Multifractal structures of mass and growth probability distributions in diffusion-limited aggregation on hierarchical lattices

Takashi Nagatani

College of Engineering, Shizuoka University, Hamamatsu 432, Japan

(Received 22 February 1988)

The scaling structures of the mass and growth probability distributions in the generalized diffusion-limited aggregation (η model) on the hierarchical lattices are derived by making use of the real-space renormalization-group method. In the dielectric breakdown language, the process is formulated as a random multiplicative process of current on the surface of the aggregate. The mass and growth probability distributions are related to the bulk and surface structures of the aggregate. The mass scaling of the clusters generated by the processes is found and compared with the scaling structure of the growth probability distribution. The dependences of the α - f spectra of the mass and growth probability distributions are shown on the parameter η describing the different dielectric breakdown models.

I. INTRODUCTION

Recently, there has been increasing interest in the physical mechanisms governing the geometrical structure in diffusion-limited aggregation (DLA).¹⁻⁷ It is well known that an aggregate cannot be fully characterized only by its fractal dimension.⁸ Meakin *et al.*,⁹ Halsey *et al.*¹⁰ and Amitrano *et al.*¹¹ have recently shown that the surface of DLA requires an infinite hierarchy of fractal dimensions for its characterization. They have presented the multifractal structure of the growth probability distribution from the numerical experiments. Nagatani¹²⁻¹⁵ has presented a real-space renormalization-group method for calculating the multifractality of the growth probability distribution. The multifractality has been manifested through nongeometrical properties. Very recently, Tl and Vicsek¹⁶ showed a geometrical multifractality of the mass distribution in the deterministic fractal models. They suggested that large fluctuations of the fractal dimension with increasing size might be a sign of geometrical multifractality. Meakin and Havlin¹⁷ performed the computer simulation for some aggregation models and suggested that an infinite family of exponents was necessary to describe the scaling structure of the mass distribution in DLA.

In this paper we study the scaling structure of the mass distribution in relation to the growth probability distribution on hierarchical lattices. We describe the DLA process in a discrete electrostatic language. The DLA process is described as the dielectric breakdown model which is a fragmentation phenomenon¹⁸ of current on the surface of the aggregate. The fragmentation process on the hierarchical lattice is exactly expressed in terms of a random multiplicative process of current on the surface. The evolution criterion is given by the minimum principle of the dissipation energy in the purely dissipative system.¹⁹ The criterion leads to the renormalization-group transformation equation of the surface conductance and the existence of a stable fixed point of its equation. The

scaling structure of the growth probability distribution is derived from the multiplicative process of current and the renormalization equation of the surface conductance. The scaling property of the mass distribution is derived from counting the distinct configurations of spanning clusters. The relation between the mass distribution and the growth probability distribution is shown in terms of the α - f spectra. The organization of the paper is as follows. In Sec. II we present the model which describes the generalized diffusion-limited aggregation on the hierarchical lattices. In Sec. III the fragmentation process is expressed by the random multiplicative process of current on the surface of the aggregate. In Sec. IV the evolution criterion of DLA is presented. In Sec. V the renormalization procedure is presented. In Sec. VI both multifractalities of mass and growth probability distributions are calculated. Section VII presents the Conclusion.

II. MODEL

We consider the diffusion-limited aggregations on the two hierarchical lattices: diamond and Wheatstone-bridge lattices. We describe DLA in the dielectric break-

FIG. 1. The second generations of the hierarchical lattices: (a) diamond lattice and (b) Wheatstone-bridge lattice.

FIG. 2. Illustration of the dielectric breakdown model. A constant voltage is applied between the bottom and the top. The breakdown proceeds one by one from the bottom to the top according to the growth probability proportional to the current on the perimeter bond. The thick, wavy, and thin lines indicate, respectively, break, perimeter, and unbroken bonds.

down language. The two hierarchical lattices are shown in Figs. 1(a) and 1(b). Figures 1(a) and 1(b) show, respectively, the second generations of the diamond and Wheatstone-bridge lattices. A constant voltage is applied between the bottom and top at the N th generation where N is a sufficiently large number. The dielectric breakdown proceeds from the bottom to the top. Figure 2 shows the illustration of the breakdown model. The thick lines indicate break bonds: superconducting bonds which construct the aggregate. The bonds on the perimeter of the aggregate are represented by the wavy lines. So the electrostatic problem is solved under the boundary condition in Fig. 2. A growth probability proportional to the current is then assigned to the perimeter bond. The interface proceeds to the top according to the growth probability. In the generalized DLA (η model), the growth probability P_i at the growing-perimeter bond i is given by

$$P_i \sim (I_i)^\eta, \quad (1)$$

FIG. 3. The fragmentation process of current on a part of the surface of the aggregate before and after renormalization.

FIG. 4. The current fraction within the cell in the fragmentation process shown by Fig. 3.

where I_i is the local current at the growth bond i . When $\eta = 1$ this gives DLA.

III. MULTIPLICATIVE PROCESS OF CURRENT

The dielectric breakdown process is a fragmentation phenomenon of current on the surface of the aggregate. The fragmentation process on the hierarchical lattice will be exactly expressed in terms of a random multiplicative process of current fraction. We consider the fragmentation of current on a part of the surface of the aggregate which is shown in Fig. 3(a). It is renormalized to that shown by Fig. 3(b) by making use of the renormalization procedure explained in Sec. V. The following relationship between the currents I_n and I_{n-1} before and after renormalization holds:

$$\begin{aligned} I_{n,1} &= i_{1,a} I_{n-1,1}, \\ I_{n,2} &= i_{a,2} I_{n-1,1}, \\ I_{n,3} &= i_{b,1} I_{n-1,2}, \\ I_{n,4} &= i_{b,2} I_{n-1,2}, \end{aligned} \quad (2)$$

with $i_{a,1} + i_{a,2} = 1$ and $i_{b,1} + i_{b,2} = 1$, where i indicates the current fraction within the cell shown by Fig. 4, and the labeling is shown in Figs. 3 and 4. It is found that the current fragmentation on the hierarchical lattice can be exactly described by the multiplicative process of the current fraction within the cell.

IV. EVOLUTION CRITERION AND FIXED POINT

In contrast to the theorem of minimum free energy in an equilibrium state, the nonequilibrium steady states are characterized by an extremum principle according to which the dissipation energy has its minimum value for a purely dissipative system without convection.¹⁹ The dynamics of the formation of DLA can be assumed to be determined by a "quasistationary" diffusion field.²⁰ The process is a purely dissipative system. The process is characterized by the theorem of minimum dissipation energy.

We consider the dissipation energy in place of the free

energy (or Hamiltonian). In the lattice, the dissipation energy is given by

$$H_d = \sum_i \sigma_i E_i^2 = \sigma_t E_0^2, \quad (3)$$

where the first represents the summation of dissipation energy on each bond over total bonds within the system, and the second indicates the total dissipation energy in the representation of the total conductance σ_t and the total voltage drop E_0 . The total voltage drop E_0 is given by the applied voltage. We note that energy is dissipated over all the bonds in the exterior of the aggregate in DLA. Thus the dissipation energy form (3) describes the interaction between neighboring potentials by this simple gradient term E_i . The total current, carrying through the surface layer of the aggregate, is given by $J_t = \sum_{i_s} \sigma_{i_s} E_{i_s}$ where the sum indicates the summation over all the bonds on the surface layer. We can express the dissipation energy in terms of the surface conductance,

$$H_d = E_0 \left[\sum_{i_s} \sigma_{i_s} E_{i_s} \right], \quad (4)$$

where σ_{i_s} indicates the conductance of the bonds on the perimeter of the aggregate. By making use of a decimation method (see Sec. V), one can write the dissipation energy in terms of the coarse-grained variables σ'_{i_s}, E'_{i_s} on the new lattice with the lattice spacing b (b is the scale factor),

$$H_d = E_0 \left[\sum_{i_s} \sigma'_{i_s} E'_{i_s} \right]. \quad (5)$$

Thus the short-range interaction between neighboring potentials by the gradient term is renormalized. The surface conductance σ is transformed to σ' . This surface conductance plays the role of the so-called coupling constant in the Hamiltonian of phase transition. We can obtain the renormalization-group equation

$$\sigma' = R(\sigma). \quad (6)$$

After many repeated renormalizations, the dissipation energy is expressed by

$$H_d = \sigma_t E_0^2 \text{ with } \sigma_t = \sigma_N(\sigma_{N-1}). \quad (7)$$

Under the boundary condition of the constant voltage ($E_0 = \text{const}$) we consider a perturbation of the dissipation energy. The variation of the dissipation energy is given by

$$\begin{aligned} \delta H_d &= \delta \sigma_t E_0^2 \\ &= (\partial \sigma_N / \partial \sigma_{N-1}) \\ &\quad \times (\partial \sigma_{N-1} / \partial \sigma_{N-2}) \cdots (\partial \sigma_1 / \partial \sigma_0) \delta \sigma_0 E_0^2. \end{aligned} \quad (8)$$

If the following condition is satisfied, the dissipation energy approaches to a minimum value:

$$\sigma^* = R(\sigma^*), \quad 0 < |\partial \sigma' / \partial \sigma|_{\sigma=\sigma^*} < 1. \quad (9)$$

The theorem of minimum dissipation energy is consistent with the condition that the renormalization-group (RG) equation (6) has a stable fixed point. The aggregate grows in such a way that the total conductance of the system approaches to a minimum value. We find that the evolution criterion of DLA is given by the condition that the RG equation of the surface conductance has a stable fixed point.

V. RENORMALIZATION

We consider the renormalization procedure for deriving the renormalization-group equation for the surface conductance: $\sigma' = R(\sigma)$. We distinguish between three types of bonds on the lattice before and after a renormalization procedure: (a) interior bonds which are occupied by superconducting bonds and construct the aggregate, (b) growth bonds which are on the surface of the aggregate and can be successively grown, and (c) exterior bonds which construct the electrostatic field and are in the exterior of the aggregate except for the growth bonds. The interior, growth, and exterior bonds are respectively indicated by the thick, wavy, and thin lines in figures. We partition all the space of the hierarchical lattice into cells of size $b=2$ (b is the scale factor), each containing a single generator. After a renormalization transformation these cells play the role of “renormalized” bonds. The n th generation of the hierarchical lattice is transformed to the $(n-1)$ th generation. The renormalized bonds are then classified into the three types of bonds, similarly to bonds before the renormalization. The renormalization procedure is indicated by Fig. 5. If the cell is spanned with the bonds occupied by the interior bond, the renormalized bond is then considered to be interior [Fig. 5(a)]. If the cell is not spanned with the interior bond and is

FIG. 5. Illustration of the renormalization for the diamond lattice. The cells in the upper side are renormalized to the bonds in the bottom side. There are three types of bonds: interior bonds indicated by thick lines, growth bonds by wavy lines, and exterior bonds by thin lines. Examples of the distinct configurations are shown in (a), (b), and (c), which are respectively renormalized as interior, growth, and exterior bonds.

FIG. 6. All distinct configurations of the cell that it is possible to renormalize as the growth bond for the diamond lattice. The configuration (1) is constructed by adding a break bond onto the growth bonds 1 or 2 in the configuration (0). In addition, by adding a break bond onto the growth bond 2 in the configuration (1), the configuration (2) is constructed.

nearest neighbor to the aggregate, then the cell is renormalized as the growth bond on the surface [Fig. 5(b)]. When the cell is constructed by the exterior bond only and is not nearest neighbor with the aggregate cluster, the cell is renormalized as the exterior bond [Fig. 5(c)]. The conductance of the cell to be renormalized as the exterior bond does not change before and after the renormalization because all the bonds within the cell are occupied by exterior bonds with the unit conductance (where we choose the conductance of bond before renormalization as the unit). We consider the conductance of the cell that it is possible to renormalize as the growth bond. We define the conductance of the growth bond as the surface conductance. After many repeated renormalizations, the surface conductance must approach to a finite value, not equal to zero according to the evolution criterion of DLA. We note that the nonlocal nature of the electrostatic field is taken into account as the surface conductance. We assume that the DLA process occurs stepwise: the breakdown proceeds one by one, and more bonds than one never break down simultaneously. We first consider the diamond lattice and secondly the Wheatstone-bridge lattice.

For the diamond lattice Fig. 6 shows all the configurations of the cell for which it is possible to renormalize as the growth bond. Let us consider the configurational probability C_α with which a particular configuration α appears. The distinct configurations are labeled by α ($\alpha=0,1,2$) in Fig. 6. The configuration (1) is constructed by adding a break bond to the configuration (0). The probability with which a break bond adds onto the growth bonds 1 or 2 in configuration (0) is given by the growth probabilities $p_{0,1}$ or $p_{0,2}$ of the growth bonds 1 or 2 in the configuration (0). In addition, by adding a break bond to the configuration (1), the configuration (2) occurs. We here assume that the breakdown proceeds one by one. The configurational probabilities C_α are given by

$$\begin{aligned} C_1 &= C_0(p_{0,1} + p_{0,2}), \\ C_2 &= C_1 p_{1,2}, \end{aligned} \quad (10)$$

where $p_{0,1} = p_{0,2} = \frac{1}{2}$. The configurational probability C_0

FIG. 7. All distinct configurations for the renormalized growth bond in the Wheatstone-bridge lattice.

is determined from the normalization condition

$$\sum_{\alpha} C_{\alpha} = C_0 + C_1 + C_2 = 1. \quad (11)$$

The growth probability $p_{\alpha,i}$ on the growth bond i within the cell α is proportional to the electric field E_i (or current) on the growth bond. Consider the electrostatic problem for cells which can be renormalized as the growth bond. The electric fields on the growth bonds within a cell are determined by the conductance of the growth bond and the configuration of the cell. In the configuration labeled by α (see Fig. 6), the growth probabilities $p_{\alpha,i}$ of growth bonds i are given by

$$\begin{aligned} p_{0,1} &= p_{0,2} = \frac{1}{2}, \\ p_{1,1} &= (1 + \sigma) / (2 + \sigma), \quad p_{1,2} = 1 / (2 + \sigma), \\ p_{2,1} &= p_{2,2} = \frac{1}{2}, \end{aligned} \quad (12)$$

where σ indicates the conductance of the growth bond, and the conductance of the break and unbroken bonds are respectively given by infinite and unit values. The conductance σ'_α of the cell with the configuration α is renormalized as follows:

$$\begin{aligned} \sigma'_0 &= 2\sigma / (1 + \sigma), \\ \sigma'_1 &= \sigma + \sigma / (1 + \sigma), \\ \sigma'_2 &= 2\sigma. \end{aligned} \quad (13)$$

The renormalized conductance σ' of the growth bond will be assumed to be given by the most probable value

$$\sigma' = \exp(C_0 \ln \sigma'_0 + C_1 \ln \sigma'_1 + C_2 \ln \sigma'_2). \quad (14)$$

The relationships (13) and (14) present the renormalization-group equation $\sigma' = R(\sigma)$. Equations (10)–(14) are simultaneously solved. We find a stable fixed point $\sigma^* = 2.123$ —from $\sigma^* = R(\sigma^*)$.

Similarly we derive the renormalization-group equation for the Wheatstone-bridge lattice. Figure 7 shows all the configurations of the cell for which it is possible to renormalize as the growth bond. The configurational probabilities C_α with which a particular configuration α appears are given by

$$\begin{aligned} C_1 &= C_0(p_{0,1} + p_{0,2}), \\ C_2 &= C_1(p_{1,2} + p_{1,3}), \end{aligned} \quad (15)$$

where $C_0 + C_1 + C_2 = 1$. The growth probability $p_{\alpha,i}$ on the growth bond i within the cell α is given by

$$\begin{aligned} p_{0,1} &= p_{0,2} = \frac{1}{2}, \\ p_{1,1} &= (1 + 2\sigma)/(3 + 2\sigma), \\ p_{1,2} &= p_{1,3} = 1/(3 + 2\sigma), \\ p_{2,1} &= p_{2,2} = \frac{1}{2}. \end{aligned} \quad (16)$$

The conductance σ'_α of the cell with the configuration α is renormalized as follows:

$$\begin{aligned} \sigma'_0 &= 2\sigma/(1 + \sigma), \\ \sigma'_1 &= \sigma + 2\sigma/(1 + 2\sigma), \\ \sigma'_2 &= 2\sigma. \end{aligned} \quad (17)$$

The renormalized conductance σ' of the growth bond is given by

$$\sigma' = \exp(C_0 \ln \sigma'_0 + C_1 \ln \sigma'_1 + C_2 \ln \sigma'_2). \quad (18)$$

The relationships (15)–(18) present the RG equation for the Wheatstone-bridge lattice. We find a stable fixed point $\sigma^* = 2.270$ —for the Wheatstone-bridge lattice.

We consider the η model introduced by Niemeyer *et al.*²¹ The growth probability P_i at the growing-perimeter bond i is given by $P_i \sim (I_i)^\eta$ where I_i is the electric current on the perimeter bond. For the diamond lattice, the growth probability $p_{\alpha,i}$ on the growth bond i , within the cell α is given in place of Eq. (12),

$$\begin{aligned} p_{0,1} &= p_{0,2} = \frac{1}{2}, \\ p_{1,1} &= 1/[1 + (1 + \sigma)^{-\eta}], \quad p_{1,2} = 1 - p_{1,1}, \\ p_{2,1} &= p_{2,2} = \frac{1}{2}. \end{aligned} \quad (12')$$

For the Wheatstone-bridge lattice, we obtain in place of Eq. (16),

$$\begin{aligned} p_{0,1} &= p_{0,2} = \frac{1}{2}, \\ p_{1,1} &= 1/[1 + 2(1 + 2\sigma)^{-\eta}], \\ p_{1,2} &= p_{1,3} = (1 - p_{1,1})/2, \\ p_{2,1} &= p_{2,2} = \frac{1}{2}. \end{aligned} \quad (16')$$

Thus we can obtain the renormalization-group equation of the surface conductance for the η model, only by replacing the growth probability of DLA.

VI. MULTIFRACTALITIES

We derive the multifractal structure of the mass distribution in DLA. The mass distribution is derived from the growth probability distribution. We first consider the multifractal structure of the growth probability distribution.

A. Multifractality of growth probability distribution

We consider the growth probability on any growth bond. We define the following two growth probabilities: (a) the growth probability $P_i(L)$ over the total perimeter bonds of the aggregate with the size L and (b) the growth probability $p_{\alpha,i}$ on the growth bond i within the cell α . The growth probability $P_i(L)$ on the growth bond i is proportional to the current I_i on the growth bond i . The cell's growth probability $p_{\alpha,i}$ on the growth bond i within the cell α is also proportional to the current i_i on the growth bond i within the cell α . The relation (2) holds between the currents before and after renormalization. Therefore, before and after renormalization, the growth probability $P_i(L)$ on any growth bond i is given by

$$P_i(L) = p_{\alpha,i} P_\alpha(L/b), \quad (19)$$

where L represents the size of the system, and b is the scale factor ($b = 2$). The cell's growth probability $p_{\alpha,i}$ depends on the configuration of the cell (see Figs. 6 and 7). Each cell's growth probability $p_{\alpha,i}$ is represented by a function of the surface conductance. After many repeated renormalizations, the surface conductance σ approaches to the value σ^* at the fixed point. The cell's growth probability $p_{\alpha,i}$ approaches to that evaluated at the fixed point $p_{\alpha,i}^*$. The total growth probability $P_i(L)$ assigned to each growth bond is then given by a random multiplicative process of the cell's growth probabilities evaluated at the fixed point. In the limit of L sufficiently large, the generalized dimensions $D(q)$ are defined as

$$\sum_i P_i^q \sim L^{-(q-1)D(q)}. \quad (20)$$

An infinite hierarchy of the generalized dimensions $D(q)$ is given by

$$D(q) = -(q-1)^{-1} \ln \left\langle \sum_i p_{\alpha,i}^{*q} \right\rangle / \ln b, \quad (21)$$

FIG. 8. The generalized dimension $D(q)$ of the growth probability distributions in the diamond lattice. The curves indicate, respectively, the results for $\eta = 0, 0.5, 1, 2$, and 3 .

FIG. 9. The generalized dimension $D(q)$ of the growth probability distributions in the Wheatstone-bridge lattice. The curves indicate, respectively, the results for $\eta=0, 0.5, 1, 2$, and 3 .

where the angle brackets indicate the configurational average and $p_{\alpha,i}^*$ is given by Eqs. (12') and (16') with $\sigma=\sigma^*$ for the diamond and Wheatstone-bridge lattices. The fractal dimension of the aggregate d_f is given by $D(\infty)+1$. The surface fractal dimension d_s is also given by $D(0)$. The partition of $D(q)$ into a density of singularities $f(q)$ with singularity strength $\alpha(q)$ is introduced,

$$\begin{aligned}\alpha(q) &= d/dq[(q-1)D(q)], \\ f(q) &= q\alpha(q) - (q-1)D(q).\end{aligned}\quad (22)$$

Following the method of Turkevich and Scher,²² we obtain the fractal dimensions $d_f = D(\infty) + 1 = 1.400$ (for the diamond lattice), 1.444 (for the Wheatstone-bridge lattice) in DLA.

The values of $D(q)$ for the η model are plotted in Figs. 8 and 9 for the diamond and Wheatstone-bridge lattices.

FIG. 10. The α - f spectra of the growth probability distributions obtained from $D(q)$ in Fig. 8 for the diamond lattice.

FIG. 11. The α - f spectra of the growth probability distributions obtained from $D(q)$ in Fig. 9 for the Wheatstone-bridge lattice.

The α - f spectra are displayed in Figs. 10 and 11. As the parameter η increases, the α - f spectrum becomes a more smooth convex shape.

B. Multifractality of mass distribution

We consider the mass distribution. The following relation holds between the masses M_n and M_{n-1} after and before renormalization:

$$M_n = (1/m)M_{n-1}, \quad (23)$$

where m indicates the mass within the cell with the spanning cluster. Figure 12 shows the configurations of the spanning cluster for the diamond lattice. The configuration (a) of the spanning cluster is constructed by breaking down the growth bond 1 in the configuration (1)

FIG. 12. All distinct configurations of the spanning cluster, which are renormalized as the interior bond, shown on the bottom side for the diamond lattice. The configurations (a) and (b) are respectively constructed from the configurations of the growth bonds shown on the top side.

FIG. 13. All distinct configurations of the spanning cluster for the Wheatstone-bridge lattice. The configurations (a) and (b) are respectively constructed from the configurations of the growth bonds shown on the top side.

of the growth bond (see Figs. 6 and 12). The configuration (b) of the spanning cluster is constructed by breaking down either the growth bond 1 or 2 in the configuration (2) of the growth bond. The configurational probability, with which a particular configuration appears, is given by

$$C_{m,a} = Cp_{1,1}C_1, \quad C_{m,b} = C(p_{2,1} + p_{2,2})C_2, \quad (24)$$

where the unknown constant C is determined from the normalization condition $C_{m,a} + C_{m,b} = 1$, and the cell's growth probability $p_{\alpha,i}$ and the configurational probability of the growth bond C_α are given by Eqs. (10) and (12). Following the definition by Tél and Vicsek,¹⁶ the generalized dimensions $D_m(q)$ of the mass are defined as

$$\sum_i (M_i/M)^q \sim (1/L)^{(q-1)D_m(q)}, \quad (25)$$

FIG. 14. The generalized dimensions $D_m(q)$ of the mass distributions in the diamond lattice. The curves indicate, respectively, the results for $\eta=0, 0.5, 1, 2$, and 3 .

FIG. 15. The generalized dimensions $D_m(q)$ of the mass distributions in the Wheatstone-bridge lattice. The curves indicate, respectively, the results for $\eta=0, 0.5, 1, 2$, and 3 .

where M_i is the mass of the i th nonempty box when the aggregate with the mass M is covered by boxes of size l such that $a \ll l \ll L$ (a is the lattice constant). After and before renormalization, we obtain from (23) and (25)

$$\begin{aligned} \left\langle \sum_i M_{i,n}^q \right\rangle / \left\langle \sum_i M_{i,n-1}^q \right\rangle &= \left\langle \sum_i (1/m)^q \right\rangle \\ &= C_{m,a} 2(\frac{1}{2})^q + C_{m,b} 3(\frac{1}{3})^q. \end{aligned} \quad (26)$$

The generalized dimensions of mass are given by

$$D_m(q) = -(q-1)^{-1} \ln \left\langle \sum_i (1/m)^q \right\rangle / \ln b. \quad (27)$$

These $D_m(q)$ values lie in the range between $D_m(\infty) = \alpha_{\min} = \ln 2 / \ln 2 = 1$ and $D_m(-\infty) = \alpha_{\max} = \ln 3 / \ln 2 = 1.584$. The fractal dimension of the object are given by $D_m(0) = f_m(0) = \ln \langle m \rangle / \ln b = 1.165$ —for DLA. This value is different from the fractal dimension $[D(\infty) + 1]$ which is derived from the

FIG. 16. The α_m - f_m spectra of the mass distributions for the diamond lattice.

FIG. 17. The α_m - f_m spectra of the mass distributions for the Wheatstone-bridge lattice.

maximum growth probability. Also it is different from the maximum fractal dimension $D_m(-\infty) = \alpha_{\max}$.

Similarly, we can obtain the generalized dimensions $D_m(q)$ for the Wheatstone-bridge lattice. Figure 13 shows the configurations of the spanning cluster for the diamond lattice. The configurational probabilities $C_{m,a}$ and $C_{m,b}$ are given by Eq. (24) where the cell's growth probability $p_{\alpha,i}$ and the configurational probability of the growth bond C_α are given by Eqs. (15) and (16). The generalized dimensions of mass are given by Eq. (27) with Eq. (26). These $D_m(q)$ values lie in the same range as the diamond lattice. The fractal dimension of the object is different from the diamond lattice and is given by $D_m(0) = f_m(0) = \ln \langle m \rangle / \ln b = 1.179$ —for DLA. Figures 14 and 15 show the generalized dimensions for the diamond and Wheatstone-bridge lattices. The $D_m(q)$ curve becomes larger with decreasing η . The fractal dimension of the object $D_m(0) = f_m(0)$ is larger with de-

FIG. 18. The comparison between the mass and growth probability distributions in the diamond lattice.

FIG. 19. The comparison between the mass and growth probability distributions in the Wheatstone-bridge lattice.

creasing η . Figures 16 and 17 show the α - f spectra for the diamond and Wheatstone-bridge lattices. For DLA, the multifractal structures of mass distributions are shown in Figs. 18 and 19 with the growth probability distributions. The maximum fractal dimension of the mass is close to the fractal dimension derived from the maximum growth probability.

The relationship $M \sim L^D$ between the mass and the linear size is approximately fulfilled with $D \simeq D_m(-\infty) = \alpha_{\max}$. The fractal dimension D is different from the fractal dimension of the object, $D_m(0)$. This is due to a large fluctuation of D with the increase of the linear size L of the cluster. The important consequence of our results agrees with that obtained by Tl and Vicsek.¹⁶ I believe that this is not a consequence of a possible approximation involved in our method.

We note that our renormalization procedure does not give an exact result for the hierarchical lattices because the renormalized conductance of the growth bond is approximated by the most probable value. In order to improve the approximation, one shall need to apply the renormalization procedure to the larger cell. By applying the renormalization to the second, third, etc. generations, one shall obtain the systematic way to improve the approximation.

VII. CONCLUSION

We show the multifractal structures of the mass and growth probability distributions for the diffusion-limited aggregation on the hierarchical lattices by making use of the real-space renormalization-group method. The system is formulated as a random multiplicative process of current on the surface of the aggregate. The renormalization-group equation for the surface conductance is derived. A stable fixed point is found in the renormalization-group equation. The scaling of the

growth probability distribution is derived. The mass scaling of the clusters is found and compared with the scaling structure of the growth probability distribution. It is found that the fractal dimension of the object $D_m(0)$

is different from the scaling exponent $[D(\infty)+1]$ derived from the maximum growth probability which is close to the maximum fractal dimension $\alpha_{\max}=D_m(-\infty)$ of the mass.

-
- ¹T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).
²P. Meakin, Phys. Rev. A **27**, 1495 (1983).
³*Kinetics of Aggregation and Gelation*, edited by F. Family and D. P. Landau (North-Holland, Amsterdam, 1984).
⁴*On Growth and Form*, edited by H. E. Stanley and N. Ostrowsky (Nijhoff, The Hague, 1985).
⁵*Fractals in Physics*, edited by L. Pietronero and E. Tosatti (North-Holland, Amsterdam, 1986).
⁶A. Coniglio, in *Statistical Physics*, edited by H. E. Stanley (North-Holland, Amsterdam, 1986), p. 51; Physica A **140**, 51 (1986).
⁷H. J. Herrmann, Phys. Rep. **136**, 154 (1986).
⁸B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
⁹P. Meakin, A. Coniglio, H. E. Stanley, and T. A. Witten, Phys. Rev. A **34**, 3325 (1986).
¹⁰T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).
¹¹C. Amitrano, A. Coniglio, and F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).
¹²T. Nagatani, J. Phys. A **20**, L381 (1987).
¹³T. Nagatani, Phys. Rev. A **36**, 5812 (1987).
¹⁴T. Nagatani, J. Phys. A **20**, 6603 (1987).
¹⁵T. Nagatani, Phys. Rev. A **37**, 3514; **37**, 4461 (1988).
¹⁶T. Tél and T. Vicsek, J. Phys. A **20**, L835 (1987).
¹⁷P. Meakin and S. Havlin, Phys. Rev. A **36**, 4428 (1987).
¹⁸L. Pietronero and A. P. Siebesma, Phys. Rev. Lett. **57**, 1098 (1986).
¹⁹P. Glansdorff and I. Prigogine, *Structure, Stability and Fluctuations* (Wiley, London, 1971).
²⁰R. C. Ball, in *On Growth and Form*, edited by H. E. Stanley and N. Ostrowsky (Nijhoff, The Hague, 1985), p. 69.
²¹L. Niemeyer, L. Pietronero, and H. J. Wiesmann, Phys. Rev. Lett. **52**, 1033 (1984).
²²L. A. Turkevich and H. Scher, Phys. Rev. Lett. **55**, 1026 (1985).