Crossover phenomena in viscous fingering with nonviscous-flow threshold

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Models for viscous fingering are presented to show a crossover between the diffusion-limited-aggregation (DLA) fractal and the dense pattern by interplay between nonviscous and viscous forces. The velocity has a characteristic property: If the velocity is less than the critical value v_c , the displaced fluid behaves as a nonviscous flow, and if the velocity is larger than v_c , the displaced fluid flow is viscous. By using the dimensional analysis, a dimensionless parameter is found to govern the crossover between the DLA fractal and the dense pattern. It is shown that the crossover length from the DLA fractal to the dense pattern increases with the applied pressure. At a very low flow rate the pattern becomes the dense structure. At a high flow rate the pattern becomes the DLA fractal. The similarity of the result with the experiment of the viscous fingering through porous media is discussed. The model can also be applied to the viscous fingering under the gravitational field.

Fractal growth phenomena in pattern formation have recently attracted considerable attention.¹⁻⁹ The viscous fingering serves as one paradigm of fractal pattern formation. In its simplest form, one injects a fluid of low viscosity into a fluid of higher viscosity, using a Hele-Shaw cell.⁴ In the limiting case with zero interfacial tension and infinite viscosity ratio, one expects to find patterns that are isomorphic to diffusion-limited aggregation (DLA). Various experiments for viscous fingering have been performed to find the fractal structure of pattern. However, in real experimental situations, various factors affect the pattern formation. For example, at a finite viscosity ratio the asymptotic behavior of viscous fingers must eventually cross over from the DLA fractal to the dense pattern.^{10,11} The crossover has been recently analyzed by using the two-parameter real-space renormalization-group method.¹² In some experiments the polymer solutions showing a non-Newtonian fluid have been used for the displaced fluid.^{13,14} The pattern shows the fractal being different from the DLA fractal.¹⁵ Furthermore, the experiments have been performed by using porous media. Lenormand¹⁶ found that by tuning the flow rate in porous media made of interconnected channels, the pattern of the injected fluid evolves continuously from invasion percolation to DLA. Arian *et al.*¹⁷ presented a dielectric breakdown model with a growth probability threshold. The model describes the viscous fingering when a wetting fluid is displaced by a nonwetting fluid. The capillary pressure prevents the nonwetting-injected fluid to enter a pore. In the model, it was found that the crossover from the DLA fractal to the needle occurred. The experimental result by Lenormand¹⁶ is not explained by the model.

In this Rapid Communication, we consider the model describing the viscous fingering when a nonwetting fluid is displaced by a wetting fluid. The capillary pressure forces the wetting-injected fluid to enter a pore. We shall show that crossover from the DLA fractal to the dense structure occurs, the dense pattern appears at a low flow rate, and the DLA fractal appears at a high flow rate. We shall construct the model for the experiment performed by

Lenormand.¹⁶ In the experiment, very viscous oil filling in the porous media made of interconnected microchannels is displaced by injected air. At a very low flow rate, air invades the porous media by capillary forces. Viscous forces are negligible. At a high flow rate viscous forces dominate and the pattern is very similar to the DLA fractal. The invasion of injected fluid by capillary forces can also exist in the Hele-Shaw cell. There are two types of capillary forces: the one acting between fluid-fluid interface and the other acting on the interface between the fluid and the glass plate. Tao, Novotny, and Kaski¹⁸ showed by computer simulation that the capillary force on the fluid-fluid interface induced crossover from the dense pattern at shorter length scales to the DLA fractal at larger length scales. Here we consider the capillary force on the interface between the fluid and the glass plate and neglect the capillary force on the fluid-fluid interface. We assume that at the fluid-fluid interface the velocity of the fluids has the following characteristic property: If the velocity is less than the critical value v_c , the displaced fluid behaves like nonviscous flow, and if the velocity is superior to v_c , the displaced fluid flow is viscous. The injected fluid has no viscosity. At the interface, the velocity is described by

$$v = -KVP - v_c |\nabla P|^{n-1} \nabla P \quad (1)$$

at the interface, where $n \rightarrow 0+$ and K is the permeability coefficient. Figure 1 shows the plot of the velocity against the pressure gradient. In the limit of $n \rightarrow 0+$, Eq. (1) has the characteristic property having the critical velocity:

$$\text{if } |v| < v_c, \text{ flow is nonviscous,} \quad (2)$$

$$\text{if } |v| > v_c, \text{ flow is viscous.}$$

The pressure field of the displaced fluid satisfies the Laplace equation

$$\nabla^2 P = 0. \quad (3)$$

Because of the lack of viscosity of the injected fluid, the boundary condition on the interface is given by the con-

FIG. 1. The characteristic property of velocity against pressure gradient. If the velocity is less than the critical value v_c , the displaced fluid behaves like nonviscous flow, and if the velocity is greater than v_c , the displaced fluid flow is viscous.

stant pressure

$$P_s = \text{const.} \quad (4)$$

The applied pressure ΔP is given by $\Delta P = P_s - P_\infty$ (P_∞ is the pressure at far field). If the first term on the right-hand side of Eq. (1) dominates, Eq. (1) reduces to the ordinary Laplacian growth. The pattern will then be similar to the DLA fractal. On the other hand, when the second term on the right-hand side of Eq. (1) dominates, Eq. (1) reduces to the η model. In the limit of $n \rightarrow 0+$ the η model produces the dense pattern.

We can extend the model to the viscous fingering under the gravitational field. Then the velocity of the displaced fluid is described by

$$v = -K\nabla P - v_c |\nabla P|^{n-1} \nabla P, \quad (5)$$

where $n \rightarrow 0+$. The pressure field of the displaced fluid satisfies

$$\nabla(K\nabla P + v_c |\nabla P|^{n-1} \nabla P) = 0. \quad (6)$$

We note that the capillary force acts on the interface but the gravitational force acts on the bulk of the displaced fluid. If the term on the right-hand side of Eq. (5) dominates, Eq. (6) reduces to the Laplace equation. The pattern will then be similar to the DLA fractal. On the other hand, when the second term on the right-hand side of Eq. (5) dominates, Eq. (6) reduces to the power-law model for the non-Newtonian fluid. The pattern formation in the non-Newtonian viscous fingering was solved by Nagatani¹⁵ with the use of the real-space renormalization-group method. The dependence of the fractal dimension upon the parameter n was derived. The pattern forming in the power-law model is similar to that of the η model. In the limiting case of $n \rightarrow 0+$, the non-Newtonian

viscous finger forms the dense pattern. The model described by Eqs. (1) and (3) shows similar behavior to that described by Eqs. (5) and (6).

We shall study the crossover between the DLA fractal and the dense pattern by using dimensional analysis. We can discuss the crossover without solving the partial differential equations. For many flow systems one can select a characteristic length and a characteristic velocity. Thus the characteristic length is usually taken to be the pattern size L because of an infinite space and the characteristic velocity, $K(P_s - P_\infty)/L$ where $(P_s - P_\infty) = \Delta P$ is the applied pressure. Once this choice has been made, we may define the following dimensionless variables and differential operations:

$$x^* = x/L, \quad y^* = y/L, \quad (7)$$

$$P^* = P/\Delta P, \quad \nabla^* = L\nabla.$$

We may rewrite Eqs. (1) (at the interface) and (3) in terms of the foregoing dimensionless variables

$$v^* = -\nabla^* P^* - \chi |\nabla^* P^*|^{n-1} \nabla^* P^*, \quad (1')$$

$$\nabla^{*2} P^* = 0, \quad (3')$$

where $\chi = v_c L^{1-n}/K\Delta P^{1-n}$. Note that in the dimensionless form of Eqs. (1') and (3') the "scale factors," that is, those variables describing the overall size and the pressure of the system, are concentrated in one dimensionless group,

$$\chi = v_c L^{1-n}/K\Delta P^{1-n}. \quad (8)$$

If χ is sufficiently small Eqs. (1') and (3') reduce to the ordinary Laplacian growth model. On the other hand, when χ is sufficiently large Eqs. (1') and (3') reduce to the η model. The dimensionless parameter χ becomes smaller and smaller with increasing applied pressure. Then pattern formation is determined by the Laplacian growth. The DLA fractal appears. At high applied pressure or high flow rate, the DLA fractal appears. Inversely, the dimensionless parameter χ becomes larger and larger with decreasing applied pressure. Then the dense pattern appears. At very low applied pressure or a low flow rate, the pattern becomes a dense structure. With constant applied pressure, the finger crosses over from the DLA fractal to the dense pattern. The crossover from the DLA fractal to the dense pattern is similar to that in the viscous fingering at a finite viscosity ratio. However, the crossover with the increase of applied pressure appears in the system described by Eqs. (1) and (3) but not in the viscous fingering at a finite viscosity ratio.

Similarly, the model Eq. (6) describing the viscous fingering under gravitational force is rewritten in terms of the foregoing dimensionless variables

$$\nabla^*(\nabla^* P^* + \chi |\nabla^* P^*|^{n-1} \nabla^* P^*) = 0. \quad (6')$$

The same dimensional parameter χ appears in Eq. (6'). The crossover phenomena appearing in Eqs. (1) and (3) occur in the system described by Eq. (6).

In order to quantify this crossover behavior, we define a crossover exponent ϕ and a crossover radius r_c . We pro-

pose the scaling ansatz

$$M(r, \chi) = r^{d_f} F(\chi r^\phi), \quad (9)$$

where M is the mass of the cluster, r the radius of gyration, d_f the fractal dimension of DLA, and $\chi = v_c L^{1-n}/K\Delta P^{1-n}$ ($n \rightarrow 0+$). The scaling function is assumed as follows:

$$F(x) \approx \begin{cases} 1, & \text{if } x \ll 1 \\ x^{(d-d_f)/\phi}, & \text{if } x \gg 1 \end{cases} \quad (10)$$

Here d is the embedding dimension. The crossover radius scales as $r_c \approx \chi^{-1/\phi}$. We cannot obtain the crossover exponent ϕ . The crossover exponent may be derived by using the real-space renormalization-group method.¹⁹ Our scaling ansatz has the characteristic property depending upon the applied pressure ΔP through χ . With the increase of the applied pressure, χ decreases and the crossover occurs from the dense pattern to the DLA fractal. We shall show that the crossover with the increase of flow rate also occurs in the viscous fingering under a gravitational field. We consider the case in which the gravitational force directs toward the flow direction. The injected fluid has no viscosity. The density of the injected fluid is negligible as compared with that of the displaced fluid. The velocity of the displaced fluid is given by

$$v = -(k/\mu)\nabla P - (k/\mu)\rho g, \quad (11)$$

where k is the permeability of the medium to the fluid, μ the viscosity of the displaced fluid, ρ the density of the displaced fluid, and g the acceleration due to gravity. From conservation of volume, we obtain

$$\nabla[(k/\mu)(\nabla P + \rho g)] = 0. \quad (12)$$

Equations (11) and (12) coincide with Eqs. (5) and (6) in

the limiting case of $n \rightarrow 0+$. K and v_c correspond to k/μ and $\rho g k/\mu$. The dimensionless parameter χ is given by

$$\chi = \rho g L / \Delta P. \quad (13)$$

The crossover in pattern formation under the gravitational field occurs from the dense pattern to the DLA fractal with increasing applied pressure. A hydrodynamic instability can occur when two superposed viscous fluids are forced by gravity and an imposed pressure gradient.^{20,21} If the steady state was one of uniform motion with velocity $V = (k/\mu)\Delta P/L$ vertically downwards, and the interface between the two fluids was horizontal, then it could be shown that the interface was unstable for small deviations if

$$(\mu_2/k_2 - \mu_1/k_1)V > (\rho_2 - \rho_1)g, \quad (14)$$

where the subscript 1 refers to the upper fluid and the subscript 2 to the lower. In the limits of $\mu_1 = 0$ and $\rho_1 = 0$, the condition for destabilizing the interface is given by

$$\chi = (\mu/k)V/\rho g > 1, \quad (15)$$

where the subscript 2 is omitted. The critical velocity v_c corresponds to the transition point of the hydrodynamic instability. The interface is stable until the velocity is superior to the critical value v_c . When the velocity is superior to the critical value, the interface becomes unstable and the pattern evolves from the dense pattern to the DLA fractal.

In summary, we present the model to show the crossover from the dense pattern to the DLA fractal with an increasing flow rate. We show that the model describes the crossover phenomena occurring in the viscous fingering through porous media. The model can also be applied to the viscous fingering under gravity.

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