is defined as a positive rotation and the sequence is defined as $\mu_m, \mu_{m-1}, \cdots, \mu_{-m+1}, \mu_{-m}$) When we consider a closed path C in the complex μ -plane, the positive branch of $\sqrt{\Delta(\mu_i)^2 - 4}$ is on the upper Riemann sheet. In order to get a correct sign of ν_i , we should take into account the phase $(-1)^{N-i}$ (eq.(12)) and k_i can be finally given by

$$k_i = \text{sign}(i)(m + 1 - |i|) \quad (-m \leq i \leq m) \quad (32)$$

and this is our main result. Although eq.(32) is obtained by the numerical calculation up to $N = 6$, we will conjecture that it will hold generally for the N -particle case and we hope it will be proved by more mathematical way. Using eq.(32), eq.(31) becomes

$$\oint \boldsymbol{\nu} \cdot d\boldsymbol{\mu} = \begin{cases} \sum_{i=1}^m (m+1-i)(I_i - I_{-i}), & \text{for } N = 2m+1 \\ \sum_{i=1}^m (m+1-i)(I_i - I_{-i}) + (m+1)I_0, & \text{for } N = 2m+2 \end{cases} \quad (33)$$

When the EBK quantization $I_i \rightarrow 2\pi(n_i + \frac{1}{2})$ is applied, the term $2\pi \times 1/2 = \pi$ coming from the Maslov index is canceled by the phase loss of each two turning points of the invariant torus. Finally one can get

$$\frac{1}{2\pi} \oint \boldsymbol{\nu} \cdot d\boldsymbol{\mu} = \begin{cases} \sum_{i=1}^m (m+1-i)(n_i - n_{-i}), & \text{for } N = 2m+1 \\ \sum_{i=1}^m (m+1-i)(n_i - n_{-i}) + (m+1)n_0, & \text{for } N = 2m+2 \end{cases} \quad (34)$$

Using the definition of N_S (eq.(13))

$$\int_{C_1} \mathbf{p} \cdot d\mathbf{x} = \frac{1}{N} \oint \boldsymbol{\nu} \cdot d\boldsymbol{\mu} = 2\pi \frac{N_S}{N} + 2\pi n_s \quad (n_s; \text{integer}) \quad (35)$$

Therefore

$$S(s\mathbf{q}, \mathbf{I}; s\mathbf{p}) = S(\mathbf{q}, \mathbf{I}; \mathbf{p}) + 2\pi \frac{N_S}{N} + 2\pi n_s + \sum_i (l_i + l'_i) I_i \quad (36)$$

The substitution $I_i = 2\pi(n_i + \frac{1}{2})$ and the subtraction of the phase loss $(l_i + l'_i)\pi$ give

$$S(s\mathbf{q}, \mathbf{I}; s\mathbf{p}) = S(\mathbf{q}, \mathbf{I}; \mathbf{p}) + 2\pi \frac{N_S}{N} + 2\pi n_s + \sum_i 2\pi(l_i + l'_i)n_i \quad (37)$$

and finally

$$\begin{aligned} \exp(iS(s\mathbf{q}, \mathbf{I}; s\mathbf{p})) &= \exp(iS(\mathbf{q}, \mathbf{I}; \mathbf{p})) \exp(2\pi i \frac{N_S}{N}) \\ &= \zeta^l \exp(iS(\mathbf{q}, \mathbf{I}; \mathbf{p})) \end{aligned} \quad (38)$$

when $N_S = l$. Eq.(38) exactly corresponds to eq.(7). Concerning the amplitude $A(\mathbf{q}, \mathbf{I})$, one can easily see that $A(\mathbf{q}, \mathbf{I}) = A(s\mathbf{q}, \mathbf{I})$ because of eq.(36).

Secondly we will consider the t operator. To be consistent with eq.(8), $t\phi(\mathbf{q}) = \phi(\mathbf{q})^*$, which determines the phase, the initial point $O(\mathbf{p}_0, \mathbf{q}_0)$ of the path C should be chosen such that $tO = (-t\mathbf{p}_0, t\mathbf{q}_0) = (\mathbf{p}_0, \mathbf{q}_0) = O$. Since the invariant torus has the property $\mathbf{I}(-t\mathbf{p}, t\mathbf{q}) = \mathbf{I}(\mathbf{p}, \mathbf{q})$, the transformation of $\phi(\mathbf{q})$ becomes

$$t\phi(\mathbf{q}) = A(t\mathbf{q}, \mathbf{I}) \exp(iS(t\mathbf{q}, \mathbf{I}; -t\mathbf{p})) \quad (39)$$

$$S(t\mathbf{q}, \mathbf{I}; -t\mathbf{p}) = \int_{(\mathbf{p}_0, \mathbf{q}_0)}^{(-t\mathbf{p}, t\mathbf{q})} \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} \quad (40)$$

Let us define a path C' on the invariant torus by applying the t operator $C' = tC$, i.e.

$$(\mathbf{p}, \mathbf{q}) \in C, \quad (\mathbf{p}', \mathbf{q}') \in C', \quad \mathbf{p}'(\mathbf{q}', \mathbf{I}) = \mathbf{p}'(t\mathbf{q}, \mathbf{I}) = -t\mathbf{p}(\mathbf{q}, \mathbf{I}) \quad (41)$$

Thus

$$\begin{aligned} S(t\mathbf{q}, \mathbf{I}; -t\mathbf{p}) &= \int_{C'} \mathbf{p}'(\mathbf{x}', \mathbf{I}) \cdot d\mathbf{x}' &= \int_C \mathbf{p}'(t\mathbf{x}, \mathbf{I}) \cdot d(t\mathbf{x}) \\ &= - \int_C t\mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d(t\mathbf{x}) &= - \int_C \mathbf{p}(\mathbf{x}, \mathbf{I}) \cdot d\mathbf{x} \\ &= -S(\mathbf{q}, \mathbf{I}; \mathbf{p}) \end{aligned} \quad (42)$$

since $Tt = 1$, and $A(t\mathbf{q}, \mathbf{I}) = A(\mathbf{q}, \mathbf{I})$. Finally one can get $t\phi(\mathbf{q}) = \phi(\mathbf{q})^*$ and show that $\phi(\mathbf{q})$ satisfies eq.(8).

In summary, we have considered the symmetry of the eigenfunction of the periodic Toda lattice in the semiclassical region by the EBK quantization and found that the integer quantum numbers have a close relationship to a structure of the invariant torus of the classical motion. However, the symmetric property is a discrete one and it does not depend on the excitation energy or the size of $\hbar$. Therefore it is concluded that this relationship is one of the distinctive features of the periodic Toda lattice. $\square$

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Table and Figure captions

Table 1 Eigenfunctions and the matrix elements of the conserved quantities A_i for each symmetry in the non-degenerate (A) / degenerate (B) case.

Fig.1 Behavior of auxiliary spectra μ_i for $N = 5$. The point P_1 is $\mathbf{p}=(-0.6, -0.4, 0.2, 0.3, 0.5)$ and $\mathbf{q}=(-0.6, -0.4, 0.0, 0.2, 0.8)$. Dots ($\bullet$) and triangles ($\triangle$) show the points of the upper and lower Riemann sheets respectively. Solid (dashed) lines show the movements of μ_i in the upper (lower) Riemann sheet.

Fig.2 Behavior of auxiliary spectra μ_i for $N = 6$. The point P_1 is $\mathbf{p}=(-0.5, -0.4, -0.1, 0.2, 0.3, 0.5)$ and $\mathbf{q}=(-0.6, -0.4, 0.0, 0.1, 0.3, 0.6)$. Descriptions are the same as Fig.1.

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Table 1A

sym.	w.f.	A_{odd}	A_{even}
A	ϕ^{A_1}	0	$\langle \phi^{A_1} A_i \phi^{A_1} \rangle$
B	ϕ^{B_1}	0	$\langle \phi^{B_1} A_i \phi^{B_1} \rangle$

Table 1B

sym.	w.f.	A_{odd}	A_{even}
A	$\frac{1}{\sqrt{2}}(\phi^{A_1} \pm i\phi^{A_2})$	$\pm \text{Im} \langle \phi^{A_2} A_i \phi^{A_1} \rangle$	$\frac{1}{2}(\langle \phi^{A_1} A_i \phi^{A_1} \rangle + \langle \phi^{A_2} A_i \phi^{A_2} \rangle)$
B	$\frac{1}{\sqrt{2}}(\phi^{B_1} \pm i\phi^{B_2})$	$\pm \text{Im} \langle \phi^{B_2} A_i \phi^{B_1} \rangle$	$\frac{1}{2}(\langle \phi^{B_1} A_i \phi^{B_1} \rangle + \langle \phi^{B_2} A_i \phi^{B_2} \rangle)$
E	$\phi^E, \quad \phi^{E*}$	$\pm \langle \phi^E A_i \phi^E \rangle$	$\langle \phi^E A_i \phi^E \rangle$

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