Resistor-network approach to growth probability for dielectric-breakdown models at a surface

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For the dielectric-breakdown model, a general method is presented for calculation of the growth probability of the growing-perimeter sites of the aggregate at a surface. The growth probability is given by solving the electrostatic problem for a superconducting aggregate in the semi-infinite normal-resistor network. The electric field at the perimeter site is obtained by solving the resistor-network problem with the lattice bond-bond Green's function. The fractal dimension D is found from the scaling assumption for the cluster-top occupancy probability.

Recently, considerable attention has been focused on the problem of structure in aggregation and growth models.¹⁻⁴ There are a number of different types of growth processes ranging from diffusion-limited aggregation⁵ (DLA) to the Eden model.⁶ The structure of the aggregate strongly depends on the dynamics of the growth process. It is well known that the structures have a strong measure of self-similarity, which is characterized by a fractal dimension D .⁷ Several analytical attempts, including mean-field theories^{8,9} and position-space renormalization-group methods,¹⁰ have been made to derive the fractal dimension. Amitrano, Coniglio, and diLiberto¹¹ and Halsey, Meakin, and Porcaccia¹² assert that a possible way to characterize a growth model fully is to determine at each time step the growth-site probability distribution $\{p_i\}$, $i \in \Gamma$, where p_i is the probability that site i becomes part of the aggregate. Turkevich and Sher¹³ and Coniglio¹⁴ show an analytical method for calculating the growth-site probability distribution, using a continuum version of the electrostatic analogy.

In this Rapid Communication, we present a general method for calculating the growth probability on the discrete-lattice model. The growth probability is given by solving the electrostatic problem for a superconductor-resistor network. We find the fractal dimension from the scaling assumption for the cluster-top occupancy probability.

We start with the dielectric-breakdown model.¹⁵ Growth occurs on the perimeter Γ of the aggregate. In this model the growth probability p_i at the growing-perimeter site i is given by $p_i \sim (E_i)^\eta$ where E_i is the local electric field at the perimeter site i obtained by solving the discrete Laplace equation with boundary conditions $\phi=0$ for the lower boundary and 1 for an upper boundary at large enough distance, where the aggregates, growing at a lower boundary, consist of superconducting bonds. When the parameter η equals 1, one recovers the DLA model. The growth probability p_i is just proportional to the flux ∇u of random walkers at perimeter site i . Thus, in order to solve for the probability p_i , we merely solve an electrostatic problem $\nabla^2 \phi = 0$ for a superconducting cluster ($\phi=0$) inside an semi-infinite normal-resistor network.

The model is an infinite two-dimensional square net of

lattice points where aggregates are constructed by superconducting bonds within the normal-resistor network (Fig. 1). We call the superconductor-resistor network the imperfect lattice shown by Fig. 1, and the normal-resistor network when no aggregates exist the perfect lattice. We consider the imperfect lattice in terms of perturbations from the perfect reference lattice, which is defined as a network of conductances g_0 across each of which is a field E_0 . The starting point is a Green's-function formalism.^{16,17} The electric field E_α of a bond α can be expressed as

$$E_\alpha = E_0 \delta_{y,\alpha} + \sum_{\beta} \Gamma_{\alpha\beta} \Delta_{\beta} E_{\beta}, \quad (1)$$

where g_α is the conductance of a bond α , Greek subscripts label the bonds, Δ_{β} is equal to $(g_{\beta} - g_0)/g_0$, and $\delta_{y,\alpha}$ is 1 if the α bond is parallel to the y direction and zero otherwise. The conductance g_α of a bond α has a large value or a finite value g_0 depending on whether it is a superconductor or a normal resistor, respectively. $\Gamma_{\alpha\beta}$ is a bond-bond Green's function that depends on the boundary condition, the separation of bonds α and β , and their relative orientation.

The evaluation of $\Gamma_{\alpha\beta}$ has been presented by Blackman.¹⁶ Let us now consider how $\Gamma_{\alpha\beta}$ must be modified if either α or β is a bond at the surface. We must distinguish

FIG. 1. Superconductor-resistor network. The bold lines indicate the superconducting bonds, consisting of the aggregates. The light lines are the normal resistors.

FIG. 2. Bonds α and β at interior and boundary positions.

between two types of bonds as shown in Fig. 2—interior bonds denoted by I , and bonds adjacent to the lower boundary denoted by L . Only if the α bond is a bond at the lower boundary is the lattice bond-bond Green's function modified:

$$\Gamma_{\alpha\beta}^{LI} = G_{j,k} - G_{j,l}, \quad (2)$$

where $G_{i,j}$ indicates the lattice Green's function between site i and site j .¹⁸

The local electric field $E_{(pg)}$ between a perimeter site p

$$\begin{pmatrix} E_a/E_0 \\ E_b/E_0 \\ E_c/E_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.0465 & -0.0451 & 0.1366 & 0.0451 \\ 0.0451 & -0.1366 & 0.1817 & 0.0 \\ 0.0451 & 0.1366 & -0.0451 & 0.0465 \end{pmatrix} \begin{pmatrix} -0.2500 & 0.2500 & 0.2500 & -0.2500 \\ 0.1817 & -0.5000 & -0.1817 & 0.1366 \\ 0.1366 & -0.1817 & -0.5000 & 0.1817 \\ -0.1817 & 0.1366 & 0.1817 & -0.5000 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (4)$$

We then obtain $E_a/E_0=1.852$, $E_b/E_0=0.872$, and $E_c/E_0=0.660$. If the growth probability is defined by $p_i = E_i/[E_a + 2E_c + 2(E_c/2)]$, the p_i 's are, respectively, $p_a=0.435$, $p_b=0.205$, and $p_c=0.078$, where for the growth probability at site c we take half of the value because a particle sticking at site c is a part of the aggregate or a distinct aggregate.

We now show that the cluster-top occupancy probability p_a determines the fractal dimension D , following Turkevich's method.¹³ Given an (h, M) cluster, which consists of M particles and has height $h(M)$, if the $(M+1)$ st random walker lands at the top portion of the cluster, the cluster grows in height: $h(M+1) = h(M) + 1$. If the $(M+1)$ st random walker lands deeper within the cluster, the cluster does not grow in height: $h(M+1)$

FIG. 3. Example of an aggregate made of four bonds. The growth sites are represented by open circles and triangles. For any given growth site there is a probability p_i of becoming part of the aggregate. The sites, indicated by triangles, may become part of the aggregate or a distinct aggregate.

and its nearest-neighbor growth site g is then given by

$$E_{(pq)}/E_0 = \delta_{y,(pq)} - \sum_{\beta} \Gamma_{(pq)\beta} \left(\sum_{\gamma} (\Gamma^{-1})_{\beta\gamma} \delta_{\gamma,y} \right), \quad (3)$$

where the summations over β and γ range over all bonds within the aggregate, and $\delta_{y,(pq)}$ is 1 if the (pq) bond is parallel to the y direction and zero otherwise.

As an example, we show the calculation for the aggregate in Fig. 3. By evaluating the lattice bond-bond Green's function, the electric fields between a perimeter site and its nearest-neighbor growth site are given by

$= h(M)$. Combining these relations

$$h(M+1) = p_a(M)[h(M)+1] + [1-p_a(M)]h(M),$$

we obtain

$$\Delta M = (1/p_a)\Delta h. \quad (5)$$

If one assumes the scaling $(1/p_a) \sim h^{D-1}$ for the cluster-top occupancy probability, the fractal dimension is given by

$$D = 1 - \ln(p_{a,2}/p_{a,1})/\ln(h_2/h_1). \quad (6)$$

We calculate the cluster-top occupancy probabilities for the three clusters shown by Fig. 4. The result is shown in Table I for the DLA ($\eta=1$) and the Eden ($\eta=0$) models. For the DLA and Eden models, we obtain fractal dimensions $D=1.645$ ($\eta=1, h_2/h_1=2$), 1.712 ($\eta=1, h_3/h_1=3$),

TABLE I. Occupancy probabilities at the cluster top for DLA and Eden models. The probabilities are presented for the aggregates shown by Figs. 4(a)–4(c).

h			
η	1 [Fig. 4(a)]	2 [Fig. 4(b)]	3 [Fig. 4(c)]
1	0.680	0.435	0.311
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$

FIG. 4. Clusters for calculating the cluster-top occupancy probability. (a) The height $h=1$; (b) $h=2$; (c) $h=3$.

FIG. 5. Dependence of the fractal dimension D on the parameter η of the dielectric-breakdown model.

and $D = 2(\eta = 0, h_2/h_1 = 2)$, $2(\eta = 0, h_3/h_1 = 3)$. These results are in agreement with the simulations.¹⁹ For the dielectric-breakdown model with any η , we find the reasonable result shown by Fig. 5, which takes $D = 1$ in the limit of $\eta = \infty$.

In summary, we present a superconductor-resistor network formulation for the dielectric-breakdown model at a surface on the square lattice. The solution for the growth probability at a perimeter is found with the use of the lattice bond-bond Green's function. The occupancy probability at the top of the cluster is obtained. From the scaling assumption for the cluster-top occupancy probability the fractal dimensions are found for the dielectric-breakdown model with any η . We find $D = 1.712$ for DLA on a 2D square lattice, $D = 2$ for the Eden model, and $D = 1$ for $\eta = \infty$.

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