Crossover phenomena in non-Newtonian viscous fingers at a finite viscosity ratio

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A viscous fingering of non-Newtonian fluids at a finite viscosity ratio is considered in order to study the effect of non-Newtonian fluid on crossover phenomena. The crossover from the fractal pattern to the dense structure is investigated by using a two-parameter position-space renormalization-group method. The global flow diagrams in two-parameter space are obtained. It is found that there are two nontrivial fixed points: the fractal point and the Eden point. When the viscosity ratio is finite, the pattern must eventually cross over to the dense structure. The dependences of the crossover phenomena on the parameter k , which describes the different non-Newtonian fluids, are shown. It is found that the non-Newtonian fluids have important effects on the fractal point and the crossover line but the crossover exponent is independent of the non-Newtonian property.

I. INTRODUCTION

Fractal growth phenomena in pattern formation have recently attracted considerable attention.¹⁻¹⁰ The diffusion-limited aggregation (DLA) model is one of the nonequilibrium growth models. Patterns forming in the diffusive systems are isomorphic to DLA, since the Laplace equation underlies the diffusive systems. Examples of pattern formation in diffusive systems include viscous fingering, electrochemical deposition, crystal growth and dielectric breakdown. There has been increasing interest in the physical mechanisms governing the geometrical structure in the DLA. The fractal nature of the pattern has been analyzed by computational, experimental and analytical methods. Several analytical attempts, including mean-field theories and renormalization-group methods, have been made to calculate the fractal dimension and the multifractal structure¹¹⁻¹³ of the growth probability distribution.¹⁴⁻¹⁷

The viscous fingering serves as one paradigm of pattern formation for DLA. In its simplest form, one injects a fluid of low viscosity into a fluid of higher viscosity, using a Hele-Shaw cell. In the limiting case with zero interfacial tension and infinite viscosity ratio, one expects to find patterns that are isomorphic to DLA. Various experiments for viscous fingering have been performed to find the fractal structure of pattern. In real experimental situations, where the viscosity of the injected fluid is not zero, the asymptotic behavior of viscous fingers must eventually cross over from the DLA fractal to the dense structure. Very recently, Lee, Coniglio, and Stanley¹⁸ succeeded in analyzing the crossover in viscous fingering at the finite viscosity ratio by using the two-parameter position-space renormalization-group method. The position-space renormalization-group method for DLA (Refs. 19-21) has been divided by Nagatani¹⁴ to derive the multifractal scaling of the growth probability distribution.

In some experiments,²²⁻²⁴ the polymer solutions with

high viscosity have been used for the displaced fluid. The polymer solution shows a non-Newtonian behavior. The fractal structure of non-Newtonian viscous fingers has not been investigated by computational method. By using the position-space renormalization-group method, Nagatani¹⁶ found that the non-Newtonian fluid had an important effect on fractal and multifractal structures in the limiting case of an injected fluid with zero viscosity. In the viscous fingering of non-Newtonian fluids at a finite viscosity ratio, the crossover from the fractal pattern to the dense structure are also expected to occur similarly to the Newtonian fluid. An open question concerns the effect of non-Newtonian fluids on the crossover phenomena.

In this paper, we develop a real-space renormalization-group method to study the crossover in non-Newtonian viscous fingering at a finite viscosity ratio. We investigate the effect of non-Newtonian fluids on the crossover from the fractal pattern to the dense structure using a finite lattice renormalization. We show the dependences of the crossover on the parameter k describing the different non-Newtonian fluids. We compare the crossover phenomena of non-Newtonian fluid with that of Newtonian fluid.

The organization of the paper is as follows. In Sec. II we present the model describing the viscous fingering of non-Newtonian fluids. In Sec. III we develop a position-space renormalization scheme. In Sec. IV we show the global flow diagrams in two-parameter space. The crossover exponents are calculated for the parameter k . Section V presents the summary.

II. MODEL

We consider the basic equations for the viscous fingering of non-Newtonian fluids at a finite viscosity ratio. In order to simplify the problem, we assume that the displaced fluid and the injected fluid are non-Newtonian, the viscosity ratio is finite, and the interfacial tension is zero.

We have

$$\begin{aligned} \mathbf{v}_1 &= -K_1 |\nabla P_1|^{k-1} \nabla P_1 \quad \text{for the injected fluid,} \\ \mathbf{v}_2 &= -K_2 |\nabla P_2|^{k-1} \nabla P_2 \quad \text{for the displaced fluid,} \end{aligned} \quad (1)$$

where v_i is the velocity, P_i the pressure, K_i the permeability coefficient, and the index $i=1,2$ indicates injected and displaced fluids. The parameter k represents the strength of non-Newtonian property. The parameter k is larger than 1 in a typical fluid.²⁴ For Newtonian fluids $k=1$. Because of the $(\nabla P_i)^k$ term in (1), for $k>1$ the preferential growth of the tips is more pronounced in the non-Newtonian case than for Newtonian growth. From conservation of volume $\text{div}(\mathbf{v}_i)=0$, the pressure field satisfies

$$\begin{aligned} \nabla \cdot (|\nabla P_1|^{k-1} \nabla P_1) &= 0 \quad \text{for the injected fluid,} \\ \nabla \cdot (|\nabla P_2|^{k-1} \nabla P_2) &= 0 \quad \text{for the displaced fluid.} \end{aligned} \quad (2)$$

The boundary conditions on the interface are given by

$$\begin{aligned} P_1 &= P_2, \\ \mathbf{v}_n &= -K_1 \mathbf{n} \cdot (|\nabla P_1|^{k-1} \nabla P_1) \\ &= -K_2 \mathbf{n} \cdot (|\nabla P_2|^{k-1} \nabla P_2), \end{aligned} \quad (3)$$

where v_n is the normal velocity of the interface and $\mathbf{n}$ is the unit vector normal to the interface. These equations completely determine the evolution of the system.

By using an electrostatic analogy, we transform the viscous fingering problem to the dielectric breakdown model. Consider a dielectric breakdown process in a material which has a characteristic property of nonlinear resistor.²⁵⁻²⁷ The part of the breakdown material is assumed to have a nonlinear property with a different conductivity. The current density satisfies

$$\begin{aligned} \mathbf{j}_1 &= -\sigma_1 |\nabla \Phi_1|^{k-1} \nabla \Phi_1 \quad \text{for the unbroken material,} \\ \mathbf{j}_2 &= -\sigma_2 |\nabla \Phi_2|^{k-1} \nabla \Phi_2 \quad \text{for the breakdown material,} \end{aligned} \quad (4)$$

where σ_1 and σ_2 are the conductivities of the unbroken and the breakdown materials, and Φ_i is the electric potential. The parameter k represents the strength of nonlinearity. For linear resistors $k=1$. From electric current conservation $\text{div}(\mathbf{j}_i)=0$, the electric potential satisfies

$$\begin{aligned} \nabla \cdot (|\nabla \Phi_1|^{k-1} \nabla \Phi_1) &= 0 \quad \text{for the unbroken material,} \\ \nabla \cdot (|\nabla \Phi_2|^{k-1} \nabla \Phi_2) &= 0 \quad \text{for the breakdown material.} \end{aligned} \quad (5)$$

The boundary conditions on the interface are given by

$$\begin{aligned} \Phi_1 &= \Phi_2, \\ \mathbf{j}_n &= -\sigma_1 \mathbf{n} \cdot (|\nabla \Phi_1|^{k-1} \nabla \Phi_1) \\ &= -\sigma_2 \mathbf{n} \cdot (|\nabla \Phi_2|^{k-1} \nabla \Phi_2). \end{aligned} \quad (6)$$

Due to the formal similarity between Eqs. (2) and (5), the fluid flow problem becomes isomorphic to the electric problem.

FIG. 1. Illustration of the dielectric breakdown model on the diamond hierarchical lattice consisting of nonlinear resistors. A constant voltage is applied between the bottom and the top. The thick lines indicate the breakdown bonds.

We consider the lattice version of the system. We consider the dielectric breakdown model on the nonlinear-resistor network.²⁵⁻²⁷ For simplicity, we consider the breakdown model on the diamond hierarchical lattice. The real-space renormalization-group method applied to the hierarchical lattice is comparatively accurate to derive the scaling behavior of the pattern growth.¹⁵ The diamond hierarchical lattice consists of nonlinear resistors. Each bond of the breakdown pattern has a different conductivity from the unbroken bond. The nonlinear resistor has the characteristic current-voltage relation²⁵⁻²⁷

$$\begin{aligned} I_1 &= \sigma_1 |V_1|^k \text{sgn} V_1 \quad \text{for the breakdown bond,} \\ I_2 &= \sigma_2 |V_2|^k \text{sgn} V_2 \quad \text{for the unbroken bond.} \end{aligned} \quad (7)$$

A constant voltage is applied between the bottom and the top on the diamond hierarchical lattice (see Fig. 1). The dielectric breakdown proceeds from the bottom to the top. Figure 1 shows the illustration of the breakdown model. The thin lines indicate the unbroken nonlinear resistors. The thick lines represent the breakdown nonlinear resistors. A growth probability proportional to the current is assigned to the perimeter bond. The interface proceeds to the top according to the growth probability. The growth probability P_i on the growing-perimeter bond i is given by

$$P_i \sim I_i, \quad (8)$$

where I_i is the current flowing through the growth bond i .

III. RENORMALIZATION-GROUP APPROACH

We describe the non-Newtonian viscous fingering problem in terms of the breakdown model on the nonlinear resistor network.²⁵⁻²⁷ We consider the renormalization procedure on the diamond hierarchical lattice.¹⁵ We shall derive the renormalization-group equations for the surface conductance and the conductance ratio. See

Ref. 15 for details. We distinguish between three types of bonds on the lattice before and after a renormalization: (i) breakdown bonds constructing the breakdown pattern, (ii) growth bonds on the perimeter of the breakdown pattern, and (iii) unbroken bonds consisting of the original nonlinear resistor. The breakdown, growth, and unbroken bonds are, respectively, indicated by the thick, wavy, and thin lines in the figures. We partition all the space of the diamond lattice into cells of size $b=2$ (b is the scale factor), each containing a single generator. After a renormalization transformation these cells play the role of “renormalized” bonds. The n th generation of the diamond lattice is transformed to the $(n-1)$ th generation. The renormalized bonds are then classified into the three types of bonds, similarly to the bond before renormalization. The conductance of the cell to be renormalized as the unbroken bond on the n th generation is renormalized to that on the $(n-1)$ th generation

$$\sigma_{0,n-1} = 2^{1-k} \sigma_{0,n}, \quad (9)$$

where $\sigma_{0,n}$ is the conductance of the unbroken bond on the n th generation. We consider the conductance of the cell that it is possible to renormalize as the growth bond. We assign an “effective conductance” for the growth bond on the perimeter. If the bond on the n th generation is the growth bond, then the effective conductance is assigned to be

$$\sigma_{0,n} \sigma_s. \quad (10)$$

Then the conductance of the cell is renormalized to

$$\sigma_{0,n-1} \sigma'_s. \quad (11)$$

We call the conductance σ_s the surface conductance. The renormalization transformation of the surface conductance constitutes one of the renormalization equations:

$$\sigma'_s = R_s(\sigma_s, \sigma_a), \quad (12)$$

where σ_a is the conductance ratio between the breakdown bond and the unbroken bond.

We consider the conductance of the cell that it is possible to renormalize as the breakdown bond. The cell is a spanning cluster in which the top is connected with the bottom by the breakdown bonds. The conductance of the breakdown bond is given on the n th generation by

$$\sigma_{0,n} \sigma_a. \quad (13)$$

Then the conductance of the spanning cluster is renormalized to

$$\sigma_{0,n-1} \sigma'_a. \quad (14)$$

The renormalization transformation of the conductance ratio constitutes the other of the renormalization equations:

$$\sigma'_a = R_a(\sigma_s, \sigma_a). \quad (15)$$

Equations (12) and (15) give the renormalization-group

FIG. 2. All distinct configurations of the cell that it is possible to renormalize as the growth bond. The thick, wavy, and thin lines indicate, respectively, breakdown, growth, and unbroken bonds.

equations. We derive the renormalization functions (12) and (15) explicitly. Figure 2 shows all the configurations of the cell for which it is possible to renormalize as the growth bond. Let us consider the configurational probability C_α with which a particular configuration α appears. The distinct configurations are labeled by α ($\alpha=a, b, c$) in Fig. 2. The configurational probabilities C_α are given (see Ref. 15 for details)

$$\begin{aligned} C_a &= 1 - C_b - C_c, \\ C_b &= C_a(p_{a,1} + p_{a,2}), \\ C_c &= C_b p_{b,2}, \end{aligned} \quad (16)$$

where $p_{a,i}$ is the growth probability of the growth bond i within the cell α . The growth probability $p_{a,i}$ on the growth bond i within the cell α is proportional to the current on the growth bond. They are given by

$$\begin{aligned} p_{a,1} &= p_{a,2} = \frac{1}{2}, \\ p_{b,1} &= \frac{(\sigma_s^{-1/k} + \sigma_a^{-1/k})^{-k}}{(\sigma_s^{-1/k} + \sigma_a^{-1/k})^{-k} + (1 + \sigma_s^{-1/k})^{-k}}, \\ p_{b,2} &= 1 - p_{b,1}, \\ p_{c,1} &= p_{c,2} = \frac{1}{2}. \end{aligned} \quad (17)$$

The surface conductance $\sigma'_{s,\alpha}$ of the cell with the configuration α is renormalized as follows:

$$\begin{aligned} \sigma'_{s,a} &= 2^k (1 + \sigma_s^{-1/k})^{-k}, \\ \sigma'_{s,b} &= 2^{k-1} (\sigma_s^{-1/k} + \sigma_a^{-1/k})^{-k} + 2^{k-1} (1 + \sigma_s^{-1/k})^{-k}, \\ \sigma'_{s,c} &= 2^k (\sigma_s^{-1/k} + \sigma_a^{-1/k})^{-k}. \end{aligned} \quad (18)$$

The renormalized conductance σ'_s of the growth bond will be assumed to be given by the most probable value

$$\sigma'_s = \exp(C_a \ln \sigma'_{s,a} + C_b \ln \sigma'_{s,b} + C_c \ln \sigma'_{s,c}). \quad (19)$$

Relationships (18) and (19) present the renormalization equation (12). In the limit of the infinite viscosity ratio ($\sigma_a \rightarrow \infty$), Eqs. (18) and (19) reduce to those of the injected fluid with zero viscosity.¹⁶

We consider the renormalization of the conductance ratio for the spanning cluster to be renormalized as the

FIG. 3. All distinct configurations of the spanning cluster, which are renormalized as the breakdown bond, are shown on the bottom side. Configurations (1) and (2) are, respectively, constructed from the configurations of the growth bonds shown on the top side.

breakdown bond. Figure 3 shows all the configurations of the spanning cluster. Configurations (1) and (2) of the spanning cluster on the bottom side are constructed from the configurations of the growth cell on the top side. Configuration (1) in Fig. 3 is constructed by adding the breakdown bond onto the growth bond 1 in configuration (b) in Fig. 2. The configurational probability $C_{a,1}$ of configuration (1) is given by

$$C_{a,1} = C_{a,0} p_{b,1} C_b. \quad (20)$$

Configuration (2) in Fig. 3 is constructed by adding the breakdown bond onto growth bonds 1 or 2 in configuration (c) in Fig. 2. The configurational probability $C_{a,2}$ of configuration (2) is given by

FIG. 4. The plot of $1/\sigma_s^*$ against k . The case of $k=1$ gives the Newtonian result.

FIG. 5. The plot of the fractal dimension d_f against k .

$$C_{a,2} = C_{a,0}(p_{c,1} + p_{c,2})C_c. \quad (21)$$

The unknown constant $C_{a,0}$ is determined by the normalization condition

$$C_{a,1} + C_{a,2} = 1. \quad (22)$$

The conductance ratio $\sigma'_{a,1}$ of the cell in Fig. 3 [configuration (1)] is given by

$$\sigma'_{a,1} = (1 + \sigma_a)/2. \quad (23)$$

The conductance ratio $\sigma'_{a,2}$ of the cell in Fig. 3 [configuration (2)] is given by

$$\sigma'_{a,2} = \sigma_a/2 + 2^{k-1}(1 + \sigma_a^{-1/k})^{-k}. \quad (24)$$

The renormalized conductance ratio σ'_a of the breakdown bond will be assumed to be given by the most probable value

$$\sigma'_a = \exp[C_{a,1} \ln \sigma'_{a,1} + C_{a,2} \ln \sigma'_{a,2}]. \quad (25)$$

Relationship (25) with (23) and (24) presents the renormalization function $\sigma'_a = R_a(\sigma_s, \sigma_a)$ for the conductance ratio of the breakdown bond. Equations (16)–(25) are simultaneously solved. We find the two nontrivial fixed points $(1,1)$ and $(1/\sigma_s^*, 0)$ for a parameter k in the parameter space $(1/\sigma_s, 1/\sigma_a)$ where σ_s^* is the value of the fixed point in the limiting case of an infinite viscosity ratio. At the fixed point $(1,1)$, the growth probabilities $p_{a,i}$ give $\frac{1}{2}$ for all the growth bonds in configurations (a)–(c). The growth probability over the whole system becomes uniform over all the surface bond. The fixed point $(1,1)$ corresponds to the Eden model. It is called the Eden point. On the other hand, the fixed point $(1/\sigma_s^*, 0)$ gives the viscous finger at the infinite viscosity ratio. We have derived the critical behavior of the viscous finger at the infinite viscosity ratio in Ref. 16. The value of σ_s^* increases with the parameter k . Figure 4 shows the plot of $1/\sigma_s^*$ against k . The pattern of the viscous fingers at the infinite viscosity ratio is the fractal. The fractal dimension d_f decreases with the parameter k . Figure 5 shows the plot of the fractal dimension d_f against the parameter k . The fixed point $(1/\sigma_s^*, 0)$ is strongly dependent upon the parameter k . The fixed point represents the fractal pattern depending upon k . It is called the fractal point.

In Sec. IV we study the stability of the fixed points in the two-parameter space $(1/\sigma_s, 1/\sigma_a)$. The global flow in the two-parameter space will be obtained.

IV. CROSSOVER FROM FRACTAL TO DENSE STRUCTURE

We consider the crossover phenomena from the fractal to the Eden model. To find the global flow diagram in the two-parameter space $(1/\sigma_s, 1/\sigma_a)$, we choose a point in the parameter space, and calculate the renormalized surface conductance and the renormalized conductance ratio by using (12) and (15), to find a new point $(1/\sigma'_s, 1/\sigma'_a)$. We repeat this process to find the next point $(1/\sigma''_s, 1/\sigma''_a)$, and continue until we approach a stable fixed point. We use some initial points and plot the renormalization flow in the phase space for representative initial points. Figure 6 shows the renormalization flow for $k=2$, obtained by using (12) and (15). From the renormalization flow, we can determine the stabilities of the two fixed points: the Eden point and the fractal point. The Eden point is a node: it is stable in every direction. The fractal point is a saddle point: it is stable in the x direction and unstable in other directions. All the renormalization flows eventually merge into the Eden point. It is found from the flow diagram that there exists a crossover from the fractal pattern (the fractal fixed point) to the dense cluster (the Eden fixed point). The crossover line can be determined by following the renormalization flow which starts from an initial point very close to the fractal fixed point. It is indicated by the thick line in Fig. 6. We can calculate the crossover in the non-Newtonian fluids. Figure 7 shows the flow diagram for $k=3$. For comparison, the flow diagram of the Newtonian fluid ($k=1$) is shown in Fig. 8. Also the flow diagram of $k=\frac{1}{2}$ is indicated in Fig. 9. The crossover lines shift with the fractal fixed point according as the parameter k . The non-Newtonian fluids have important effects on the crossover phenomena from the fractal pat-

FIG. 6. Global flow diagram in two-parameter space $(1/\sigma_s, 1/\sigma_a)$ for $k=2$. There are two fixed points: the Eden point and the fractal point. All the renormalization flows are eventually sucked into the Eden point. The crossover line from the fractal pattern to the dense structure is indicated by the thick line.

FIG. 7. Global flow diagram for $k=3$.

tern to the dense structure. In order to quantify this crossover behavior, we define a crossover exponent ϕ and a crossover radius $r_\times$. We propose the scaling ansatz along the crossover line,

$$M(r, 1/\sigma_a; k) = r^{d_f} F_k((1/\sigma_a) r^\phi), \quad (26)$$

where M is the mass of the cluster, r the radius of gyration, d the embedding dimension, and $1/\sigma_a$ the conductance ratio. The same scaling form was already proposed by Lee, Coniglio, and Stanley¹⁸ for the Newtonian fluid ($k=1$). The scaling function may be dependent upon the parameter k . However, it is found that the crossover exponent ϕ is independent upon the parameter k . We assume

$$F_k(x) \sim \begin{cases} 1 & \text{if } x \ll 1, \\ x^{(d-d_f)/\phi} & \text{if } x \gg 1. \end{cases} \quad (27)$$

Here d_f is the fractal dimension depending upon the parameter k . Furthermore, we can estimate $r_\times$. From (27), $r_\times$ should be the value of the radius which corresponds to $x=1$, that is, $r_\times \sim (1/\sigma_a)^{-1/\phi}$. The crossover exponent ϕ shall be found by linearizing the renormalization equations and calculating the eigenvalues. The linearized relation of the renormalization equations is

FIG. 8. Global flow diagram for the Newtonian fluids $k=1$.

FIG. 9. Global flow diagram for $k = \frac{1}{2}$.

given by

$$\begin{bmatrix} \delta(1/\sigma_a)' \\ \delta\sigma_s' \end{bmatrix} = \begin{bmatrix} \frac{\partial(1/R_a)}{\partial(1/\sigma_a)} & \frac{\partial(1/R_a)}{\partial\sigma_s} \\ \frac{\partial R_s}{\partial(1/\sigma_a)} & \frac{\partial R_s}{\partial\sigma_s} \end{bmatrix}_{\text{fractal}} \begin{bmatrix} \delta(1/\sigma_a) \\ \delta\sigma_s \end{bmatrix}, \quad (28)$$

where the matrix is evaluated at the fractal fixed point. The values of the matrix elements are numerically calculated from the renormalization functions. The eigenval-

ues of the matrix are obtained as $\lambda_1 = 2.0$ and $\lambda_2 = 0.6$. These are independent upon the parameter k . The eigenvalues agree with those of the Newtonian fluid. The crossover exponent ϕ is given by

$$\phi = \ln \lambda_1 / \ln b = 1.0. \quad (29)$$

The exponent ϕ is independent upon the parameter k and consistent with that of the Newtonian fluid.

V. SUMMARY

We study the crossover phenomena in the viscous fingering of non-Newtonian fluids at a finite viscosity ratio. By using the two-parameter position-space renormalization-group method, we obtain the global flow diagrams in two-parameter space. We find the crossover from the fractal pattern to the dense structure. We show the dependence of the crossover upon the parameter k describing the different non-Newtonian fluids in the flow diagram. We find that the non-Newtonian fluids have important effects on the fractal pattern and the crossover but the crossover exponent is independent upon the non-Newtonian property.

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