Fractal nature of non-Newtonian viscous fingering

Takashi Nagatani

College of Engineering, Shizuoka University, Hamamatsu 432, Japan

(Received 14 August 1989)

A renormalization-group theory is developed to study the fractal and the multifractal structures in the viscous fingering of a non-Newtonian displaced fluid at an infinite viscosity ratio. The effect of the non-Newtonian fluid on the multifractal scaling of the growth-probability distribution is investigated by using a finite lattice renormalization. The dependences of the α - f spectra of the growth-probability distribution are shown on the parameter k describing the different non-Newtonian fluids. It is found that the non-Newtonian fluid has important effects on the multifractal structure of the growth-probability distribution.

I. INTRODUCTION

The diffusion-limited aggregation (DLA) model^{1,2} is one of the nonequilibrium growth models. Patterns forming in the diffusive systems are isomorphic to DLA, since the Laplace equation underlies the diffusive systems.^{3,4} The pattern formation in the viscous fingering at an infinite viscosity ratio is a good example.⁴ The properties of DLA have been intensively investigated in the last nine years among the disciplines of physics, chemistry, biology, and mathematics.⁵⁻¹⁰ Despite its apparent simplicity, a complete understanding of some aspects of this model is still in the process of emerging. Recently, there has been much interest in multifractal phenomena.¹¹ In the last four years, multifractals have been applied to many phenomena, including turbulence, chaos, localization, resistor networks, and growth processes. It is well known that a DLA fractal cannot be fully characterized only by its fractal dimension. Halsey, Meakin, and Procaccia¹² and Amitrano, Coniglio, and di Liberto¹³ have shown that the surface of DLA requires an infinite hierarchy of fractal dimensions for its characterization. They have presented the multifractal structure of the growth-probability distribution from numerical experiments. Nagatani^{14,15} has presented a real-space renormalization-group method for calculating the multifractality. By using the renormalization-group method, Lee, Alstrom, and Stanley¹⁶ have found the existence of a phase transition in the multifractal spectrum. The real-space renormalization-group method has also been applied to the crossover phenomena between DLA and non-fractal structure.^{17,18} Wang, Shapir, and Rubinstein¹⁹ have improved the renormalization-group method and calculated the multifractality.

A fundamental statistical property of DLA is the growth-probability distribution on the perimeter of the aggregate, denoted P_i . It is known that the moments of the growth probability scale with the size L for $L \gg 1$ according to the power-law behavior

$$\sum_i P_i^q \sim L^{-(q-1)D(q)}, \quad (1)$$

where $D(q)$ is the generalized dimension.^{12,13} The fractal

dimension d_f of DLA is given by²⁰

$$d_f = D(\infty) + 1. \quad (2)$$

The fractal dimension d_f is a universal exponent that characterizes the scaling behavior of DLA. A large amount of research has focused on the computational, experimental, and theoretical determination of d_f and $D(q)$.

The viscous fingering serves as one paradigm of pattern formation for DLA.⁴ In its simplest form, one injects a fluid of low viscosity into a fluid of higher viscosity, using a Hele-Shaw cell. In the limiting case with zero interfacial tension and infinite viscosity ratio, one expects to find patterns that are isomorphic to DLA, since the Laplace equation underlies both systems. Various experiments for viscous fingering have been performed to find the fractal structure of pattern. In some experiments,^{21,22} the polymer solution with high viscosity has been used for the displaced fluid. The polymer solution shows a non-Newtonian behavior.²³ The fractal structure of non-Newtonian viscous fingers has not been investigated by computational and theoretical methods. An open question concerns the effect of non-Newtonian fluid on the fractal property of viscous fingers.

In this paper we develop a real-space renormalization-group method to study the multifractal structure of the growth-probability distribution in the viscous fingering of a non-Newtonian displaced fluid at an infinite viscosity ratio. We investigate the effect of non-Newtonian fluid on the multifractal scaling using a finite lattice renormalization. We show the dependences of the α - f spectra on the parameter k describing the different non-Newtonian fluids. We compare the multifractality of non-Newtonian fluid with that of Newtonian fluid.

The organization of the paper is as follows. In Sec. II we present the model describing the viscous fingering of non-Newtonian fluid. In Sec. III we develop a real-space renormalization scheme. We derive the multifractal structure of the growth-probability distribution by using the renormalization-group method. Section IV presents the summary.

II. MODEL

We consider the basic equations for the viscous fingering problem. In order to simplify the problem, we assume that the displaced fluid is non-Newtonian, the injected fluid is Newtonian, the viscosity ratio is infinite, and the interfacial tension is zero. We have

$$\mathbf{v}_1 = -K_1 \nabla P_1 \quad (3a)$$

for the injected fluid,

$$\mathbf{v}_2 = -K_2 |\nabla P_2|^{k-1} \nabla P_2 \quad (3b)$$

for the displaced fluid, where $\mathbf{v}_i$ is the velocity field, P_i the pressure, K_i the permeability coefficient, and the index $i=1,2$ indicates injected and displaced fluids.²³ The parameter k represents the strength of the non-Newtonian property. The parameter k is larger than 1 in a typical fluid. For Newtonian fluids, $k=1$. Because of the $(\nabla P_2)^k$ term in (3), for $k > 1$ the preferential growth of the tips is more pronounced in the non-Newtonian case than for Newtonian growth. From conservation of volume $\nabla \cdot \mathbf{v}_i = 0$, the pressure field satisfies

$$\nabla^2 P_1 = 0 \quad (4a)$$

for the injected fluid,

$$\nabla \cdot (|\nabla P_2|^{k-1} \nabla P_2) = 0 \quad (4b)$$

for the displaced fluid. The boundary conditions on the interface are given by

$$\begin{aligned} P_1 &= P_2, \\ \mathbf{v}_n &= -K_1 \hat{\mathbf{n}} \cdot \nabla P_1 = -K_2 \hat{\mathbf{n}} \cdot (|\nabla P_2|^{k-1} \nabla P_2), \end{aligned} \quad (5)$$

where $\mathbf{v}_n$ is the normal velocity of the interface and $\hat{\mathbf{n}}$ the unit vector normal to the interface. These equations completely determine the time evolution of the system.

We use an electrostatic analogy to transform the viscous fingering problem into a specific type of resistor network problem. Consider a system made of two different materials in which the current density in the one material satisfies Ohm's law and the other material has a characteristic property of a nonlinear resistor. The current density satisfies

$$\mathbf{j}_1 = -\sigma_1 \nabla \Phi_1 \quad (6a)$$

for the linear material,

$$\mathbf{j}_2 = -\sigma_2 |\nabla \Phi_2|^{k-1} \nabla \Phi_2 \quad (6b)$$

for the nonlinear material, where σ_i is the conductivity and Φ_i the electric potential. The parameter k represents the strength of nonlinearity. For linear resistors, $k=1$. From electric charge conservation $\nabla \cdot \mathbf{j}_i = 0$, the electric potential satisfies

$$\nabla^2 \Phi_1 = 0 \quad (7a)$$

for the linear material,

$$\nabla \cdot (|\nabla \Phi_2|^{k-1} \nabla \Phi_2) = 0 \quad (7b)$$

for the nonlinear material. The boundary conditions on

the interface are given by

$$\begin{aligned} \Phi_1 &= \Phi_2, \\ \mathbf{j}_n &= -\sigma_1 \hat{\mathbf{n}} \cdot \nabla \Phi_1 = -\sigma_2 \hat{\mathbf{n}} \cdot (|\nabla \Phi_2|^{k-1} \nabla \Phi_2), \end{aligned} \quad (8)$$

where $\mathbf{j}_n$ is the normal current density of the interface. Due to the formal similarity between Eqs. (4) and (7), the fluid flow problem becomes isomorphic to the electric problem.

We consider the lattice version of the system. We consider the dielectric breakdown model on the nonlinear-resistor network. The dielectric breakdown model on the linear-resistor network is extended to that on the nonlinear-resistor network. For simplicity, we consider the breakdown model on the diamond hierarchical lattice. The real-space renormalization-group method applied to the hierarchical lattice is comparatively accurate to derive the scaling behavior of the pattern growth. The diamond hierarchical lattice consists of nonlinear resistors. Each bond is made from one nonlinear resistor. The nonlinear resistor has the characteristic current-voltage relation²⁴⁻²⁶

$$I = \sigma |V|^k \text{sgn} V. \quad (9)$$

It is easy to convince oneself that the conductances defined in (9) add in parallel as usual,

$$\sigma_{12} = \sigma_1 + \sigma_2. \quad (10)$$

On the other hand, the conductance of two resistors in series is given by

$$\sigma_{12} = (\sigma_1^{-1/k} + \sigma_2^{-1/k})^{-k}. \quad (11)$$

A constant voltage is applied between the bottom and the top on the diamond hierarchical lattice (see Fig. 1). The dielectric breakdown proceeds from the bottom to the top. Figure 1 shows the illustration of the breakdown model. The thin lines indicate the unbroken, nonlinear resistors. The thick lines represent the break bonds: superconducting bonds which construct the breakdown pattern. The bonds on the perimeter of the breakdown

FIG. 1. Illustration of the dielectric breakdown model on the diamond hierarchical lattice consisting of nonlinear resistors. A constant voltage is applied between the bottom and the top. The thick, wavy, and thin lines indicate, respectively, break, growth, and unbroken bonds.

pattern are indicated by the wavy lines. So the nonlinear-resistor network problem is solved under the constant applied voltage (see Fig. 1). A growth probability proportional to the current is then assigned to the perimeter bond. The interface proceeds to the top according to the growth probability. The growth probability P_i on the growing-perimeter bond is given by

$$P_i \sim I_i, \quad (12)$$

where I_i is the local current on the growth bond i . Thus we can describe the non-Newtonian viscous fingering problem in terms of the breakdown model on the nonlinear-resistor network.

III. RENORMALIZATION-GROUP APPROACH

We consider the renormalization procedure for deriving the renormalization-group equation. See Ref. 15 for the details. We distinguish between three types of bonds on the lattice before and after a renormalization: (a) break bonds constructing the breakdown pattern, (b) growth bonds on the perimeter of the breakdown pattern, and (c) unbroken bonds consisting of the nonlinear resistor. The break, growth, and unbroken bonds are, respectively, indicated by the thick, wavy, and thin lines in figures. We partition all the space of the diamond lattice into cells of size $b=2$ (b is the scale factor), each containing a single generator. After a renormalization transformation these cells play the role of "renormalized" bonds. The n th generation of the diamond lattice is transformed to the $(n-1)$ th generation. The renormalized bonds are then classified into the three types of bonds, similarly to bonds before the renormalization. The conductance of the cell to be renormalized as the unbroken bond on the n th generation is renormalized to that on the $(n-1)$ th generation (Fig. 2)

$$\sigma_{0,n-1} = 2^{1-k} \sigma_{0,n}, \quad (13)$$

where $\sigma_{0,n}$ is the conductance of the unbroken bond on the n th generation. We consider the conductance of the cell that it is possible to renormalize as the growth bond. We assign an "effective conductance" for the growth bond on the perimeter. If the bond on the n th generation is the growth bond, then the effective conductance is assigned

$$\sigma_{0,n} \sigma_s. \quad (14)$$

FIG. 2. Renormalization process of the bond conductance to be renormalized as the unbroken bond.

Then the conductance of the cell is renormalized to

$$\sigma_{0,n-1} \sigma'_s. \quad (15)$$

We call the conductance σ_s the surface conductance. When $k=1$, the surface conductance σ_s is consistent with that of the linear-resistor network problem. After many repeated renormalizations, the surface conductance must approach to a finite value, not equal to zero according to the evolution criterion of DLA.^{14,15} The renormalization transformation of the surface conductance constitutes the renormalization equation:

$$\sigma'_s = R(\sigma_s; k). \quad (16)$$

Figure 3 shows all the configurations of the cell for which it is possible to renormalize as the growth bond. Let us consider the configurational probability C_α with which a particular configuration α appears. The distinct configurations are labeled by α ($\alpha=a, b, c$) in Fig. 3. The configuration (b) is constructed by adding a break bond onto the growth bonds 1 or 2 in configuration (a). Furthermore, by adding a break bond onto the growth bond 2 in the configuration (b), the configuration (c) occurs. The configurational probabilities C_α are given by

$$C_b = C_a(p_{a,1} + p_{a,2}), \quad (17)$$

$$C_c = C_b p_{b,2},$$

where $p_{a,i}$ is the growth probability of the growth bond i within the cell α . The configurational probability C_α is determined from the normalization condition

$$\sum_{\alpha} C_\alpha = C_a + C_b + C_c = 1. \quad (18)$$

The growth probability $p_{a,i}$ on the growth bond i within the cell α is proportional to the current on the growth bond. Consider the nonlinear-resistor network problem for the cells which can be renormalized as the growth bond. The electric fields on the nonlinear-resistor network are determined by Eqs. (9)–(11). In the configuration labeled by α (see Fig. 3), the growth probabilities $p_{a,i}$ of growth bonds are given by

$$p_{a,1} = p_{a,2} = \frac{1}{2},$$

$$p_{b,1} = \sigma_s / [\sigma_s + (1 + \sigma_s^{-1/k})^{-k}], \quad (19)$$

$$p_{b,2} = 1 - p_{b,1},$$

$$p_{c,1} = p_{c,2} = \frac{1}{2}.$$

FIG. 3. All distinct configurations of the cell that it is possible to renormalize as the growth bond.

FIG. 4. The behavior of the renormalization function of the surface conductance as a function k : $\sigma'_s = R(\sigma_s; k)$. The fixed points are indicated by the black circles for $k=1, 1.5, 2$.

FIG. 5. The surface conductance at the fixed point σ^* as a function k . The case of $k=1$ gives the DLA result.

FIG. 6. The generalized dimension $D(q)$ of the growth-probability distributions. The curves indicate, respectively, the results for $k=1, 2$, and 3 .

FIG. 7. The α - f spectra of the growth-probability distribution obtained from $D(q)$ in Fig. 6.

The conductance σ'_α of the cell with the configuration α is renormalized as follows:

$$\begin{aligned}\sigma'_a &= 2^k (1 + \sigma_s^{-1/k})^{-k}, \\ \sigma'_b &= 2^{k-1} \sigma_s + 2^{k-1} (1 + \sigma_s^{-1/k})^{-k}, \\ \sigma'_c &= 2^k \sigma_s.\end{aligned}\quad (20)$$

The renormalized conductance σ'_s of the growth bond will be assumed to be given by the most probable value

$$\sigma'_s = \exp(C_a \ln \sigma'_a + C_b \ln \sigma'_b + C_c \ln \sigma'_c). \quad (21)$$

The relationships (20) and (21) present the renormalization equation (16). Equations (17)–(21) are simultaneously solved. We find a stable fixed point σ^* from $\sigma^* = R(\sigma^*; k)$ for a k value. Figure 4 shows the behavior of the renormalization function as a parameter of k . The fixed points are indicated by the black circles. When $k=1$, the result of DLA is reproducible. Figure 5 shows the surface conductance at the fixed point as a function of k . The surface conductance at the fixed point increases with k . By using the surface conductance at the fixed point, we derive the multifractal structure of the growth-probability distribution. An infinite hierarchy of the generalized dimension $D(q)$ is given by

FIG. 8. The fractal dimension as a function of k . The case of $k=1$ gives the DLA result.

$$D(q) = -(q-1)^{-1} \ln \left[\sum_{\alpha} C_{\alpha}^* \sum_i (p_{\alpha,i}^*)^q \right] / \ln b, \quad (22)$$

where C_{α}^* is the configurational probability evaluated at the fixed point, and $p_{\alpha,i}^*$ the growth probability of the cell evaluated at the fixed point. The values of $D(q)$ are plotted in Fig. 6 for $k=1, 2$, and 3 . The α - f spectra of the growth-probability distribution obtained from $D(q)$ in Fig. 6 are displayed in Fig. 7. The fractal dimension d_f obtained from Eq. (2) is shown in Fig. 8 as a function of k . It is found that the fractal dimension decreases with the increase of k . In the limit of $k \rightarrow \infty$, the shape of the breakdown pattern becomes the needle structure with the fractal dimension $d_f=1$. This is due to the preferential growth of the tips more pronounced in the nonlinear case than for the ordinary case. The behavior of the fractal dimension and the multifractal structure of the growth-probability distribution is similar to that of the η model with the increase of η . It is shown that the non-

Newtonian fluid has important effects on the fractal nature of the pattern.

IV. SUMMARY

We develop the real-space renormalization-group method to study the fractal nature of the growth pattern in the non-Newtonian viscous fingering. We describe the fluid flow problem in terms of the dielectric breakdown model on the nonlinear-resistor network. We derive the multifractal structure of the growth-probability distribution on the diamond hierarchical lattice for the parameter k describing the different non-Newtonian fluids. The effect of the non-Newtonian fluid on the multifractal scaling is investigated by using a finite lattice renormalization. It is shown that the non-Newtonian fluid has important effects on the multifractal structure of the growth-probability distribution.

¹T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981); Phys. Rev. B **27**, 5686 (1983).

²P. Meakin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1988), Vol. 12, p. 336.

³L. Pietronero and H. J. Wiesmann, J. Stat. Phys. **36**, 909 (1984).

⁴T. Vicsek, *Fractal Growth Phenomena* (World-Scientific, Singapore, 1989).

⁵*Kinetics of Aggregation and Gelation*, edited by F. Family and D. P. Landau (North-Holland, Amsterdam, 1984).

⁶*On Growth and Form*, edited by H. E. Stanley and N. Ostrowsky (Nijhoff, The Hague, 1985).

⁷*Fractals in Physics*, edited by L. Pietronero and E. Tosatti (North-Holland, Amsterdam, 1986).

⁸R. Jullien and R. Botet, *Aggregation and Fractal Aggregates* (World-Scientific, Singapore, 1987).

⁹J. Feder, *Fractals* (Plenum, New York, 1988).

¹⁰*Random Fluctuations and Pattern Growth*, edited by H. E. Stanley and N. Ostrowsky (Kluwer Academic, Dordrecht, 1988).

¹¹H. E. Stanley and P. Meakin, Nature (London) **335**, 405 (1988).

¹²T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**,

854 (1986).

¹³C. Amitrano, A. Coniglio, and F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).

¹⁴T. Nagatani, J. Phys. A **20**, L381 (1987); Phys. Rev. A **36**, 5812 (1987); J. Phys. A **20**, 6603 (1987).

¹⁵T. Nagatani, Phys. Rev. A **38**, 2632 (1988).

¹⁶J. Lee, P. Alstrom, and H. E. Stanley, Phys. Rev. Lett. **61**, 2945 (1988); Phys. Rev. A **39**, 6545 (1989).

¹⁷J. Lee, A. Coniglio, and H. E. Stanley (unpublished).

¹⁸T. Nagatani (unpublished).

¹⁹X. R. Wang, Y. Shapir, and M. Rubinstein, Phys. Rev. A **39**, 5974 (1989); J. Phys. A **22**, L507 (1989).

²⁰L. A. Turkevich and H. Scher, Phys. Rev. Lett. **55**, 1026 (1985).

²¹J. Nittmann, G. Daccord, and H. E. Stanley, Nature (London) **314**, 141 (1985).

²²G. Daccord, J. Nittmann, and H. E. Stanley, Phys. Rev. Lett. **56**, 336 (1986).

²³T. Vicsek, Ref. 4, p. 306.

²⁴S. W. Kenkel and J. P. Straley, Phys. Rev. Lett. **49**, 767 (1982).

²⁵J. P. Straley and S. W. Kenkel, Phys. Rev. B **29**, 6299 (1984).

²⁶R. Blumenfeld and A. Aharony, J. Phys. A **18**, L443 (1985).