Diffusion-limited aggregation on hierarchical lattice with multifractal heterogeneity

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A renormalization-group theory is developed to study the multifractal structure of the growth probability distribution in the diffusion-limited aggregation on the multifractal lattice proposed by Meakin [Phys. Rev. A **36**, 2833 (1987)]. The hierarchical lattices with two classes of multifractal heterogeneity are constructed using a diamond generator with two sets of weights $P_1=1$, $P_2=Q$, $P_3=Q^2$, and $P_4=Q^3$ and $P_1=1$, $P_2=1$, $P_3=Q$, and $P_4=Q$. Multifractal scaling exponents of the growth probability distribution are calculated using a finite-lattice renormalization. The effect of the multifractal heterogeneity on the growth probability distribution is considered. It is shown that the multifractal heterogeneity has important effects on the multifractal structure of the growth probability distribution.

I. INTRODUCTION

The diffusion-limited-aggregation (DLA) model is a nonequilibrium growth model. The properties of DLA have been intensively investigated in the past nine years among the disciplines of physics, chemistry, biology, and mathematics.¹⁻¹⁰ Despite its apparent simplicity, a complete understanding of some aspects of this model is still in the process of emerging. A fundamental statistical property of the DLA is the growth probability distribution on the perimeter of the aggregate, denoted as P_i . It is well known that the moments of this growth probability scale with the size L for $L \gg 1$ according to the power-law behavior

$$\sum_i P_i^q \sim L^{-(q-1)D(q)}, \quad (1)$$

where $D(q)$ is the generalized dimension.^{11,12} The fractal dimension d_f of the DLA aggregate is given by¹³

$$d_f = D(\infty) + 1. \quad (2)$$

The fractal dimension d_f is a universal exponent that characterizes the scaling behavior of the DLA. A large amount of research has focused on the computational,^{1,2,11,12,14} experimental,¹⁵⁻²¹ and theoretical²²⁻²⁷ determination of d_f and $D(q)$ in homogeneous media or perfect lattices.

Recently, there has been much interest in multifractal phenomena.²⁸ Multifractality was discovered in the context of turbulence. In the past four years, multifractals have received much attention and have been applied to many phenomena, including turbulence,^{29,30} chaos,^{31,32} localization,^{33,34} resistor network,³⁵⁻³⁹ and growth processes.^{11,12,27} Halsey, Meakin, and Procaccia¹¹ and Amitrano, Coniglio, and Liberto¹² have found the multifractal structure of the growth probability distribution in DLA by the computational method. Nagatani^{27,40} has presented a real-space renormalization-group method for calculating the multifractality of the growth probability distribution. By using the renormalization-group

method, Lee, Alstrom, and Stanley⁴¹ have found the existence of a phase transition in the multifractal spectrum. The real-space renormalization-group method has also been applied to the crossover from the DLA fractal to the compact structure.⁴² Wang, Shapir, and Rubinstein⁴³ have presented the improved renormalization-group method and calculated the multifractal structures for the DLA on the deterministic fractal lattice. Very recently, to study the effect of heterogeneity on the scaling property of the DLA, Meakin⁴⁴ investigated the DLA on a two-dimensional multifractal lattice by using computer simulation. It was found that the effective fractal dimension of the DLA patterns was decreased as the magnitude of the heterogeneity was increased. However, the multifractal structure of the growth probability distribution was not investigated. On the multifractal lattices, Eden growth,⁴⁵ random walks,⁴⁶ and self-avoiding random walks⁴⁷ were investigated using computational and renormalization-group methods. It was shown that the multifractal disorder could have important effects on scaling behaviors.

In this paper, we study the effect of multifractal heterogeneity on the diffusion-limited aggregation on the diamond hierarchical lattice. We describe the DLA process in terms of the dielectric breakdown model on the heterogeneous-resistor network. The resistor network is characterized by the multifractal lattice illustrated in Fig. 1 and defined in Sec. II. The multifractal lattices of this type were introduced to model scaling phenomena in fluid turbulence. They were recently used as substrates for DLA model, Eden growth, random walks, and self-avoiding random walks. We derive the multifractal structure of the growth probability distribution by using the real-space renormalization-group approach which was devised for the DLA on the regular (or perfect) lattice. We compare the multifractality of the growth probability on the perfect lattice with that on the multifractal lattice.

The organization of the paper is as follows. In Sec. II we present the model that describes the DLA process on the multifractal lattice. In Sec. III we develop a real-

space renormalization scheme. We derive the multifractal structure of the growth probability distribution by using the renormalization-group method. In Sec. IV we consider the DLA on another multifractal lattice. Section V presents the summary.

II. MODEL

We present the dielectric breakdown model on a multifractal lattice. The multifractal lattice is generated from an iterative construction process. The first stage of the construction consists of the regular assignment of four weights P_1, P_2, P_3 , and P_4 to the four bonds of the base set for the diamond hierarchical lattice [Fig. 1(a)]. These four numbers ($0 < P_i < 1$) can be considered to be proportional to weights representing a heterogeneity of the conductance. In the second stage of the construction, each bond is subdivided into four sub-bonds. The weight associated with each sub-bond is determined by multiplying the quadrant weight by P_1, P_2, P_3 , and P_4 in a regular order [Fig. 1(b)]. This process is continued until the smallest length scale (one lattice unit) is reached. The asymptotic limit of this process defines a multifractal measure on a two-dimensional lattice. The heterogeneous configuration is the geometric realization of the bond weights resulting from the regular assignment of the basic weights (P_1, P_2, P_3, P_4) according to the definition of the multifractal lattice. We consider a class of heterogeneity, denoted by $P_1 = 1, P_2 = Q, P_3 = Q^2$, and $P_4 = Q^3$. The parameter Q is a real number between 0 and 1. The heterogeneity was already devised for the DLA on the multifractal lattice by Meakin, who performed a computer simulation for the random heterogeneity. We consider a simpler version of the heterogeneity, the deterministic multifractal lattice, since the real-space renormalization-group approach is more easily applied to the lattice. The conductance σ_n of the bonds of the n generations is given by

$$\sigma_n = \sigma_1^i \sigma_2^j \sigma_3^k \sigma_4^l, \quad (3)$$

with $i + j + k + l = n$, where

$$\begin{aligned} \sigma_1 &= (1+Q)/(1+Q^2), \\ \sigma_2 &= (1+Q)/(1+Q^2), \\ \sigma_3 &= Q(1+Q)/(1+Q^2), \\ \sigma_4 &= Q^2(1+Q)/(1+Q^2). \end{aligned} \quad (4)$$

FIG. 1. Generation of the multifractality on the diamond hierarchical lattice; (a) first stage, (b) second stage.

Then the conductance of the n generations is proportional to the multifractal measure $P_1^i P_2^j P_3^k P_4^l$. The conductances $\sigma_1, \sigma_2, \sigma_3$, and σ_4 of the bonds of the base set are also proportional to P_1, P_2, P_3 , and P_4 . The total conductance of the base set is set to equal to the unit value (Fig. 4). Thus we can obtain the heterogeneous resistor network with the multifractal conductance. We consider the dielectric breakdown model on the heterogeneous resistor network. A constant voltage is applied between the bottom and the top at the N th generation where N is a sufficiently large number. The dielectric breakdown proceeds from the bottom to the top. Figure 2 shows the illustration of the breakdown model. The thick lines indicate break bonds—superconducting bonds that construct the aggregate. The bonds on the perimeter of the aggregate are represented by the wavy lines. So the resistor network problem is solved under the boundary condition in Fig. 2. A growth probability proportional to the current is then assigned to the perimeter bond. The interface proceeds to the top according to the growth probability. The growth probability P_i at the growing-perimeter bond i is given by

$$P_i \sim I_i, \quad (5)$$

where I_i is the local current at the growth bond i . Thus we can describe DLA on the multifractal lattice in terms of the breakdown model on the heterogeneous resistor network.

III. RENORMALIZATION-GROUP APPROACH

We consider the renormalization procedure for deriving the renormalization-group equation. We distinguish between three types of bonds on the lattice before and after a renormalization: (a) interior bonds, which are occupied by superconducting bonds and construct the aggregate; (b) growth bonds, which are on the surface of the aggregate and can be successively grown, and (c) exterior bonds, which construct the electric field and are in the

FIG. 2. Illustration of the dielectric breakdown model. A constant voltage is applied between the bottom and the top. The bond conductance on the lattice is proportional to the multifractal measure defined in Fig. 1. The thick, wavy, and thin lines indicate, respectively, break, growth, and unbroken bonds.

exterior of the aggregate except for the growth bonds. The interior, growth, and exterior bonds are respectively indicated by the thick, wavy, and thin lines in figures. We partition all the space of the hierarchical lattice into cells of size $b=2$ (b is the scale factor), each containing a single generator. After a renormalization transformation, these cells play the role of “renormalized” bonds. The n th generation of the hierarchical lattice is transformed to the $(n-1)$ th generation. The renormalized bonds are then classified into the three types of bonds, similarly to bonds before the renormalization. The renormalization procedure is indicated by Fig. 3. If the cell is spanned with the bonds occupied by the interior bond, the renormalized bond is then considered to be interior [Fig. 3(a)]. If the cell is not spanned with the interior bond and is the nearest neighbor to the aggregate, then the cell is renormalized as the growth bond on the surface [Fig. 3(b)]. When the cell is constructed by the exterior bond only and is not nearest neighbor with the aggregate, the cell is renormalized as the exterior bond [Fig. 3(c)]. The conductance of the cell to be renormalized as the exterior bond on the n th generation is renormalized to that on the $(n-1)$ th generation (Fig. 4). We consider the conductance of the cell, that it is possible to renormalize as the growth bond. We assign an “effective conductance” for the growth bond at the perimeter. If the bond i on the n th generation is the growth bond, then the effective conductance of the bond is assigned

$$\sigma_{n-1}\sigma_i\sigma_s, \quad (6)$$

where $\sigma_i = \sigma_1, \sigma_2, \sigma_3$, or σ_4 . The conductance of the cell is renormalized to

$$\sigma_{n-1}\sigma'_s. \quad (7)$$

We refer to the conductance σ_s as the surface conductance. When $Q=1$, the surface conductance σ_s is con-

FIG. 3. Illustration of the renormalization for the diamond lattice. The cells in the upper side are renormalized to the bonds in the bottom side. The bonds on the lattice are classified into three types of bonds: interior bonds indicated by thick lines, growth bonds by wavy lines, and exterior bonds by thin lines. Examples of the distinct configurations are shown in (a), (b), and (c), which are, respectively, renormalized as interior, growth, and exterior bonds.

FIG. 4. Renormalization process of the bond conductance to be renormalized as the unbroken bond.

sistent with that of the perfect lattice. After many repeated renormalizations, the surface conductance must approach a finite value, not equal to zero according to the evolution criterion of DLA. The renormalization transformation of the surface conductance constitutes the renormalization equation. Figure 5 shows all the configurations of the cell for which it is possible to renormalize as the growth bond. Let us consider the configurational probability C_α with which a particular configuration α appears. The distinct configurations are labeled by α ($\alpha=0,1,2,3$) in Fig. 5. The configuration (1) is constructed by adding a break bond onto the growth bond 2 in configuration (0). The configuration (2) is constructed by adding a break bond onto the growth bond 3 in configuration (0). The probability with which a break bond adds onto the growth bonds 2 or 3 in configuration (0) is given by the growth probabilities $p_{0,2}$ or $p_{0,3}$ of the growth bonds 2 or 3 in the configuration (0). In addition, by adding a break bond onto the growth bond 3 in the configuration (1) or onto the growth bond 2 in the configuration (2), the configuration (3) occurs. We here assume that the breakdown proceeds one by one. The configurational probabilities C_α are given by

FIG. 5. All the distinct configurations of the cell that are possible to renormalize as the growth bond. The configurations (1) or (2) are constructed by adding a break bond onto the growth bonds 2 or 3 in the configuration (0). In addition, by adding a break bond onto the growth bond 3 in the configuration (1) or onto the growth bond 2 in the configuration (2), the configuration (3) is constructed.

$$\begin{aligned}
C_1 &= C_0 p_{0,2}, \\
C_2 &= C_0 p_{0,3}, \\
C_3 &= C_1 p_{1,3} + C_2 p_{2,2}.
\end{aligned} \tag{8}$$

The configurational probability C_0 is determined from the normalization condition

$$\sum_{\alpha} C_{\alpha} = C_0 + C_1 + C_2 + C_3 = 1. \tag{9}$$

The growth probability $p_{\alpha,i}$ on the growth bond i within the cell α is proportional to the current I_i on the growth bond. Consider the resistor network problem for the cells that can be renormalized as the growth bond. The electric fields on the resistor network are determined by the conductances of the bonds and the configuration of the cell. In the configuration labeled by α (see Fig. 5), the growth probabilities $p_{\alpha,i}$ of growth bonds are given by

$$\begin{aligned}
p_{0,2} &= \sigma_1 \sigma_2 (\sigma_4 + \sigma_3 \sigma_s) / [\sigma_1 \sigma_2 (\sigma_4 + \sigma_3 \sigma_s) + \sigma_3 \sigma_4 (\sigma_1 + \sigma_2 \sigma_s)], \\
p_{0,3} &= \sigma_3 \sigma_4 (\sigma_1 + \sigma_2 \sigma_s) / [\sigma_1 \sigma_2 (\sigma_4 + \sigma_3 \sigma_s) + \sigma_3 \sigma_4 (\sigma_1 + \sigma_2 \sigma_s)], \\
p_{1,1} &= \sigma_1 (\sigma_4 + \sigma_3 \sigma_s) / [\sigma_3 \sigma_4 + \sigma_1 (\sigma_4 + \sigma_3 \sigma_s)], \\
p_{1,3} &= \sigma_3 \sigma_4 / [\sigma_3 \sigma_4 + \sigma_1 (\sigma_4 + \sigma_3 \sigma_s)], \\
p_{2,2} &= \sigma_1 \sigma_2 / [\sigma_1 \sigma_2 + \sigma_4 (\sigma_1 + \sigma_2 \sigma_s)], \\
p_{2,4} &= \sigma_4 (\sigma_1 + \sigma_2 \sigma_s) / [\sigma_1 \sigma_2 + \sigma_4 (\sigma_1 + \sigma_2 \sigma_s)], \\
p_{3,1} &= \sigma_1 / (\sigma_1 + \sigma_4), \\
p_{3,4} &= \sigma_4 / (\sigma_1 + \sigma_4),
\end{aligned} \tag{10}$$

where σ_s indicates the surface conductance. The conductance σ'_{α} of the cell with the configuration α is renormalized as follows:

$$\begin{aligned}
\sigma'_0 &= \sigma_1 \sigma_2 \sigma_s / (\sigma_1 + \sigma_2 \sigma_s) + \sigma_3 \sigma_4 \sigma_s / (\sigma_4 + \sigma_3 \sigma_s), \\
\sigma'_1 &= \sigma_1 \sigma_s + \sigma_3 \sigma_4 \sigma_s / (\sigma_4 + \sigma_3 \sigma_s), \\
\sigma'_2 &= \sigma_4 \sigma_s + \sigma_1 \sigma_2 \sigma_s / (\sigma_1 + \sigma_2 \sigma_s), \\
\sigma'_3 &= \sigma_1 \sigma_s + \sigma_4 \sigma_s.
\end{aligned} \tag{11}$$

The renormalized conductance σ' of the growth bond will be assumed to be given by the most probable value

$$\sigma' = \exp(C_0 \ln \sigma'_0 + C_1 \ln \sigma'_1 + C_2 \ln \sigma'_2 + C_3 \ln \sigma'_3). \tag{12}$$

The relationships (11) and (12) present the renormalization equation $\sigma' = R(\sigma)$. Equations (8)–(12) are simultaneously solved. We find a stable fixed point σ^* from $\sigma^* = R(\sigma^*)$ for a Q value. Figure 6 shows the behavior of the renormalization function as a parameter of Q . The fixed points are indicated by the black circles. When $Q=1$, the result of DLA on the perfect lattice is reproducible. Figure 7 shows the surface conductance at the fixed point as a function of Q . The surface conductance at the fixed point increases with the decrease of Q . By using the surface conductance at the fixed point, we derive

FIG. 6. Behavior of the renormalization function of the surface conductance as a parameter of Q : $\sigma'_s = R(\sigma_s)$. The fixed points are indicated by the closed circles for $Q=1, 0.7, 0.5$.

the multifractal structure of the growth probability distribution. The growth probability over the total perimeter of the aggregate is represented by the multiplicative process of the cell's growth probability. Each cell's growth probability is represented by a function of the surface conductance. After many repeated renormalizations, the surface conductance σ approaches the value σ^* at the fixed point. The cell's growth probability $p_{\alpha,i}$ approaches that evaluated at the fixed point $p_{\alpha,i}^*$. An infinite hierarchy of the generalized dimensions $D(q)$ is given by

$$D(q) = -(q-1)^{-1} \ln \left[\sum_{\alpha} C_{\alpha}^* \sum_i (p_{\alpha,i}^*)^q \right] / \ln b, \tag{13}$$

where C_{α}^* is the configurational probability evaluated at the fixed point, and $p_{\alpha,i}^*$ the growth probability of the cell evaluated at the fixed point. The values of $D(q)$ are plotted in Fig. 8 for $Q=1, 0.7$, and 0.5 . The α - f spectra of the growth probability distribution obtained from $D(q)$ in Fig. 8 are displayed in Fig. 9. The fractal dimension d_f obtained from Eq. (2) is shown in Fig. 10 as a function of Q . It is found that the fractal dimension decreases with Q . In the limit of $Q=0$, the shape of the aggregate becomes the needle structure with the fractal dimension $d_f=1$. The case $Q=1$ gives the DLA on the perfect lattice.

FIG. 7. Surface conductance at the fixed point σ^* as a function of Q . The case of $Q=1$ gives the DLA on the perfect lattice.

FIG. 8. Generalized dimension $D(q)$ of the growth probability distributions. The curves indicate, respectively, the results for $Q=1, 0.7$, and 0.5 .

FIG. 9. The α - f spectra of the growth probability distribution obtained from $D(q)$ in Fig. 8.

FIG. 10. Fractal dimension as a function of Q . The case of $Q=1$ gives the DLA on the perfect lattice.

tice. The fractal dimension on the perfect lattice is $d_f=1.40$. The approach of the fractal structure to the needle with the decrease of Q is reducible to the geometrical structure of the resistor network. The channel flowing current is more and more narrow with the decrease of Q because the conductance of the channel increases (see Fig. 1). The behavior of the fractal dimension and the multifractal structure of the growth probability distribution is similar to that of the η model with the increase of η . It is shown that the multifractal heterogeneity has important effects on the fractal nature of the DLA.

IV. DLA ON ANOTHER MULTIFRACTAL LATTICE

We consider the DLA on another multifractal lattice. We consider another class of heterogeneity, denoted by $P_1=1, P_2=1, P_3=Q$, and $P_4=Q$. The heterogeneity was already devised for random walks on the multifractal lattice by Meakin, who performed a computer simulation. We refer to the multifractal set as type 2. The set of weights $P_1=1, P_2=Q, P_3=Q^2$, and $P_4=Q^3$ is called type 1. We derive the multifractal structure of the growth probability distribution. We compare the multifractal structure with that derived in Sec. III. We discuss the multifractal structures of the growth probability distribution for two classes of multifractal heterogeneity. The conductance of the n generation is proportional to multifractal measure $P_1^i P_2^j P_3^k P_4^l$ with $i+j+k+l=n$. The conductances $\sigma_1, \sigma_2, \sigma_3$, and σ_4 of the bonds of the base set are also proportional to P_1, P_2, P_3 , and P_4 . The total conductance of the base set is set to equal to the unit value (Fig. 4). The conductance σ_n of the bonds of the n generation is given by

$$\sigma_n = \sigma_1^i \sigma_2^j \sigma_3^k \sigma_4^l, \quad (14)$$

where

$$\begin{aligned} \sigma_1 &= 2/(1+Q), \\ \sigma_2 &= 2/(1+Q), \\ \sigma_3 &= 2Q/(1+Q), \\ \sigma_4 &= 2Q/(1+Q). \end{aligned} \quad (15)$$

Thus we can obtain the heterogeneous resistor network with another class of the multifractal heterogeneity. We apply the renormalization-group method to the lattice. With substitution of (15) into (10)–(13), the multifractal structure of the growth probability distribution is obtained. Figure 11 shows the α - f spectra of the growth probability distribution for $Q=1, 0.7$, and 0.5 . The dependence of the α - f spectra on Q is of less significance than that in Fig. 9. Figure 12 shows the fractal dimension as a function of Q . For comparison, the fractal dimension for the set of weights $P_1=1, P_2=Q, P_3=Q^2$, and $P_4=Q^3$ is indicated by the broken line. The dependence of the fractal dimension of the type 2 upon Q is also of less significance than that of type 1. We study the origin of the significant difference between the two classes of multifractal heterogeneity. We derive the α - f spectra

FIG. 11. The α - f spectra of the growth probability distribution for another multifractal set.

FIG. 12. Fractal dimension as a function of Q , corresponding to another multifractal set in Fig. 11. The dashed line indicates the fractal dimension for the type 1 of heterogeneity.

FIG. 13. The α - f spectra of the multifractal heterogeneity for the type 1.

FIG. 14. The α - f spectra of the multifractal heterogeneity for the type 2.

of the multifractal heterogeneity. The generalized dimension of the multifractal lattices is given by

$$D(q) = -(q-1)^{-1} \ln(P_1'^q + P_2'^q + P_3'^q + P_4'^q) / \ln 2, \quad (16)$$

where

$$P_1' = P_1 / (P_1 + P_2 + P_3 + P_4),$$

$$P_2' = P_2 / (P_1 + P_2 + P_3 + P_4),$$

$$P_3' = P_3 / (P_1 + P_2 + P_3 + P_4),$$

$$P_4' = P_4 / (P_1 + P_2 + P_3 + P_4).$$

Figures 13 and 14 show respectively the α - f spectra of types 1 and 2 of multifractal lattices. The dependence of the α - f spectra on Q for the type 2 is of less significance than that of the type 1. The strength of the multifractal heterogeneity will be proportional to the spread of the α - f spectrum. The fractal nature of DLA is closely related to the α - f spectra of the multifractal heterogeneity. The α - f spectra of the growth probability distribution are spread more widely with the spread of the α - f spectra of the heterogeneity. It is found that the fractal dimension decreases with increase of the spread of the α - f spectra of the heterogeneity.

V. SUMMARY

We develop the renormalization-group theory to study the fractal nature of DLA on the multifractal lattices. We derive the multifractal structure of the growth probability distribution on the diamond hierarchical lattice with two classes of multifractal heterogeneity. We find that the α - f spectra of the growth probability distribution is closely correlated with the α - f spectra of the heterogeneity. We calculate the fractal dimension as a function of Q . The fractal dimension approaches from that on the perfect lattice ($Q=1$) to 1, with the decrease of Q . We show that the multifractal heterogeneity has important effects on the multifractal structure of the growth probability distribution.

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