Starting functions in representation theory of algebras

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Starting functions in representation theory of algebras

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Starting functions in representation theory of algebras

Ken Nakashima

Department of Mathematics,
Graduate School of Science and Technology,
Shizuoka University, 836 Ohya, Suruga-ku,
Shizuoka, 422-8529, Japan
E-mail address: gehotan@gmail.com

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Introduction

Throughout this paper $\Bbbk$ is an algebraically closed field, and all vector spaces, algebras and linear maps are assumed to be finite-dimensional $\Bbbk$-vector spaces, finite-dimensional $\Bbbk$-algebras and $\Bbbk$-linear maps, respectively. Further all modules over an algebra considered here are assumed to be finite-dimensional modules. We denote the set of non-negative integers by $\mathbb{N}_0$.

0.1. Starting functions. We first define starting functions which are a key tool in this paper. For a module X over an algebra A, we often identify the isocalss $[X]$ of X with X itself. In particular, the set Γ_0 of varticies of the AR-quiver Γ of A is identified with a complete list of indecomposable A-modules.

Definition 0.1. Let A be an algebra and Γ the AR-quiver of A. Then for an indecomposable A-module X, the *starting function* $\mathrm{s}_X : \Gamma_0 \to \mathbb{N}_0$ of X is defined by

$$\mathrm{s}_X(Y) := \dim_{\Bbbk} \operatorname{Hom}_A(X, Y)$$

for all $Y \in \Gamma_0$.

Starting functions have the following property.

Proposition 0.2. *Let A be an algebra, Γ the AR-quiver of A,*

$$\begin{tikzcd}
 & M_1 \arrow[ddr] & \\
 & M_2 \arrow[dr] \arrow[d, dotted, no head] & \\
L \arrow[uur] \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & {} \arrow[d, dotted, no head] & N \\
 & M_n \arrow[ur] &
\end{tikzcd}$$

a mesh in Γ, and X an indecomposable A-module with $X \ncong N$. Then we have

$$\mathrm{s}_X(N) = \sum_{i=1}^{n} \mathrm{s}_X(M_i) - \mathrm{s}_X(L).$$

Starting functions were introduced by Gabriel to compute AR-quivers in [**6**], and were developed such as in [**9**],[**10**],[**4**],[**3**],[**2**, 6.6] and [**1**]. In this paper, we give two results obtained by using starting functions.

0.2. The first result (Chapter 1). This part is a generalization of Hironobu Suzuki's Master thesis [**11**] that dealt with representation-finite self-injective algebras of type A in a combinatorial way. Throughout Chapter 1 n is a positive integer, all algebras considered here are assumed to be basic, connected, finite-dimensional associative $\Bbbk$-algebras and all modules are finite-dimensional *right* modules.

Let Δ be a Dynkin graph of type A, D, E with the set $\Delta_0 := \{1, \dots, n\}$ of vertices. By Riedtmann [**10**, 2.5] the computation of the Auslander-Reiten quiver (AR-quiver for short) Γ_Λ of a representation-finite standard self-injective algebra Λ of type Δ is reduced to that of stable AR-quiver ${}_s\Gamma_\Lambda$ of Λ and the configuration $\mathcal{C}_\Lambda$ of Λ as the isomorphism $\Gamma_\Lambda \cong ({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda}$ shows. The stable AR-quiver ${}_s\Gamma_\Lambda$ is given by the orbit category presentation of Λ, namely if $\Lambda \cong \hat{A}/G$ for some tilted algebra A of type Δ, then ${}_s\Gamma_\Lambda \cong \mathbb{Z}\Delta/G$. Therefore to recover the AR-quiver Γ_Λ it suffices to compute the configuration $\mathcal{C}_\Lambda$ by using information of A. Set $\mathbf{C}(\Delta)$ to be the set of configurations on the translation quiver $\mathbb{Z}\Delta$ (see Definition 1.6), and $\mathbf{T}(\Delta)$ to be the set of isoclasses of tilted algebras of type Δ. Bretscher, Läser and Riedtmann gave a bijection $c\colon \mathbf{T}(\Delta) \to \mathbf{C}(\Delta)$ in [**4**], which makes it possible to compute $\mathcal{C}_\Lambda$ as the equivalence class of $c(A)$. Hence we can compute Γ_Λ using these data. But the map c is not given in a direct way, it needs a long computation of a function on $\mathbb{Z}\Delta$. In this paper we will give an easier way to calculate the map c by giving a map sending each projective indecomposable A-module over a tilted algebra A in $\mathbf{T}(\Delta)$ to an element of the configuration $c(A)$ in $\mathbf{C}(\Delta)$.

We fix an orientation of each Dynkin graph Δ to have a quiver $\vec{\Delta}$ as in the following table.

Δ	A_n $(n \geq 1)$	D_n $(n \geq 4)$	E_n $(n = 6, 7, 8)$
$\vec{\Delta}$	$\underset{1}{\circ} \to \underset{2}{\circ} \to \cdots \to \underset{n}{\circ}$	$\underset{1}{\circ} \to \cdots \to \underset{n-2}{\overset{\overset{n}{\circ}\uparrow}{\circ}} \to \underset{n-1}{\circ}$	$\underset{1}{\circ} \to \cdots \to \underset{n-3}{\overset{\overset{n}{\circ}\uparrow}{\circ}} \to \underset{n-2}{\circ} \to \underset{n-1}{\circ}$
m_Δ	n	$2n-3$	11, 17, 29, respectively

This orientation of Δ gives us a coordinate system on the set $(\mathbb{Z}\Delta)_0 := \mathbb{Z} \times \Delta_0$ of vertices of $\mathbb{Z}\Delta := \mathbb{Z}\vec{\Delta}$ as presented in [**4**, fig. 1] and in [**6**, Fig. 13].

Let A be a tilted algebra of type Δ. Then by identifying A with the $(0,0)$-entry of the repetitive category $\hat{A}$, the vertex set of AR-quiver Γ_A is embedded into the vertex set of the stable AR-quiver ${}_s\Gamma_{\hat{A}}$ $(\cong \mathbb{Z}\Delta)$ of $\hat{A}$. Further the configuration $\mathcal{C} := c(A)$ of $\mathbb{Z}\Delta$ computed in [**4**] is given by the vertices of $\mathbb{Z}\Delta$ corresponding to radicals of projective indecomposable $\hat{A}$-modules. Note that the configuration $\mathcal{C}$ has a period m_Δ listed in the table, thus $\mathcal{C} = \tau^{m_\Delta \mathbb{Z}}\mathcal{F}$ for some subset $\mathcal{F}$ of $\mathcal{C}$. By $\mathcal{P} = \{(p(i), i) \mid i \in \Delta_0\}$ we denote the set of images of the projective vertices of Γ_A in $\mathbb{Z}\Delta$ and set

$$\mathbb{N}\mathcal{P} := \{(m, i) \in (\mathbb{Z}\Delta)_0 \mid p(i) \leq m, i \in \Delta_0\}.$$

As is well-known, there exists the *Nakayama permutation* $\hat{\nu}$ on $(\mathbb{Z}\Delta)_0$ that is defined by the isomorphism

$$\mathbb{k}(\mathbb{Z}\Delta)(x, \text{-}) \cong D(\mathbb{k}(\mathbb{Z}\Delta)(\text{-}, \hat{\nu}x))$$

for all $x \in (\mathbb{Z}\Delta)_0$, where D is the $\mathbb{k}$-dual functor $\mathrm{Hom}_{\mathbb{k}}(\text{-}, \mathbb{k})$. The explicit formula of $\hat{\nu}$ is given in [**6**, pp. 48–50]. (Note that it should be corrected as $\hat{\nu}(p, q) = (p + q + 2, 6 - q)$ if $q \leq 5$ when $\Delta = E_6$ as pointed out in [**4**, 1.1]). In this paper we will define a map $\nu' \colon \mathcal{P} \to \mathbb{N}\mathcal{P}$ using supports of the functions $\dim_{\mathbb{k}} \mathbb{k}(\mathbb{Z}\Delta)(x, \text{-}) \colon \mathbb{N}\mathcal{P} \to \mathbb{Z}$, so-called the *starting functions* from $x \in \mathbb{N}\mathcal{P}$ (cf. [**6**, Fig. 15]). Then ν' has the following property.

Lemma 0.3. *Let $x \in \mathcal{P}$ and P be the projective indecomposable A-module corresponding to x. Then $\nu' x$ corresponds to the simple module* top P.

In this paper, we make use of modules over the algebra

$$B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$$

to compute an $\mathcal{F}$ above (the configuration (see Definition 3.1) of B gives $\mathcal{F}$.) We will define a map $\nu := \nu_B$ from the set of isoclasses of simple A-modules to $\mathcal{C}$, which coincides with the restriction of the Nakayama permutation $\hat{\nu}$ if A is hereditary.

LEMMA 0.4. *Assume that a vertex $x \in \mathbb{Z}\Delta$ corresponds to a simple A-module S and let Q be the injective hull of S over $\hat{A}$. Then $\nu(x)$ corresponds to* $\operatorname{rad} Q$*, and hence $\nu(x) \in \mathcal{C}$.*

Combining the lemmas above we obtain the following.

PROPOSITION 0.5. *If $x \in \mathcal{P}$, then $\nu(\nu' x) \in \mathcal{C}$.*

This leads us to the following definition.

DEFINITION 0.6. We define a map $c_A \colon \mathcal{P} \to \mathcal{C}$ by $c_A(x) := \nu(\nu' x)$ for all $x \in \mathcal{P}$.

The image of the map c_A gives us an $\mathcal{F}$ above, namely we have the following.

THEOREM 0.7. *The map c_A is an injection, and we have $c(A) = \tau^{m_\Delta \mathbb{Z}} \operatorname{Im} c_A$.*

COROLLARY 0.8. *If A is hereditary, then $c_A = \hat{\nu}\nu'$ and we have $c(A) = \tau^{m_\Delta \mathbb{Z}} \operatorname{Im} \hat{\nu}\nu'$.*

0.3. The second result (Chapter 2). Throughout Chapter 2 all modules over an algebra considered here are assumed to be finite-dimensional *left* modules. Let A be an algebra, $\mathcal{L}$ a complete set of representatives of isoclasses of indecomposable A-modules. Then the Krull-Schmidt theorem states the following. For each A-module M, there exists a unique map $\boldsymbol{d}_M \colon \mathcal{L} \to \mathbb{N}_0$ such that

$$M \cong \bigoplus_{L \in \mathcal{L}} L^{(\boldsymbol{d}_M(L))},$$

which is called an *indecomposable decomposition* of M. Therefore, $M \cong N$ if and only if $\boldsymbol{d}_M = \boldsymbol{d}_N$ for all A-modules M and N, i.e., the map $\boldsymbol{d}_M$ is a complete invariant of M under isomorphisms. Note that since M is finite-dimensional, the support $\operatorname{supp}(\boldsymbol{d}_M) := \{L \in \mathcal{L} \mid \boldsymbol{d}_M(L) \neq 0\}$ of $\boldsymbol{d}_M$ is a finite set. We call such a theory a *decomposition theory* that computes the indecomposable decomposition of a module. The Auslander-Reiten theory was developed since 1970s in representation theory of algebras. In many cases it enabled us to compute the Auslander-Reiten quiver (AR-quiver for short) of A that is a combinatorial description of the category of modules over A, the vertex set of which can be identified with the list $\mathcal{L}$, and which is constructed by gluing all meshes that is a visual form of almost split sequences over A. Thus all information on almost split sequences over A are encoded in the AR-quiver in a visual way. Namely, if $0 \to X \to Y \to Z \to 0$ is an almost split sequence, and $Y = \bigoplus_{i=1}^{n} Y_i^{(a_i)}$ $(n \geq 1)$ is an indecomposable decomposition of Y with Y_i pairwise non-isomorphic and $a_i \geq 1$ for all i, then we express it by the quiver

$$\begin{tikzcd}
& Y_1 \arrow[dr, bend left, "a_1\text{arrows}"] \arrow[dr, bend right] & \\
X \arrow[ur, bend left, "a_1\text{arrwos}"] \arrow[ur, bend right] \arrow[rr, dashed, "\vdots" description] \arrow[dr, bend right, "a_n\text{arrows}"'] \arrow[dr, bend left] & & Z \\
& Y_n \arrow[ur, bend right, "a_n\text{arrwos}"'] \arrow[ur, bend left] &
\end{tikzcd}$$

with a broken line between X and Z (note here that also both X, Z are indecomposable by definition of almost split sequences). The correspondence $\tau \colon Z \mapsto X$ is called the *AR-translation*. For example, it has the forms

$$\begin{tikzcd}
& Y_1 \arrow[dr] & \\
X \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & Z \\
& Y_2 \arrow[ur] &
\end{tikzcd} \text{ if } n = 2, a_1 = a_2 = 1, \text{ and } \begin{tikzcd}
& Y_1 \arrow[dr, bend left] \arrow[dr, bend right] & \\
X \arrow[ur, bend left] \arrow[ur, bend right] \arrow[rr, dashed, no head] & & Z
\end{tikzcd} \text{ if } n = 1, a_1 = 2.$$

The purpose of Chapter 2 is to develope a decomposition theory by using the knowledge of AR-quivers. Thus in the case that $\mathcal{L}$ is already computed and all almost split sequences are known, we aim to compute

(I) $\boldsymbol{d}_M$ and

(II) a finite set S_M such that $\operatorname{supp}(\boldsymbol{d}_M) \subseteq S_M \subseteq \mathcal{L}$

for all A-modules M. Note that (II) is needed to give a finite algorithm. If A is representation-finite (i.e., if the set $\mathcal{L}$ is finite), then the problem (II) is trivial because we can take $S_M := \mathcal{L}$.

In the topological data analysis, to analyse a point cloud C, a set of points in $\mathbb{R}^d$ for some fixed positive integer d, some important informations on C are encoded in the persistent homology M_C, which is just a module over the path algebra $\Lambda_n = \Bbbk Q_n$ of a quiver Q_n of the form

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

of Dynkin type A_n for some positive integer n. Therefore to understand the point cloud C we can use the knowledge of the map $\boldsymbol{d}_{M_C}$, which is nothing but the "persistence diagram" of C, where usually the values of $\boldsymbol{d}_{M_C}(L)$ $(L \in \mathcal{L})$ is presented by colors on $\mathcal{L}$, and $\mathcal{L}$ is expressed by a set of lattice points in a triangle. More precisely, the list $\mathcal{L}$ is given by $\{M(b,d) \mid 1 \leq b \leq d \leq n\}$ thanks to Gabriel's theorem on representations of Dynkin quivers, where $M(b,d)$ is given by

$$0 \to \cdots \to 0 \to \Bbbk \xrightarrow{1} \Bbbk \xrightarrow{1} \cdots \xrightarrow{1} \Bbbk \to 0 \to \cdots \to 0$$

with $\Bbbk$ starting at the vertex b and stopping at d. Therefore there exists a 1-1 correspondence between $\mathcal{L}$ and the set $\{(b,d) \mid 1 \leq b \leq d \leq n\}$, which is a subset of $\mathbb{Z}^2$ forming a triangle (See for instance papers [**21**] and [**16**]). Note that this set of vertices together with horizontal and vertical edges connecting them can be regarded as the underlying graph of the AR-quiver of Λ_n(See Example 2.7). To analyse property of a set of point clouds, e.g., a motion of a point cloud, persistent homologies were generalized to persistence modules M, which turn out to be modules over an algebra of the form $\Lambda_m \otimes_{\Bbbk} \Lambda_n$, where we allow any orientation of Q_m and Q_n, namely their underlying graphs have the form

$$1 \;\text{——}\; 2 \;\text{——}\; \cdots \;\text{——}\; l$$

of type A_l for $l = m, n$. Also in this case we need to compute the persistence diagram $\boldsymbol{d}_M$ to investigate the set of point clouds. It was done in [**18**] for the case $(m, n) = (2, 3)$. Our argument here can be applied to have a decomposition theory for persistence modules.

EXAMPLE 0.9. The decomposition theory for polynomial algebras in one variable $A = \Bbbk[x]$ is already well known. A finite-dimensional A-module is a pair (V, f) of a finite-dimensional $\Bbbk$-vector space V and an endomorphism f of V, and by fixing a basis of V we may regard $V = \Bbbk^d$ for $d := \dim V$ and f as a square matrix M of size d. In this way we identify (V, f) with M. In this case we may have $\mathcal{L} = \{J_i(\lambda) \mid i \geq 1, \lambda \in \Bbbk\}$, where $J_i(\lambda)$ is the Jordan cell of size $i \geq 1$ with eigenvalue $\lambda \in \Bbbk$. Let Λ be the set of all distinct eigenvalues of M and set $M_\lambda = M - \lambda E_d$ for $\lambda \in \Lambda$. Then the following is well known.

THEOREM 0.10. *The problems* (I) *and* (II) *are solved as follows.*

A solution to (I)**:** *Let $i \in \mathbb{N}$ and $\lambda \in \Lambda$. Then*

$$\boldsymbol{d}_M(J_i(\lambda)) = \begin{cases} d + \operatorname{rank} M_\lambda^2 - 2\operatorname{rank} M_\lambda & \text{if } i = 1; \text{ and} \\ \operatorname{rank} M_\lambda^{i+1} + \operatorname{rank} M_\lambda^{i-1} - 2\operatorname{rank} M_\lambda^i & \text{if } i \geq 2; \end{cases} \tag{0.1}$$

(*Note that by setting M_λ^0 to be the identity matrix of size d, the first equality has the same form as the second.*)

A solution to (II)**:** $S_M = \{J_i(\lambda) \mid i \leq d, \lambda \in \Lambda\}$.

In this paper, we will solve the problem (I) in the decomposition theory for any finite-dimensional algebra A. This turns out to be an extension of the result for $A = \Bbbk[x]$ above. In particular, for the Kronecker algebra $A = \Bbbk Q$ with $Q = (1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2)$, we will give an explicit formula for the problem (I) and a solution to the problem (II).

Decomposition theory is based on the approach as follows. Let A be a directed algebra. Then there is a complete set of isoclasses of indecomposable A-module $\{M_1, \cdots, M_n\}$ such that $\operatorname{Hom}_A(M_i, M_j) \neq 0$ implies $i \leq j$. An example of this numbering is given as follows.

EXAMPLE 0.11.

$$\text{Let } (Q, I) := \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & \circlearrowright & 3 \arrow[dl] \\ & 4 & \end{tikzcd} \quad \text{and } A := Q/I.$$

Then the AR-quiver Γ of A is as follows.

$$\begin{tikzcd}
& M_2 \arrow[rr, dash, dashed] \arrow[ddr] & & M_5 \arrow[rr, dash, dashed] \arrow[ddr] & & M_9 \arrow[ddr] & \\
& & & M_6 \arrow[dr] & & & \\
M_1 \arrow[uur] \arrow[dr] \arrow[rr, dash, dashed] & & M_4 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & M_8 \arrow[uur] \arrow[dr] \arrow[rr, dash, dashed] & & M_{11} \\
& M_3 \arrow[ur] \arrow[rr, dash, dashed] & & M_7 \arrow[ur] \arrow[rr, dash, dashed] & & M_{10} \arrow[ur] &
\end{tikzcd}$$

Let $M \in \operatorname{mod} A$. Assume $M \cong \bigoplus_{i=1}^{n} M_i^{(a_i)}$ where $a_i \in \mathbb{N}_0$. Define $b_j := \dim \operatorname{Hom}_A(M, M_j)$ for each $j \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
b_j &= \dim \operatorname{Hom}_A(\bigoplus_{i=1}^{n} M_i^{(a_i)}, M_j) \\
&= \sum_{i=1}^{n} a_i \cdot \dim \operatorname{Hom}_A(M_i, M_j) \\
&= \sum_{i=1}^{n} a_i \cdot s_{M_i}(M_j) \\
&= (a_1, \ldots, a_n) \begin{pmatrix} s_{M_1}(M_j) \\ \vdots \\ s_{M_n}(M_j) \end{pmatrix}.
\end{aligned}$$

Hense, we obtain

$$(b_1, \ldots, b_n) = (a_1, \ldots, a_n) \begin{pmatrix} s_{M_1}(M_1) & \cdots & s_{M_1}(M_n) \\ \vdots & \ddots & \vdots \\ s_{M_n}(M_1) & \cdots & s_{M_n}(M_n) \end{pmatrix}.$$

We set U_Γ to be the matrix on the right hand side whose (i, j)-entry has the value $s_{M_i}(M_j)$ of starting function s_{M_i}. Since $s_{M_i}(M_j) \neq 0$ implies $i \leq j$, U_Γ is an upper triangular matrix, which is invertible because the diagonal entries are equal to 1. Thus, we obtain

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) U_\Gamma^{-1}.$$

Hence, in order to realize the decomposition theory, we find it important to study U_Γ^{-1}. It is very interesting to see that U_Γ^{-1} is given by the information of AR-quiver Γ as follows.

Definition 0.12 (AR-matrix). Let A be an algebra, Γ the AR-quiver of A. Then the *AR-matrix* $V_\Gamma = [v_{ij}]_{i,j}$ of A is defined by

$$v_{ij} := \begin{cases} 1 & (j = i \text{ or } M_j \cong \tau^{-1}(M_i)) \\ -c & (M_i \xrightarrow{c \text{ arrows}} M_j \text{ in } \Gamma) \\ 0 & (\text{otherwise}) \end{cases}.$$

PROPOSITION 0.13. *V_Γ is the inverse of U_Γ.*

REMARK 0.14. We gave three deferent proofs of this proposition. The first one calculated cofactor matrices, the second one checked the equality $U_\Gamma V_\Gamma = E$, and the third one used the fact that U_Γ is the Cartan matrix of the module category of A (cf. Remark0.16(1) belows).

EXAMPLE 0.15. Let A and Γ be as in Example 1.11. Then we have

$$U_\Gamma = [s_{M_i}(M_j)]_{i,j} = \begin{bmatrix} 1&1&1&1&0&1&0&0&0&0&0\\ 0&1&0&1&0&1&1&1&1&0&0\\ 0&0&1&1&1&1&0&1&0&1&0\\ 0&0&0&1&1&1&1&2&1&1&0\\ 0&0&0&0&1&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&0&0\\ 0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \text{ and}$$

$$V_\Gamma = [v_{ij}]_{i,j} = \begin{bmatrix} 1&-1&-1&1&0&0&0&0&0&0&0\\ 0&1&0&-1&1&0&0&0&0&0&0\\ 0&0&1&-1&0&0&1&0&0&0&0\\ 0&0&0&1&-1&-1&-1&1&0&0&0\\ 0&0&0&0&1&0&0&-1&1&0&0\\ 0&0&0&0&0&1&0&-1&0&0&0\\ 0&0&0&0&0&0&1&-1&0&1&0\\ 0&0&0&0&0&0&0&1&-1&-1&1\\ 0&0&0&0&0&0&0&0&1&0&-1\\ 0&0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

It is easy to check $U_\Gamma V_\Gamma = E_{11} = V_\Gamma U_\Gamma$.

REMARK 0.16. After submitting the paper we are pointed out by Emerson Escolor and the referee that there was already a similar investigation [**17**] by Dowbor and Mróz in the literature, which we did not know before. Thus this work was done independently. We here list some relationships between their results and ours.

(1) They also have the same statement as Theorem 2.4 and its dual version, namely a solution to (I). Their proof is similar to the first version of ours using a "Cartan matrix" of the module category of an algebra A and an AR-matrix of A as its inverse, but the proof presented here does not use them and is much simplified by using the minimal projective resolutions of simple functors that are given by almost split sequences and sink maps into indecomposable injective modules, which also give the matrix V_Γ in Proposition 0.13 (Proposition 2.3).
(2) Our Theorem 3.3 gives an explicit way of computation of the map $\boldsymbol{d}_M$ for a module M by using ranks of matrices constructed by the structure matrices of M, while they did not give such formulas explicitly.

(3) To solve the problem (II) we used traces and rejects, which are easily computed and give us a decomposition of a module into the preprojective part, the preinjective part, the regular part with parameter ∞, and the regular part without parameter ∞. This together with Theorem 3.3 gives an effective computation of the indecomposable decomposition of a module M. For instance, if the preprojective part or the preinjective part of M is zero, it avoids unnecessary computations of the decomposition for those parts, in contrast, such computations are done in their algorithm repeatedly.

(4) Proposition 4.4 in [**17**] gives another way to compute regular direct summands, which seems to be interesting.

(5) They investigated also the cases of general $\tilde{A}$-quivers and representation-finite string algebras.

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CHAPTER 1

Tilted algebras and configurations of self-injective algebras of Dynkin type

In this chapter, we give an easier way to calculate a bijection from the set of isoclasses of tilted algebras of Dynkin type Δ to the set of configurations on the translation quiver $\mathbb{Z}\Delta$. Section 1 is devoted to preparations. In Section 2 we will give the complete list of indecomposable projectives and indecomposable injectives over a triangular matrix algebra B defined there. In Section 3 we state and prove the main results. Throughout this chapter, all modules are assumed to be finite-dimensional *right* modules.

1. Preliminaries

1.1. Algebras and categories. A category $\mathcal{C}$ is called a $\Bbbk$-*category* if the morphism sets $\mathcal{C}(x, y)$ are $\Bbbk$-vector spaces, and the compositions $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ are $\Bbbk$-bilinear for all $x, y, z \in \mathcal{C}_0$ ($\mathcal{C}_0$ is the class of objects of $\mathcal{C}$, we sometimes write $x \in \mathcal{C}$ for $x \in \mathcal{C}_0$). In the sequel *all categories are assumed to be* $\Bbbk$*-categories* unless stated otherwise.

To construct repetitive categories and to make use of a covering theory we need to extend the range of considerations from algebras to categories. First we regard an algebra as a special type of categories by constructing a category $\operatorname{cat} A$ from an algebra A as follows.

(1) We fix a decomposition $1 = e_1 + \cdots + e_n$ of the identity element 1 of A as a sum of orthogonal primitive idempotents.
(2) We set the object class of $\operatorname{cat} A$ to be the set $\{e_1, \dots, e_n\}$.
(3) For each pair (e_i, e_j) of objects, we set $(\operatorname{cat} A)(e_i, e_j) := e_j A e_i$.
(4) We define the composition of $\operatorname{cat} A$ by the multiplication of A.

The obtained category $\operatorname{cat} A$ is uniquely determined up to isomorphism not depending on the decomposition of 1. The category $C = \operatorname{cat} A$ is a small category having the following three properties.

(1) Distinct objects are not isomorphic.
(2) For each object x of C the algebra $C(x, x)$ is local.
(3) For each pair (x, y) of objects of C the morphism space $C(x, y)$ is finite-dimensional.

A small category with these three properties is called a *spectroid*[1] and its objects are sometimes called *points*. A spectroid with only a finite number of points is called *finite*. The category $\operatorname{cat} A$ is a finite spectroid. Conversely we can construct a matrix algebra

[1]a terminology used in [**7**]

from a finite spectroid C as follows.

$$\operatorname{alg} C := \{(m_{yx})_{x,y\in C} \mid m_{yx} \in C(x,y), \forall x,y \in C\}.$$

Here we have $\operatorname{alg}\operatorname{cat} A \cong A, \operatorname{cat}\operatorname{alg} C \cong C$. Therefore we can identify the class of algebras and the class of finite spectroids by using cat and alg.

A spectroid C is called *locally bounded* if for each point x the set $\{y \in C \mid C(x,y) \neq 0 \text{ or } C(y,x) \neq 0\}$ is a finite set. Of course algebras (= finite spectroids) are locally bounded. In the range of locally bounded spectroids we can freely construct repetitive categories or consider coverings.

Remark 1.1. We can construct the "path-category" $\Bbbk Q$ from a locally finite quiver Q in the same way as in the definition of the path-algebra. The only difference is in the following definition of compositions: For paths μ, ν with[2] $s(\mu) \neq t(\nu)$, it was defined as $\mu\nu = 0$ in the path-algebra, but in contrast the composition $\mu\nu$ is not defined in the path-category.

A locally bounded spectroid C is also presented as the form $\Bbbk Q/I$ for some locally finite quiver Q and for some ideal I of the path-category $\Bbbk Q$ such that I is included in the ideal of $\Bbbk Q$ generated by the set of paths of length 2. Here the quiver Q is uniquely determined by C up to isomorphism. This Q is called *the quiver* of C.

A (right) *module* over a spectroid C is a contravariant functor $C \to \operatorname{Mod}\Bbbk$. From a usual (right) module over an algebra A we can construct a contravariant functor $\operatorname{cat} A \to \operatorname{Mod}\Bbbk$ by the correspondence $e_i \mapsto Me_i$ for each point e_i in $\operatorname{cat} A$, and $f \mapsto (\cdot f\colon Me_j \to Me_i)$ for each $f \in e_jAe_i = (\operatorname{cat} A)(e_i, e_j)$. Conversely, from a contravariant functor $F\colon \operatorname{cat} A \to \operatorname{Mod}\Bbbk$ we can construct an A-module $\bigoplus_{i=1}^n F(e_i)$; and these constructions are inverse to each other. In this way we can identify A-modules and modules over $\operatorname{cat} A$.

The set of projective indecomposable modules over a spectroid C is given by $\{C(\text{-}, x)\}_{x\in C}$ up to isomorphism, and finitely generated projective C-modules are nothing but finite direct sums of these. Using this we can define finitely generated modules or finitely presented modules over C by the same way as those over algebras. By $\operatorname{mod} C$ we denote the full subcategory of $\operatorname{Mod} C$ consisting of finitely generated C-modules.

The dimension of a C-module M is defined to be the dimension of $\bigoplus_{x\in C} M(x)$. When C is locally bounded, a C-module is finitely presented if and only if it is finitely generated if and only if it is finite-dimensional.

1.2. Repetitive category.

Definition 1.2. Let A be an algebra with a basic set of local idempotents $\{e_1, \dots, e_n\}$.

(1) The *repetitive category* $\hat{A}$ of A is a spectroid defined as follows.

Objects: $\hat{A}_0 := \{x^{[i]} := (x, i) \mid x \in \{e_1, \dots, e_n\}, i \in \mathbb{Z}\}$.

Morphisms: Let $x^{[i]}, y^{[j]} \in \hat{A}_0$. Then we set

$$\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} := (f, i) \mid f \in A(x,y)\} & (j = i) \\ \{\varphi^{[i]} := (\varphi, i) \mid \varphi \in DA(y,x)\} & (j = i+1) \\ 0 & \text{otherwise.} \end{cases}$$

[2]Here $s(\mu)$ and $t(\nu)$ stand for the source of μ and the target of ν and compositions are written from the right to the left.

Compositions: The composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is defined as follows.

(i) If $j = i, k = j$, then we use the composition of A:
$$A(y, z) \times A(x, y) \to A(x, z).$$

(ii) If $j = i, k = j + 1$, then we use the right A-module structure of $DA(\text{-}, ?)$:
$$DA(z, y) \times A(x, y) \to DA(z, x).$$

(iii) If $j = i+1, k = j$, then we use the left A-module structure of $DA(\text{-}, ?)$:
$$A(y, z) \times DA(y, x) \to DA(z, x).$$

(iv) Otherwise the composition is zero.

(2) For each $i \in \mathbb{Z}$, we denote by $A^{[i]}$ the full subcategory of $\hat{A}$ whose object class is $\{x^{[i]} \mid x \in \{e_1, \dots, e_n\}\}$.

(3) We define the *Nakayama automorphism* ν_A of $\hat{A}$ as follows: for each $i \in \mathbb{Z}, x, y \in A, f \in A(x, y)$ and $\phi \in DA(y, x)$,
$$\nu_A(x^{[i]}) := x^{[i+1]}, \nu_A(f^{[i]}) := f^{[i+1]}, \nu_A(\varphi^{[i]}) := \varphi^{[i+1]}.$$

Remark 1.3. (1) The repetitive category of an algebra A is locally bounded. (2) The set of all $\mathbb{Z} \times \mathbb{Z}$-matrices with only a finite number of nonzero entries whose diagonal entries belong to A, $(i + 1, i)$ entries belong to DA for all $i \in \mathbb{Z}$, and other entries are zero forms an infinite-dimensional algebra without identity element, which is called the *repetitive algebra* of A. The repetitive category $\hat{A}$ is nothing but this repetitive algebra regarded as a spectroid in a similar way. This is not an algebra (= a finite spectroid) any more, but a locally bounded spectroid.

Definition 1.4 (Gabriel [**5**]). Let C be a locally bounded spectroid with a free[3] action of a group G. Then we define the *orbit category* C/G of C by G as follows.

(1) The objects of C/G are the G-orbits Gx of objects x of C.

(2) For each pair Gx, Gy of objects of C/G we set
$$(C/G)(Gx, Gy) := \left\{ ({}_bf_a)_{a,b} \in \prod_{(a,b)\in Gx\times Gy} C(a, b) \;\middle|\; {}_{gb}f_{ga} = g({}_bf_a), \text{ for all } g \in G \right\}.$$

(3) The composition is defined by
$$({}_dh_c)_{c,d} \cdot ({}_bf_a)_{a,b} := \left(\sum_{b\in Gy} {}_dh_b \cdot {}_bf_a \right)_{a,d}.$$
for all $({}_bf_a)_{a,b} \in (C/G)(Gx, Gy), ({}_dh_c)_{c,d} \in (C/G)(Gy, Gz)$. Note that each entry of the right hand side is a finite sum because C is locally bounded.

A functor $F : C \to C'$ is called a *Galois covering* with group G if it is isomorphic to the canonical functor $\pi : C \to C/G$, namely if there exists an isomorphism $H : C/G \to C'$ such that $F = H\pi$.

[3] $1 \neq g \in G$, $x \in C_0$ implies $gx \neq x$

Remark 1.5. Recall that a spectroid C is said to be *self-injective* in case $C(\text{-}, x)$ is injective in $\operatorname{mod} C$ and $C(x, \text{-})$ is injective in $\operatorname{mod} C^{\mathrm{op}}$ for all $x \in C_0$. If A is an algebra and a group G acts freely on the category $\hat{A}$, then $\hat{A}/G$ turns out to be a self-injective spectroid. In particular, when $\hat{A}/G$ is a finite spectroid, it becomes a self-injective algebra. In this way we can construct a great number of self-injective algebras.

Definition 1.6. From a quiver Q we can construct a translation quiver $\mathbb{Z}Q$ as follows.

- $(\mathbb{Z}Q)_0 := \mathbb{Z} \times Q_0$,
- $(\mathbb{Z}Q)_1 := \mathbb{Z} \times Q_1 \cup \{(i, \alpha') \mid i \in \mathbb{Z}, \alpha \in Q_1\}$,
- We define the sources and the targets of arrows by

$$(i, \alpha) \colon (i, s(\alpha)) \to (i, t(\alpha)),\ (i, \alpha') \colon (i, t(\alpha)) \to (i+1, s(\alpha))$$

  for all $(i, \alpha) \in \mathbb{Z} \times Q_1$.
- We take the bijection $\tau \colon (\mathbb{Z}Q)_0 \to (\mathbb{Z}Q)_0, (i, x) \mapsto (i-1, x)$ as the translation.

In addition, we can define a polarization by $(i+1, \alpha) \mapsto (i, \alpha')$, $(i, \alpha') \mapsto (i, \alpha)$. Note that by construction the translation quiver $\mathbb{Z}Q$ does not have any projective or injective vertices.

For example,

$$Q = \begin{tikzcd} & 1 \\ 2 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & 3 \end{tikzcd} \quad \text{gives} \quad \mathbb{Z}Q = \begin{tikzcd}[column sep=small] \cdots \arrow[rr, dash, dashed] & & (-1,1) \arrow[rr, dash, dashed] \arrow[dr, "{(-1,\alpha')}"] & & (0,1) \arrow[rr, dash, dashed] \arrow[dr, "{(0,\alpha')}"] & & (1,1) \arrow[r, dash] & \cdots \\ \cdots \arrow[r, dash] & (-1,2) \arrow[rr, dash, dashed] \arrow[ur, "{(-1,\alpha)}"] \arrow[dr, "{(-1,\beta)}"'] & & (0,2) \arrow[rr, dash, dashed] \arrow[ur, "{(0,\alpha)}"] \arrow[dr, "{(0,\beta)}"'] & & (1,2) \arrow[rr, dash, dashed] \arrow[ur, "{(1,\alpha)}"] \arrow[dr, "{(1,\beta)}"'] & & \cdots \\ \cdots \arrow[rr, dash, dashed] & & (-1,3) \arrow[rr, dash, dashed] \arrow[ur, "{(-1,\beta')}"'] & & (0,3) \arrow[rr, dash, dashed] \arrow[ur, "{(0,\beta')}"'] & & (1,3) \arrow[r, dash] & \cdots \end{tikzcd}.$$

Remark 1.7. When Q is a Dynkin quiver with the underlying graph Δ, the isoclass of $\mathbb{Z}Q$ does not depend on orientations of Δ, therefore we set $\mathbb{Z}\Delta := \mathbb{Z}Q$.

2. Triangular Matrix Algebras

In this section we will give the complete list of indecomposable projectives and indecomposable injectives over a triangular matrix algebra B defined in (2.1) below.

Definition 2.1. Let R and S be algebras, M be an S-R-bimodule. We define a category $\mathcal{C} = \mathcal{C}(R, S, M)$ as follows.

Objects: $\mathcal{C}_0 := \{(X, Y, f) \mid X_R \in \operatorname{mod} R, Y_S \in \operatorname{mod} S, f \in \operatorname{Hom}_R(Y \otimes_S M, X)\}$.
Morphisms: Let $(X, Y, f), (X', Y', f') \in \mathcal{C}_0$. Then we set

$$\mathcal{C}((X, Y, f), (X', Y', f')) := \left\{ (\phi_0, \phi_1) \in \operatorname{Hom}_R(X, X') \times \operatorname{Hom}_S(Y, Y') \;\middle|\; \begin{tikzcd} Y \otimes_S M \arrow[r, "f"] \arrow[d, "\phi_1 \otimes 1_M"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\phi_0"] \\ Y' \otimes_S M \arrow[r, "f'"'] & X' \end{tikzcd} \right\}.$$

Compositions: Let $(X, Y, f), (X', Y', f'), (X'', Y'', f'') \in \mathcal{C}_0$ and let

$$(\phi_0, \phi_1) \in \mathcal{C}((X, Y, f), (X', Y', f')), (\phi_0', \phi_1') \in \mathcal{C}((X', Y', f'), (X'', Y'', f'')).$$

Then we set

$$(\phi'_0, \phi'_1)(\phi_0, \phi_1) := (\phi'_0\phi_0, \phi'_1\phi_1) \in \mathcal{C}((X, Y, f), (X'', Y'', f'')).$$

Then the following is well known.

PROPOSITION 2.2. *Let R and S be algebras, M be an S-R-bimodule, and set $T := \begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$. Then*

$$\operatorname{mod} T \simeq \mathcal{C}(R, S, M).$$

Recall that an equivalence $F\colon \operatorname{mod} T \to \mathcal{C}(R, S, M)$ is given as follows.

Objects: For each $L \in (\operatorname{mod} T)_0$,

$$F(L) := (L\varepsilon_1, L\varepsilon_2, f_L),$$

where $\varepsilon_1 := \begin{bmatrix} 1_R & 0 \\ 0 & 0 \end{bmatrix}, \varepsilon_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1_S \end{bmatrix}$ and $f_L\colon L\varepsilon_2 \otimes_S M \to L\varepsilon_1$ is defined by $f_L(l\varepsilon_2 \otimes m) := l \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ for all $l \in L$ and $m \in M$.

Morphisms: For each $\alpha \in \operatorname{Hom}_T(L, L')$,

$$F(\alpha) := (\alpha\,|_{L\varepsilon_1}, \alpha\,|_{L\varepsilon_2}).$$

Let A be a tilted algebra of type Δ, and set

$$B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}, \quad \mathcal{C} := \mathcal{C}(A, A, DA). \tag{2.1}$$

Then we have $\operatorname{mod} B \simeq \mathcal{C}$ by Proposition 2.2. By this equivalence, we identify $\operatorname{mod} B$ with $\mathcal{C}$.

Let $\{e_1, \dots, e_n\}$ be a complete set of orthogonal local idempotents of A. Then as is easily seen $\{e_1^{[0]}, \dots, e_n^{[0]}, e_1^{[1]}, \dots, e_n^{[1]}\}$ is a complete set of orthogonal local idempotents of B, where we regard the objects $e_i^{[0]}$ of $A^{[0]}$ (resp. $e_i^{[1]}$ of $A^{[1]}$) as the elements $\begin{bmatrix} e_i & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & 0 \\ 0 & e_i \end{bmatrix}$) of B for all $i \in \{1, \dots, n\}$. Hence $\{e_1^{[0]}B, \dots, e_n^{[0]}B, e_1^{[1]}B, \dots, e_n^{[1]}B\}$ is a complete set of isoclasses of projective indecomposable B-modules.

PROPOSITION 2.3. *For each $i = 1, \dots, n$, we have*

$$F(e_i^{[0]}B) \cong (e_iA, 0, 0),$$
$$F(e_i^{[1]}B) \cong (e_i(DA), e_iA, \mathrm{can}).$$

PROOF.

$$F(e_i^{[0]}B) = (e_i^{[0]}B\varepsilon_1, e_i^{[0]}B\varepsilon_2, f_{e_i^{[0]}B}) = \left(\begin{bmatrix} e_iA & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right) \cong (e_iA, 0, 0),$$

$$F(e_i^{[1]}B) = (e_i^{[1]}B\varepsilon_1, e_i^{[1]}B\varepsilon_2, f_{e_i^{[1]}B}), = \left(\begin{bmatrix} 0 & 0 \\ e_i(DA) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & e_iA \end{bmatrix}, f_{e_i^{[1]}B} \right) \cong (e_i(DA), e_iA, \mathrm{can}),$$

where

$$\begin{tikzcd}
\begin{bmatrix} 0 & 0 \\ 0 & e_iA \end{bmatrix} \otimes_A DA \arrow[rr, "f_{e_i^{[1]}B}"] \arrow[d, "\wr"] \arrow[drr, phantom, "\circlearrowleft"] & & \begin{bmatrix} 0 & 0 \\ e_i(DA) & 0 \end{bmatrix} \arrow[d, "\wr"'] \\
e_iA \otimes_A DA \arrow[rr, dashed, "\mathrm{can}"'] & & e_i(DA).
\end{tikzcd}$$

$\square$

In addition $\{D(Be_1^{[0]}), \dots, D(Be_n^{[0]}), D(Be_1^{[1]}), \dots, D(Be_n^{[1]})\}$ is a complete set of iso-classes of injective indecomposable B-modules.

LEMMA 2.4. *For each $i = 1, \dots, n$, we have*

(1) $D\begin{bmatrix} Ae_i & 0 \\ (DA)e_i & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ D(Ae_i) & e_iA \end{bmatrix}$, *and*

(2) $D\begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix}$.

PROOF. (1) Define a map $\phi\colon \begin{bmatrix} 0 & 0 \\ D(Ae_i) & e_iA \end{bmatrix} \to D\begin{bmatrix} Ae_i & 0 \\ (DA)e_i & 0 \end{bmatrix}$ by

$$\begin{bmatrix} 0 & 0 \\ \alpha & a \end{bmatrix} \mapsto (\begin{bmatrix} b & 0 \\ \beta & 0 \end{bmatrix} \mapsto \alpha(b) + \beta(a))$$

for all $a \in e_iA, \alpha \in D(Ae_i), b \in Ae_i$ and $\beta \in (DA)e_i$. Then it is easy to check that ϕ is a homomorphism of right B-modules and that ϕ is injective. Since the dimensions of the left hand side and the right hand side are equal, ϕ is an isomorphism.

(2) Define a map $\psi\colon \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix} \to D\begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix}$ by

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \mapsto (\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \mapsto \alpha(a)),$$

which is easily seen to be an isomorphism. $\square$

PROPOSITION 2.5. *For each $i = 1, \dots, n$, we have*

$$F(D(Be_i^{[0]})) \cong (e_i(DA), e_iA, \mathrm{can}) \cong e_i^{[1]}B,$$
$$F(D(Be_i^{[1]})) \cong (0, e_i(DA), 0).$$

PROOF. Since $Be_i^{[0]} = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}\begin{bmatrix} e_i & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Ae_i & 0 \\ (DA)e_i & 0 \end{bmatrix}$, we have

$$D(Be_i^{[0]}) = D\begin{bmatrix} Ae_i & 0 \\ (DA)e_i & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ D(Ae_i) & e_iA \end{bmatrix}$$

by Lemma 2.4(1). Hence

$$F(D(Be_i^{[0]})) \cong F\begin{bmatrix} 0 & 0 \\ D(Ae_i) & e_iA \end{bmatrix} \cong (e_i(DA), e_iA, \mathrm{can}) \cong e_i^{[1]}B.$$

Since $Be_i^{[1]} = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & e_i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix}$, we have

$$D(Be_i^{[1]}) = D \begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix}$$

by Lemma 2.4(2). Hence $F(D(Be_i^{[1]})) = F \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix} \cong (0, e_i(DA), 0)$. □

3. Configurations

Throughout the rest of this paper Λ is a *standard representation-finite self-injective algebra*. If a module M is both projective and injective, we say that M is *projective-injective* for short.

3.1. Recover of AR-quivers from stable AR-quivers and configurations.

Definition 3.1. Let C be a locally bounded spectroid with the AR-quiver Γ_C. Then the set

$$\mathcal{C}_C := \{[\operatorname{rad} P] \in \Gamma_C \mid P : \text{projective-injective } C\text{-module}\}$$

is called the *configuration* of C.

In this section we compute the configuration of Λ.

Definition 3.2. Let Γ be a stable translation quiver, and $\mathcal{C}$ a subset of Γ_0. Then we define a translation quiver $\Gamma_{\mathcal{C}}$ by

$$(\Gamma_{\mathcal{C}})_0 := \Gamma_0 \sqcup \{p_x \mid x \in \mathcal{C}\},$$
$$(\Gamma_{\mathcal{C}})_1 := \Gamma_1 \sqcup \{x \to p_x,\ p_x \to \tau^{-1}x\},$$

where the translation of $\Gamma_{\mathcal{C}}$ is the same as that of Γ. In particular, p_x are projective-injective vertices for all $x \in \mathcal{C}$.

Remark 3.3. (1) Let C be a self-injective locally bounded spectroid. Then the quiver of the stable category $\underline{\operatorname{mod}}\, C$ of $\operatorname{mod} C$ is the full subquiver ${}_s\Gamma_C$ of Γ_C with

$$({}_s\Gamma_C)_0 := \{x \mid x \text{ is a stable vertex of } \Gamma_C\}$$

(namely ${}_s\Gamma_C$ is obtained from Γ_C by removing all projective vertices), which is a stable translation quiver.

(2) It holds that $\mathcal{C}_\Lambda \subseteq ({}_s\Gamma_\Lambda)_0$, and by Riedtmann [**10**, 2.5] we have

$$({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong \Gamma_\Lambda. \tag{3.1}$$

Thus we can recover the AR-quiver from the stable AR-quiver by using configurations.

Theorem 3.4. *Let Λ be a standard representation-finite self-injective algebra and Δ the Dynkin type of Λ. Then the following hold.*

(1) (*Waschbüsch* [**8, 12**]) *There exist a tilted algebra A of type Δ and an automorphism ϕ of $\hat{A}$ without fixed vertices such that $\Lambda \cong \hat{A}/\langle\phi\rangle$.*

(2) (*Riedtmann* [**9**]) *There is an isomorphism* $f\colon {}_s\Gamma_{\hat{A}} \to \mathbb{Z}\Delta$. *Denote also by* ϕ *the automorphism of* ${}_s\Gamma_{\hat{A}}$ *induced from* ϕ *canonically, and define an automorphism* ϕ' *of* $\mathbb{Z}\Delta$ *by the following commutative diagram:*

$$\begin{tikzcd} {}_s\Gamma_{\hat{A}} \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}\Delta \arrow[d, "\phi'"] \\ {}_s\Gamma_{\hat{A}} \arrow[r, "f"'] & \mathbb{Z}\Delta. \end{tikzcd}$$

Then we have ${}_s\Gamma_\Lambda \cong {}_s\Gamma_{\hat{A}}/\langle\phi\rangle \cong \mathbb{Z}\Delta/\langle\phi'\rangle$.

By the formula (3.1) to compute Γ_Λ it is enough to solve the following problem.

Problem 1. Let Λ be a standard representation-finite self-injective algebra, which has the form $\hat{A}/\langle\phi\rangle$ for some tilted algebra A of Dynkin type and an automorphism ϕ of $\hat{A}$ by Theorem 3.4. Then compute $\mathcal{C}_\Lambda$ from A.

Remark 3.5. Let $f'\colon {}_s\Gamma_\Lambda \to \mathbb{Z}\Delta/\langle\phi'\rangle$ be an isomorphism, and set $\mathcal{C} := f'(\mathcal{C}_\Lambda)$. Then we have

$$\Gamma_\Lambda \cong ({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong (\mathbb{Z}\Delta/\langle\phi'\rangle)_{\mathcal{C}}.$$

Thus we can compute Γ_Λ by Theorem 3.4(2) if we can obtain the set $\mathcal{C}$.

Theorem 3.6 (Gabriel [**5**, Theorem 3.6]). *Let R be a locally representation-finite and locally bounded $\Bbbk$-category, and G a group consisting of automorphisms of R such that G acts freely on R. Then the AR-quiver Γ_R of R has an induced G-action, and we have* $\Gamma_R/G \cong \Gamma_{R/G}$.

Corollary 3.7. *Let A be a tilted algebra of Dynkin type, and ϕ an automorphism of $\hat{A}$ without fixed vertices. Then we have*

$$\mathcal{C}_{\hat{A}}/\langle\phi\rangle \cong \mathcal{C}_\Lambda.$$

Therefore to solve Problem 1 it is enough to consider the following.

Problem 2. In the same setting as in Problem 1, compute $\mathcal{C}_{\hat{A}}$ from A.

Throughout the rest of this section

(1) let A be a tilted algebra of Dynkin type Δ, and set

(2) $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$.

By (1), Γ_A has a section $\mathcal{S}$ whose underlying graph is isomorphic to Δ.

3.2. Relationships between $\hat{A}$, B and A. We set as follows:

$$I_{0,1} = \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{0,1\}, j \in \{1,\dots,n\}\rangle,$$
$$I_0 = \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{0\}, j \in \{1,\dots,n\}\rangle,$$
$$I_1 = \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{1\}, j \in \{1,\dots,n\}\rangle.$$

Then $\hat{A}/I_{0,1} \cong B,\ \hat{A}/I_0 \cong A^{[0]}(\cong A)$ and $\hat{A}/I_1 \cong A^{[1]}(\cong A)$. We also have

$$B \Big/ \begin{bmatrix} 0 & 0 \\ DA & 0 \end{bmatrix} \cong A^{[0]} \times A^{[1]}.$$

We have the following surjective algebra homomorphisms

$$\begin{tikzcd}
 & & & A^{[0]} \\
\hat{A} \arrow[r, two heads] \arrow[urrr, two heads] \arrow[drrr, two heads] & B \arrow[r, two heads] & A^{[0]} \times A^{[1]} \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & & & A^{[1]},
\end{tikzcd}$$

which induce the following embeddings of categories

$$\begin{tikzcd}
 & & \operatorname{mod} A^{[0]} \arrow[dl, hook'] \\
\operatorname{mod} \hat{A} & \operatorname{mod} B \arrow[l, hook', "\sigma"'] & \\
 & & \operatorname{mod} A^{[1]}. \arrow[ul, hook]
\end{tikzcd}$$

We regard $\operatorname{mod} A \subseteq \operatorname{mod} B$ by the embedding $\operatorname{mod} A = \operatorname{mod} A^{[0]} \hookrightarrow \operatorname{mod} B$. The embeddings above give us the following embeddings of vertex sets of AR-quivers:

$$\begin{tikzcd}
 & & (\Gamma_{A^{[0]}})_0 = (\Gamma_A)_0 \arrow[dl, hook', "\sigma_0"'] \\
(\Gamma_{\hat{A}})_0 & (\Gamma_B)_0 \arrow[l, hook', "\sigma"'] & \\
 & & (\Gamma_{A^{[1]}})_0. \arrow[ul, hook, "\sigma_1"]
\end{tikzcd}$$

We define an ideal $\Bbbk(\mathbb{Z}\Delta)^+$ of the mesh category $\Bbbk(\mathbb{Z}\Delta)$ as follows:

$$\Bbbk(\mathbb{Z}\Delta)^+ := \langle (\mathbb{Z}\Delta)_1 + I_{\mathbb{Z}\Delta} \rangle,$$

where $I_{\mathbb{Z}\Delta}$ is the mesh ideal of the translation quiver $\mathbb{Z}\Delta$. Then the values of $m_\Delta := \min\{m \in \mathbb{N} \mid (\Bbbk(\mathbb{Z}\Delta)^+)^i = 0, \forall i \geq m\}$ are known to be as follows:

$$m_\Delta = \begin{cases} n & (\Delta = A_n) \\ 2n-3 & (\Delta = D_n) \\ 11 & (\Delta = E_6) \\ 17 & (\Delta = E_7) \\ 29 & (\Delta = E_8) \end{cases}.$$

We see that the following two propositions hold by [**4**, Sect. 2].

Proposition 3.8. *Let $i = 0, 1$.*

(1) *The full subquiver $\mathcal{S}_B^{[i]}$ of Γ_B with the vertex set $\sigma_i(\mathcal{S}_0)$ forms a section of ${}_s\Gamma_B$.*
(2) *The full subquiver $\mathcal{S}_{\hat{A}}^{[i]}$ of $\Gamma_{\hat{A}}$ with the vertex set $\sigma\sigma_i(\mathcal{S}_0)$ forms a section of ${}_s\Gamma_{\hat{A}}$.*

Remark 3.9. A quiver Q without oriented cycles will be regarded as a poset by the order defined as follows:

$$\text{For each } x, y \in Q_0, x \preceq y :\Leftrightarrow \text{there is a path in } Q \text{ from } x \text{ to } y.$$

Definition 3.10. (1) We set $\mathcal{H}_B$ to be the full subquiver of Γ_B defined by the set

$$(\mathcal{H}_B)_0 := \{x \in (\Gamma_B)_0 \mid a \preceq x \preceq b \text{ for some } a \in (\mathcal{S}_B^{[0]})_0, b \in (\mathcal{S}_B^{[1]})_0\}$$

of vertices.

(2) We set $\mathcal{H}_{\hat{A}}^{[0,1]}$ to be the full subquiver of $\Gamma_{\hat{A}}$ defined by the set

$$(\mathcal{H}_{\hat{A}}^{[0,1]})_0 := \{x \in (\Gamma_{\hat{A}})_0 \mid a \preceq x \preceq b \text{ for some } a \in (\mathcal{S}_{\hat{A}}^{[0]})_0, b \in (\mathcal{S}_{\hat{A}}^{[1]})_0\}$$

of vertices.

Proposition 3.11. (1) *The map $\sigma\colon (\Gamma_B)_0 \to (\Gamma_{\hat{A}})_0$ is uniquely extended to a quiver isomorphism $\mathcal{H}_B \to \mathcal{H}_{\hat{A}}^{[0,1]}$.*
(2) *We have $\mathcal{S}_{\hat{A}}^{[1]} = \tau^{-m_\Delta}\mathcal{S}_{\hat{A}}^{[0]}$. We set $\mathcal{S}_{\hat{A}}^{[n]} := \tau^{-nm_\Delta}\mathcal{S}_{\hat{A}}^{[0]}$ for all $n \in \mathbb{Z}$.*
(3) *Set $\mathcal{H}_{\hat{A}}^{[n,n+1]} := \tau^{-nm_\Delta}(\mathcal{H}_{\hat{A}}^{[0,1]})$ for all $n \in \mathbb{Z}$. Then for each $i = 0, 1$*

$$(\Gamma_{\hat{A}})_i = \bigcup_{n \in \mathbb{Z}} (\mathcal{H}_{\hat{A}}^{[n,n+1]})_i$$
$$(\mathcal{S}_{\hat{A}}^{[n+1]})_i = (\mathcal{H}_{\hat{A}}^{[n,n+1]})_i \cap (\mathcal{H}_{\hat{A}}^{[n+1,n+2]})_i$$

Roughly speaking, $\Gamma_{\hat{A}}$ is obtained by connecting infinite copies of $\mathcal{H}_B$ on both sides.

Example 3.12. Let A be the path algebra of the following quiver.

$$\begin{tikzcd} 1^{[0]} \arrow[r] & 2^{[0]} \arrow[r] & 3^{[0]} \end{tikzcd}$$

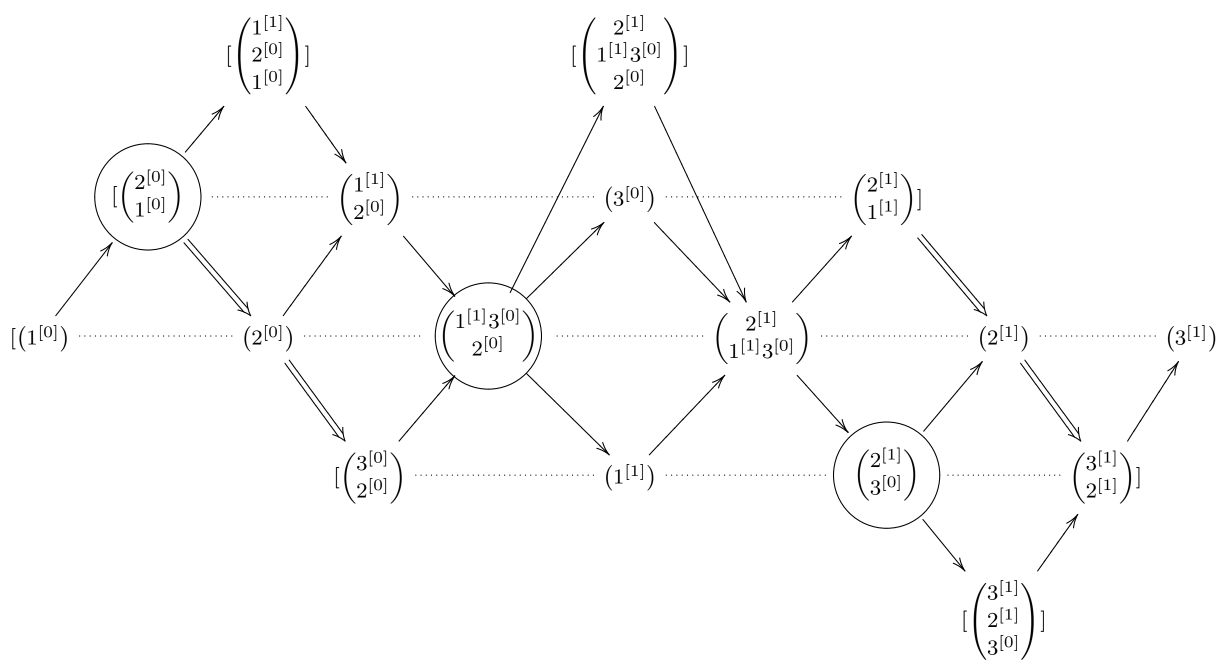
Then Γ_A is given as follows (double arrows represent a section).

$$\begin{tikzcd}[column sep=small]
 & {[}\begin{pmatrix} 2^{[0]} \\ 1^{[0]} \end{pmatrix}{]} \arrow[dr, Rightarrow] & & & \\
{[}(1^{[0]}) \arrow[ur] \arrow[rr, dotted, no head] & & (2^{[0]}) \arrow[dr, Rightarrow] \arrow[rr, dotted, no head] & & (3^{[0]}){]} \\
 & & & {[}\begin{pmatrix} 3^{[0]} \\ 2^{[0]} \end{pmatrix}{]} \arrow[ur] &
\end{tikzcd}$$

Therefore A is a tilted algebra of type A_3. Moreover $B = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} = \begin{bmatrix} A^{[0]} & 0 \\ (DA)^{[0]} & A^{[1]} \end{bmatrix}$ is an algebra given by following quiver with relations.

$$\begin{tikzcd}
1^{[0]} \arrow[r] & 2^{[0]} \arrow[r] \arrow[dl] \arrow[d, phantom, "\circlearrowright"] & 3^{[0]} \arrow[dl] \\
1^{[1]} \arrow[r] & 2^{[1]} \arrow[r] & 3^{[1]}
\end{tikzcd}$$

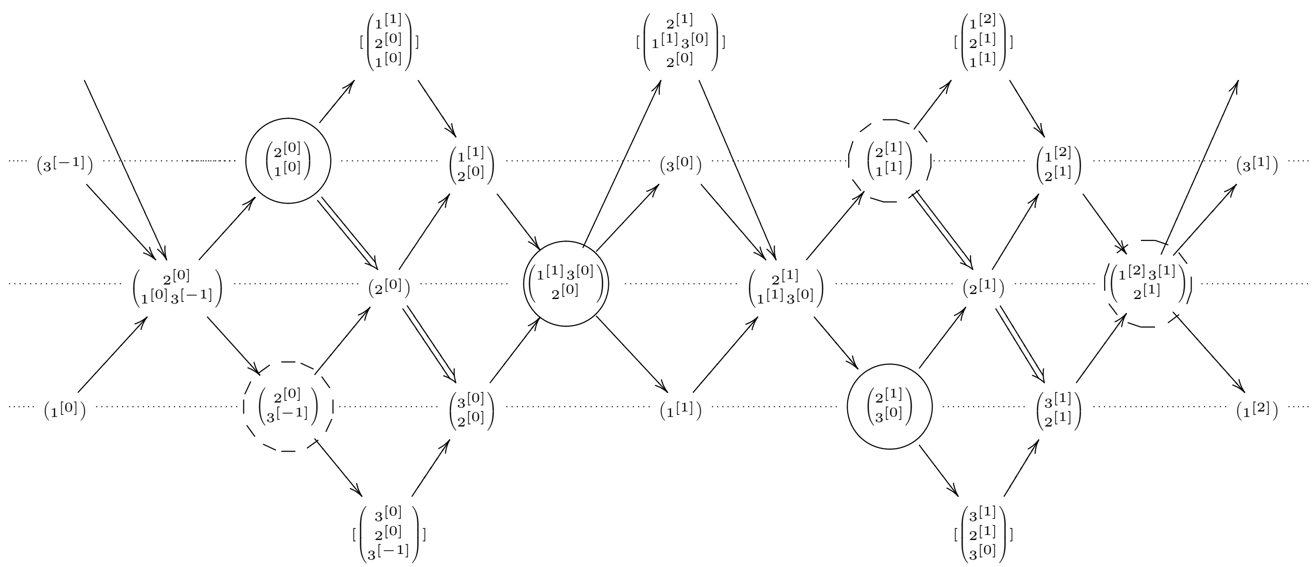
Then Γ_B is given as follows (elements of $\mathcal{C}_B$ are encircled).



In the above, $\mathcal{H}_B$ is given by the full subquiver consisting of vertices between the left section and the right section. $\hat{A}$ is given by the following quiver with relations.

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
1^{[-1]} \arrow[r] & 2^{[-1]} \arrow[r] \arrow[dl] \arrow[d, phantom, "\circlearrowright"] & 3^{[-1]} \arrow[dl] \\
1^{[0]} \arrow[r] & 2^{[0]} \arrow[r] \arrow[dl] \arrow[d, phantom, "\circlearrowright"] & 3^{[0]} \arrow[dl] \\
1^{[1]} \arrow[r] & 2^{[1]} \arrow[r] & 3^{[1]} \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

Then $\Gamma_{\hat{A}}$ is as follows (each element of $\mathcal{C}_{\hat{A}}$ is encircled by a broken or solid line, in particular solid circles present elements of $\mathcal{C}_B$). In this case we have $m_\Delta = 3$.



The following is immediate from Proposition 3.11.

COROLLARY 3.13. *We have* $\mathcal{C}_{\hat{A}} = \tau^{\mathbb{Z}m_\Delta}\sigma(\mathcal{C}_B)$.

By this corollary, Problem 2 is reduced to the following.

PROBLEM 3. Let A be a tilted algebra of Dynkin type Δ, and B as above. Then give the configuration $\mathcal{C}_B$ from A.

3.3. Configuration of B**.** The purpose of this subsection is to solve Problem 3.

DEFINITION 3.14. (1) We define an ideal $\mathcal{PI}$ of $\operatorname{mod} B$ as follows and set $\widetilde{\operatorname{mod}}\, B := (\operatorname{mod} B)/\mathcal{PI}$. For each $X, Y \in (\operatorname{mod} B)_0$

$$\mathcal{PI}(X,Y) := \{f \in \operatorname{Hom}_B(X,Y) \,|\, f \text{ factors through a projective-injective } B\text{-module}\}$$

Let $(\tilde{?})\colon \operatorname{mod} B \to \widetilde{\operatorname{mod}}\, B$ be the canonical functor and set

$$\widetilde{\operatorname{Hom}}_B(\tilde{X}, \tilde{Y}) := (\widetilde{\operatorname{mod}}\, B)(\tilde{X}, \tilde{Y})$$

for all $X, Y \in \operatorname{mod} B$. Thus $\tilde{X} = X$ for all $X \in (\operatorname{mod} B)_0$ and $\tilde{f} = f + \mathcal{PI}(X,Y)$ for all $f \in \operatorname{Hom}_B(X,Y)$.

(2) We denote by $\operatorname{mod}_{\mathcal{PI}} B$ the full subcategory of $\operatorname{mod} B$ consisting of B-modules without projective-injective direct summands.

(3) Let X and $Y \in \operatorname{mod}_{\mathcal{PI}} B$. Then it is well known that $\mathcal{PI}(X,Y) \subseteq \operatorname{rad}_B(X,Y)$. We set $\widetilde{\operatorname{rad}}_B(X,Y) := \operatorname{rad}_B(X,Y)/\mathcal{PI}(X,Y)$.

DEFINITION 3.15. We define the full translation subquiver $\tilde{\Gamma}_B$ of Γ_B by

$$(\tilde{\Gamma}_B)_0 := \{X \in (\Gamma_B)_0 \mid X \text{ is } \textit{not} \text{ projective-injective. }\}.$$

Moreover we set

$$\operatorname{supp}(s_X) := \{Y \in (\tilde{\Gamma}_B)_0 \mid s_X(Y) \neq 0\},$$

where the map $s_X\colon (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s_X(Y) := \dim \widetilde{\mathrm{Hom}}_B(\tilde{X}, \tilde{Y})$ $(Y \in (\tilde{\Gamma}_B)_0)$ for all $X \in (\tilde{\Gamma}_B)_0$.

Definition 3.16. Let P be a projective indecomposable A-module, and $\operatorname{rad} P = \bigoplus_{i=1}^r R_i$ with R_i indecomposable for all i. Then we define a full subquiver $\mathcal{R}_P$ of $\tilde{\Gamma}_B$ by

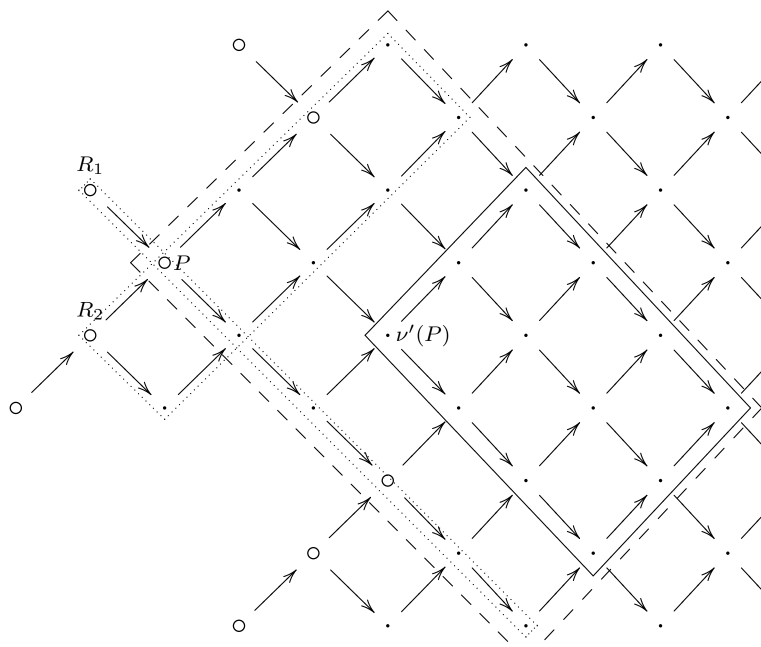
$$(\mathcal{R}_P)_0 := \operatorname{supp}(s_P) \setminus \left(\bigcup_{i=1}^r \operatorname{supp}(s_{R_i}) \right).$$

Definition 3.17. We regard the subquiver $\mathcal{R}_P$ as a poset by Remark 3.9. For a projective indecomposable A-module P if $\min \mathcal{R}_P$ exists, we define

$$\nu'(P) := \min \mathcal{R}_P,$$

otherwise we do not define the notation $\nu'(P)$.

Example 3.18. In the following figure, the vertices inside broken lines form $\operatorname{supp}(s_P)$ and those inside doted lines form $(\bigcup_{i=1}^r \operatorname{supp}(s_{R_i}))$. Therefore the subquiver $\mathcal{R}_P$ consists of the vertices inside solid lines, and $\nu'(P)$ is the minimum element of $\mathcal{R}_P$. Projective vertices are presented by white circles $\circ$.



Proposition 3.19. *Let P be a projective indecomposable A-module. Then $\nu'(P)$ is always defined and $\nu'(P) \cong \operatorname{top} P$.*

Proof. We set $J := \operatorname{rad} A$. We have $P \cong e_i A$ for some i. It is enough to show that $\operatorname{top} P$ is the minimum element of the poset $\mathcal{R}_P$. First we show that $\operatorname{top} P \in \mathcal{R}_P$, equivalently that $\operatorname{top} P \in \operatorname{supp}(s_P)$ but $\operatorname{top} P \notin \bigcup_{j=1}^r \operatorname{supp}(s_{R_j})$.

(i) $\underline{\operatorname{top} P \in \operatorname{supp}(s_P)}$. Let $\pi\colon P \to \operatorname{top} P$ be the canonical epimorphism in $\operatorname{mod} A$. It is enough to show that $\tilde{\pi} \neq 0$. Assume $\tilde{\pi} = 0$. Then π factors through a projective-injective

B-module Q in mod B, namely there is $(\alpha, \beta) \in \mathrm{Hom}_B(P, Q) \times \mathrm{Hom}_B(Q, \mathrm{top}\, P)$ such that $\pi = \beta\alpha$.

$$\begin{tikzcd} P \arrow[rr, two heads, "\pi"] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{top}\, P \\ & Q \arrow[ur, two heads, "\beta"'] & \end{tikzcd}$$

Since π is an epimorphism, so is β. Moreover

$$\beta(\mathrm{rad}\, Q) = \beta(Q\, \mathrm{rad}\, B) = \beta(Q)\, \mathrm{rad}\, B \cong (\mathrm{top}\, P)(\mathrm{rad}\, B) = 0$$

because $\mathrm{top}\, P$ is simple.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{rad}\, Q \arrow[r, hook] & Q \arrow[r, two heads] \arrow[d, "\beta"] & \mathrm{top}\, Q \arrow[r] \arrow[dl, dotted, two heads, "\gamma"] & 0 \\ & & \mathrm{top}\, P & & \end{tikzcd}$$

By the universality of cokernel, there is a unique $\gamma \in \mathrm{Hom}_B(\mathrm{top}\, Q, \mathrm{top}\, P)$ such that the above diagram is commutative. Since β is an epimorphism, so is γ. Then we have an exact sequence

$$0 \to \mathrm{Ker}\, \gamma \hookrightarrow \mathrm{top}\, Q \xrightarrow{\gamma} \mathrm{top}\, P \to 0.$$

Since Q is a projective-injective B-module, $\mathrm{top}\, Q$ has the form $\bigoplus_{j=1}^{n} (e_j^{[1]} B / e_j^{[1]}\, \mathrm{rad}\, B)^{m_j}$ for some $(m_j)_j \in \mathbb{Z}^n \backslash \{(0)_j\}$ by Propositions 2.3 and 2.5. Further since $\mathrm{top}\, Q$ is semisimple, the exact sequence above splits, namely $\mathrm{top}\, P$ is a direct summand of $\mathrm{top}\, Q$. But $\mathrm{top}\, P \cong e_i^{[0]} B / e_i^{[0]}\, \mathrm{rad}\, B$, a contradiction. Hence we must have $\tilde{\pi} \neq 0$.

(ii) $\underline{\mathrm{top}\, P \notin \bigcup_{j=1}^{r} \mathrm{supp}(s_{R_j})}$.

Assume $\mathrm{top}\, P \in \bigcup_{j=1}^{r} \mathrm{supp}(s_{R_j})$ ($= \mathrm{supp}(\mathrm{rad}\, P)$). Then $\widetilde{\mathrm{Hom}}_B(\mathrm{rad}\, P, \mathrm{top}\, P) \neq 0$, and there is an $f \in \mathrm{Hom}_B(\mathrm{rad}\, P, \mathrm{top}\, P)$ such that $\tilde{f} \neq 0$. Since $\mathrm{top}\, P$ is simple, f is an epimorphism. Taking the following diagram into account, we see that $\mathrm{top}\, P$ is a direct summand of $\mathrm{top}(\mathrm{rad}\, P)$ by the same argument as above.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{rad}(\mathrm{rad}\, P) \arrow[r, hook] & \mathrm{rad}\, P \arrow[r, two heads] \arrow[d, "f"] & \mathrm{top}(\mathrm{rad}\, P) \arrow[r] \arrow[dl, dotted, two heads, "\gamma'"] & 0 \\ & & \mathrm{top}\, P & & \end{tikzcd}$$

Since we have

$$\mathrm{top}\, P = e_i A / e_i J, \ \mathrm{top}(\mathrm{rad}\, P) = e_i J / e_i J^2,$$

$e_i A / e_i J$ is a direct summand of $e_i J / e_i J^2$. Then we have $e_i J e_i / e_i J^2 e_i \neq 0$ because $0 \neq e_i A e_i / e_i J e_i \hookrightarrow e_i J e_i / e_i J^2 e_i$. Thus there is a loop at the vertex i in the quiver of A. But it is impossible because A is a tilted algebra. Hence $\mathrm{top}\, P \in \mathcal{R}_P$.

(iii) $\underline{\mathrm{top}\, P \text{ is the minimum element in } \mathcal{R}_P.}$ It is enough to show that $\widetilde{\mathrm{Hom}}_B(\mathrm{top}\, P, X) \neq 0$ for all $X \in \mathcal{R}_P$. Since $X \in \mathrm{supp}(s_P)$ and $X \notin \bigcup_{j=1}^{r} \mathrm{supp}(s_{R_j})$, we have $\widetilde{\mathrm{Hom}}_B(P, X) \neq 0$ and $\widetilde{\mathrm{Hom}}_B(\mathrm{rad}\, P, X) = \widetilde{\mathrm{Hom}}_B(\bigoplus_{j=1}^{r} R_j, X) = \bigoplus_{j=1}^{r} \widetilde{\mathrm{Hom}}_B(R_j, X) = 0$. We take $\alpha \in$

$\operatorname{Hom}_B(P, X)$ such that $\tilde{\alpha} \neq 0$. Since $\widetilde{\operatorname{Hom}}_B(\operatorname{rad} P, X) = 0$, there is a projective-injective B-module Q such that $\alpha\sigma = hg$ for some g and h below.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad} P \arrow[r, hook, "\sigma"] \arrow[d, dashed, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] & \operatorname{top} P \arrow[r] & 0 \\
& Q \arrow[r, dashed, "h"'] & X & &
\end{tikzcd}$$

Since Q is injective, there is some $t \in \operatorname{Hom}_B(P, Q)$ such that $g = t\sigma$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad} P \arrow[r, hook, "\sigma"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft", near start] & P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] \arrow[dl, dashed, "t"] & \operatorname{top} P \arrow[r] & 0 \\
& Q \arrow[r, "h"'] & X & &
\end{tikzcd}$$

Since $\alpha\sigma = hg = ht\sigma$, we have $(\alpha - ht)\sigma = 0$. By the universality of the cokernel, there is a unique $\alpha' \in \operatorname{Hom}_B(\operatorname{top} P, X)$ such that $\alpha - ht = \alpha'\pi$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad} P \arrow[r, hook, "\sigma"] & P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha-ht"'] \arrow[dr, phantom, "\circlearrowleft", near start] & \operatorname{top} P \arrow[r] \arrow[dl, dashed, "\alpha'"] & 0 \\
& & X & &
\end{tikzcd}$$

Here α' is nonzero in $\widetilde{\operatorname{mod}}\, B$. Indeed, if α' factors through a projective-injective B-module Q', then $\alpha' = h'g'$ for some g' and h' in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad} P \arrow[r, hook, "\sigma"] & P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha-ht"'] \arrow[dr, phantom, "\circlearrowleft", near start] & \operatorname{top} P \arrow[r] \arrow[dl, dashed, "\alpha'"] \arrow[d, dashed, "g'"] \arrow[dl, phantom, "\circlearrowleft", near end, shift right=3] & 0 \\
& & X & Q' \arrow[l, dashed, "h'"] &
\end{tikzcd}$$

Since $\alpha - ht = \alpha'\pi = h'g'\pi$, we have $\alpha = h'g'\pi + ht = \begin{bmatrix} h' & h \end{bmatrix} \begin{bmatrix} g'\pi \\ t \end{bmatrix}$.

$$\begin{tikzcd}
P \arrow[rr, "\alpha"] \arrow[dr, "\begin{bmatrix} g'\pi \\ t \end{bmatrix}"'] \arrow[rr, phantom, "\circlearrowleft", shift right=6] & & X \\
& Q' \oplus Q \arrow[ur, "\begin{bmatrix} h' & h \end{bmatrix}"'] &
\end{tikzcd}$$

Thus $\tilde{\alpha} = 0$, a contradiction. Hence $\tilde{\alpha}' \neq 0$ and $\widetilde{\operatorname{Hom}}_B(\operatorname{top} P, X) \neq 0$. $\square$

We will give an alternative definition of the map ν' below, which is easier to compute than the first one.

Definition 3.20. Let $P \in \operatorname{mod} B$ be projective.

(1) Let $\mathcal{P}_P$ be the full subcategory of $\operatorname{mod} B$ consisting of projective modules Q such that P is not a direct summand of Q.

(2) We define an ideal $\mathcal{I}_P$ of $\operatorname{mod} B$ and the factor category $\underline{\operatorname{mod}}^P B := \operatorname{mod} B/\mathcal{I}_P$ of $\operatorname{mod} B$ by setting

$$\mathcal{I}_P(X,Y) := \{f \in \operatorname{Hom}_B(X,Y) \,|\, f \text{ factors through an object in } \mathcal{P}_P\},$$

and set

$$\underline{\operatorname{Hom}}_B^P(X,Y) := \operatorname{Hom}_B(X,Y)/\mathcal{I}_P(X,Y)$$

for all $X, Y \in \operatorname{mod} B$. Let $\underline{(?)}\colon \operatorname{mod} B \to \underline{\operatorname{mod}}^P B$ be the canonical functor. Thus $\underline{X} = X$ for all $X \in (\operatorname{mod} B)_0$ and $\underline{f} = f + \mathcal{I}_P(X,Y)$ for all $f \in \operatorname{Hom}_B(X,Y)$.

Let

$$\operatorname{supp}(s'_P) := \{X \in (\tilde{\Gamma}_B)_0 \mid s'_P(X) \neq 0\} \subseteq (\tilde{\Gamma}_B)_0$$

where the map $s'_P\colon (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s'_P(X) := \dim \underline{\operatorname{Hom}}_B^P(P,X)$ $(X \in (\tilde{\Gamma}_B)_0)$ for all $P \in (\tilde{\Gamma}_B)_0$.

Lemma 3.21. *Let Q and X be in $\operatorname{mod} B$. If Q is projective and there is an epimorphism $Q \to X$, then the projective cover of X is a direct summand of Q.*

Proof. By composing an epimorphism $Q \to X$ and the canonical epimorphism $X \to \operatorname{top} X$, we obtain a nonzero morphism $Q \to \operatorname{top} X$, which induced a retraction $\operatorname{top} Q \to \operatorname{top} X$. By taking projective covers, we have a retraction $Q \to P(\operatorname{top} X) = P(X)$, where $P(Y)$ denotes the projective cover of a B-module Y. □

Lemma 3.22. *If $f\colon X \to \operatorname{top} P$ is nonzero in $\operatorname{mod} B$, then $\underline{f} \neq 0$.*

Proof. Since $\operatorname{top} P$ is simple, f is an epimorphism. Assume that $\underline{f} = 0$. Then there is a $Q \in \mathcal{P}_P$ such that f factor through Q, namely there is a pair $(\alpha, \beta) \in \operatorname{Hom}_B(X,Q) \times \operatorname{Hom}_B(Q, \operatorname{top} P)$ such that $f = \beta\alpha$.

$$\begin{tikzcd}
X \arrow[rr, two heads, "f"] \arrow[dr, "\alpha"'] & \circlearrowleft & \operatorname{top} P \\
& Q \arrow[ur, two heads, "\beta"'] &
\end{tikzcd}$$

Since Q is projective and β is an epimorphism, P is a direct summand of Q by Lemma 3.21, a contradiction. □

Proposition 3.23. *Let P be a projective indecomposable A-module. Then the poset $\operatorname{supp}(s'_P)$ has the maximum element and we have*

$$\max \operatorname{supp}(s'_P) \cong \operatorname{top} P.$$

Thus $\nu'(P) = \max \operatorname{supp}(s'_P)$.

Proof. It is enough to show that $\operatorname{top} P$ is the maximum element of the poset $\operatorname{supp}(s'_P)$.
(i) First we show that $\operatorname{top} P \in \operatorname{supp}(s'_P)$, equivalently that $\underline{\operatorname{Hom}}_B^P(P, \operatorname{top} P) \neq 0$.

Let $\pi \in \operatorname{Hom}_B(P, \operatorname{top} P)$ be the canonical epimorphism. Then since π is nonzero, we have $\underline{\pi} \neq 0$ by Lemma 3.22.

(ii) Next we show that $\underline{\operatorname{Hom}}_B^P(X, \operatorname{top} P) \neq 0$ for all $X \in \operatorname{supp}(s'_P)$.

Take arbitrary $X \in \operatorname{supp}(s'_P)$. Then there is an $f \in \operatorname{Hom}_B(P, X)$ such that $\underline{f} \neq 0$. Let $P(X)$ be a projective cover of X. Since P is projective, there is $g \in \operatorname{Hom}_B(P, P(X))$ such that $f = \pi g$.

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "g"'] \\ P(X) \arrow[r, two heads, "\pi"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

Since $\underline{f} \neq 0$, we have $P(X) \notin \mathcal{P}_P$. Thus P is a direct summand of $P(X)$. Then $\operatorname{top} P$ is a direct summand of $\operatorname{top} P(X) \cong \operatorname{top} X$. By composing the canonical epimorphism $X \to \operatorname{top} X$ and a retraction $\operatorname{top} X \to \operatorname{top} P$, we obtain a nonzero morphism $h\colon X \to \operatorname{top} P$. By Lemma 3.22, we have $\underline{h} \neq 0$.

□

Next we define a map sending a simple A-module to an element of the configuration.

Lemma 3.24. *Let S be a simple A-module, and Q the injective hull of S in* $\operatorname{mod} B$. *Then the left $(\widetilde{\operatorname{mod}} B)$-module $\widetilde{\operatorname{Hom}}_B(S, \text{-})$ has a simple socle, and*

$$\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}) \cong \widetilde{\operatorname{Hom}}_B(\operatorname{rad} Q, \text{-}) / \widetilde{\operatorname{rad}}_B(\operatorname{rad} Q, \text{-}).$$

Proof. Note that Q is projective-injective by Proposition 2.5. Since $\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-})$ is a semisimple $\widetilde{\operatorname{mod}} B$-module, it has the form

$$\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}) \cong \bigoplus_{i=1}^{d} \operatorname{top} \widetilde{\operatorname{Hom}}_B(Y_i, \text{-}) \tag{3.2}$$

for some indecomposable B-modules Y_i ($i \in \{1, \dots, d\}$). Let $i \in \{1, \dots, d\}$ and put $Y := Y_i$. Then $(\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}))(Y) \neq 0$. Since $(\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}))(Y) \neq 0$, we can take an element $f \in \operatorname{Hom}_B(S, Y)$ such that $\tilde{f} \neq 0$ but $\tilde{\alpha}\tilde{f} = 0$ for all $\tilde{\alpha} \in \widetilde{\operatorname{rad}}_B(Y, Z)$. Consider the following diagram in $\operatorname{mod} B$.

$$\begin{tikzcd} S \arrow[r, tail, "f"] \arrow[d, hook, "\rho"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[ddl, dashed, "g"] \\ \operatorname{rad} Q \arrow[d, hook, "\sigma"'] & \\ Q & \end{tikzcd}$$

Then f is a monomorphism because S is simple and $f \neq 0$. Since Q is injective, there is a homomorphism $g \in \operatorname{Hom}_B(Y, Q)$ such that $\sigma\rho = gf$. If g is an epimorphism, then g is a retraction because Q is projective. Thus Q is a direct summand of the indecomposable module Y, and hence $Y \cong Q$. Then $\tilde{f} = 0$, a contradiction. Therefore g cannot be an epimorphism.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, hook, "\rho"'] & Y \arrow[dl, dashed, "h"'] \arrow[ddl, "g"] \\
\operatorname{rad} Q \arrow[d, hook, "\sigma"'] \arrow[d, phantom, "\circlearrowleft", shift left=3] & \\
Q &
\end{tikzcd}$$

Hence there is a homomorphism $h \in \operatorname{Hom}_B(Y, \operatorname{rad} Q)$ such that $g = \sigma h$. Then $\sigma\rho = \sigma h f$, and we have $\rho = hf$ because σ is a monomorphism.

Assume that Y is not isomorphic to $\operatorname{rad} Q$. Then $h \in \operatorname{rad}_B(Y, \operatorname{rad} Q)$, and $\tilde{h} \in \widetilde{\operatorname{rad}}_B(Y, \operatorname{rad} Q)$, we have $\tilde{\rho} = \tilde{h}\tilde{f} = 0$. Therefore there is a projective-injective B-module P such that $\rho = \beta\alpha$ for some $\alpha \in \operatorname{Hom}_B(S, P)$ and $\beta \in \operatorname{Hom}_B(P, \operatorname{rad} Q)$.

$$\begin{tikzcd}
S \arrow[rr, hook, "\sigma\rho"] \arrow[dr, "\alpha"] \arrow[ddr, phantom, "\circlearrowleft"] \arrow[dd, hook, "\rho"'] & \arrow[d, phantom, "\circlearrowleft"] & Q \arrow[dl, dashed, "\gamma"] \\
& P \arrow[dl, "\beta"] & \\
\operatorname{rad} Q & &
\end{tikzcd}$$

Since P is injective, there is a morphism $\gamma \in \operatorname{Hom}_B(Q, P)$ such that $\alpha = \gamma\sigma\rho$. Thus we have $\rho = \beta\gamma\sigma\rho$. Since $\beta\gamma$ is not a monomorphism, $(\beta\gamma)(S) = 0$, that is, $\rho = \beta\gamma\sigma\rho = 0$, a contradiction. Hence we must have $Y \cong \operatorname{rad} Q$. Then by (3.2) we have $\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}) \cong (\operatorname{top} \widetilde{\operatorname{Hom}}_B(\operatorname{rad} Q, \text{-}))^{(d)}$. Since $\operatorname{rad} Q$ is indecomposable, we have $\operatorname{End}_B(\operatorname{rad} Q)/\operatorname{rad}\operatorname{End}_B(\operatorname{rad} Q) \cong \Bbbk$. Then

$$\begin{aligned}
[\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-})](\operatorname{rad} Q) &\cong (\widetilde{\operatorname{End}}_B(\operatorname{rad} Q)/\operatorname{rad}\widetilde{\operatorname{End}}_B(\operatorname{rad} Q))^{(d)} \\
&\cong (\operatorname{End}_B(\operatorname{rad} Q)/\operatorname{rad}\operatorname{End}_B(\operatorname{rad} Q))^{(d)} \cong \Bbbk^{(d)}.
\end{aligned}$$

Here we have $d = 1$ because

$$\begin{aligned}
1 \leq d = \dim(\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}))(\operatorname{rad} Q) &\leq \dim \widetilde{\operatorname{Hom}}_B(S, \operatorname{rad} Q) \\
\leq \dim \operatorname{Hom}_B(S, \operatorname{rad} Q) &= \dim \operatorname{Hom}_B(S, S) = 1.
\end{aligned}$$

Thus $\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-})$ is simple, and $\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S, \text{-}) \cong \widetilde{\operatorname{Hom}}_B(\operatorname{rad} Q, \text{-})/\widetilde{\operatorname{rad}}_B(\operatorname{rad} Q, \text{-})$. $\square$

It follows by the lemma above that the poset $\operatorname{supp}(s_S)$ has the maximum element for each simple A-module S. We then set $\nu_B(S)$ to be the maximum element. The following is immediate.

Proposition 3.25. *Let S be a simple A-module, and Q the injective hull of S in* $\operatorname{mod} B$. *Then we have $\nu_B(S) \cong \operatorname{rad} Q$.*

We finally obtain the following by Propositions 3.23 and 3.25.

THEOREM 3.26. *Let $\mathcal{P}$ be a complete set of representatives of isoclasses of indecomposable projective A-modules. Then we have*

$$\mathcal{C}_B = \nu_B(\nu'(\mathcal{P})).$$

Hence as is stated before, $\mathcal{C}_\Lambda$ is obtained as follows.

THEOREM 3.27.

$$\mathcal{C}_\Lambda = \mathcal{C}_{\hat{A}}/\langle\phi\rangle = (\tau^{\mathbb{Z}m_\Delta}\sigma(\mathcal{C}_B))/\langle\phi\rangle = (\tau^{\mathbb{Z}m_\Delta}\sigma\nu_B\nu'(\mathcal{P}))/\langle\phi\rangle.$$

CHAPTER 2

Decomposition theory of modules: the case of Kronecker algebra

In this chapter, we give a general formula that computes the indecomposable decomposition of any finite-dimensional module over any finite-dimensional algebra. We presented two problems (I) and (II), and explained why decomposition theory is required in Introduction. We give a general solution of the problem (I) in Section 2, and apply it to the Kronecker algebra in Section 3. Moreover, We consider problem (II) for the Kronecker algebra in Section 4. Fundamental facts on the Kronecker algebra are collected in Section 1. Throughout this chapter, all modules are assumed to be finite-dimensional *left* modules.

1. Kronecker algebra

Let m, n be non-negative integers. Then we denote by $\operatorname{Mat} m, n$ the vector space of $m \times n$ matrices over $\Bbbk$, and by E_n the identity matrix of size n (for $n \geq 1$). By the isomorphism $\operatorname{Mat} m, n \to \operatorname{Hom}_{\Bbbk}(\Bbbk^n, \Bbbk^m)$ sending each $M \in \operatorname{Mat} m, n$ to the linear map given by the left multiplication by M we identify $\operatorname{Mat} m, n$ with $\operatorname{Hom}_{\Bbbk}(\Bbbk^n, \Bbbk^m)$, and regard each $M \in \operatorname{Mat} m, n$ as the corresponding linear map $\Bbbk^n \to \Bbbk^m$. If m or n is zero, we denote the matrices corresponding to the zero maps $\Bbbk^n \to \Bbbk^m$ by $\mathsf{J}_{m,n}$, respectively and call them *empty matrices*.

The Kronecker algebra A is a path algebra of the quiver $Q = (1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2)$, and the category $\operatorname{mod} A$ of finite-dimensional A-modules is equivalent to the category $\operatorname{rep} Q$ of finite-dimensional representations of Q over $\Bbbk$. We usually identify these categories. Recall that a representation M of Q is a diagram $M(1) \overset{M(\alpha)}{\underset{M(\beta)}{\rightrightarrows}} M(2)$ of vector spaces and linear maps, and the *dimension vector* of M is defined to be the pair $\underline{\dim}\, M := (\dim M(1), \dim M(2))$. When $\underline{\dim}\, M = (d_1, d_2)$, without loss of generality we may set $M(i) = \Bbbk^{d_i}$ for $i = 1, 2$ and $M(\alpha), M(\beta) \in \operatorname{Mat} d_2, d_1$. We denote M by the pair of matrices $(M(\alpha), M(\beta))$.

We here list well known facts on the Kronecker algebra (see Ringel [**20**, 3.2] for instance).

Theorem 1.1. *For the Kronecker algebra A the following statements hold.*

(1) *The list $\mathcal{L}$ of indecomposables is given as follows.*

Preprojective indecomposables: $\mathcal{P} := \left\{ P_n := \left(\begin{bmatrix} E_{n-1} \\ {}^t\mathbf{0} \end{bmatrix}, \begin{bmatrix} {}^t\mathbf{0} \\ E_{n-1} \end{bmatrix} \right) \middle| \, n \geq 1 \right\}$,

Preinjective indecomposables: $\mathcal{I} := \{ I_n := ([E_{n-1}, \mathbf{0}], [\mathbf{0}, E_{n-1}]) \mid n \geq 1 \}$, *Regular indecomposables:*

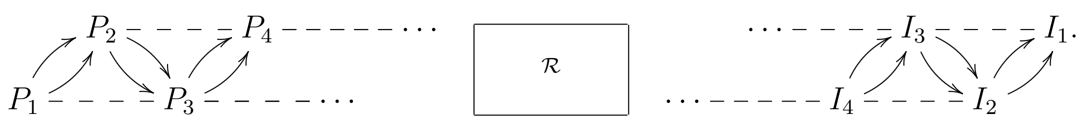
$$\mathcal{R} := \{R_n(\lambda) := (E_n, J_n(\lambda)), R_n(\infty) := (J_n(0), E_n) \mid n \geq 1, \lambda \in \Bbbk\},$$

where $\mathbf{0}$ *is the* $n \times 1$ *matrix with all entries 0. Note that*

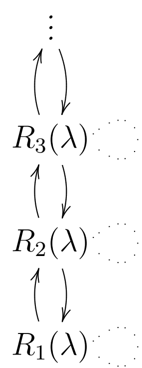
$$\underline{\dim}\, P_n = (n-1, n), \underline{\dim}\, I_n = (n, n-1), \underline{\dim}\, R_n(\lambda) = (n, n)$$

for all $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{P}^1(\Bbbk) = \Bbbk \cup \{\infty\}$.

(2) *The Auslander-Reiten quiver (AR-quiver for short) of* A *has the following form:*



In the above the rectangle part $\mathcal{R}$ *is given as the disjoint union of a family* $(\mathcal{R}_\lambda)_{\lambda \in \mathbb{P}^1(\Bbbk)}$ *of "homogeneous tubes"* $\mathcal{R}_\lambda$ *that has the form*



where dotted loops mean that for all $n \in \mathbb{N}$ *the Auslander-Reiten translation* τ *sends* $R_n(\lambda)$ *to itself:* $\tau R_n(\lambda) = R_n(\lambda)$.

(3) *Let* $X, Y \in \mathcal{L}$. *If* $\operatorname{Hom}_A(X, Y) \neq 0$, *then* X *is "on the left" of* Y, *i.e., one of the following occurs:*

(i) $X \cong P_m, Y \cong P_n$ *with* $m \leq n$,
(ii) $X \in \mathcal{P}, Y \in \mathcal{R} \cup \mathcal{I}$,
(iii) $X \cong R_m(\lambda), Y \cong R_n(\mu)$ *with* $\lambda = \mu$,
(iv) $X \in \mathcal{R}, Y \in \mathcal{I}$, *or*
(v) $X \cong I_m, Y \cong I_n$ *with* $m \geq n$.

Remark 1.2. (1) Let $m, n \in \mathbb{Z}$ with $m \leq n$. Then we note that there exists a monomorphism $P_m \to P_n$ and an epimorphism $I_n \to I_m$.

(2) Now for $(a_1, a_2), (b_1, b_2) \in \mathbb{Z}^2$ we define $(a_1, a_2) \leq (b_1, b_2)$ if and only if $a_i \leq b_i$ for $i = 1$ and 2. Then if there exists a monomorphism $T \to U$ (or an epimorphism $U \to T$) in $\operatorname{mod} A$, we have $\underline{\dim}\, T \leq \underline{\dim}\, U$.

2. Simple functors: a solution to the problem (I) in general

In this section we give a solution to the problem (I) by using Auslander-Reiten theory for an arbitrary algebra A.

Definition 2.1. For an indecomposable A-module L we set

$$\mathcal{S}_L := \operatorname{Hom}_A(L,\text{-})/\operatorname{rad}\operatorname{Hom}_A(L,\text{-}) : \operatorname{mod} A \to \operatorname{mod} \Bbbk.$$

It is well-known that $\mathcal{S}_L$ is a simple functor.

Lemma 2.2. *Let M be an A-module. Then for any indecomposable A-module L we have*

$$\boldsymbol{d}_M(L) = \dim \mathcal{S}_L(M).$$

Proof. Since L is indecomposable, $\operatorname{End}_A(L)$ is a local algebra. Therefore $\mathcal{S}_L(L) = \operatorname{End}_A(L)/\operatorname{rad}(\operatorname{End}_A(L))$ is a finite-dimensional skew field over the algebraically closed field $\Bbbk$, and hence $\mathcal{S}_L(L) \cong \Bbbk$. If $X \not\cong L$, then $\operatorname{End}_A(L) = \operatorname{rad}(\operatorname{End}_A(L))$, and $\mathcal{S}_L(X) = 0$. Thus

$$\mathcal{S}_L(X) \cong \begin{cases} \Bbbk & \text{if } X \cong L \\ 0 & \text{if } X \not\cong L \end{cases}$$

for all indecomposable A-modules X. Therefore, the indecomposable decomposition

$$M \cong \bigoplus_{L \in \mathcal{L}} L^{(\boldsymbol{d}_M(L))}$$

of M gives us

$$\mathcal{S}_L(M) \cong \Bbbk^{(\boldsymbol{d}_M(L))},$$

which shows the assertion. □

Recall the following fundamental statement in the Auslander-Reiten theory (see Auslander-Reiten [**15**] or Assem-Simson-Skowroński [**14**, IV, 6.11.]):

Proposition 2.3. *Let L be an indecomposable A-module. When L is non-injective, let $0 \to L \xrightarrow{f} \bigoplus_{X \in J_L} X^{(a(X))} \xrightarrow{g} \tau^{-1}L \to 0$ be an almost split sequence starting at L with $J_L \subseteq \mathcal{L}$ and $a(X) \geq 1$ $(X \in J_L)$. When L is injective, let $f\colon L \to L/\operatorname{soc} L = \bigoplus_{X \in J_L} X^{(a(X))}$ be the canonical epimorphism (note that $J_L = \emptyset$ if L is simple injective). Then the simple functor $\mathcal{S}_L$ has a minimal projective resolution*

$$0 \to \operatorname{Hom}_A(\tau^{-1}L,\text{-}) \xrightarrow{\operatorname{Hom}_A(g,\text{-})} \bigoplus_{X \in J_L} \operatorname{Hom}_A(X,\text{-})^{(a(X))} \xrightarrow{\operatorname{Hom}_A(f,\text{-})} \operatorname{Hom}_A(L,\text{-}) \xrightarrow{can} \mathcal{S}_L \to 0,$$

where $g = 0$ and $\tau^{-1}L = 0$ if L is injective.

Proposition 2.3 together with Lemma 2.2 readily gives us the following.

Theorem 2.4. *Let M be an A-module. Then for any indecomposable A-module L we have*

$$\boldsymbol{d}_M(L) = \dim \operatorname{Hom}_A(L, M) - \sum_{X \in J_L} a(X) \dim \operatorname{Hom}_A(X, M) + \dim \operatorname{Hom}_A(\tau^{-1}L, M).$$

Remark 2.5. When an algebra A is of the form $\Bbbk Q/I$ for some quiver Q and some ideal I of $\Bbbk Q$, it is possible to compute $\dim \operatorname{Hom}_A(H, M)$ for every $H, M \in \operatorname{mod} A$ by using the rank of a suitable matrix as follows, and thus $\boldsymbol{d}_M(L)$ in Theorem 2.4 is computable. First regard A-modules H and M as representations $(H(i), H(\alpha))_{i\in Q_0,\alpha\in Q_1}$ and $(M(i), M(\alpha))_{i\in Q_0,\alpha\in Q_1}$ of Q, respectively. Then by definition we have

$$\operatorname{Hom}_A(H, M) = \{(f_i)_{i\in Q_0} \in \prod_{i\in Q_0} \operatorname{Hom}_{\Bbbk}(H(i), M(i)) \ \mid M(\alpha)f_i = f_j H(\alpha), \forall \alpha : i \to j \text{ in } Q_1\}. \tag{2.1}$$

Therefore

$$\operatorname{Hom}_A(H, M) \cong \{\boldsymbol{x} \in \Bbbk^N \mid B\boldsymbol{x} = 0\},$$

where $N := \sum_{i\in Q_0} \dim H(i) \dim M(i)$ and B is a $|Q_1| \times N$-matrix given as the coefficient matrix of the homogeneous system of linear equations $M(\alpha)f_i - f_j H(\alpha) = 0$ for f_i. Hence we obtain the equality:

$$\dim \operatorname{Hom}_A(H, M) = N - \operatorname{rank} B.$$

EXAMPLE 2.6. Let $A := \Bbbk[x]$ be the polynomial algebra in one variable. Although it is an infinite-dimensional algebra, the category $\operatorname{mod} A$ of finite-dimensional A-modules is well understood because $\Bbbk[x]$ is a principal ideal domain, and we can apply Auslander-Reiten theory to $\operatorname{mod} A$. It is easy to give all almost split sequences over $\Bbbk[x]$. Namely, they are given as follows:

$$\begin{gathered} 0 \to J_1(\lambda) \to J_2(\lambda) \to J_1(\lambda) \to 0, \\ 0 \to J_i(\lambda) \to J_{i-1}(\lambda) \oplus J_{i+1}(\lambda) \to J_i(\lambda) \to 0 \end{gathered} \tag{2.2}$$

for all $i \geq 2$ and $\lambda \in \Bbbk$. This is verified by the similar argument used in the Nakayama algebra case (cf. [**14**, 4.1 Theorem]). The reader may notice a similarity between (0.1) and (2.2), which will become clear now. Let $M = (\Bbbk^n, M)$ be an A-module. Then we have

$$\dim \operatorname{Hom}_A(J_i(\lambda), M) = n - \operatorname{rank} M_\lambda^i, \tag{2.3}$$

which together with Theorem 2.4 and the formula (2.2) yields the formula (0.1).

Indeed, let $X \in \operatorname{Mat} n, i$, and put X_j to be the j-th column of X $(j = 1, \dots, i)$. Then by (2.1) $X \in \operatorname{Hom}_A(J_i(\lambda), M)$ iff $MX = XJ_i(\lambda) = X(\lambda E_i + J_i(0)) = \lambda X + XJ_i(0)$ iff $M_\lambda X = XJ_i(0)$ iff $M_\lambda(X_1, \dots, X_i) = (0, X_1, \dots, X_{i-1})$ iff M_λ maps X_j's as follows

$$X_i \mapsto X_{i-1} \mapsto \cdots \mapsto X_1 \mapsto 0.$$

Hence the correspondence $X \mapsto X_i$ yields the isomorphism (the inverse is given by the correspondence $v \mapsto [M_\lambda^{i-1}v, \dots, M_\lambda v, v]$)

$$\operatorname{Hom}_A(J_i(\lambda), M) \cong \{v \in \Bbbk^n \mid M_\lambda^i v = 0\} = \operatorname{Ker} M_\lambda^i,$$

which shows the equality (2.3).

EXAMPLE 2.7. Let n be a positive integer and set $A := \Bbbk Q$, where Q is a Dynkin quiver $1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$ of type A_n. Decomposition Theory for modules over this algebra has important applications in the topological data analysis (See Introduction). Let $M := (M_i)_{i=1}^n := (\Bbbk^{(a_1)} \xrightarrow{M_1} \Bbbk^{(a_2)} \xrightarrow{M_2} \cdots \xrightarrow{M_{n-1}} \Bbbk^{(a_n)})$ be a representation of Q

(i.e. an A-module). Then the morphism space $\mathrm{Hom}_A(M(b,d),M)$ is the set of sequences $(f_i : M(b,d)(i) \to \Bbbk^{(a_i)})_{i=1}^{n}$ that make the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \arrow[r, "0"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, phantom, "\circlearrowleft"] & \Bbbk \arrow[r, "1"] \arrow[d, "f_b"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \arrow[r, "1"] \arrow[dr, phantom, "\circlearrowleft"] & \Bbbk \arrow[r, "0"] \arrow[d, "f_d"'] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \arrow[r, "0"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[d, "0"'] \\
\Bbbk^{(a_1)} \arrow[r, "M_1"'] & \cdots \arrow[r, "M_{b-2}"'] & \Bbbk^{(a_{b-1})} \arrow[r, "M_{b-1}"'] & \Bbbk^{(a_b)} \arrow[r, "M_b"'] & \cdots \arrow[r, "M_{d-1}"'] & \Bbbk^{(a_d)} \arrow[r, "M_d"'] & \Bbbk^{(a_{d+1})} \arrow[r, "M_{d+1}"'] & \cdots \arrow[r, "M_{n-1}"'] & \Bbbk^{(a_n)},
\end{tikzcd}$$

where $M(b,d)(i) := \begin{cases} \Bbbk & (b \le i \le d) \\ 0 & (otherwise) \end{cases}$. In particular, if $d = n$ (namely $M(b,d)$ is projective), then

$$\mathrm{Hom}_A(M(b,n),M) \cong \{(f_i)_{i=b}^{n} \mid M_b f_b = f_{b+1}, \ldots, M_{n-1} f_{n-1} = f_n\} \cong \Bbbk^{(a_b)},$$

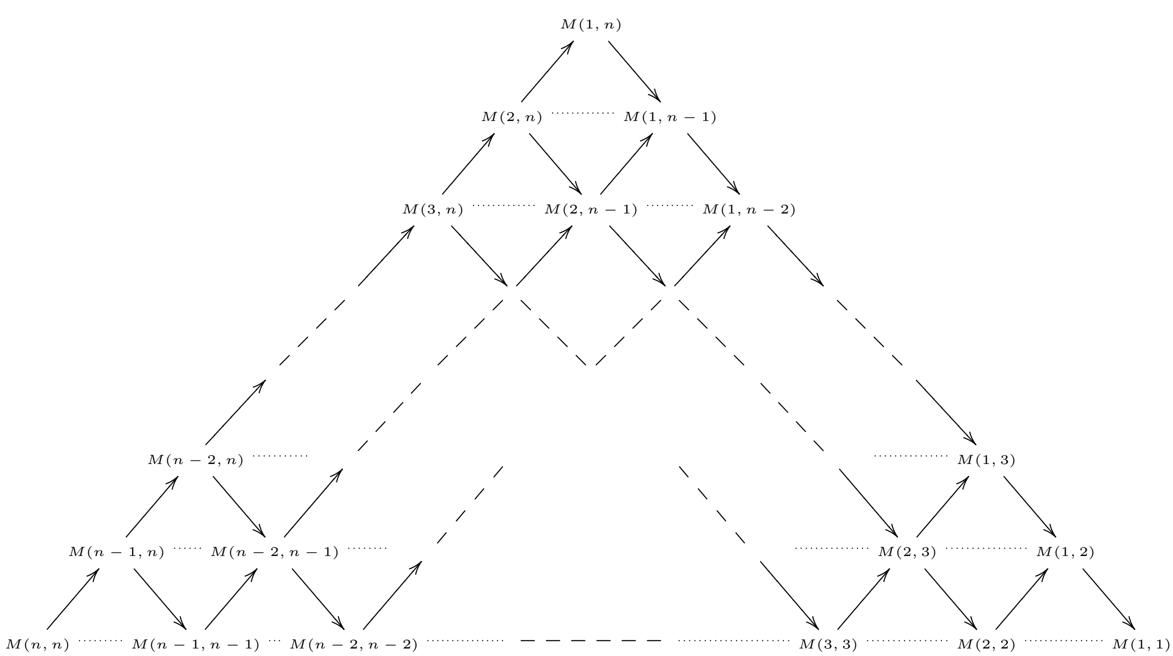
and if $d \le n-1$, then

$$\begin{aligned}
\mathrm{Hom}_A(M(b,d),M) &\cong \{(f_i)_{i=b}^{d} \mid M_b f_b = f_{b+1}, \ldots, M_{d-1} f_{d-1} = f_d, M_d f_d = 0\} \\
&\cong \{f_b \in \Bbbk^{(a_b)} \mid M_d M_{d-1} \cdots M_b f_b = 0\}.
\end{aligned}$$

Hence we obtain

$$\dim \mathrm{Hom}_A(M(b,d),M) = a_b - \mathrm{rank}(M_d M_{d-1} \cdots M_b), \tag{2.4}$$

where we set $M_n := 0$. Since the AR-quiver Γ_A of A is of the following form:



the formula (2.4) and Theorem 2.4 give us the formula

$$\boldsymbol{d}_M(M(b,d)) = R(b-1,d) - R(b,d),$$

where we set $M_0 := 0$ and $M_n := 0$ and

$$R(b,d) := \begin{cases} \operatorname{rank}(M_d \cdots M_b) - \operatorname{rank}(M_{d-1} \cdots M_b) & (b < d) \\ \operatorname{rank}(M_d \cdots M_b) - a_b & (b = d) \end{cases}$$

for each $(b,d) \in \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq j \leq n\}$.

3. Solution to the problem (I) for the Kronecker algebra

Throughout the rest of this paper A is the Kronecker algebra. To apply Theorem 2.4 we compute the dimensions of the spaces $\operatorname{Hom}_A(L, M)$ for all $L \in \mathcal{L}$ and $M \in \operatorname{mod} A$ following Remark 2.5.

Definition 3.1. Let M be an A-module. We first define the following matrices with $n \geq 1$, $\lambda \in \Bbbk$ (note that $P_1(M) = \mathsf{J}_{0,1}$ is an empty matrix).

$$P_n(M) := \left.\overbrace{\begin{bmatrix} M(\beta) & M(\alpha) & 0 & 0 & \cdots & 0 \\ 0 & M(\beta) & M(\alpha) & 0 & \ddots & \vdots \\ 0 & 0 & M(\beta) & M(\alpha) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & M(\beta) & M(\alpha) \end{bmatrix}}^{n \text{ blocks}}\right\} n-1 \text{ blocks},$$

$$I_n(M) := \left.\overbrace{\begin{bmatrix} M(\beta) & 0 & 0 & \cdots & 0 \\ M(\alpha) & M(\beta) & 0 & \ddots & \vdots \\ 0 & M(\alpha) & M(\beta) & \ddots & 0 \\ 0 & 0 & M(\alpha) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & M(\beta) \\ 0 & 0 & \cdots & 0 & M(\alpha) \end{bmatrix}}^{n \text{ blocks}}\right\} n+1 \text{ blocks},$$

$$R_n(\lambda, M) := \left.\overbrace{\begin{bmatrix} M_\lambda(\alpha,\beta) & 0 & 0 & \cdots & 0 \\ M(\alpha) & M_\lambda(\alpha,\beta) & 0 & \ddots & \vdots \\ 0 & M(\alpha) & M_\lambda(\alpha,\beta) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M(\alpha) & M_\lambda(\alpha,\beta) \end{bmatrix}}^{n \text{ blocks}}\right\} n \text{ blocks}, \textit{ and}$$

$$R_n(\infty, M) := \left.\overbrace{\begin{bmatrix} M(\alpha) & 0 & 0 & \cdots & 0 \\ -M(\beta) & M(\alpha) & 0 & \ddots & \vdots \\ 0 & -M(\beta) & M(\alpha) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -M(\beta) & M(\alpha) \end{bmatrix}}^{n \text{ blocks}}\right\} n \text{ blocks},$$

where we put $M_\lambda(\alpha,\beta) := \lambda M(\alpha) - M(\beta)$, and we define the following numbers.

$$
\begin{aligned}
p_1(M) &:= 0, p_n(M) := \operatorname{rank} P_n(M)\ (n \geq 2),\\
i_0(M) &:= 0, i_n(M) := \operatorname{rank} I_n(M)\ (n \geq 1),\\
r_n(\lambda, M) &:= \operatorname{rank} R_n(\lambda, M)\ (n \geq 1, \lambda \in \mathbb{P}^1(\Bbbk)).
\end{aligned}
$$

Using the data above we can compute the dimensions of Hom spaces $\operatorname{Hom}_A(L, M)$ with L indecomposable as follows.

Proposition 3.2. *Let M be an A-module. Then we have the following formulas:*

$$
\begin{aligned}
\dim \operatorname{Hom}_A(P_n, M) &= \begin{cases} (n-1)d_1 - p_{n-1}(M) & (n \geq 2)\\ d_2 & (n = 1)\end{cases}\\
\dim \operatorname{Hom}_A(I_n, M) &= nd_1 - i_n(M) \quad (n \geq 1)\\
\dim \operatorname{Hom}_A(R_n(\lambda), M) &= nd_1 - r_n(\lambda, M) \quad (n \geq 1, \lambda \in \mathbb{P}^1(\Bbbk))
\end{aligned}
$$

Proof. Assume that $n \geq 2$. Let $(X, Y) \in \operatorname{Mat} d_1, n-1 \times \operatorname{Mat} d_2, n$, and put X_i (resp. Y_i) to be i-th column of X $(i = 1, \ldots, n-1)$ (resp. Y $(i = 1, \ldots, n)$). Then by (2.1) $(X, Y) \in \operatorname{Hom}_A(P_n, M)$ iff

$$
M(\alpha)X = Y \begin{bmatrix} E_{n-1} \\ 0 \end{bmatrix}, \quad M(\beta)X = Y \begin{bmatrix} 0 \\ E_{n-1} \end{bmatrix}
$$

iff

$$
\begin{cases} M(\alpha)X_1 = Y_1, M(\alpha)X_2 = Y_2, \ldots, M(\alpha)X_{n-1} = Y_{n-1}\\ M(\beta)X_1 = Y_2, M(\beta)X_2 = Y_3, \ldots, M(\beta)X_{n-1} = Y_n \end{cases}
$$

iff

$$
\begin{array}{rc}
 & \overbrace{\hphantom{M(\alpha)\quad M(\alpha)\quad \ddots \quad M(\alpha)}}^{n-1 \text{ blocks}} \quad \overbrace{\hphantom{-E_{d_2}\quad -E_{d_2}\quad \ddots \quad -E_{d_2}\quad 0}}^{n \text{ blocks}} \\
\begin{array}{r} n-1 \text{ blocks}\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right. \\ n-1 \text{ blocks}\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right. \end{array} &
\left[\begin{array}{cccc|ccccc}
M(\alpha) & & & & -E_{d_2} & & & & 0\\
 & M(\alpha) & & & & -E_{d_2} & & & 0\\
 & & \ddots & & & & \ddots & & \vdots\\
 & & & M(\alpha) & & & & -E_{d_2} & 0\\ \hline
M(\beta) & & & & 0 & -E_{d_2} & & & \\
 & M(\beta) & & & 0 & & -E_{d_2} & & \\
 & & \ddots & & \vdots & & & \ddots & \\
 & & & M(\beta) & 0 & & & & -E_{d_2}
\end{array}\right]
\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = 0
\end{array}
$$

Let B be the coefficient matrix of this equation. Then a direct calculation shows that B is equivalent to $P_{n-1}(M) \oplus E_{nd_2}$. Therefore $\operatorname{rank} B = nd_2 + p_{n-1}(M)$, which shows that $\dim \operatorname{Hom}_A(P_n, M) = (n-1)d_1 + nd_2 - \operatorname{rank} B = (n-1)d_1 - p_{n-1}(M)$, as desired. The remaining formulas are proved similarly. □

Propositions 3.2 and Theorem 2.4 give us a solution to the problem (I) for the Kronecker algebra as follows.

THEOREM 3.3. *Let M be an A-module. Then we have the following formulas:*

$$\boldsymbol{d}_M(P_n) = \begin{cases} 2p_n(M) - p_{n-1}(M) - p_{n+1}(M) & (n \geq 2) \\ d_2 - p_2(M) & (n = 1), \end{cases}$$

$$\boldsymbol{d}_M(I_n) = \begin{cases} 2i_{n-1}(M) - i_n(M) - i_{n-2}(M) & (n \geq 2) \\ d_1 - i_1(M) & (n = 1), \end{cases}$$

$$\boldsymbol{d}_M(R_n(\lambda)) = \begin{cases} r_{n-1}(\lambda, M) + r_{n+1}(\lambda, M) - 2r_n(\lambda, M) & (n \geq 2) \\ r_2(\lambda, M) - 2r_1(\lambda, M) & (n = 1). \end{cases}$$

Here we note that $\boldsymbol{d}_M(P_1)$ and $\boldsymbol{d}_M(I_1)$ have obvious menanings that $\boldsymbol{d}_M(P_1) = \dim \operatorname{Coker}[M(\beta)\ M(\alpha)]$ and $\boldsymbol{d}_M(I_1) = \dim \operatorname{Ker} \begin{bmatrix} M(\beta) \\ M(\alpha) \end{bmatrix}$.

PROOF. Note that by Theorem 1.1(2) we know all the almost split sequences for the Kronecker algebra. Therefor we can apply Theorem 2.4. We first compute $\boldsymbol{d}_M(P_1)$ and $\boldsymbol{d}_M(I_1)$. Noting that $\dim \operatorname{Hom}_A(P_2, M) = d_1 - p_1(M) = d_1$ the almost split sequence starting at P_1 that is given by the mesh starting at P_1 in the AR-quiver shows that

$$\begin{aligned} \boldsymbol{d}_M(P_1) &= \dim \operatorname{Hom}_A(P_1, M) - 2 \dim \operatorname{Hom}_A(P_2, M) + \dim \operatorname{Hom}_A(P_3, M) \\ &= d_2 - 2d_1 + 2d_1 - p_2(M) = d_2 - p_2(M) \\ &= d_2 - \operatorname{rank}[M(\beta)\ M(\alpha)] = \dim \operatorname{Coker}[M(\beta)\ M(\alpha)]. \end{aligned}$$

Now since I_1 is simple and injective, we have $I_1/\operatorname{soc} I_1 = 0$ and $\tau^{-1} I_1 = 0$. Hence

$$\begin{aligned} \boldsymbol{d}_M(I_1) &= \dim \operatorname{Hom}_A(I_1, M) = d_1 - i_1(M) \\ &= d_1 - \operatorname{rank} \begin{bmatrix} M(\beta) \\ M(\alpha) \end{bmatrix} = \dim \operatorname{Ker} \begin{bmatrix} M(\beta) \\ M(\alpha) \end{bmatrix}. \end{aligned}$$

Next we compute $\boldsymbol{d}_M(P_n)$ for $n \geq 2$.

$$\begin{aligned} \boldsymbol{d}_M(P_n) &= \dim \operatorname{Hom}_A(P_n, M) - 2 \dim \operatorname{Hom}_A(P_{n+1}, M) + \dim \operatorname{Hom}_A(P_{n+2}, M) \\ &= (n-1)d_1 - p_{n-1}(M) - 2(nd_1 - p_n(M)) + (n+1)d_1 - p_{n+1}(M) \\ &= 2p_n(M) - p_{n-1}(M) - p_{n+1}(M), \end{aligned}$$

as desired. The remaining cases are proved similarly. □

4. Solution to the problem (II) for the Kronecker algebra

Let $F \colon \bigoplus_{L \in \mathcal{L}} L^{(\boldsymbol{d}_M(L))} \to M$ be an isomorphism. Then we have

$$M = P_M \oplus R_M \oplus I_M,$$

where P_M, R_M and I_M are the images of $\bigoplus_{L \in \mathcal{P}} L^{(\boldsymbol{d}_M(L))}$, $\bigoplus_{L \in \mathcal{R}} L^{(\boldsymbol{d}_M(L))}$ and $\bigoplus_{L \in \mathcal{I}} L^{(\boldsymbol{d}_M(L))}$ by F, respectively. To compute P_M, R_M and I_M we here use the trace and reject in a module of a class of modules (see Anderson–Fuller [**13**] for details). Let $\mathcal{U}$ be a class of modules in $\operatorname{mod} A$ and $M \in \operatorname{mod} A$. Recall that the *trace* $\mathsf{Tr}_M(\mathcal{U})$ of $\mathcal{U}$ in M and the *reject* $\mathsf{Rej}_M(\mathcal{U})$ of $\mathcal{U}$ in M are defined by

$$\mathsf{Tr}_M(\mathcal{U}) := \sum\{\operatorname{Im} f \mid f \in \operatorname{Hom}_A(U, M) \text{ for some } U \in \mathcal{U}\}, \text{and}$$
$$\mathsf{Rej}_M(\mathcal{U}) := \bigcap\{\operatorname{Ker} f \mid f \in \operatorname{Hom}_A(M, U) \text{ for some } U \in \mathcal{U}\}.$$

When $\mathcal{U} = \{U\}$ is a singleton, we set $\mathsf{Tr}_M(U) := \mathsf{Tr}_M(\mathcal{U})$ and $\mathsf{Rej}_M(U) := \mathsf{Rej}_M(\mathcal{U})$. We cite the following from [**13**, 8.18 Proposition].

LEMMA 4.1. *Let $(M_i)_{i\in I}$ be a family of A-modules indexed by a set I and $\mathcal{U}$ a class of modules in* $\operatorname{mod} A$*. Then we have*

$$\mathsf{Tr}_{\bigoplus_{i\in I} M_i}(\mathcal{U}) = \bigoplus_{i\in I} \mathsf{Tr}_{M_i}(\mathcal{U}) \quad \textit{and} \quad \mathsf{Rej}_{\bigoplus_{i\in I} M_i}(\mathcal{U}) = \bigoplus_{i\in I} \mathsf{Rej}_{M_i}(\mathcal{U}).$$

PROPOSITION 4.2 (Calculation of $R_M \oplus I_M$). *If $\{f_1, \dots, f_a\}$ is a basis of* $\operatorname{Hom}_A(M, P_{d_2})$, *then we have*

$$\bigcap_{i=1}^{a} \operatorname{Ker} f_i = R_M \oplus I_M \quad \textit{and hence} \quad P_M \cong M/\left(\bigcap_{i=1}^{a} \operatorname{Ker} f_i\right).$$

PROOF. By assumption it is obvious that $\bigcap_{i=1}^{a} \operatorname{Ker} f_i = \mathsf{Rej}_M(P_{d_2})$. Therefore, it is enough to show that

$$\mathsf{Rej}_M(P_{d_2}) = R_M \oplus I_M. \tag{4.1}$$

By Lemma 4.1 we have

$$\mathsf{Rej}_M(P_{d_2}) = \mathsf{Rej}_{P_M \oplus R_M \oplus I_M}(P_{d_2}) = \mathsf{Rej}_{P_M}(P_{d_2}) \oplus \mathsf{Rej}_{R_M}(P_{d_2}) \oplus \mathsf{Rej}_{I_M}(P_{d_2}).$$

By Theorem 1.1(3) we have $\operatorname{Hom}_A(R_M, P_{d_2}) = 0$ and $\operatorname{Hom}_A(I_M, P_{d_2}) = 0$, which shows that

$$\mathsf{Rej}_{R_M}(P_{d_2}) = R_M \quad \text{and} \quad \mathsf{Rej}_{I_M}(P_{d_2}) = I_M.$$

If a preprojective indecomposable module P_i is a direct summand of M, then it follows from $(i-1, i) = \underline{\dim}\, P_i \le \underline{\dim}\, M = (d_1, d_2)$ that $i \le d_2$ (see Remark 1.2(2)). Therefore, we have $P_M = \bigoplus_{i=1}^{d_2} P_i^{(a_i)}$ for some $a_i \ge 0$ (we identify P_i with $F(P_i)$), and then $\mathsf{Rej}_{P_M}(P_{d_2}) = \bigoplus_{i=1}^{d_2} (\mathsf{Rej}_{P_i}(P_{d_2}))^{(a_i)}$. Now if $i \le d_2$, then by Remark 1.2(1) we have a monomorphism $P_i \to P_{d_2}$, which shows that $\mathsf{Rej}_{P_i}(P_{d_2}) = 0$ for all $i \le d_2$, and therefore

$$\mathsf{Rej}_{P_M}(P_{d_2}) = 0.$$

Hence the equality (4.1) holds. $\square$

PROPOSITION 4.3 (Calculation of I_M). *If $\{g_1, \dots, g_b\}$ is a basis of* $\operatorname{Hom}_A(I_{d_1}, R_M \oplus I_M)$, *then we have*

$$\sum_{i=1}^{b} \operatorname{Im} g_i = I_M.$$

PROOF. By assumption it is obvious that $\sum_{i=1}^{b} \operatorname{Im} g_i = \mathsf{Tr}_{R_M \oplus I_M}(I_{d_1})$. Therefore it is enough to show that

$$\mathsf{Tr}_{R_M \oplus I_M}(I_{d_1}) = I_M. \tag{4.2}$$

By Lemma 4.1 we have

$$\mathsf{Tr}_{R_M \oplus I_M}(I_{d_1}) = \mathsf{Tr}_{R_M}(I_{d_1}) \oplus \mathsf{Tr}_{I_M}(I_{d_1}).$$

By Theorem 1.1(3) we have $\mathrm{Hom}_A(I_{d_1}, R_M) = 0$, which shows that

$$\mathsf{Tr}_{R_M}(I_{d_1}) = 0.$$

If a preinjective indecomposable module I_i is a direct summand of M, then it follows from $(i, i-1) = \underline{\dim}\, I_i \leq \underline{\dim}\, M = (d_1, d_2)$ that $i \leq d_1$. Therefore we have $I_M = \bigoplus_{i=1}^{d_1} I_i^{(b_i)}$ for some $b_i \geq 0$ (we identify I_i with $F(I_i)$), and then $\mathsf{Tr}_{I_M}(I_{d_1}) = \bigoplus_{i=1}^{d_1}(\mathsf{Tr}_{I_i}(I_{d_1}))^{(b_i)}$. Now if $i \leq d_1$, then we have an epimorphism $I_{d_1} \to I_i$, which shows that $\mathsf{Tr}_{I_i}(I_{d_1}) = I_i$ for all $i \leq d_1$, and therefore

$$\mathsf{Tr}_{I_M}(I_{d_1}) = I_M.$$

Hence the equality (4.2) holds. □

By Propositions 4.2 and 4.3 we have the following.

Proposition 4.4 (Calculation of R_M). *Let $\{f_1, \ldots, f_a\}$ a basis of $\mathrm{Hom}_A(M, P_{d_2})$ and $\{g_1, \ldots, g_b\}$ a basis of $\mathrm{Hom}_A(I_{d_1}, \bigcap_{i=1}^a \mathrm{Ker}\, f_i)$. Then we have*

$$R_M \cong \left(\bigcap_{i=1}^{a} \mathrm{Ker}\, f_i\right) / \left(\sum_{i=1}^{b} \mathrm{Im}\, g_i\right).$$

By this isomorphism we identify R_M with the right hand side. Since $R_M = (R_M(\alpha), R_M(\beta))$ is the direct sum of regular indecomposable modules, both $R_M(\alpha)$ and $R_M(\beta)$ are square matrices, say of size d. Put $R(\infty) := \mathsf{Tr}_{R_M}(R_d(\infty))$. Note that $\mathsf{Tr}_{R_M}(R_d(\infty)) = \mathsf{Tr}_{R_M}(\bigoplus_{n=1}^d R_n(\infty))$ because there exists an epimorphism $R_n(\infty) \to R_m(\infty)$ for $n \geq m$. Then $R_M = R(\infty) \oplus R'$ for some A-submodule $R' = (X', Y')$ of R_M such that R' has no direct summand of the form $R_n(\infty)$ for any n by Theorem 1.1(3)(iii). [Decompose R_M into indecomposables of the form $R_n(\lambda)$ with $n \geq 1, \lambda \in \mathbb{P}^1(\Bbbk)$. Then R' is given by the direct sum of those direct summands of the form $R_n(\lambda)$ with $\lambda \neq \infty$ because $R(\infty)$ is given by the direct sum of summands of the form $R_n(\infty)$.] Since the matrix X' is invertible, we have

$$R' \cong (E_l, (X')^{-1}Y')$$

for some $l \leq d$. Therefore, the set Λ of eigenvalues of $(X')^{-1}Y'$ is finite.

Then by Propositions 4.2, 4.3 and 4.4, we obtain the following.

Theorem 4.5. *Set*

$$S_M := \{P_i, I_j, R_k(\lambda) \mid 1 \leq i \leq d_2, 1 \leq j \leq d_1, 1 \leq k \leq d, \lambda \in \Lambda \cup \{\infty\}\}.$$

Then this gives a solution to the problem (II) *for the Kronecker algebra.*

Remark 4.6. Note that if $R(\infty) = 0$, then we can replace S_M by

$$\{P_i, I_j, R_k(\lambda) \mid 1 \leq i \leq d_2, 1 \leq j \leq d_1, 1 \leq k \leq d, \lambda \in \Lambda\}.$$

5. Examples for the Kronecker algebra

(1) For a preprojective module $M = P_3 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ with $\underline{\dim}\, M = (2,3)$, we will compute $p_n(M)$ ($n \in \mathbb{N}$) and then we will give $\boldsymbol{d}_M(P_n)$ ($n \in \mathbb{N}$). By Definition 3.1 we have $p_1(M) = 0$,

$$p_2(M) = \operatorname{rank} \left[\begin{array}{c|c} M(\beta) & M(\alpha) \end{array} \right] = \operatorname{rank} \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \right] = 3,$$

$$p_3(M) = \operatorname{rank} \left[\begin{array}{c|c|c} M(\beta) & M(\alpha) & 0 \\ \hline 0 & M(\beta) & M(\alpha) \end{array} \right] = \operatorname{rank} \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & & \\ 1 & 0 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & & \\ \hline & & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 0 & 1 \\ & & 0 & 1 & 0 & 0 \end{array} \right] = 6,$$

$$p_4(M) = \operatorname{rank} \left[\begin{array}{c|c|c|c} M(\beta) & M(\alpha) & 0 & 0 \\ \hline 0 & M(\beta) & M(\alpha) & 0 \\ \hline 0 & 0 & M(\beta) & M(\alpha) \end{array} \right] = \operatorname{rank} \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 0 & & & & \\ 1 & 0 & 0 & 1 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ \hline & & 0 & 0 & 1 & 0 & & \\ & & 1 & 0 & 0 & 1 & & \\ & & 0 & 1 & 0 & 0 & & \\ \hline & & & & 0 & 0 & 1 & 0 \\ & & & & 1 & 0 & 0 & 1 \\ & & & & 0 & 1 & 0 & 0 \end{array} \right] = 8,$$

and

$$p_5(M) = \operatorname{rank} \left[\begin{array}{c|c|c|c|c} M(\beta) & M(\alpha) & 0 & 0 & 0 \\ \hline 0 & M(\beta) & M(\alpha) & 0 & 0 \\ \hline 0 & 0 & M(\beta) & M(\alpha) & 0 \\ \hline 0 & 0 & 0 & M(\beta) & M(\alpha) \end{array} \right]$$

$$= \operatorname{rank} \left[\begin{array}{cc|cc|cc|cc|cc} 0 & 0 & 1 & 0 & & & & & & \\ 1 & 0 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ \hline & & 0 & 0 & 1 & 0 & & & & \\ & & 1 & 0 & 0 & 1 & & & & \\ & & 0 & 1 & 0 & 0 & & & & \\ \hline & & & & 0 & 0 & 1 & 0 & & \\ & & & & 1 & 0 & 0 & 1 & & \\ & & & & 0 & 1 & 0 & 0 & & \\ \hline & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & 1 & 0 & 0 & 1 \\ & & & & & & 0 & 1 & 0 & 0 \end{array} \right] = 10.$$

Similarly, we have $p_n(M) = 2n$ for $n \geq 3$. Hence by Theorem 3.3 we have

$$\begin{aligned}\boldsymbol{d}_M(P_1) &= 3 - p_2(M) = 0,\\ \boldsymbol{d}_M(P_2) &= 2p_2(M) - p_1(M) - p_3(M) = 6 - 0 - 6 = 0,\\ \boldsymbol{d}_M(P_3) &= 2p_3(M) - p_2(M) - p_4(M) = 12 - 3 - 8 = 1,\end{aligned}$$

and for $n \geq 4$,

$$\boldsymbol{d}_M(P_n) = 2p_n(M) - p_{n-1}(M) - p_{n+1}(M) = 2 \cdot 2n - 2(n-1) - 2(n+1) = 0.$$

Thus we can confirm $\boldsymbol{d}_M(P_3) = 1$ and $\boldsymbol{d}_M(P_n) = 0$ for $n \neq 3$.

(2) For a module $M = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, 0_{2,2}\right) = P_1 \oplus R_1(1) \oplus I_1$ with $\underline{\dim}\, M = (2,2)$, we will compute $\mathsf{Rej}_M(P_2)$ and $\mathsf{Tr}_{\mathsf{Rej}_M(P_2)}(I_2)$. Recall that $P_2 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. If $(X, Y) \in \mathrm{Hom}_A(M, P_2)$, then we have $X = 0_{1,2}, Y = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ for some $a, b \in \Bbbk$, and we can take $\left\{f_1 = \left(0_{1,2}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), f_2 = \left(0_{1,2}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right\}$ as a basis of $\mathrm{Hom}_A(M, P_2)$. Hence we have

$$\mathsf{Rej}_M(P_2) = \mathrm{Ker}\, f_1 \cap \mathrm{Ker}\, f_2 = \left(\begin{bmatrix} 1 & 0 \end{bmatrix}, 0_{1,2}\right) = R_1(1) \oplus I_1$$

with $\underline{\dim}\, \mathsf{Rej}_M(P_2) = (2,1)$ and have $M/\mathsf{Rej}_M(P_2) \cong P_1$. Moreover, recall that $I_2 = \left(\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right)$. If $(X, Y) \in \mathrm{Hom}_A(I_2, \mathsf{Rej}_M(P_2))$, then we have $X = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}, Y = 0_{1,1}$ for some $c, d \in \Bbbk$, and we can also take $\left\{g_1 = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, 0_{1,1}\right), g_2 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0_{1,1}\right)\right\}$ as a basis of $\mathrm{Hom}_A(I_2, \mathsf{Rej}_M(P_2))$. Therefore, we have

$$\mathsf{Tr}_{\mathsf{Rej}_M(P_2)}(I_2) = \mathrm{Im}\, g_1 + \mathrm{Im}\, g_2 = (\mathsf{J}_{0,1}, \mathsf{J}_{0,1}) = I_1$$

with $\underline{\dim}\, \mathsf{Tr}_{\mathsf{Rej}_M(P_2)}(I_2) = (1,0)$, and $\mathsf{Rej}_M(P_2)/\mathsf{Tr}_{\mathsf{Rej}_M(P_2)}(I_2) \cong R_1(1)$. Thus we can confirm the process to get $S_M = \{P_1, P_2, I_1, I_2, R_1(1)\}$ in Section 4.

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