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## Starting functions in representation theory of algebras

| メタデータ | 言語：eng |
| :--- | :--- |
|  | 出版者： |
|  | 公開日：2017－06－07 |
|  | キーワード（Ja）： |
|  | キーワード（En）： |
|  | 作成者：Nakashima，Ken |
|  | メールアドレス： |
|  | 所属： |
| URL | https：／／doi．org／10．14945／00010201 |

## 静岡大学博士論文

# Starting functions in representation theory 

 of algebras
## 多元環の表現論における出発関数

## 中島 健

大学院自然科学系教育部<br>情報科学専攻

2016年12月

# Starting functions in representation theory of algebras 

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## Introduction

Throughout this paper $\mathbb{k}$ is an algebraically closed field, and all vector spaces, algebras and linear maps are assumed to be finite-dimensional $\mathbb{k}$-vector spaces, finite-dimensional $\mathbb{k}$-algebras and $\mathbb{k}$-linear maps, respectively. Further all modules over an algebra considered here are assumed to be finite-dimensional modules. We denote the set of non-negative integers by $\mathbb{N}_{0}$.
0.1. Starting functions. We first define starting functions which are a key tool in this paper. For a module $X$ over an algebra $A$, we often identify the isocalss $[X]$ of $X$ with $X$ itself. In particular, the set $\Gamma_{0}$ of varticies of the AR-quiver $\Gamma$ of $A$ is identified with a complete list of indecomposable $A$-modules.

Definition 0.1. Let $A$ be an algebra and $\Gamma$ the AR-quiver of $A$. Then for an indecomposable $A$-module $X$, the starting function $\mathrm{s}_{X}: \Gamma_{0} \rightarrow \mathbb{N}_{0}$ of $X$ is defined by

$$
\mathrm{s}_{X}(Y):=\operatorname{dim}_{\mathrm{k}} \operatorname{Hom}_{A}(X, Y)
$$

for all $Y \in \Gamma_{0}$.
Starting functions have the following property.
Proposition 0.2. Let $A$ be an algebra, $\Gamma$ the $A R$-quiver of $A$,

a mesh in $\Gamma$, and $X$ an indecomposable $A$-module with $X \not \approx N$. Then we have

$$
\mathrm{s}_{X}(N)=\sum_{i=1}^{n} \mathrm{~s}_{X}\left(M_{i}\right)-\mathrm{s}_{X}(L)
$$

Starting functions were introduced by Gabriel to compute AR-quivers in [6], and were developed such as in $[\mathbf{9}],[\mathbf{1 0}],[\mathbf{4}],[\mathbf{3}],[\mathbf{2}, 6.6]$ and $[\mathbf{1}]$. In this paper, we give two results obtained by using starting functions.
0.2. The first result (Chapter 1). This part is a generalization of Hironobu Suzuki's Master thesis [11] that dealt with representation-finite self-injective algebras of type A in a combinatorial way. Throughout Chapter $1 n$ is a positive integer, all algebras considered here are assumed to be basic, connected, finite-dimensional associative $\mathbb{k}$-algebras and all modules are finite-dimensional right modules.

Let $\Delta$ be a Dynkin graph of type A, D, E with the set $\Delta_{0}:=\{1, \ldots, n\}$ of vertices. By Riedtmann [10, 2.5] the computation of the Auslander-Reiten quiver (AR-quiver for short) $\Gamma_{\Lambda}$ of a representation-finite standard self-injective algebra $\Lambda$ of type $\Delta$ is reduced to that of stable AR-quiver ${ }_{s} \Gamma_{\Lambda}$ of $\Lambda$ and the configuration $\mathcal{C}_{\Lambda}$ of $\Lambda$ as the isomorphism $\Gamma_{\Lambda} \cong\left({ }_{s} \Gamma_{\Lambda}\right)_{\mathcal{C}_{\Lambda}}$ shows. The stable AR-quiver ${ }_{s} \Gamma_{\Lambda}$ is given by the orbit category presentation of $\Lambda$, namely if $\Lambda \cong \hat{A} / G$ for some tilted algebra $A$ of type $\Delta$, then ${ }_{s} \Gamma_{\Lambda} \cong \mathbb{Z} \Delta / G$. Therefore to recover the AR-quiver $\Gamma_{\Lambda}$ it suffices to compute the configuration $\mathcal{C}_{\Lambda}$ by using information of $A$. Set $\mathbf{C}(\Delta)$ to be the set of configurations on the translation quiver $\mathbb{Z} \Delta$ (see Definition 1.6), and $\mathbf{T}(\Delta)$ to be the set of isoclasses of tilted algebras of type $\Delta$. Bretscher, Läser and Riedtmann gave a bijection $c: \mathbf{T}(\Delta) \rightarrow \mathbf{C}(\Delta)$ in [4], which makes it possible to compute $\mathcal{C}_{\Lambda}$ as the equivalence class of $c(A)$. Hence we can compute $\Gamma_{\Lambda}$ using these data. But the map $c$ is not given in a direct way, it needs a long computation of a function on $\mathbb{Z} \Delta$. In this paper we will give an easier way to calculate the map $c$ by giving a map sending each projective indecomposable $A$-module over a tilted algebra $A$ in $\mathbf{T}(\Delta)$ to an element of the configuration $c(A)$ in $\mathbf{C}(\Delta)$.

We fix an orientation of each Dynkin graph $\Delta$ to have a quiver $\vec{\Delta}$ as in the following table.

| $\Delta$ | $A_{n}(n \geq 1)$ | $D_{n}(n \geq 4)$ | $E_{n}(n=6,7,8)$ |
| :---: | :---: | :---: | :---: |
| $\vec{\Delta}$ | $\stackrel{\stackrel{ }{1}}{\longrightarrow}{ }_{2}^{0} \longrightarrow \cdots \longrightarrow{ }_{n}^{0}$ |  |  |
| $m_{\Delta}$ | $n$ | $2 n-3$ | 11, 17, 29, respectively |

This orientation of $\Delta$ gives us a coordinate system on the set $(\mathbb{Z} \Delta)_{0}:=\mathbb{Z} \times \Delta_{0}$ of vertices of $\mathbb{Z} \Delta:=\mathbb{Z} \vec{\Delta}$ as presented in [4, fig. 1] and in [6, Fig. 13].

Let $A$ be a tilted algebra of type $\Delta$. Then by identifying $A$ with the $(0,0)$-entry of the repetitive category $\hat{A}$, the vertex set of AR -quiver $\Gamma_{A}$ is embedded into the vertex set of the stable AR-quiver ${ }_{s} \Gamma_{\hat{A}}(\cong \mathbb{Z} \Delta)$ of $\hat{A}$. Further the configuration $\mathcal{C}:=c(A)$ of $\mathbb{Z} \Delta$ computed in $[\mathbf{4}]$ is given by the vertices of $\mathbb{Z} \Delta$ corresponding to radicals of projective indecomposable $\hat{A}$-modules. Note that the configuration $\mathcal{C}$ has a period $m_{\Delta}$ listed in the table, thus $\mathcal{C}=\tau^{m \Delta \mathbb{Z}} \mathcal{F}$ for some subset $\mathcal{F}$ of $\mathcal{C}$. By $\mathcal{P}=\left\{(p(i), i) \mid i \in \Delta_{0}\right\}$ we denote the set of images of the projective vertices of $\Gamma_{A}$ in $\mathbb{Z} \Delta$ and set

$$
\mathbb{N} \mathcal{P}:=\left\{(m, i) \in(\mathbb{Z} \Delta)_{0} \mid p(i) \leq m, i \in \Delta_{0}\right\} .
$$

As is well-known, there exists the Nakayama permutation $\hat{\nu}$ on $(\mathbb{Z} \Delta)_{0}$ that is defined by the isomorphism

$$
\mathbb{k}(\mathbb{Z} \Delta)(x,-) \cong D(\mathbb{k}(\mathbb{Z} \Delta)(-, \hat{\nu} x))
$$

for all $x \in(\mathbb{Z} \Delta)_{0}$, where $D$ is the $\mathbb{k}$-dual functor $\operatorname{Hom}_{\mathbb{k}}(-, \mathbb{k})$. The explicit formula of $\hat{\nu}$ is given in [6, pp. 48-50]. (Note that it should be corrected as $\hat{\nu}(p, q)=(p+q+2,6-q)$ if $q \leq 5$ when $\Delta=E_{6}$ as pointed out in [4,1.1]). In this paper we will define a map $\nu^{\prime}: \mathcal{P} \rightarrow \mathbb{N} \mathcal{P}$ using supports of the functions $\operatorname{dim}_{\mathbb{k}} \mathbb{k}(\mathbb{Z} \Delta)(x,-): \mathbb{N} \mathcal{P} \rightarrow \mathbb{Z}$, so-called the starting functions from $x \in \mathbb{N} \mathcal{P}$ (cf. [6, Fig. 15]). Then $\nu^{\prime}$ has the following property.

Lemma 0.3. Let $x \in \mathcal{P}$ and $P$ be the projective indecomposable $A$-module corresponding to $x$. Then $\nu^{\prime} x$ corresponds to the simple module top $P$.

In this paper, we make use of modules over the algebra

$$
B:=\left[\begin{array}{cc}
A & 0 \\
D A & A
\end{array}\right]
$$

to compute an $\mathcal{F}$ above (the configuration (see Definition 3.1) of $B$ gives $\mathcal{F}$.) We will define a map $\nu:=\nu_{B}$ from the set of isoclasses of simple $A$-modules to $\mathcal{C}$, which coincides with the restriction of the Nakayama permutation $\hat{\nu}$ if $A$ is hereditary.

Lemma 0.4. Assume that a vertex $x \in \mathbb{Z} \Delta$ corresponds to a simple $A$-module $S$ and let $Q$ be the injective hull of $S$ over $\hat{A}$. Then $\nu(x)$ corresponds to $\operatorname{rad} Q$, and hence $\nu(x) \in \mathcal{C}$.

Combining the lemmas above we obtain the following.
Proposition 0.5. If $x \in \mathcal{P}$, then $\nu\left(\nu^{\prime} x\right) \in \mathcal{C}$.
This leads us to the following definition.
Definition 0.6. We define a map $c_{A}: \mathcal{P} \rightarrow \mathcal{C}$ by $c_{A}(x):=\nu\left(\nu^{\prime} x\right)$ for all $x \in \mathcal{P}$.
The image of the map $c_{A}$ gives us an $\mathcal{F}$ above, namely we have the following.
THEOREM 0.7. The map $c_{A}$ is an injection, and we have $c(A)=\tau^{m_{\Delta}} \mathbb{Z} \operatorname{Im} c_{A}$.
Corollary 0.8. If $A$ is hereditary, then $c_{A}=\hat{\nu} \nu^{\prime}$ and we have $c(A)=\tau^{m \Delta \mathbb{Z}} \operatorname{Im} \hat{\nu} \nu^{\prime}$.
0.3. The second result (Chapter 2). Throughout Chapter 2 all modules over an algebra considered here are assumed to be finite-dimensional left modules. Let $A$ be an algebra, $\mathcal{L}$ a complete set of representatives of isoclasses of indecomposable $A$-modules. Then the Krull-Schmidt theorem states the following. For each $A$-module $M$, there exists a unique map $\boldsymbol{d}_{M}: \mathcal{L} \rightarrow \mathbb{N}_{0}$ such that

$$
M \cong \bigoplus_{L \in \mathcal{L}} L^{\left(\boldsymbol{d}_{M}(L)\right)}
$$

which is called an indecomposable decomposition of $M$. Therefore, $M \cong N$ if and only if $\boldsymbol{d}_{M}=\boldsymbol{d}_{N}$ for all $A$-modules $M$ and $N$, i.e., the map $\boldsymbol{d}_{M}$ is a complete invariant of $M$ under isomorphisms. Note that since $M$ is finite-dimensional, the support $\operatorname{supp}\left(\boldsymbol{d}_{M}\right):=$ $\left\{L \in \mathcal{L} \mid \boldsymbol{d}_{M}(L) \neq 0\right\}$ of $\boldsymbol{d}_{M}$ is a finite set. We call such a theory a decomposition theory that computes the indecomposable decomposition of a module. The Auslander-Reiten theory was developed since 1970s in representation theory of algebras. In many cases it enabled us to compute the Auslander-Reiten quiver (AR-quiver for short) of $A$ that is a combinatorial description of the category of modules over $A$, the vertex set of which can be identified with the list $\mathcal{L}$, and which is constructed by gluing all meshes that is a visual form of almost split sequences over $A$. Thus all information on almost split sequences over $A$ are encoded in the AR-quiver in a visual way. Namely, if $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an almost split sequence, and $Y=\bigoplus_{i=1}^{n} Y_{i}^{\left(a_{i}\right)} \quad(n \geq 1)$ is an indecomposable decomposition
of $Y$ with $Y_{i}$ pairwise non-isomorphic and $a_{i} \geq 1$ for all $i$, then we express it by the quiver

with a broken line between $X$ and $Z$ (note here that also both $X, Z$ are indecomposable by definition of almost split sequences). The correspondence $\tau: Z \mapsto X$ is called the $A R$-translation. For example, it has the forms


The purpose of Chapter 2 is to develope a decomposition theory by using the knowledge of AR-quivers. Thus in the case that $\mathcal{L}$ is already computed and all almost split sequences are known, we aim to compute
(I) $\boldsymbol{d}_{M}$ and
(II) a finite set $S_{M}$ such that $\operatorname{supp}\left(\boldsymbol{d}_{M}\right) \subseteq S_{M} \subseteq \mathcal{L}$
for all $A$-modules $M$. Note that (II) is needed to give a finite algorithm. If $A$ is representation-finite (i.e., if the set $\mathcal{L}$ is finite), then the problem (II) is trivial because we can take $S_{M}:=\mathcal{L}$.

In the topological data analysis, to analyse a point cloud $C$, a set of points in $\mathbb{R}^{d}$ for some fixed positive integer $d$, some important informations on $C$ are encoded in the persistent homology $M_{C}$, which is just a module over the path algebra $\Lambda_{n}=\mathbb{k} Q_{n}$ of a quiver $Q_{n}$ of the form

$$
1 \xrightarrow{\alpha_{1}} 2 \xrightarrow{\alpha_{2}} \cdots \xrightarrow{\alpha_{n-1}} n
$$

of Dynkin type $A_{n}$ for some positive integer $n$. Therefore to understand the point cloud $C$ we can use the knowledge of the map $\boldsymbol{d}_{M_{C}}$, which is nothing but the "persistence diagram" of $C$, where usually the values of $\boldsymbol{d}_{M_{C}}(L) \quad(L \in \mathcal{L})$ is presented by colors on $\mathcal{L}$, and $\mathcal{L}$ is expressed by a set of lattice points in a triangle. More precisely, the list $\mathcal{L}$ is given by $\{M(b, d) \mid 1 \leq b \leq d \leq n\}$ thanks to Gabriel's theorem on representations of Dynkin quivers, where $M(b, d)$ is given by

$$
0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathbb{k} \xrightarrow{1} \mathbb{k} \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{k} \rightarrow 0 \rightarrow \cdots \rightarrow 0
$$

with $\mathbb{k}$ starting at the vertex $b$ and stopping at $d$. Therefore there exists a 1-1 correspondence between $\mathcal{L}$ and the set $\{(b, d) \mid 1 \leq b \leq d \leq n\}$, which is a subset of $\mathbb{Z}^{2}$ forming a triangle (See for instance papers [21] and [16]). Note that this set of vertices together with
horizontal and vertical edges connecting them can be regarded as the underlying graph of the AR-quiver of $\Lambda_{n}$ (See Example 2.7). To analyse property of a set of point clouds, e.g., a motion of a point cloud, persistent homologies were generalized to persistence modules $M$, which turn out to be modules over an algebra of the form $\Lambda_{m} \otimes_{\mathbb{k}} \Lambda_{n}$, where we allow any orientation of $Q_{m}$ and $Q_{n}$, namely their underlying graphs have the form

$$
1-2-\cdots-l
$$

of type $A_{l}$ for $l=m, n$. Also in this case we need to compute the persistence diagram $\boldsymbol{d}_{M}$ to investigate the set of point clouds. It was done in [18] for the case $(m, n)=(2,3)$. Our argument here can be applied to have a decomposition theory for persistence modules.

Example 0.9. The decomposition theory for polynomial algebras in one variable $A=\mathbb{k}[x]$ is already well known. A finite-dimensional $A$-module is a pair $(V, f)$ of a finitedimensional $\mathbb{k}$-vector space $V$ and an endomorphism $f$ of $V$, and by fixing a basis of $V$ we may regard $V=\mathbb{k}^{d}$ for $d:=\operatorname{dim} V$ and $f$ as a square matrix $M$ of size $d$. In this way we identify $(V, f)$ with $M$. In this case we may have $\mathcal{L}=\left\{J_{i}(\lambda) \mid i \geq 1, \lambda \in \mathbb{k}\right\}$, where $J_{i}(\lambda)$ is the Jordan cell of size $i \geq 1$ with eigenvalue $\lambda \in \mathbb{k}$. Let $\Lambda$ be the set of all distinct eigenvalues of $M$ and set $M_{\lambda}=M-\lambda E_{d}$ for $\lambda \in \Lambda$. Then the following is well known.

Theorem 0.10. The problems (I) and (II) are solved as follows.
A solution to (I): Let $i \in \mathbb{N}$ and $\lambda \in \Lambda$. Then

$$
\boldsymbol{d}_{M}\left(J_{i}(\lambda)\right)= \begin{cases}d+\operatorname{rank} M_{\lambda}^{2}-2 \operatorname{rank} M_{\lambda} & \text { if } i=1 ; \text { and }  \tag{0.1}\\ \operatorname{rank} M_{\lambda}^{i+1}+\operatorname{rank} M_{\lambda}^{i-1}-2 \operatorname{rank} M_{\lambda}^{i} & \text { if } i \geq 2 ;\end{cases}
$$

(Note that by setting $M_{\lambda}^{0}$ to be the identity matrix of size $d$, the first equality has the same form as the second.)
A solution to (II): $S_{M}=\left\{J_{i}(\lambda) \mid i \leq d, \lambda \in \Lambda\right\}$.
In this paper, we will solve the problem (I) in the decomposition theory for any finitedimensional algebra $A$. This turns out to be an extension of the result for $A=\mathbb{k}[x]$ above. In particular, for the Kronecker algebra $A=\mathbb{k} Q$ with $Q=(\underset{\beta}{2} 2)$, we will give an explicit formula for the problem (I) and a solution to the problem (II).

Decomposition theory is based on the approach as follows. Let $A$ be a directed algebra. Then there is a complete set of isoclasses of indecomposable $A$-module $\left\{M_{1}, \cdots, M_{n}\right\}$ such that $\operatorname{Hom}_{A}\left(M_{i}, M_{j}\right) \neq 0$ implies $i \leq j$. An example of this numbering is given as follows.

Example 0.11.

$$
\text { Let }(Q, I):=2 ڭ_{4}^{\star}{ }_{4}^{1} 3 \text { and } A:=Q / I \text {. }
$$

Then the AR-quiver $\Gamma$ of $A$ is as follows.


Let $M \in \bmod A$. Assume $M \cong \bigoplus_{i=1}^{n} M_{i}^{\left(a_{i}\right)}$ where $a_{i} \in \mathbb{N}_{0}$. Define $b_{j}:=\operatorname{dim} \operatorname{Hom}_{A}\left(M, M_{j}\right)$ for each $j \in\{1, \ldots, n\}$. Then

$$
\begin{aligned}
b_{j} & =\operatorname{dim} \operatorname{Hom}_{A}\left(\bigoplus_{i=1}^{n} M_{i}^{\left(a_{i}\right)}, M_{j}\right) \\
& =\sum_{i=1}^{n} a_{i} \cdot \operatorname{dim} \operatorname{Hom}_{A}\left(M_{i}, M_{j}\right) \\
& =\sum_{i=1}^{n} a_{i} \cdot s_{M_{i}}\left(M_{j}\right) \\
& =\left(a_{1}, \ldots, a_{n}\right)\left(\begin{array}{c}
s_{M_{1}}\left(M_{j}\right) \\
\vdots \\
s_{M_{n}}\left(M_{j}\right)
\end{array}\right) .
\end{aligned}
$$

Hense, we obtain

$$
\left(b_{1}, \ldots, b_{n}\right)=\left(a_{1}, \ldots, a_{n}\right)\left(\begin{array}{ccc}
s_{M_{1}}\left(M_{1}\right) & \cdots & s_{M_{1}}\left(M_{n}\right) \\
\vdots & \ddots & \vdots \\
s_{M_{n}}\left(M_{1}\right) & \cdots & s_{M_{n}}\left(M_{n}\right)
\end{array}\right) .
$$

We set $U_{\Gamma}$ to be the matrix on the right hand side whose $(i, j)$-entry has the value $s_{M_{i}}\left(M_{j}\right)$ of starting function $s_{M_{i}}$. Since $s_{M_{i}}\left(M_{j}\right) \neq 0$ implies $i \leq j, U_{\Gamma}$ is an upper triangular matrix, which is invertible because the diagonal entries are equal to 1 . Thus, we obtain

$$
\left(a_{1}, \ldots, a_{n}\right)=\left(b_{1}, \ldots, b_{n}\right) U_{\Gamma}^{-1}
$$

Hence, in order to realize the decomposition theory, we find it important to study $U_{\Gamma}^{-1}$. It is very interesting to see that $U_{\Gamma}^{-1}$ is given by the information of AR-quiver $\Gamma$ as follows.

Definition 0.12 (AR-matrix). Let $A$ be an algebra, $\Gamma$ the AR-quiver of $A$. Then the AR-matrix $V_{\Gamma}=\left[v_{i j}\right]_{i, j}$ of $A$ is defined by

$$
v_{i j}:=\left\{\begin{array}{ll}
1 & \left(j=i \text { or } M_{j} \cong \tau^{-1}\left(M_{i}\right)\right) \\
-c & \left(M_{i} \xrightarrow{c \text { arrows }} M_{j} \text { in } \Gamma\right) \\
0 & (\text { otherwise })
\end{array} .\right.
$$

Proposition 0.13. $V_{\Gamma}$ is the inverse of $U_{\Gamma}$.
Remark 0.14 . We gave three deferent proofs of this proposition. The first one calculated cofactor matrices, the second one checked the equality $U_{\Gamma} V_{\Gamma}=E$, and the third one used the fact that $U_{\Gamma}$ is the Cartan matrix of the module category of $A$ (cf. Remark0.16(1) belows).

Example 0.15. Let $A$ and $\Gamma$ be as in Example 1.11. Then we have

$$
\begin{gathered}
U_{\Gamma}=\left[s_{M_{i}}\left(M_{j}\right)\right]_{i, j}=\left[\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \text { and } \\
V_{\Gamma}=\left[v_{i j}\right]_{i, j}=\left[\begin{array}{cccccccccc}
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\
0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1
\end{array}\right] .
\end{gathered}
$$

It is easy to check $U_{\Gamma} V_{\Gamma}=E_{11}=V_{\Gamma} U_{\Gamma}$.
Remark 0.16. After submitting the paper we are pointed out by Emerson Escolor and the referee that there was already a similar investigation [17] by Dowbor and Mróz in the literature, which we did not know before. Thus this work was done independently. We here list some relationships between their results and ours.
(1) They also have the same statement as Theorem 2.4 and its dual version, namely a solution to (I). Their proof is similar to the first version of ours using a "Cartan matrix" of the module category of an algebra $A$ and an AR-matrix of $A$ as its inverse, but the proof presented here does not use them and is much simplified by using the minimal projective resolutions of simple functors that are given by almost split sequences and sink maps into indecomposable injective modules, which also give the matrix $V_{\Gamma}$ in Proposition 0.13 (Proposition 2.3).
(2) Our Theorem 3.3 gives an explicit way of computation of the map $\boldsymbol{d}_{M}$ for a module $M$ by using ranks of matrices constructed by the structure matrices of $M$, while they did not give such formulas explicitly.
(3) To solve the problem (II) we used traces and rejects, which are easily computed and give us a decomposition of a module into the preprojective part, the preinjective part, the regular part with parameter $\infty$, and the regular part without parameter $\infty$. This together with Theorem 3.3 gives an effective computation of the indecomposable decomposition of a module $M$. For instance, if the preprojective part or the preinjective part of $M$ is zero, it avoids unnecessary computations of the decomposition for those parts, in contrast, such computations are done in their algorithm repeatedly.
(4) Proposition 4.4 in [ $\mathbf{1 7}]$ gives another way to compute regular direct summands, which seems to be interesting.
(5) They investigated also the cases of general $\tilde{A}$-quivers and representation-finite string algebras.

## Acknowledgments

The author would like to express his sincere gratitude to his supervisor Professor Hideto Asashiba for providing him this precious study opportunity as a Ph.D. student and for giving him earnest guidance. He would also like to thank Dr. Michio Yoshiwaki for helpful advice and encouragement. Chapter 1 is a joint work with Asashiba, and Chapter 2 is a joint work with Asashiba and Yoshiwaki. The starting idea of Chapter 2 was made when the coauthor Asashiba was at the lecture on the Kronecker canonical form of singular mixed matrix pencils by S. Iwata at CREST meeting in February, 2016. In fact, later the authors recognized that Iwata-Shimizu [19] had given relationships between values of ranks $p_{n}(M)$ (resp. $i_{n}(M), r_{n}(0, M)$, and $\left.r_{n}(\infty, M)\right)$ in Definition 3.1 and the values of $\boldsymbol{d}_{M}(L)$ for $L=P_{n}$ (resp. $I_{n}, R_{n}(0)$, and $\left.R_{n}(\infty)\right)$ in Theorem 1.1. However, they did not mention regular indecomposables $R_{n}(\lambda)$ with $\lambda \neq 0, \infty$. The authors would like to thank S. Iwata for his lecture and M. Takamatsu for informing them of the paper [19]. After submitting the paper E. Escolar and the referee pointed out that the paper [17] was already published dealing with the same problem with similar results, for which they are very thankful. This work is partially supported by Grant-in-Aid for Scientific Research 25610003 and 25287001 from JSPS (Japan Society for the Promotion of Science), and by JST (Japan Science and Technology Agency) CREST Mathematics (15656429).

## CHAPTER 1

## Tilted algebras and configurations of self-injective algebras of Dynkin type

In this chapter, we give an easier way to calculate a bijection from the set of isoclasses of tilted algebras of Dynkin type $\Delta$ to the set of configurations on the translation quiver $\mathbb{Z} \Delta$. Section 1 is devoted to preparations. In Section 2 we will give the complete list of indecomposable projectives and indecomposable injectives over a triangular matrix algebra $B$ defined there. In Section 3 we state and prove the main results. Throughout this chapter, all modules are assumed to be finite-dimensional right modules.

## 1. Preliminaries

1.1. Algebras and categories. A category $\mathcal{C}$ is called a $\mathbb{k}$-category if the morphism sets $\mathcal{C}(x, y)$ are $\mathbb{k}$-vector spaces, and the compositions $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \rightarrow \mathcal{C}(x, z)$ are $\mathbb{k}$-bilinear for all $x, y, z \in \mathcal{C}_{0}\left(\mathcal{C}_{0}\right.$ is the class of objects of $\mathcal{C}$, we sometimes write $x \in \mathcal{C}$ for $x \in \mathcal{C}_{0}$ ). In the sequel all categories are assumed to be $\mathbb{k}$-categories unless stated otherwise.

To construct repetitive categories and to make use of a covering theory we need to extend the range of considerations from algebras to categories. First we regard an algebra as a special type of categories by constructing a category cat $A$ from an algebra $A$ as follows.
(1) We fix a decomposition $1=e_{1}+\cdots+e_{n}$ of the identity element 1 of $A$ as a sum of orthogonal primitive idempotents.
(2) We set the object class of cat $A$ to be the set $\left\{e_{1}, \ldots, e_{n}\right\}$.
(3) For each pair $\left(e_{i}, e_{j}\right)$ of objects, we set $(\operatorname{cat} A)\left(e_{i}, e_{j}\right):=e_{j} A e_{i}$.
(4) We define the composition of cat $A$ by the multiplication of $A$.

The obtained category cat $A$ is uniquely determined up to isomorphism not depending on the decomposition of 1 . The category $C=\operatorname{cat} A$ is a small category having the following three properties.
(1) Distinct objects are not isomorphic.
(2) For each object $x$ of $C$ the algebra $C(x, x)$ is local.
(3) For each pair $(x, y)$ of objects of $C$ the morphism space $C(x, y)$ is finite-dimensional.

A small category with these three properties is called a spectroid ${ }^{1}$ and its objects are sometimes called points. A spectroid with only a finite number of points is called finite. The category cat $A$ is a finite spectroid. Conversely we can construct a matrix algebra

[^0]from a finite spectroid $C$ as follows.
$$
\operatorname{alg} C:=\left\{\left(m_{y x}\right)_{x, y \in C} \mid m_{y x} \in C(x, y), \forall x, y \in C\right\}
$$

Here we have alg cat $A \cong A$, cat alg $C \cong C$. Therefore we can identify the class of algebras and the class of finite spectroids by using cat and alg.

A spectroid $C$ is called locally bounded if for each point $x$ the set $\{y \in C \mid C(x, y) \neq$ 0 or $C(y, x) \neq 0\}$ is a finite set. Of course algebras ( $=$ finite spectroids) are locally bounded. In the range of locally bounded spectroids we can freely construct repetitive categories or consider coverings.

Remark 1.1. We can construct the "path-category" $\mathbb{k} Q$ from a locally finite quiver $Q$ in the same way as in the definition of the path-algebra. The only difference is in the following definition of compositions: For paths $\mu, \nu$ with $^{2} s(\mu) \neq t(\nu)$, it was defined as $\mu \nu=0$ in the path-algebra, but in contrast the composition $\mu \nu$ is not defined in the path-category.

A locally bounded spectroid $C$ is also presented as the form $\mathbb{k} Q / I$ for some locally finite quiver $Q$ and for some ideal $I$ of the path-category $\mathbb{k} Q$ such that $I$ is included in the ideal of $\mathbb{k} Q$ generated by the set of paths of length 2 . Here the quiver $Q$ is uniquely determined by $C$ up to isomorphism. This $Q$ is called the quiver of $C$.

A (right) module over a spectroid $C$ is a contravariant functor $C \rightarrow \operatorname{Mod} \mathbb{k}$. From a usual (right) module over an algebra $A$ we can construct a contravariant functor cat $A \rightarrow$ Mod $\mathbb{k}$ by the correspondence $e_{i} \mapsto M e_{i}$ for each point $e_{i}$ in cat $A$, and $f \mapsto\left(\cdot f: M e_{j} \rightarrow\right.$ $\left.M e_{i}\right)$ for each $f \in e_{j} A e_{i}=(\operatorname{cat} A)\left(e_{i}, e_{j}\right)$. Conversely, from a contravariant functor $F$ : cat $A \rightarrow \operatorname{Mod} \mathbb{k}$ we can construct an $A$-module $\bigoplus_{i=1}^{n} F\left(e_{i}\right)$; and these constructions are inverse to each other. In this way we can identify $A$-modules and modules over cat $A$.

The set of projective indecomposable modules over a spectroid $C$ is given by $\{C(-, x)\}_{x \in C}$ up to isomorphism, and finitely generated projective $C$-modules are nothing but finite direct sums of these. Using this we can define finitely generated modules or finitely presented modules over $C$ by the same way as those over algebras. By $\bmod C$ we denote the full subcategory of Mod $C$ consisting of finitely generated $C$-modules.

The dimension of a $C$-module $M$ is defined to be the dimension of $\bigoplus_{x \in C} M(x)$. When $C$ is locally bounded, a $C$-module is finitely presented if and only if it is finitely generated if and only if it is finite-dimensional.

### 1.2. Repetitive category.

Definition 1.2. Let $A$ be an algebra with a basic set of local idempotents $\left\{e_{1}, \ldots, e_{n}\right\}$.
(1) The repetitive category $\hat{A}$ of $A$ is a spectroid defined as follows.

Objects: $\hat{A}_{0}:=\left\{x^{[i]}:=(x, i) \mid x \in\left\{e_{1}, \ldots, e_{n}\right\}, i \in \mathbb{Z}\right\}$.
Morphisms: Let $x^{[i]}, y^{[j]} \in \hat{A}_{0}$. Then we set

$$
\hat{A}\left(x^{[i]}, y^{[j]}\right):= \begin{cases}\left\{f^{[i]}:=(f, i) \mid f \in A(x, y)\right\} & (j=i) \\ \left\{\varphi^{[i]}:=(\varphi, i) \mid \varphi \in D A(y, x)\right\} & (j=i+1) \\ 0 & \text { otherwise. }\end{cases}
$$

[^1]Compositions: The composition $\hat{A}\left(y^{[j]}, z^{[k]}\right) \times \hat{A}\left(x^{[i]}, y^{[j]}\right) \rightarrow \hat{A}\left(x^{[i]}, z^{[k]}\right)$ is defined as follows.
(i) If $j=i, k=j$, then we use the composition of $A$ :

$$
A(y, z) \times A(x, y) \rightarrow A(x, z)
$$

(ii) If $j=i, k=j+1$, then we use the right $A$-module structure of $D A(-, ?)$ :

$$
D A(z, y) \times A(x, y) \rightarrow D A(z, x)
$$

(iii) If $j=i+1, k=j$, then we use the left $A$-module structure of $D A(-, ?)$ :

$$
A(y, z) \times D A(y, x) \rightarrow D A(z, x)
$$

(iv) Otherwise the composition is zero.
(2) For each $i \in \mathbb{Z}$, we denote by $A^{[i]}$ the full subcategory of $\hat{A}$ whose object class is $\left\{x^{[i]} \mid x \in\left\{e_{1}, \ldots, e_{n}\right\}\right\}$.
(3) We define the Nakayama automorphism $\nu_{A}$ of $\hat{A}$ as follows: for each $i \in \mathbb{Z}, x, y \in$ $A, f \in A(x, y)$ and $\phi \in D A(y, x)$,

$$
\nu_{A}\left(x^{[i]}\right):=x^{[i+1]}, \nu_{A}\left(f^{[i]}\right):=f^{[i+1]}, \nu_{A}\left(\varphi^{[i]}\right):=\varphi^{[i+1]} .
$$

Remark 1.3. (1) The repetitive category of an algebra $A$ is locally bounded. (2) The set of all $\mathbb{Z} \times \mathbb{Z}$-matrices with only a finite number of nonzero entries whose diagonal entries belong to $A,(i+1, i)$ entries belong to $D A$ for all $i \in \mathbb{Z}$, and other entries are zero forms an infinite-dimensional algebra without identity element, which is called the repetitive algebra of $A$. The repetitive category $\hat{A}$ is nothing but this repetitive algebra regarded as a spectroid in a similar way. This is not an algebra ( $=$ a finite spectroid) any more, but a locally bounded spectroid.

Definition 1.4 (Gabriel [5]). Let $C$ be a locally bounded spectroid with a free ${ }^{3}$ action of a group $G$. Then we define the orbit category $C / G$ of $C$ by $G$ as follows.
(1) The objects of $C / G$ are the $G$-orbits $G x$ of objects $x$ of $C$.
(2) For each pair $G x, G y$ of objects of $C / G$ we set
$(C / G)(G x, G y):=\left\{\left({ }_{b} f_{a}\right)_{a, b} \in \prod_{(a, b) \in G x \times G y} C(a, b) \mid{ }_{g b} f_{g a}=g\left({ }_{b} f_{a}\right)\right.$, for all $\left.g \in G\right\}$.
(3) The composition is defined by

$$
\left({ }_{d} h_{c}\right)_{c, d} \cdot\left({ }_{b} f_{a}\right)_{a, b}:=\left(\sum_{b \in G y}{ }_{d} h_{b} \cdot{ }_{b} f_{a}\right)_{a, d} .
$$

for all $\left({ }_{b} f_{a}\right)_{a, b} \in(C / G)(G x, G y),\left({ }_{d} h_{c}\right)_{c, d} \in(C / G)(G y, G z)$. Note that each entry of the right hand side is a finite sum because $C$ is locally bounded.
A functor $F: C \rightarrow C^{\prime}$ is called a Galois covering with group $G$ if it is isomorphic to the canonical functor $\pi: C \rightarrow C / G$, namely if there exists an isomorphism $H: C / G \rightarrow C^{\prime}$ such that $F=H \pi$.

[^2]Remark 1.5. Recall that a spectroid $C$ is said to be self-injective in case $C(-, x)$ is injective in $\bmod C$ and $C(x,-)$ is injective in $\bmod C^{\mathrm{op}}$ for all $x \in C_{0}$. If $A$ is an algebra and a group $G$ acts freely on the category $\hat{A}$, then $\hat{A} / G$ turns out to be a self-injective spectroid. In particular, when $\hat{A} / G$ is a finite spectroid, it becomes a self-injective algebra. In this way we can construct a great number of self-injective algebras.

Definition 1.6. From a quiver $Q$ we can construct a translation quiver $\mathbb{Z} Q$ as follows.

- $(\mathbb{Z} Q)_{0}:=\mathbb{Z} \times Q_{0}$,
- $(\mathbb{Z} Q)_{1}:=\mathbb{Z} \times Q_{1} \cup\left\{\left(i, \alpha^{\prime}\right) \mid i \in \mathbb{Z}, \alpha \in Q_{1}\right\}$,
- We define the sources and the targets of arrows by

$$
(i, \alpha):(i, s(\alpha)) \rightarrow(i, t(\alpha)),\left(i, \alpha^{\prime}\right):(i, t(\alpha)) \rightarrow(i+1, s(\alpha))
$$

for all $(i, \alpha) \in \mathbb{Z} \times Q_{1}$.

- We take the bijection $\tau:(\mathbb{Z} Q)_{0} \rightarrow(\mathbb{Z} Q)_{0},(i, x) \mapsto(i-1, x)$ as the translation. In addition, we can define a polarization by $(i+1, \alpha) \mapsto\left(i, \alpha^{\prime}\right),\left(i, \alpha^{\prime}\right) \mapsto(i, \alpha)$. Note that by construction the translation quiver $\mathbb{Z} Q$ does not have any projective or injective vertices.

For example,


Remark 1.7. When $Q$ is a Dynkin quiver with the underlying graph $\Delta$, the isoclass of $\mathbb{Z} Q$ does not depend on orientations of $\Delta$, therefore we set $\mathbb{Z} \Delta:=\mathbb{Z} Q$.

## 2. Triangular Matrix Algebras

In this section we will give the complete list of indecomposable projectives and indecomposable injectives over a triangular matrix algebra $B$ defined in (2.1) below.

Definition 2.1. Let $R$ and $S$ be algebras, $M$ be an $S$ - $R$-bimodule. We define a category $\mathcal{C}=\mathcal{C}(R, S, M)$ as follows.

Objects: $\mathcal{C}_{0}:=\left\{(X, Y, f) \mid X_{R} \in \bmod R, Y_{S} \in \bmod S, f \in \operatorname{Hom}_{R}\left(Y \otimes_{S} M, X\right)\right\}$.
Morphisms: Let $(X, Y, f),\left(X^{\prime}, Y^{\prime}, f^{\prime}\right) \in \mathcal{C}_{0}$. Then we set

Compositions: Let $(X, Y, f),\left(X^{\prime}, Y^{\prime}, f^{\prime}\right),\left(X^{\prime \prime}, Y^{\prime \prime}, f^{\prime \prime}\right) \in \mathcal{C}_{0}$ and let

$$
\left(\phi_{0}, \phi_{1}\right) \in \mathcal{C}\left((X, Y, f),\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)\right),\left(\phi_{0}^{\prime}, \phi_{1}^{\prime}\right) \in \mathcal{C}\left(\left(X^{\prime}, Y^{\prime}, f^{\prime}\right),\left(X^{\prime \prime}, Y^{\prime \prime}, f^{\prime \prime}\right)\right)
$$

Then we set

$$
\left(\phi_{0}^{\prime}, \phi_{1}^{\prime}\right)\left(\phi_{0}, \phi_{1}\right):=\left(\phi_{0}^{\prime} \phi_{0}, \phi_{1}^{\prime} \phi_{1}\right) \in \mathcal{C}\left((X, Y, f),\left(X^{\prime \prime}, Y^{\prime \prime}, f^{\prime \prime}\right)\right)
$$

Then the following is well known.
Proposition 2.2. Let $R$ and $S$ be algebras, $M$ be an $S$ - $R$-bimodule, and set $T:=$ $\left[\begin{array}{cc}R & 0 \\ M & S\end{array}\right]$. Then

$$
\bmod T \simeq \mathcal{C}(R, S, M)
$$

Recall that an equivalence $F: \bmod T \rightarrow \mathcal{C}(R, S, M)$ is given as follows.
Objects: For each $L \in(\bmod T)_{0}$,

$$
F(L):=\left(L \varepsilon_{1}, L \varepsilon_{2}, f_{L}\right)
$$

where $\varepsilon_{1}:=\left[\begin{array}{cc}1_{R} & 0 \\ 0 & 0\end{array}\right], \varepsilon_{2}:=\left[\begin{array}{cc}0 & 0 \\ 0 & 1_{S}\end{array}\right]$ and $f_{L}: L \varepsilon_{2} \otimes_{S} M \rightarrow L \varepsilon_{1}$ is defined by $f_{L}\left(l \varepsilon_{2} \otimes m\right):=l\left[\begin{array}{cc}0 & 0 \\ m & 0\end{array}\right]$ for all $l \in L$ and $m \in M$.
Morphisms: For each $\alpha \in \operatorname{Hom}_{T}\left(L, L^{\prime}\right)$,

$$
F(\alpha):=\left(\left.\alpha\right|_{L \varepsilon_{1}},\left.\alpha\right|_{L \varepsilon_{2}}\right)
$$

Let $A$ be a tilted algebra of type $\Delta$, and set

$$
B:=\left[\begin{array}{cc}
A & 0  \tag{2.1}\\
D A & A
\end{array}\right], \quad \mathcal{C}:=\mathcal{C}(A, A, D A)
$$

Then we have $\bmod B \simeq \mathcal{C}$ by Proposition 2.2. By this equivalence, we identify $\bmod B$ with $\mathcal{C}$.

Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be a complete set of orthogonal local idempotents of $A$. Then as is easily seen $\left\{e_{1}^{[0]}, \ldots, e_{n}^{[0]}, e_{1}^{[1]}, \ldots, e_{n}^{[1]}\right\}$ is a complete set of orthogonal local idempotents of $B$, where we regard the objects $e_{i}^{[0]}$ of $A^{[0]}$ (resp. $e_{i}^{[1]}$ of $A^{[1]}$ ) as the elements $\left[\begin{array}{cc}e_{i} & 0 \\ 0 & 0\end{array}\right]$ (resp. $\left[\begin{array}{cc}0 & 0 \\ 0 & e_{i}\end{array}\right]$ ) of $B$ for all $i \in\{1, \ldots, n\}$. Hence $\left\{e_{1}^{[0]} B, \ldots, e_{n}^{[0]} B, e_{1}^{[1]} B, \ldots, e_{n}^{[1]} B\right\}$ is a complete set of isoclasses of projective indecomposable $B$-modules.

Proposition 2.3. For each $i=1, \ldots$, $n$, we have

$$
\begin{aligned}
& F\left(e_{i}^{[0]} B\right) \cong\left(e_{i} A, 0,0\right) \\
& F\left(e_{i}^{[1]} B\right) \cong\left(e_{i}(D A), e_{i} A, \text { can }\right)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& F\left(e_{i}^{[0]} B\right)=\left(e_{i}^{[0]} B \varepsilon_{1}, e_{i}^{[0]} B \varepsilon_{2}, f_{e_{i}^{[0]} B}\right)=\left(\left[\begin{array}{cc}
e_{i} A & 0 \\
0 & 0
\end{array}\right],\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right], 0\right) \cong\left(e_{i} A, 0,0\right), \\
& F\left(e_{i}^{[1]} B\right)=\left(e_{i}^{[1]} B \varepsilon_{1}, e_{i}^{[1]} B \varepsilon_{2}, f_{e_{i}^{[1]} B}\right)=\left(\left[\begin{array}{cc}
0 & 0 \\
e_{i}(D A) & 0
\end{array}\right],\left[\begin{array}{cc}
0 & 0 \\
0 & e_{i} A
\end{array}\right], f_{e_{i}^{[1]} B}\right) \cong\left(e_{i}(D A), e_{i} A, \text { can }\right),
\end{aligned}
$$

where

$$
\begin{gathered}
{\left[\begin{array}{cc}
0 & 0 \\
0 & e_{i} A
\end{array}\right] \otimes_{A} D A \xrightarrow{f_{e_{i}^{[1]} B}}\left[\begin{array}{cc}
0 & 0 \\
e_{i}(D A) & 0
\end{array}\right]} \\
\downarrow^{\imath} \\
e_{i} A \otimes_{A} D A--\overbrace{\text { can }^{2}}-\cdots e_{i}(D A) .
\end{gathered}
$$

In addition $\left\{D\left(B e_{1}^{[0]}\right), \ldots, D\left(B e_{n}^{[0]}\right), D\left(B e_{1}^{[1]}\right), \ldots, D\left(B e_{n}^{[1]}\right)\right\}$ is a complete set of isoclasses of injective indecomposable $B$-modules.

Lemma 2.4. For each $i=1, \ldots, n$, we have
(1) $D\left[\begin{array}{cc}A e_{i} & 0 \\ (D A) e_{i} & 0\end{array}\right] \cong\left[\begin{array}{cc}0 & 0 \\ D\left(A e_{i}\right) & e_{i} A\end{array}\right]$, and
(2) $D\left[\begin{array}{cc}0 & 0 \\ 0 & A e_{i}\end{array}\right] \cong\left[\begin{array}{cc}0 & 0 \\ 0 & D\left(A e_{i}\right)\end{array}\right]$.

Proof. (1) Define a map $\phi:\left[\begin{array}{cc}0 & 0 \\ D\left(A e_{i}\right) & e_{i} A\end{array}\right] \rightarrow D\left[\begin{array}{cc}A e_{i} & 0 \\ (D A) e_{i} & 0\end{array}\right]$ by

$$
\left[\begin{array}{cc}
0 & 0 \\
\alpha & a
\end{array}\right] \mapsto\left(\left[\begin{array}{cc}
b & 0 \\
\beta & 0
\end{array}\right] \mapsto \alpha(b)+\beta(a)\right)
$$

for all $a \in e_{i} A, \alpha \in D\left(A e_{i}\right), b \in A e_{i}$ and $\beta \in(D A) e_{i}$. Then it is easy to check that $\phi$ is a homomorphism of right $B$-modules and that $\phi$ is injective. Since the dimensions of the left hand side and the right hand side are equal, $\phi$ is an isomorphism.
(2) Define a map $\psi:\left[\begin{array}{cc}0 & 0 \\ 0 & D\left(A e_{i}\right)\end{array}\right] \rightarrow D\left[\begin{array}{cc}0 & 0 \\ 0 & A e_{i}\end{array}\right]$ by

$$
\left[\begin{array}{cc}
0 & 0 \\
0 & \alpha
\end{array}\right] \mapsto\left(\left[\begin{array}{ll}
0 & 0 \\
0 & a
\end{array}\right] \mapsto \alpha(a)\right),
$$

which is easily seen to be an isomorphism.
Proposition 2.5. For each $i=1, \ldots$, $n$, we have

$$
\begin{aligned}
& F\left(D\left(B e_{i}^{[0]}\right)\right) \cong\left(e_{i}(D A), e_{i} A, \text { can }\right) \cong e_{i}^{[1]} B, \\
& F\left(D\left(B e_{i}^{[1]}\right)\right) \cong\left(0, e_{i}(D A), 0\right) .
\end{aligned}
$$

Proof. Since $B e_{i}^{[0]}=\left[\begin{array}{cc}A & 0 \\ D A & A\end{array}\right]\left[\begin{array}{cc}e_{i} & 0 \\ 0 & 0\end{array}\right]=\left[\begin{array}{cc}A e_{i} & 0 \\ (D A) e_{i} & 0\end{array}\right]$, we have

$$
D\left(B e_{i}^{[0]}\right)=D\left[\begin{array}{cc}
A e_{i} & 0 \\
(D A) e_{i} & 0
\end{array}\right] \cong\left[\begin{array}{cc}
0 & 0 \\
D\left(A e_{i}\right) & e_{i} A
\end{array}\right]
$$

by Lemma 2.4(1). Hence

$$
F\left(D\left(B e_{i}^{[0]}\right)\right) \cong F\left[\begin{array}{cc}
0 & 0 \\
D\left(A e_{i}\right) & e_{i} A
\end{array}\right] \cong\left(e_{i}(D A), e_{i} A, \operatorname{can}\right) \cong e_{i}^{[1]} B .
$$

Since $B e_{i}^{[1]}=\left[\begin{array}{cc}A & 0 \\ D A & A\end{array}\right]\left[\begin{array}{cc}0 & 0 \\ 0 & e_{i}\end{array}\right]=\left[\begin{array}{cc}0 & 0 \\ 0 & A e_{i}\end{array}\right]$, we have

$$
D\left(B e_{i}^{[1]}\right)=D\left[\begin{array}{cc}
0 & 0 \\
0 & A e_{i}
\end{array}\right] \cong\left[\begin{array}{cc}
0 & 0 \\
0 & D\left(A e_{i}\right)
\end{array}\right]
$$

by Lemma 2.4(2). Hence $F\left(D\left(B e_{i}^{[1]}\right)\right)=F\left[\begin{array}{cc}0 & 0 \\ 0 & D\left(A e_{i}\right)\end{array}\right] \cong\left(0, e_{i}(D A), 0\right)$.

## 3. Configurations

Throughout the rest of this paper $\Lambda$ is a standard representation-finite self-injective algebra. If a module $M$ is both projective and injective, we say that $M$ is projectiveinjective for short.

### 3.1. Recover of AR-quivers from stable AR-quivers and configurations.

Definition 3.1. Let $C$ be a locally bounded spectroid with the AR-quiver $\Gamma_{C}$. Then the set

$$
\mathcal{C}_{C}:=\left\{[\operatorname{rad} P] \in \Gamma_{C} \mid P: \text { projective-injective } C \text {-module }\right\}
$$

is called the configuration of $C$.
In this section we compute the configuration of $\Lambda$.
Definition 3.2. Let $\Gamma$ be a stable translation quiver, and $\mathcal{C}$ a subset of $\Gamma_{0}$. Then we define a translation quiver $\Gamma_{\mathcal{C}}$ by

$$
\begin{aligned}
\left(\Gamma_{\mathcal{C}}\right)_{0} & :=\Gamma_{0} \sqcup\left\{p_{x} \mid x \in \mathcal{C}\right\} \\
\left(\Gamma_{\mathcal{C}}\right)_{1} & :=\Gamma_{1} \sqcup\left\{x \rightarrow p_{x}, p_{x} \rightarrow \tau^{-1} x\right\}
\end{aligned}
$$

where the translation of $\Gamma_{\mathcal{C}}$ is the same as that of $\Gamma$. In particular, $p_{x}$ are projectiveinjective vertices for all $x \in \mathcal{C}$.

Remark 3.3. (1) Let $C$ be a self-injective locally bounded spectroid. Then the quiver of the stable category $\bmod C$ of $\bmod C$ is the full subquiver ${ }_{s} \Gamma_{C}$ of $\Gamma_{C}$ with

$$
\left({ }_{s} \Gamma_{C}\right)_{0}:=\left\{x \mid x \text { is a stable vertex of } \Gamma_{C}\right\}
$$

(namely ${ }_{s} \Gamma_{C}$ is obtained from $\Gamma_{C}$ by removing all projective vertices), which is a stable translation quiver.
(2) It holds that $\mathcal{C}_{\Lambda} \subseteq\left({ }_{s} \Gamma_{\Lambda}\right)_{0}$, and by Riedtmann [10, 2.5] we have

$$
\begin{equation*}
\left({ }_{s} \Gamma_{\Lambda}\right)_{\mathcal{C}_{\Lambda}} \cong \Gamma_{\Lambda} . \tag{3.1}
\end{equation*}
$$

Thus we can recover the AR-quiver from the stable AR-quiver by using configurations.
Theorem 3.4. Let $\Lambda$ be a standard representation-finite self-injective algebra and $\Delta$ the Dynkin type of $\Lambda$. Then the following hold.
(1) (Waschbüsch $[\mathbf{8}, \mathbf{1 2}])$ There exist a tilted algebra $A$ of type $\Delta$ and an automorphism $\phi$ of $\hat{A}$ without fixed vertices such that $\Lambda \cong \hat{A} /\langle\phi\rangle$.

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(2) (Riedtmann [9]) There is an isomorphism $f:{ }_{s} \Gamma_{\hat{A}} \rightarrow \mathbb{Z} \Delta$. Denote also by $\phi$ the automorphism of ${ }_{s} \Gamma_{\hat{A}}$ induced from $\phi$ canonically, and define an automorphism $\phi^{\prime}$ of $\mathbb{Z} \Delta$ by the following commutative diagram:


Then we have ${ }_{s} \Gamma_{\Lambda} \cong{ }_{s} \Gamma_{\hat{A}} /\langle\phi\rangle \cong \mathbb{Z} \Delta /\left\langle\phi^{\prime}\right\rangle$.
By the formula (3.1) to compute $\Gamma_{\Lambda}$ it is enough to solve the following problem.
Problem 1. Let $\Lambda$ be a standard representation-finite self-injective algebra, which has the form $\hat{A} /\langle\phi\rangle$ for some tilted algebra $A$ of Dynkin type and an automorphism $\phi$ of $\hat{A}$ by Theorem 3.4. Then compute $\mathcal{C}_{\Lambda}$ from $A$.

REmARK 3.5. Let $f^{\prime}:{ }_{s} \Gamma_{\Lambda} \rightarrow \mathbb{Z} \Delta /\left\langle\phi^{\prime}\right\rangle$ be an isomorphism, and set $\mathcal{C}:=f^{\prime}\left(\mathcal{C}_{\Lambda}\right)$. Then we have

$$
\Gamma_{\Lambda} \cong\left({ }_{s} \Gamma_{\Lambda}\right)_{\mathcal{C}_{\Lambda}} \cong\left(\mathbb{Z} \Delta /\left\langle\phi^{\prime}\right\rangle\right)_{\mathcal{C}} .
$$

Thus we can compute $\Gamma_{\Lambda}$ by Theorem 3.4(2) if we can obtain the set $\mathcal{C}$.
Theorem 3.6 (Gabriel [5, Theorem 3.6]). Let $R$ be a locally representation-finite and locally bounded $\mathbb{k}$-category, and $G$ a group consisting of automorphisms of $R$ such that $G$ acts freely on $R$. Then the $A R$-quiver $\Gamma_{R}$ of $R$ has an induced $G$-action, and we have $\Gamma_{R} / G \cong \Gamma_{R / G}$.

Corollary 3.7. Let $A$ be a tilted algebra of Dynkin type, and $\phi$ an automorphism of $\hat{A}$ without fixed vertices. Then we have

$$
\mathcal{C}_{\hat{A}} /\langle\phi\rangle \cong \mathcal{C}_{\Lambda} .
$$

Therefore to solve Problem 1 it is enough to consider the following.
Problem 2. In the same setting as in Problem 1, compute $\mathcal{C}_{\hat{A}}$ from $A$.
Throughout the rest of this section
(1) let $A$ be a tilted algebra of Dynkin type $\Delta$, and set
(2) $B:=\left[\begin{array}{cc}A & 0 \\ D A & A\end{array}\right]$.

By (1), $\Gamma_{A}$ has a section $\mathcal{S}$ whose underlying graph is isomorphic to $\Delta$.
3.2. Relationships between $\hat{A}, B$ and $A$. We set as follows:

$$
\begin{aligned}
I_{0,1} & =\left\langle e_{j}^{[i]} \mid i \in \mathbb{Z} \backslash\{0,1\}, j \in\{1, \ldots, n\}\right\rangle, \\
I_{0} & =\left\langle e_{j}^{[i]} \mid i \in \mathbb{Z} \backslash\{0\}, j \in\{1, \ldots, n\}\right\rangle, \\
I_{1} & =\left\langle e_{j}^{[i]} \mid i \in \mathbb{Z} \backslash\{1\}, j \in\{1, \ldots, n\}\right\rangle .
\end{aligned}
$$

Then $\hat{A} / I_{0,1} \cong B, \hat{A} / I_{0} \cong A^{[0]}(\cong A)$ and $\hat{A} / I_{1} \cong A^{[1]}(\cong A)$. We also have

$$
B /\left[\begin{array}{cc}
0 & 0 \\
D A & 0
\end{array}\right] \cong A^{[0]} \times A^{[1]}
$$

We have the following surjective algebra homomorphisms

which induce the following embeddings of categories


We regard $\bmod A \subseteq \bmod B$ by the embedding $\bmod A=\bmod A^{[0]} \longrightarrow \bmod B$. The embeddings above give us the following embeddings of vertex sets of AR-quivers:


We define an ideal $\mathbb{k}(\mathbb{Z} \Delta)^{+}$of the mesh category $\mathbb{k}(\mathbb{Z} \Delta)$ as follows:

$$
\mathbb{k}(\mathbb{Z} \Delta)^{+}:=\left\langle(\mathbb{Z} \Delta)_{1}+I_{\mathbb{Z}}\right\rangle
$$

where $I_{\mathbb{Z} \Delta}$ is the mesh ideal of the translation quiver $\mathbb{Z} \Delta$. Then the values of $m_{\Delta}:=$ $\min \left\{m \in \mathbb{N} \mid\left(\mathbb{k}(\mathbb{Z} \Delta)^{+}\right)^{i}=0, \forall i \geq m\right\}$ are known to be as follows:

$$
m_{\Delta}=\left\{\begin{array}{ll}
n & \left(\Delta=A_{n}\right) \\
2 n-3 & \left(\Delta=D_{n}\right) \\
11 & \left(\Delta=E_{6}\right) \\
17 & \left(\Delta=E_{7}\right) \\
29 & \left(\Delta=E_{8}\right)
\end{array} .\right.
$$

We see that the following two propositions hold by [4, Sect. 2].

Proposition 3.8. Let $i=0,1$.
(1) The full subquiver $\mathcal{S}_{B}^{[i]}$ of $\Gamma_{B}$ with the vertex set $\sigma_{i}\left(\mathcal{S}_{0}\right)$ forms a section of ${ }_{s} \Gamma_{B}$.
(2) The full subquiver $\mathcal{S}_{\hat{A}}^{[i]}$ of $\Gamma_{\hat{A}}$ with the vertex set $\sigma \sigma_{i}\left(\mathcal{S}_{0}\right)$ forms a section of ${ }_{s} \Gamma_{\hat{A}}$.

Remark 3.9. A quiver $Q$ without oriented cycles will be regarded as a poset by the order defined as follows:

For each $x, y \in Q_{0}, x \preceq y: \Leftrightarrow$ there is a path in $Q$ from $x$ to $y$.
Definition 3.10. (1) We set $\mathcal{H}_{B}$ to be the full subquiver of $\Gamma_{B}$ defined by the set

$$
\left(\mathcal{H}_{B}\right)_{0}:=\left\{x \in\left(\Gamma_{B}\right)_{0} \mid a \preceq x \preceq b \text { for some } a \in\left(\mathcal{S}_{B}^{[0]}\right)_{0}, b \in\left(\mathcal{S}_{B}^{[1]}\right)_{0}\right\}
$$

of vertices.
(2) We set $\mathcal{H}_{\hat{A}}^{[0,1]}$ to be the full subquiver of $\Gamma_{\hat{A}}$ defined by the set

$$
\left(\mathcal{H}_{\hat{A}}^{[0,1]}\right)_{0}:=\left\{x \in\left(\Gamma_{\hat{A}}\right)_{0} \mid a \preceq x \preceq b \text { for some } a \in\left(\mathcal{S}_{\hat{A}}^{[0]}\right)_{0}, b \in\left(\mathcal{S}_{\hat{A}}^{[1]}\right)_{0}\right\}
$$

of vertices.
Proposition 3.11. (1) The map $\sigma:\left(\Gamma_{B}\right)_{0} \rightarrow\left(\Gamma_{\hat{A}}\right)_{0}$ is uniquely extended to a quiver isomorphism $\mathcal{H}_{B} \rightarrow \mathcal{H}_{\hat{A}}^{[0,1]}$.
(2) We have $\mathcal{S}_{\hat{A}}^{[1]}=\tau^{-m_{\Delta}} \mathcal{S}_{\hat{A}}^{[0]}$. We set $\mathcal{S}_{\hat{A}}^{[n]}:=\tau^{-n m_{\Delta}} \mathcal{S}_{\hat{A}}^{[0]}$ for all $n \in \mathbb{Z}$.
(3) Set $\mathcal{H}_{\hat{A}}^{[n, n+1]}:=\tau^{-n m_{\Delta}}\left(\mathcal{H}_{\hat{A}}^{[0,1]}\right)$ for all $n \in \mathbb{Z}$. Then for each $i=0,1$

$$
\begin{aligned}
\left(\Gamma_{\hat{A}}\right)_{i} & =\bigcup_{n \in \mathbb{Z}}\left(\mathcal{H}_{\hat{A}}^{[n, n+1]}\right)_{i} \\
\left(\mathcal{S}_{\hat{A}}^{[n+1]}\right)_{i} & =\left(\mathcal{H}_{\hat{A}}^{[n, n+1]}\right)_{i} \cap\left(\mathcal{H}_{\hat{A}}^{[n+1, n+2]}\right)_{i}
\end{aligned}
$$

Roughly speaking, $\Gamma_{\hat{A}}$ is obtained by connecting infinite copies of $\mathcal{H}_{B}$ on both sides.
Example 3.12. Let $A$ be the path algebra of the following quiver.

$$
1^{[0]} \longrightarrow 2^{[0]} \longrightarrow 3^{[0]}
$$

Then $\Gamma_{A}$ is given as follows (double arrows represent a section).


Therefore $A$ is a tilted algebra of type $A_{3}$. Moreover $B=\left[\begin{array}{cc}A & 0 \\ D A & A\end{array}\right]=\left[\begin{array}{cc}A^{[0]} & 0 \\ (D A)^{[0]} & \left.A^{[1]}\right]\end{array}\right]$ is an algebra given by following quiver with relations.


Then $\Gamma_{B}$ is given as follows (elements of $\mathcal{C}_{B}$ are encircled).


In the above, $\mathcal{H}_{B}$ is given by the full subquiver consisting of vertices between the left section and the right section. $\hat{A}$ is given by the following quiver with relations.


Then $\Gamma_{\hat{A}}$ is as follows (each element of $\mathcal{C}_{\hat{A}}$ is encircled by a broken or solid line, in particular solid circles present elements of $\mathcal{C}_{B}$ ). In this case we have $m_{\Delta}=3$.


The following is immediate from Proposition 3.11.
Corollary 3.13. We have $\mathcal{C}_{\hat{A}}=\tau^{\mathbb{Z} m_{\Delta}} \sigma\left(\mathcal{C}_{B}\right)$.
By this corollary, Problem 2 is reduced to the following.
Problem 3. Let $A$ be a tilted algebra of Dynkin type $\Delta$, and $B$ as above. Then give the configuration $\mathcal{C}_{B}$ from $A$.
3.3. Configuration of $B$. The purpose of this subsection is to solve Problem 3.

Definition 3.14. (1) We define an ideal $\mathcal{P} \mathcal{I}$ of $\bmod B$ as follows and set $\widetilde{\bmod } B:=$ $(\bmod B) / \mathcal{P} \mathcal{I}$. For each $X, Y \in(\bmod B)_{0}$
$\mathcal{P} \mathcal{I}(X, Y):=\left\{f \in \operatorname{Hom}_{B}(X, Y) \mid f\right.$ factors through a projective-injective $B$-module $\}$
Let $(\tilde{?}): \bmod B \rightarrow \widetilde{\bmod } B$ be the canonical functor and set

$$
\left.\widetilde{\operatorname{Hom}}_{B}(\tilde{X}, \tilde{Y}):=\widetilde{\bmod B} B\right)(\tilde{X}, \tilde{Y})
$$

for all $X, Y \in \bmod B$. Thus $\tilde{X}=X$ for all $X \in(\bmod B)_{0}$ and $\tilde{f}=f+\mathcal{P} \mathcal{I}(X, Y)$ for all $f \in \operatorname{Hom}_{B}(X, Y)$.
(2) We denote by $\bmod _{\mathcal{P} \mathcal{I}} B$ the full subcategory of $\bmod B$ consisting of $B$-modules without projective-injective direct summands.
(3) Let $X$ and $Y \in \bmod _{\mathcal{P} \mathcal{I}} B$. Then it is well known that $\mathcal{P} \mathcal{I}(X, Y) \subseteq \operatorname{rad}_{B}(X, Y)$. We set $\operatorname{rad}_{B}(X, Y):=\operatorname{rad}_{B}(X, Y) / \mathcal{P} \mathcal{I}(X, Y)$.

Definition 3.15. We define the full translation subquiver $\tilde{\Gamma}_{B}$ of $\Gamma_{B}$ by

$$
\left(\tilde{\Gamma}_{B}\right)_{0}:=\left\{X \in\left(\Gamma_{B}\right)_{0} \mid X \text { is not projective-injective. }\right\}
$$

Moreover we set

$$
\operatorname{supp}\left(s_{X}\right):=\left\{Y \in\left(\tilde{\Gamma}_{B}\right)_{0} \mid s_{X}(Y) \neq 0\right\}
$$

where the map $s_{X}:\left(\tilde{\Gamma}_{B}\right)_{0} \rightarrow \mathbb{Z}_{\geq 0}$ is defined by $s_{X}(Y):=\operatorname{dim} \widetilde{\operatorname{Hom}}_{B}(\tilde{X}, \tilde{Y})\left(Y \in\left(\tilde{\Gamma}_{B}\right)_{0}\right)$ for all $X \in\left(\tilde{\Gamma}_{B}\right)_{0}$.

Definition 3.16. Let $P$ be a projective indecomposable $A$-module, and $\operatorname{rad} P=$ $\bigoplus_{i=1}^{r} R_{i}$ with $R_{i}$ indecomposable for all $i$. Then we define a full subquiver $\mathcal{R}_{P}$ of $\tilde{\Gamma}_{B}$ by

$$
\left(\mathcal{R}_{P}\right)_{0}:=\operatorname{supp}\left(s_{P}\right) \backslash\left(\bigcup_{i=1}^{r} \operatorname{supp}\left(s_{R_{i}}\right)\right)
$$

Definition 3.17. We regard the subquiver $\mathcal{R}_{P}$ as a poset by Remark 3.9. For a projective indecomposable $A$-module $P$ if $\min \mathcal{R}_{P}$ exists, we define

$$
\nu^{\prime}(P):=\min \mathcal{R}_{P},
$$

otherwise we do not define the notation $\nu^{\prime}(P)$.
Example 3.18. In the following figure, the vertices inside broken lines form $\operatorname{supp}\left(s_{P}\right)$ and those inside doted lines form $\left(\bigcup_{i=1}^{r} \operatorname{supp}\left(s_{R_{i}}\right)\right)$. Therefore the subquiver $\mathcal{R}_{P}$ consists of the vertices inside solid lines, and $\nu^{\prime}(P)$ is the minimum element of $\mathcal{R}_{P}$. Projective vertices are presented by white circles o.


Proposition 3.19. Let $P$ be a projective indecomposable $A$-module. Then $\nu^{\prime}(P)$ is always defined and $\nu^{\prime}(P) \cong$ top $P$.

Proof. We set $J:=\operatorname{rad} A$. We have $P \cong e_{i} A$ for some $i$. It is enough to show that top $P$ is the minimum element of the poset $\mathcal{R}_{P}$. First we show that top $P \in \mathcal{R}_{P}$, equivalently that top $P \in \operatorname{supp}\left(s_{P}\right)$ but top $P \notin \bigcup_{j=1}^{r} \operatorname{supp}\left(s_{R_{j}}\right)$.
(i) top $P \in \operatorname{supp}\left(s_{P}\right)$. Let $\pi: P \rightarrow \operatorname{top} P$ be the canonical epimorphism in $\bmod A$. It is enough to show that $\tilde{\pi} \neq 0$. Assume $\tilde{\pi}=0$. Then $\pi$ factors through a projective-injective
$B$-module $Q$ in $\bmod B$, namely there is $(\alpha, \beta) \in \operatorname{Hom}_{B}(P, Q) \times \operatorname{Hom}_{B}(Q, \operatorname{top} P)$ such that $\pi=\beta \alpha$.


Since $\pi$ is an epimorphism, so is $\beta$. Moreover

$$
\beta(\operatorname{rad} Q)=\beta(Q \operatorname{rad} B)=\beta(Q) \operatorname{rad} B \cong(\operatorname{top} P)(\operatorname{rad} B)=0
$$

because top $P$ is simple.


By the universality of cokernel, there is a unique $\gamma \in \operatorname{Hom}_{B}(\operatorname{top} Q$, $\operatorname{top} P)$ such that the above diagram is commutative. Since $\beta$ is an epimorphism, so is $\gamma$. Then we have an exact sequence

$$
0 \rightarrow \operatorname{Ker} \gamma \hookrightarrow \operatorname{top} Q \xrightarrow{\gamma} \operatorname{top} P \rightarrow 0 .
$$

Since $Q$ is a projective-injective $B$-module, top $Q$ has the form $\bigoplus_{j=1}^{n}\left(e_{j}^{[1]} B / e_{j}^{[1]} \operatorname{rad} B\right)^{m_{j}}$ for some $\left(m_{j}\right)_{j} \in \mathbb{Z}^{n} \backslash\left\{(0)_{j}\right\}$ by Propositions 2.3 and 2.5. Further since top $Q$ is semisimple, the exact sequence above splits, namely top $P$ is a direct summand of top $Q$. But top $P \cong$ $e_{i}^{[0]} B / e_{i}^{[0]} \operatorname{rad} B$, a contradiction. Hence we must have $\tilde{\pi} \neq 0$.
(ii) top $P \notin \bigcup_{j=1}^{r} \operatorname{supp}\left(s_{R_{j}}\right)$.

Assume top $P \in \bigcup_{j=1}^{r} \operatorname{supp}\left(s_{R_{j}}\right)(=\operatorname{supp}(\operatorname{rad} P))$. Then $\widetilde{\operatorname{Hom}}_{B}(\operatorname{rad} P, \operatorname{top} P) \neq 0$, and there is an $f \in \operatorname{Hom}_{B}(\operatorname{rad} P$, top $P)$ such that $\tilde{f} \neq 0$. Since top $P$ is simple, $f$ is an epimorphism. Taking the following diagram into account, we see that top $P$ is a direct summand of top $(\operatorname{rad} P)$ by the same argument as above.


Since we have

$$
\operatorname{top} P=e_{i} A / e_{i} J, \operatorname{top}(\operatorname{rad} P)=e_{i} J / e_{i} J^{2},
$$

$e_{i} A / e_{i} J$ is a direct summand of $e_{i} J / e_{i} J^{2}$. Then we have $e_{i} J e_{i} / e_{i} J^{2} e_{i} \neq 0$ because $0 \neq$ $e_{i} A e_{i} / e_{i} J e_{i} \hookrightarrow e_{i} J e_{i} / e_{i} J^{2} e_{i}$. Thus there is a loop at the vertex $i$ in the quiver of A. But it is impossible because $A$ is a tilted algebra. Hence top $P \in \mathcal{R}_{P}$.
(iii) top $P$ is the minimum element in $\mathcal{R}_{P}$. It is enough to show that $\widetilde{\operatorname{Hom}}_{B}(\operatorname{top} P, X) \neq$ 0 for all $X \in \mathcal{R}_{P}$. Since $X \in \operatorname{supp}\left(s_{P}\right)$ and $X \notin \bigcup_{j=1}^{r} \operatorname{supp}\left(s_{R_{j}}\right)$, we have $\widetilde{\operatorname{Hom}}_{B}(P, X) \neq 0$ and $\widetilde{\operatorname{Hom}}_{B}(\operatorname{rad} P, X)=\widetilde{\operatorname{Hom}}_{B}\left(\bigoplus_{j=1}^{r} R_{j}, X\right)=\bigoplus_{j=1}^{r} \widetilde{\operatorname{Hom}}_{B}\left(R_{j}, X\right)=0$. We take $\alpha \in$
$\operatorname{Hom}_{B}(P, X)$ such that $\tilde{\alpha} \neq 0$. Since $\widetilde{\operatorname{Hom}}_{B}(\operatorname{rad} P, X)=0$, there is a projective-injective $B$-module $Q$ such that $\alpha \sigma=h g$ for some $g$ and $h$ below.


Since $Q$ is injective, there is some $t \in \operatorname{Hom}_{B}(P, Q)$ such that $g=t \sigma$.


Since $\alpha \sigma=h g=h t \sigma$, we have $(\alpha-h t) \sigma=0$. By the universality of the cokernel, there is a unique $\alpha^{\prime} \in \operatorname{Hom}_{B}(\operatorname{top} P, X)$ such that $\alpha-h t=\alpha^{\prime} \pi$.


Here $\alpha^{\prime}$ is nonzero in $\widetilde{\bmod } B$. Indeed, if $\alpha^{\prime}$ factors through a projective-injective $B$-module $Q^{\prime}$, then $\alpha^{\prime}=h^{\prime} g^{\prime}$ for some $g^{\prime}$ and $h^{\prime}$ in the following diagram.


Since $\alpha-h t=\alpha^{\prime} \pi=h^{\prime} g^{\prime} \pi$, we have $\alpha=h^{\prime} g^{\prime} \pi+h t=\left[\begin{array}{ll}h^{\prime} & h\end{array}\right]\left[\begin{array}{c}g^{\prime} \pi \\ t\end{array}\right]$.


Thus $\tilde{\alpha}=0$, a contradiction. Hence $\tilde{\alpha}^{\prime} \neq 0$ and $\widetilde{\operatorname{Hom}}_{B}(\operatorname{top} P, X) \neq 0$.
We will give an alternative definition of the map $\nu^{\prime}$ below, which is easier to compute than the first one.

Definition 3.20. Let $P \in \bmod B$ be projective.
(1) Let $\mathcal{P}_{P}$ be the full subcategory of $\bmod B$ consisting of projective modules $Q$ such that $P$ is not a direct summand of $Q$.
(2) We define an ideal $\mathcal{I}_{P}$ of $\bmod B$ and the factor category $\bmod ^{P} B:=\bmod B / \mathcal{I}_{P}$ of $\bmod B$ by setting

$$
\mathcal{I}_{P}(X, Y):=\left\{f \in \operatorname{Hom}_{B}(X, Y) \mid f \text { factors through an object in } \mathcal{P}_{P}\right\},
$$

and set

$$
\underline{\operatorname{Hom}}_{B}^{P}(X, Y):=\operatorname{Hom}_{B}(X, Y) / \mathcal{I}_{P}(X, Y)
$$

for all $X, Y \in \bmod B$. Let $(?): \bmod B \rightarrow \underline{\bmod }^{P} B$ be the canonical functor. Thus $\underline{X}=X$ for all $X \in(\bmod B)_{0}$ and $\underline{f}=f+\mathcal{I}_{P}(X, Y)$ for all $f \in \operatorname{Hom}_{B}(X, Y)$.

Let

$$
\operatorname{supp}\left(s_{P}^{\prime}\right):=\left\{X \in\left(\tilde{\Gamma}_{B}\right)_{0} \mid s_{P}^{\prime}(X) \neq 0\right\} \subseteq\left(\tilde{\Gamma}_{B}\right)_{0}
$$

where the map $s_{P}^{\prime}:\left(\tilde{\Gamma}_{B}\right)_{0} \rightarrow \mathbb{Z}_{\geq 0}$ is defined by $s_{P}^{\prime}(X):=\operatorname{dim} \underline{\operatorname{Hom}}_{B}^{P}(P, X)\left(X \in\left(\tilde{\Gamma}_{B}\right)_{0}\right)$ for all $P \in\left(\tilde{\Gamma}_{B}\right)_{0}$.

Lemma 3.21. Let $Q$ and $X$ be in $\bmod B$. If $Q$ is projective and there is an epimorphism $Q \rightarrow X$, then the projective cover of $X$ is a direct summand of $Q$.

Proof. By composing an epimorphism $Q \rightarrow X$ and the canonical epimorphism $X \rightarrow$ top $X$, we obtain a nonzero morphism $Q \rightarrow \operatorname{top} X$, which induced a retraction $\operatorname{top} Q \rightarrow$ top $X$. By taking projective covers, we have a retraction $Q \rightarrow P(\operatorname{top} X)=P(X)$, where $P(Y)$ denotes the projective cover of a $B$-module $Y$.

Lemma 3.22. If $f: X \rightarrow \operatorname{top} P$ is nonzero in $\bmod B$, then $\underline{f} \neq 0$.
Proof. Since top $P$ is simple, $f$ is an epimorphism. Assume that $\underline{f}=0$. Then there is a $Q \in \mathcal{P}_{P}$ such that $f$ factor through $Q$, namely there is a pair $(\alpha, \beta) \in \operatorname{Hom}_{B}(X, Q) \times$ $\operatorname{Hom}_{B}(Q, \operatorname{top} P)$ such that $f=\beta \alpha$.


Since $Q$ is projective and $\beta$ is an epimorphism, $P$ is a direct summand of $Q$ by Lemma 3.21, a contradiction.

Proposition 3.23. Let $P$ be a projective indecomposable $A$-module. Then the poset $\operatorname{supp}\left(s_{P}^{\prime}\right)$ has the maximum element and we have

$$
\max \operatorname{supp}\left(s_{P}^{\prime}\right) \cong \operatorname{top} P
$$

Thus $\nu^{\prime}(P)=\max \operatorname{supp}\left(s_{P}^{\prime}\right)$.
Proof. It is enough to show that top $P$ is the maximum element of the poset $\operatorname{supp}\left(s_{P}^{\prime}\right)$. (i) First we show that top $P \in \operatorname{supp}\left(s_{P}^{\prime}\right)$, equivalently that $\operatorname{Hom}_{B}^{P}(P$, top $P) \neq 0$.

Let $\pi \in \operatorname{Hom}_{B}(P, \operatorname{top} P)$ be the canonical epimorphism. Then since $\pi$ is nonzero, we have $\underline{\pi} \neq 0$ by Lemma 3.22.
(ii) Next we show that $\operatorname{Hom}_{B}^{P}(X, \operatorname{top} P) \neq 0$ for all $X \in \operatorname{supp}\left(s_{P}^{\prime}\right)$.

Take arbitrary $X \in \operatorname{supp}\left(s_{P}^{\prime}\right)$. Then there is an $f \in \operatorname{Hom}_{B}(P, X)$ such that $f \neq 0$. Let $P(X)$ be a projective cover of $X$. Since $P$ is projective, there is $g \in \operatorname{Hom}_{B}(P, \bar{P}(X))$ such that $f=\pi g$.


Since $\underline{f} \neq 0$, we have $P(X) \notin \mathcal{P}_{P}$. Thus $P$ is a direct summand of $P(X)$. Then top $P$ is a direct summand of top $P(X) \cong$ top $X$. By composing the canonical epimorphism $X \rightarrow$ top $X$ and a retraction top $X \rightarrow \operatorname{top} P$, we obtain a nonzero morphism $h: X \rightarrow \operatorname{top} P$. By Lemma 3.22, we have $\underline{h} \neq 0$.

Next we define a map sending a simple $A$-module to an element of the configuration.
Lemma 3.24. Let $S$ be a simple $A$-module, and $Q$ the injective hull of $S$ in $\bmod B$. Then the left $(\widetilde{\bmod } B)$-module $\widetilde{\operatorname{Hom}}_{B}(S,-)$ has a simple socle, and

$$
\operatorname{soc} \widetilde{\operatorname{Hom}}_{B}(S,-) \cong \widetilde{\operatorname{Hom}}_{B}(\operatorname{rad} Q,-) / \widetilde{\operatorname{rad}}_{B}(\operatorname{rad} Q,-)
$$

Proof. Note that $Q$ is projective-injective by Proposition 2.5. Since soc $\widetilde{\operatorname{Hom}}_{B}(S,-)$ is a semisimple $\widetilde{\bmod } B$-module, it has the form

$$
\begin{equation*}
\operatorname{soc} \widetilde{\operatorname{Hom}}_{B}(S,-) \cong \bigoplus_{i=1}^{d} \operatorname{top} \widetilde{\operatorname{Hom}}_{B}\left(Y_{i},-\right) \tag{3.2}
\end{equation*}
$$

for some indecomposable $B$-modules $Y_{i}(i \in\{1, \ldots, d\})$. Let $i \in\{1, \ldots, d\}$ and put $Y:=Y_{i}$. Then $\left(\operatorname{soc} \widetilde{\operatorname{Hom}}_{B}(S,-)\right)(Y) \neq 0$. Since $\left(\operatorname{soc} \widetilde{\operatorname{Hom}}_{B}(S,-)\right)(Y) \neq 0$, we can take an element $f \in \operatorname{Hom}_{B}(S, Y)$ such that $\tilde{f} \neq 0$ but $\tilde{\alpha} \tilde{f}=0$ for all $\tilde{\alpha} \in \widetilde{\operatorname{rad}_{B}}(Y, Z)$. Consider the following diagram in $\bmod B$.


Then $f$ is a monomorphism because $S$ is simple and $f \neq 0$. Since $Q$ is injective, there is a homomorphism $g \in \operatorname{Hom}_{B}(Y, Q)$ such that $\sigma \rho=g f$. If $g$ is an epimorphism, then $g$ is a retraction because $Q$ is projective. Thus $Q$ is a direct summand of the indecomposable module $Y$, and hence $Y \cong Q$. Then $\tilde{f}=0$, a contradiction. Therefore $g$ cannot be an epimorphism.


Hence there is a homomorphism $h \in \operatorname{Hom}_{B}(Y, \operatorname{rad} Q)$ such that $g=\sigma h$. Then $\sigma \rho=\sigma h f$, and we have $\rho=h f$ because $\sigma$ is a monomorphism.

Assume that $Y$ is not isomorphic to $\operatorname{rad} Q$. Then $h \in \operatorname{rad}_{B}(Y, \operatorname{rad} Q)$, and $\tilde{h} \in$ $\widetilde{\operatorname{rad}}_{B}(Y, \operatorname{rad} Q)$, we have $\tilde{\rho}=\tilde{h} \tilde{f}=0$. Therefore there is a projective-injective $B$-module $P$ such that $\rho=\beta \alpha$ for some $\alpha \in \operatorname{Hom}_{B}(S, P)$ and $\beta \in \operatorname{Hom}_{B}(P, \operatorname{rad} Q)$.


Since $P$ is injective, there is a morphism $\gamma \in \operatorname{Hom}_{B}(Q, P)$ such that $\alpha=\gamma \sigma \rho$. Thus we have $\rho=\beta \gamma \sigma \rho$. Since $\beta \gamma$ is not a monomorphism, $(\beta \gamma)(S)=0$, that is, $\rho=$ $\beta \gamma \sigma \rho=0$, a contradiction. Hence we must have $Y \cong \operatorname{rad} Q$. Then by (3.2) we have soc $\widetilde{\operatorname{Hom}}_{B}(S,-) \cong\left(\operatorname{top} \widetilde{\operatorname{Hom}}_{B}(\operatorname{rad} Q,-)\right)^{(d)}$. Since $\operatorname{rad} Q$ is indecomposable, we have $\operatorname{End}_{B}(\operatorname{rad} Q) / \operatorname{rad} \operatorname{End}_{B}(\operatorname{rad} Q) \cong \mathbb{k}$. Then

$$
\begin{aligned}
{\left[\operatorname{soc}{\widetilde{\operatorname{Hom}_{B}}}^{(S,-)](\operatorname{rad} Q)}\right.} & \cong\left(\widetilde{\operatorname{End}}_{B}(\operatorname{rad} Q) / \operatorname{rad} \widetilde{\operatorname{End}}_{B}(\operatorname{rad} Q)\right)^{(d)} \\
& \cong\left(\operatorname{End}_{B}(\operatorname{rad} Q) / \operatorname{rad} \operatorname{End}_{B}(\operatorname{rad} Q)\right)^{(d)} \cong \mathbb{k}^{(d)}
\end{aligned}
$$

Here we have $d=1$ because

$$
\begin{aligned}
1 \leq d & =\operatorname{dim}\left(\operatorname{soc}{\widetilde{\operatorname{Hom}_{B}}}^{(S,-))(\operatorname{rad} Q) \leq \operatorname{dim} \widetilde{\operatorname{Hom}}_{B}(S, \operatorname{rad} Q)}\right. \\
& \leq \operatorname{dim} \operatorname{Hom}_{B}(S, \operatorname{rad} Q)=\operatorname{dim} \operatorname{Hom}_{B}(S, S)=1
\end{aligned}
$$

Thus soc $\widetilde{\operatorname{Hom}}_{B}(S,-)$ is simple, and soc $\widetilde{\operatorname{Hom}}_{B}(S,-) \cong \widetilde{\operatorname{Hom}}_{B}(\operatorname{rad} Q,-) / \widetilde{\operatorname{rad}}_{B}(\operatorname{rad} Q,-)$.

It follows by the lemma above that the poset $\operatorname{supp}\left(s_{S}\right)$ has the maximum element for each simple $A$-module $S$. We then set $\nu_{B}(S)$ to be the maximum element. The following is immediate.

Proposition 3.25. Let $S$ be a simple $A$-module, and $Q$ the injective hull of $S$ in $\bmod B$. Then we have $\nu_{B}(S) \cong \operatorname{rad} Q$.

We finally obtain the following by Propositions 3.23 and 3.25.

Theorem 3.26. Let $\mathcal{P}$ be a complete set of representatives of isoclasses of indecomposable projective $A$-modules. Then we have

$$
\mathcal{C}_{B}=\nu_{B}\left(\nu^{\prime}(\mathcal{P})\right)
$$

Hence as is stated before, $\mathcal{C}_{\Lambda}$ is obtained as follows.
Theorem 3.27.

$$
\mathcal{C}_{\Lambda}=\mathcal{C}_{\hat{A}} /\langle\phi\rangle=\left(\tau^{\mathbb{Z} m_{\Delta}} \sigma\left(\mathcal{C}_{B}\right)\right) /\langle\phi\rangle=\left(\tau^{\mathbb{Z} m_{\Delta}} \sigma \nu_{B} \nu^{\prime}(\mathcal{P})\right) /\langle\phi\rangle .
$$

## CHAPTER 2

## Decomposition theory of modules: the case of Kronecker algebra

In this chapter, we give a general formula that computes the indecomposable decomposition of any finite-dimensional module over any finite-dimensional algebra. We presented two problems (I) and (II), and explained why decomposition theory is required in Introduction. We give a general solution of the problem (I) in Section 2, and apply it to the Kronecker algebra in Section 3. Moreover, We consider problem (II) for the Kronecker algebra in Section 4. Fundamental facts on the Kronecker algebra are collected in Section 1. Throughout this chapter, all modules are assumed to be finite-dimensional left modules.

## 1. Kronecker algebra

Let $m, n$ be non-negative integers. Then we denote by Mat $m, n$ the vector space of $m \times n$ matrices over $\mathbb{k}$, and by $E_{n}$ the identity matrix of size $n$ (for $n \geq 1$ ). By the isomorphism Mat $m, n \rightarrow \operatorname{Hom}_{\mathbb{k}}\left(\mathbb{k}^{n}, \mathbb{k}^{m}\right)$ sending each $M \in \operatorname{Mat} m, n$ to the linear map given by the left multiplication by $M$ we identify Mat $m, n$ with $\operatorname{Hom}_{\mathbb{k}}\left(\mathbb{k}^{n}, \mathbb{k}^{m}\right)$, and regard each $M \in$ Mat $m, n$ as the corresponding linear map $\mathbb{k}^{n} \rightarrow \mathbb{k}^{m}$. If $m$ or $n$ is zero, we denote the matrices corresponding to the zero maps $\mathbb{k}^{n} \rightarrow \mathbb{k}^{m}$ by $J_{m, n}$, respectively and call them empty matrices.

The Kronecker algebra $A$ is a path algebra of the quiver $Q=(1 \underset{\beta}{\stackrel{\alpha}{\longrightarrow}} 2)$, and the category $\bmod A$ of finite-dimensional $A$-modules is equivalent to the category rep $Q$ of finitedimensional representations of $Q$ over $\mathbb{k}$. We usually identify these categories. Recall that a representation $M$ of $Q$ is a diagram $M(1) \xrightarrow[M(\beta)]{\xrightarrow{M(\alpha)}} M(2)$ of vector spaces and linear maps,
 When $\operatorname{dim} M=\left(d_{1}, d_{2}\right)$, without loss of generality we may set $M(i)=\mathbb{K}^{d_{i}}$ for $i=1,2$ and $M(\alpha), M(\beta) \in \operatorname{Mat} d_{2}, d_{1}$. We denote $M$ by the pair of matrices $(M(\alpha), M(\beta))$.

We here list well known facts on the Kronecker algebra (see Ringel [20,3.2] for instance).

Theorem 1.1. For the Kronecker algebra A the following statements hold.
(1) The list $\mathcal{L}$ of indecomposables is given as follows.

Preprojective indecomposables: $\mathcal{P}:=\left\{P_{n}: \left.=\left(\left[\begin{array}{c}E_{n-1} \\ { }^{t} \mathbf{0}\end{array}\right],\left[\begin{array}{c}{ }^{t} \mathbf{0} \\ E_{n-1}\end{array}\right]\right) \right\rvert\, n \geq 1\right\}$,
Preinjective indecomposables: $\mathcal{I}:=\left\{I_{n}:=\left(\left[E_{n-1}, \mathbf{0}\right],\left[\mathbf{0}, E_{n-1}\right]\right) \mid n \geq 1\right\}$,

Regular indecomposables:

$$
\mathcal{R}:=\left\{R_{n}(\lambda):=\left(E_{n}, J_{n}(\lambda)\right), R_{n}(\infty):=\left(J_{n}(0), E_{n}\right) \mid n \geq 1, \lambda \in \mathbb{k}\right\}
$$

where $\mathbf{0}$ is the $n \times 1$ matrix with all entries 0 . Note that

$$
\underline{\operatorname{dim}} P_{n}=(n-1, n), \underline{\operatorname{dim}} I_{n}=(n, n-1), \underline{\operatorname{dim}} R_{n}(\lambda)=(n, n)
$$

for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{P}^{1}(\mathbb{k})=\mathbb{k} \cup\{\infty\}$.
(2) The Auslander-Reiten quiver (AR-quiver for short) of $A$ has the following form:


In the above the rectangle part $\mathcal{R}$ is given as the disjoint union of a family $\left(\mathcal{R}_{\lambda}\right)_{\lambda \in \mathbb{P}^{1}(\mathbb{k})}$ of "homogeneous tubes" $\mathcal{R}_{\lambda}$ that has the form

where dotted loops mean that for all $n \in \mathbb{N}$ the Auslander-Reiten translation $\tau$ sends $R_{n}(\lambda)$ to itself: $\tau R_{n}(\lambda)=R_{n}(\lambda)$.
(3) Let $X, Y \in \mathcal{L}$. If $\operatorname{Hom}_{A}(X, Y) \neq 0$, then $X$ is "on the left" of $Y$, i.e., one of the following occurs:
(i) $X \cong P_{m}, Y \cong P_{n}$ with $m \leq n$,
(ii) $X \in \mathcal{P}, Y \in \mathcal{R} \cup \mathcal{I}$,
(iii) $X \cong R_{m}(\lambda), Y \cong R_{n}(\mu)$ with $\lambda=\mu$,
(iv) $X \in \mathcal{R}, Y \in \mathcal{I}$, or
(v) $X \cong I_{m}, Y \cong I_{n}$ with $m \geq n$.

Remark 1.2. (1) Let $m, n \in \mathbb{Z}$ with $m \leq n$. Then we note that there exists a monomorphism $P_{m} \rightarrow P_{n}$ and an epimorphism $I_{n} \rightarrow I_{m}$.
(2) Now for $\left(a_{1}, a_{2}\right),\left(b_{1}, b_{2}\right) \in \mathbb{Z}^{2}$ we define $\left(a_{1}, a_{2}\right) \leq\left(b_{1}, b_{2}\right)$ if and only if $a_{i} \leq b_{i}$ for $i=1$ and 2. Then if there exists a monomorphism $T \rightarrow U$ (or an epimorphism $U \rightarrow T$ ) in $\bmod A$, we have $\underline{\operatorname{dim}} T \leq \underline{\operatorname{dim}} U$.

## 2. Simple functors: a solution to the problem (I) in general

In this section we give a solution to the problem (I) by using Auslander-Reiten theory for an arbitrary algebra $A$.

Definition 2.1. For an indecomposable $A$-module $L$ we set

$$
\mathcal{S}_{L}:=\operatorname{Hom}_{A}(L,-) / \operatorname{rad} \operatorname{Hom}_{A}(L,-): \bmod A \rightarrow \bmod \mathbb{k} .
$$

It is well-known that $\mathcal{S}_{L}$ is a simple functor.
Lemma 2.2. Let $M$ be an $A$-module. Then for any indecomposable $A$-module $L$ we have

$$
\boldsymbol{d}_{M}(L)=\operatorname{dim} \mathcal{S}_{L}(M)
$$

Proof. Since $L$ is indecomposable, $\operatorname{End}_{A}(L)$ is a local algebra. Therefore $\mathcal{S}_{L}(L)=$ $\operatorname{End}_{A}(L) / \operatorname{rad}\left(\operatorname{End}_{A}(L)\right)$ is a finite-dimensional skew field over the algebraically closed field $\mathbb{k}$, and hence $\mathcal{S}_{L}(L) \cong \mathbb{k}$. If $X \neq L$, then $\operatorname{End}_{A}(L)=\operatorname{rad}\left(\operatorname{End}_{A}(L)\right.$, and $\mathcal{S}_{L}(X)=0$. Thus

$$
\mathcal{S}_{L}(X) \cong\left\{\begin{array}{cc}
\mathbb{k} & \text { if } X \cong L \\
0 & \text { if } X \not \approx L
\end{array}\right.
$$

for all indecomposable $A$-modules $X$. Therefore, the indecomposable decomposition

$$
M \cong \bigoplus_{L \in \mathcal{L}} L^{\left(\boldsymbol{d}_{M}(L)\right)}
$$

of $M$ gives us

$$
\mathcal{S}_{L}(M) \cong \mathbb{k}^{\left(\boldsymbol{d}_{M}(L)\right)},
$$

which shows the assertion.
Recall the following fundamental statement in the Auslander-Reiten theory (see AuslanderReiten [15] or Assem-Simson-Skowroński [14, IV, 6.11.]):

Proposition 2.3. Let $L$ be an indecomposable $A$-module. When $L$ is non-injective, let $0 \rightarrow L \xrightarrow{f} \bigoplus_{X \in J_{L}} X^{(a(X))} \xrightarrow{g} \tau^{-1} L \rightarrow 0$ be an almost split sequence starting at $L$ with $J_{L} \subseteq \mathcal{L}$ and $a(X) \geq 1\left(X \in J_{L}\right)$. When $L$ is injective, let $f: L \rightarrow L / \operatorname{soc} L=\bigoplus_{X \in J_{L}} X^{(a(X))}$ be the canonical epimorphism (note that $J_{L}=\emptyset$ if $L$ is simple injective). Then the simple functor $\mathcal{S}_{L}$ has a minimal projective resolution

$$
0 \rightarrow \operatorname{Hom}_{A}\left(\tau^{-1} L,-\right) \xrightarrow{\operatorname{Hom}_{A}(g,-)} \bigoplus_{X \in J_{L}} \operatorname{Hom}_{A}(X,-)^{(a(X))} \xrightarrow{\operatorname{Hom}_{A}(f,-)} \operatorname{Hom}_{A}(L,-) \xrightarrow{c a n} \mathcal{S}_{L} \rightarrow 0
$$

where $g=0$ and $\tau^{-1} L=0$ if $L$ is injective.
Proposition 2.3 together with Lemma 2.2 readily gives us the following.
Theorem 2.4. Let $M$ be an $A$-module. Then for any indecomposable $A$-module $L$ we have

$$
\boldsymbol{d}_{M}(L)=\operatorname{dim} \operatorname{Hom}_{A}(L, M)-\sum_{X \in J_{L}} a(X) \operatorname{dim} \operatorname{Hom}_{A}(X, M)+\operatorname{dim} \operatorname{Hom}_{A}\left(\tau^{-1} L, M\right)
$$

Remark 2.5. When an algebra $A$ is of the form $\mathbb{k} Q / I$ for some quiver $Q$ and some ideal $I$ of $\mathbb{k} Q$, it is possible to compute $\operatorname{dim} \operatorname{Hom}_{A}(H, M)$ for every $H, M \in \bmod A$ by
using the rank of a suitable matrix as follows, and thus $\boldsymbol{d}_{M}(L)$ in Theorem 2.4 is computable. First regard $A$-modules $H$ and $M$ as representations $(H(i), H(\alpha))_{i \in Q_{0}, \alpha \in Q_{1}}$ and $(M(i), M(\alpha))_{i \in Q_{0}, \alpha \in Q_{1}}$ of $Q$, respectively. Then by definition we have
$\operatorname{Hom}_{A}(H, M)=\left\{\left(f_{i}\right)_{i \in Q_{0}} \in \prod_{i \in Q_{0}} \operatorname{Hom}_{\mathbb{k}}(H(i), M(i)) \mid M(\alpha) f_{i}=f_{j} H(\alpha), \forall \alpha: i \rightarrow j\right.$ in $\left.Q_{1}\right\}$.

Therefore

$$
\operatorname{Hom}_{A}(H, M) \cong\left\{\boldsymbol{x} \in \mathbb{k}^{N} \mid B \boldsymbol{x}=0\right\},
$$

where $N:=\sum_{i \in Q_{0}} \operatorname{dim} H(i) \operatorname{dim} M(i)$ and $B$ is a $\left|Q_{1}\right| \times N$-matrix given as the coefficient matrix of the homogeneous system of linear equations $M(\alpha) f_{i}-f_{j} H(\alpha)=0$ for $f_{i}$. Hence we obtain the equality:

$$
\operatorname{dim}_{\operatorname{Hom}_{A}}(H, M)=N-\operatorname{rank} B
$$

Example 2.6. Let $A:=\mathbb{k}[x]$ be the polynomial algebra in one variable. Although it is an infinite-dimensional algebra, the category $\bmod A$ of finite-dimensional $A$-modules is well understood because $\mathbb{k}[x]$ is a principal ideal domain, and we can apply AuslanderReiten theory to $\bmod A$. It is easy to give all almost split sequences over $\mathbb{k}[x]$. Namely, they are given as follows:

$$
\begin{gather*}
0 \rightarrow J_{1}(\lambda) \rightarrow J_{2}(\lambda) \rightarrow J_{1}(\lambda) \rightarrow 0 \\
0 \rightarrow J_{i}(\lambda) \rightarrow J_{i-1}(\lambda) \oplus J_{i+1}(\lambda) \rightarrow J_{i}(\lambda) \rightarrow 0 \tag{2.2}
\end{gather*}
$$

for all $i \geq 2$ and $\lambda \in \mathbb{k}$. This is verified by the similar argument used in the Nakayama algebra case (cf. [14, 4.1 Theorem]). The reader may notice a similarity between (0.1) and (2.2), which will become clear now. Let $M=\left(\mathbb{k}^{n}, M\right)$ be an $A$-module. Then we have

$$
\begin{equation*}
\operatorname{dim} \operatorname{Hom}_{A}\left(J_{i}(\lambda), M\right)=n-\operatorname{rank} M_{\lambda}^{i}, \tag{2.3}
\end{equation*}
$$

which together with Theorem 2.4 and the formula (2.2) yields the formula (0.1).
Indeed, let $X \in \operatorname{Mat} n, i$, and put $X_{j}$ to be the $j$-th column of $X(j=1, \ldots, i)$. Then by (2.1) $X \in \operatorname{Hom}_{A}\left(J_{i}(\lambda), M\right)$ iff $M X=X J_{i}(\lambda)=X\left(\lambda E_{i}+J_{i}(0)\right)=\lambda X+X J_{i}(0)$ iff $M_{\lambda} X=X J_{i}(0)$ iff $M_{\lambda}\left(X_{1}, \ldots, X_{i}\right)=\left(0, X_{1}, \ldots, X_{i-1}\right)$ iff $M_{\lambda}$ maps $X_{j}$ 's as follows

$$
X_{i} \mapsto X_{i-1} \mapsto \cdots \mapsto X_{1} \mapsto 0
$$

Hence the correspondence $X \mapsto X_{i}$ yields the isomorphism (the inverse is given by the correspondence $\left.v \mapsto\left[M_{\lambda}^{i-1} v, \ldots, M_{\lambda} v, v\right]\right)$

$$
\operatorname{Hom}_{A}\left(J_{i}(\lambda), M\right) \cong\left\{v \in \mathbb{k}^{n} \mid M_{\lambda}^{i} v=0\right\}=\operatorname{Ker} M_{\lambda}^{i},
$$

which shows the equality (2.3).
Example 2.7. Let $n$ be a positive integer and set $A:=\mathbb{k} Q$, where $Q$ is a Dynkin quiver $1 \xrightarrow{\alpha_{1}} 2 \xrightarrow{\alpha_{2}} \cdots \xrightarrow{\alpha_{n-1}} n$ of type $A_{n}$. Decomposition Theory for modules over this algebra has important applications in the topological data analysis (See Introduction). Let $M:=\left(M_{i}\right)_{i=1}^{n}:=\left(\mathbb{K}^{\left(a_{1}\right)} \xrightarrow{M_{1}} \mathbb{k}^{\left(a_{2}\right)} \xrightarrow{M_{2}} \cdots \xrightarrow{M_{n-1}} \mathbb{k}^{\left(a_{n}\right)}\right)$ be a representation of $Q$
(i.e. an $A$-module). Then the morphism space $\operatorname{Hom}_{A}(M(b, d), M)$ is the set of sequences $\left(f_{i}: M(b, d)(i) \rightarrow \mathbb{k}^{\left(a_{i}\right)}\right)_{i=1}^{n}$ that make the following diagram commutative:

$$
\begin{aligned}
& 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} \mathbb{k} \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{k} \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \\
& 0_{\downarrow} \downarrow \quad 0 \quad 0 \quad 0_{\downarrow} \quad 0 \quad f_{b} \downarrow \text { O } \quad 0 f_{d} \downarrow \quad 0 \quad{ }_{\downarrow} \downarrow \text { O } \quad 0 \quad 0 \downarrow
\end{aligned}
$$

where $M(b, d)(i):=\left\{\begin{array}{cc}\mathbb{k} & (b \leq i \leq d) \\ 0 & (\text { otherwise })\end{array}\right.$. In particular, if $d=n$ (namely $M(b, d)$ is projective), then

$$
\operatorname{Hom}_{A}(M(b, n), M) \cong\left\{\left(f_{i}\right)_{i=b}^{n} \mid M_{b} f_{b}=f_{b+1}, \ldots, M_{n-1} f_{n-1}=f_{n}\right\} \cong \mathbb{k}^{\left(a_{b}\right)}
$$

and if $d \leq n-1$, then

$$
\begin{aligned}
\operatorname{Hom}_{A}(M(b, d), M) & \cong\left\{\left(f_{i}\right)_{i=b}^{d} \mid M_{b} f_{b}=f_{b+1}, \ldots, M_{d-1} f_{d-1}=f_{d}, M_{d} f_{d}=0\right\} \\
& \cong\left\{f_{b} \in \mathbb{k}^{\left(a_{b}\right)} \mid M_{d} M_{d-1} \cdots M_{b} f_{b}=0\right\}
\end{aligned}
$$

Hence we obtain

$$
\begin{equation*}
\operatorname{dim} \operatorname{Hom}_{A}(M(b, d), M)=a_{b}-\operatorname{rank}\left(M_{d} M_{d-1} \cdots M_{b}\right), \tag{2.4}
\end{equation*}
$$

where we set $M_{n}:=0$. Since the AR-quiver $\Gamma_{A}$ of $A$ is of the following form:

the formula (2.4) and Theorem 2.4 give us the formula

$$
\boldsymbol{d}_{M}(M(b, d))=R(b-1, d)-R(b, d),
$$

where we set $M_{0}:=0$ and $M_{n}:=0$ and

$$
R(b, d):=\left\{\begin{array}{lr}
\operatorname{rank}\left(M_{d} \cdots M_{b}\right)-\operatorname{rank}\left(M_{d-1} \cdots M_{b}\right) & (b<d) \\
\operatorname{rank}\left(M_{d} \cdots M_{b}\right)-a_{b} & (b=d)
\end{array}\right.
$$

for each $(b, d) \in\left\{(i, j) \in \mathbb{Z}^{2} \mid 1 \leq i \leq j \leq n\right\}$.

## 3. Solution to the problem (I) for the Kronecker algebra

Throughout the rest of this paper $A$ is the Kronecker algebra. To apply Theorem 2.4 we compute the dimensions of the spaces $\operatorname{Hom}_{A}(L, M)$ for all $L \in \mathcal{L}$ and $M \in \bmod A$ following Remark 2.5.

Definition 3.1. Let $M$ be an $A$-module. We first define the following matrices with $n \geq 1, \lambda \in \mathbb{k}$ (note that $P_{1}(M)=\mathrm{J}_{0,1}$ is an empty matrix).

$$
\begin{aligned}
& I_{n}(M):=\left[\begin{array}{ccccc}
\left.\begin{array}{ccccc}
M(\beta) & 0 & 0 & \cdots & 0 \\
M(\alpha) & M(\beta) & 0 & \ddots & \vdots \\
0 & M(\alpha) & M(\beta) & \ddots & 0 \\
0 & 0 & M(\alpha) & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & M(\beta) \\
0 & 0 & \cdots & 0 & M(\alpha)
\end{array}\right]
\end{array}\right\} n+1 \text { blocks }, \\
& R_{n}(\lambda, M):=\left[\begin{array}{ccccc}
\left.\begin{array}{ccccc}
M_{\lambda}(\alpha, \beta) & 0 & 0 & \cdots & 0 \\
M(\alpha) & M_{\lambda}(\alpha, \beta) & 0 & \ddots & \vdots \\
0 & M(\alpha) & M_{\lambda}(\alpha, \beta) & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & M(\alpha) & M_{\lambda}(\alpha, \beta)
\end{array}\right]
\end{array}\right\} n \text { blocks } \quad \text { band } \\
& \left.R_{n}(\infty, M):=\left[\begin{array}{ccccc}
\left.\begin{array}{ccccc}
M(\alpha) & 0 & 0 & \cdots & 0 \\
-M(\beta) & M(\alpha) & 0 & \ddots & \vdots \\
0 & -M(\beta) & M(\alpha) & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -M(\beta) & M(\alpha)
\end{array}\right]
\end{array}\right)\right\} n \text { blocks },
\end{aligned}
$$

where we put $M_{\lambda}(\alpha, \beta):=\lambda M(\alpha)-M(\beta)$, and we define the following numbers.

$$
\begin{aligned}
p_{1}(M) & :=0, p_{n}(M):=\operatorname{rank} P_{n}(M)(n \geq 2), \\
i_{0}(M) & :=0, i_{n}(M):=\operatorname{rank} I_{n}(M)(n \geq 1), \\
r_{n}(\lambda, M) & :=\operatorname{rank} R_{n}(\lambda, M)\left(n \geq 1, \lambda \in \mathbb{P}^{1}(\mathbb{k})\right) .
\end{aligned}
$$

Using the data above we can compute the dimensions of Hom spaces $\operatorname{Hom}_{A}(L, M)$ with $L$ indecomposable as follows.

Proposition 3.2. Let $M$ be an $A$-module. Then we have the following formulas:

$$
\left.\begin{array}{rl}
\operatorname{dim} \operatorname{Hom}_{A}\left(P_{n}, M\right) & = \begin{cases}(n-1) d_{1}-p_{n-1}(M) & (n \geq 2) \\
d_{2} & (n=1)\end{cases} \\
\operatorname{dim} \operatorname{Hom}_{A}\left(I_{n}, M\right) & =n d_{1}-i_{n}(M) \quad(n \geq 1)
\end{array}\right] \begin{array}{ll}
\operatorname{dim} \operatorname{Hom}_{A}\left(R_{n}(\lambda), M\right) & =n d_{1}-r_{n}(\lambda, M) \quad\left(n \geq 1, \lambda \in \mathbb{P}^{1}(\mathbb{k})\right)
\end{array}
$$

Proof. Assume that $n \geq 2$. Let $(X, Y) \in \operatorname{Mat} d_{1}, n-1 \times \operatorname{Mat} d_{2}, n$, and put $X_{i}$ (resp. $\left.Y_{i}\right)$ to be $i$-th column of $X(i=1, \ldots, n-1)$ (resp. $Y(i=1, \ldots, n)$ ). Then by (2.1) $(X, Y) \in \operatorname{Hom}_{A}\left(P_{n}, M\right)$ iff

$$
M(\alpha) X=Y\left[\begin{array}{c}
E_{n-1} \\
0
\end{array}\right], \quad M(\beta) X=Y\left[\begin{array}{c}
0 \\
E_{n-1}
\end{array}\right]
$$

iff

$$
\begin{aligned}
& \left\{\begin{array}{l}
M(\alpha) X_{1}=Y_{1}, M(\alpha) X_{2}=Y_{2}, \ldots, M(\alpha) X_{n-1}=Y_{n-1} \\
M(\beta) X_{1}=Y_{2}, M(\beta) X_{2}=Y_{3}, \ldots, M(\beta) X_{n-1}=Y_{n}
\end{array}\right.
\end{aligned}
$$

iff

Let $B$ be the coefficient matrix of this equation. Then a direct calculation shows that $B$ is equivalent to $P_{n-1}(M) \oplus E_{n d_{2}}$. Therefore rank $B=n d_{2}+p_{n-1}(M)$, which shows that $\operatorname{dim} \operatorname{Hom}_{A}\left(P_{n}, M\right)=(n-1) d_{1}+n d_{2}-\operatorname{rank} B=(n-1) d_{1}-p_{n-1}(M)$, as desired. The remaining formulas are proved similarly.

Propositions 3.2 and Theorem 2.4 give us a solution to the problem (I) for the Kronecker algebra as follows.

Theorem 3.3. Let $M$ be an $A$-module. Then we have the following formulas:

$$
\begin{aligned}
\boldsymbol{d}_{M}\left(P_{n}\right) & = \begin{cases}2 p_{n}(M)-p_{n-1}(M)-p_{n+1}(M) & (n \geq 2) \\
d_{2}-p_{2}(M) & (n=1),\end{cases} \\
\boldsymbol{d}_{M}\left(I_{n}\right) & = \begin{cases}2 i_{n-1}(M)-i_{n}(M)-i_{n-2}(M) & (n \geq 2) \\
d_{1}-i_{1}(M) & (n=1),\end{cases} \\
\boldsymbol{d}_{M}\left(R_{n}(\lambda)\right) & = \begin{cases}r_{n-1}(\lambda, M)+r_{n+1}(\lambda, M)-2 r_{n}(\lambda, M) & (n \geq 2) \\
r_{2}(\lambda, M)-2 r_{1}(\lambda, M) & (n=1) .\end{cases}
\end{aligned}
$$

Here we note that $\boldsymbol{d}_{M}\left(P_{1}\right)$ and $\boldsymbol{d}_{M}\left(I_{1}\right)$ have obvious menanings that $\boldsymbol{d}_{M}\left(P_{1}\right)=\operatorname{dim} \operatorname{Coker}[M(\beta) M(\alpha)]$ and $\boldsymbol{d}_{M}\left(I_{1}\right)=\operatorname{dim} \operatorname{Ker}\left[\begin{array}{l}M(\beta) \\ M(\alpha)\end{array}\right]$.

Proof. Note that by Theorem 1.1(2) we know all the almost split sequences for the Kronecker algebra. Therefor we can apply Theorem 2.4. We first compute $\boldsymbol{d}_{M}\left(P_{1}\right)$ and $\boldsymbol{d}_{M}\left(I_{1}\right)$. Noting that $\operatorname{dim} \operatorname{Hom}_{A}\left(P_{2}, M\right)=d_{1}-p_{1}(M)=d_{1}$ the almost split sequence starting at $P_{1}$ that is given by the mesh starting at $P_{1}$ in the AR-quiver shows that

$$
\begin{aligned}
\boldsymbol{d}_{M}\left(P_{1}\right) & =\operatorname{dim} \operatorname{Hom}_{A}\left(P_{1}, M\right)-2 \operatorname{dim} \operatorname{Hom}_{A}\left(P_{2}, M\right)+\operatorname{dim} \operatorname{Hom}_{A}\left(P_{3}, M\right) \\
& =d_{2}-2 d_{1}+2 d_{1}-p_{2}(M)=d_{2}-p_{2}(M) \\
& =d_{2}-\operatorname{rank}[M(\beta) M(\alpha)]=\operatorname{dim} \operatorname{Coker}[M(\beta) M(\alpha)] .
\end{aligned}
$$

Now since $I_{1}$ is simple and injective, we have $I_{1} / \operatorname{soc} I_{1}=0$ and $\tau^{-1} I_{1}=0$. Hence

$$
\begin{aligned}
\boldsymbol{d}_{M}\left(I_{1}\right) & =\operatorname{dim} \operatorname{Hom}_{A}\left(I_{1}, M\right)=d_{1}-i_{1}(M) \\
& =d_{1}-\operatorname{rank}\left[\begin{array}{l}
M(\beta) \\
M(\alpha)
\end{array}\right]=\operatorname{dim} \operatorname{Ker}\left[\begin{array}{l}
M(\beta) \\
M(\alpha)
\end{array}\right] .
\end{aligned}
$$

Next we compute $\boldsymbol{d}_{M}\left(P_{n}\right)$ for $n \geq 2$.

$$
\begin{aligned}
\boldsymbol{d}_{M}\left(P_{n}\right) & =\operatorname{dim} \operatorname{Hom}_{A}\left(P_{n}, M\right)-2 \operatorname{dim} \operatorname{Hom}_{A}\left(P_{n+1}, M\right)+\operatorname{dim} \operatorname{Hom}_{A}\left(P_{n+2}, M\right) \\
& =(n-1) d_{1}-p_{n-1}(M)-2\left(n d_{1}-p_{n}(M)\right)+(n+1) d_{1}-p_{n+1}(M) \\
& =2 p_{n}(M)-p_{n-1}(M)-p_{n+1}(M),
\end{aligned}
$$

as desired. The remaining cases are proved similarly.

## 4. Solution to the problem (II) for the Kronecker algebra

Let $F: \bigoplus_{L \in \mathcal{L}} L^{\left(\boldsymbol{d}_{M}(L)\right)} \rightarrow M$ be an isomorphism. Then we have

$$
M=P_{M} \oplus R_{M} \oplus I_{M},
$$

where $P_{M}, R_{M}$ and $I_{M}$ are the images of $\bigoplus_{L \in \mathcal{P}} L^{\left(\boldsymbol{d}_{M}(L)\right)}, \bigoplus_{L \in \mathcal{R}} L^{\left(\boldsymbol{d}_{M}(L)\right)}$ and $\bigoplus_{L \in \mathcal{I}} L^{\left(\boldsymbol{d}_{M}(L)\right)}$ by $F$, respectively. To compute $P_{M}, R_{M}$ and $I_{M}$ we here use the trace and reject in a module of a class of modules (see Anderson-Fuller [13] for details). Let $\mathcal{U}$ be a class of
modules in $\bmod A$ and $M \in \bmod A$. Recall that the $\operatorname{trace} \operatorname{Tr}_{M}(\mathcal{U})$ of $\mathcal{U}$ in $M$ and the reject $\operatorname{Rej}_{M}(\mathcal{U})$ of $\mathcal{U}$ in $M$ are defined by

$$
\begin{aligned}
\operatorname{Tr}_{M}(\mathcal{U}) & :=\sum\left\{\operatorname{Im} f \mid f \in \operatorname{Hom}_{A}(U, M) \text { for some } U \in \mathcal{U}\right\}, \text { and } \\
\operatorname{Rej}_{M}(\mathcal{U}) & :=\bigcap\left\{\operatorname{Ker} f \mid f \in \operatorname{Hom}_{A}(M, U) \text { for some } U \in \mathcal{U}\right\}
\end{aligned}
$$

When $\mathcal{U}=\{U\}$ is a singleton, we set $\operatorname{Tr}_{M}(U):=\operatorname{Tr}_{M}(\mathcal{U})$ and $\operatorname{Rej}_{M}(U):=\operatorname{Rej}_{M}(\mathcal{U})$. We cite the following from [13, 8.18 Proposition].

Lemma 4.1. Let $\left(M_{i}\right)_{i \in I}$ be a family of $A$-modules indexed by a set $I$ and $\mathcal{U}$ a class of modules in $\bmod A$. Then we have

$$
\operatorname{Tr}_{\oplus_{i \in I} M_{i}}(\mathcal{U})=\bigoplus_{i \in I} \operatorname{Tr}_{M_{i}}(\mathcal{U}) \quad \text { and } \quad \operatorname{Rej}_{\oplus_{i \in I} M_{i}}(\mathcal{U})=\bigoplus_{i \in I} \operatorname{Rej}_{M_{i}}(\mathcal{U})
$$

Proposition 4.2 (Calculation of $\left.R_{M} \oplus I_{M}\right)$. If $\left\{f_{1}, \ldots, f_{a}\right\}$ is a basis of $\operatorname{Hom}_{A}\left(M, P_{d_{2}}\right)$, then we have

$$
\bigcap_{i=1}^{a} \operatorname{Ker} f_{i}=R_{M} \oplus I_{M} \quad \text { and hence } \quad P_{M} \cong M /\left(\bigcap_{i=1}^{a} \operatorname{Ker} f_{i}\right) .
$$

Proof. By assumption it is obvious that $\bigcap_{i=1}^{a} \operatorname{Ker} f_{i}=\operatorname{Rej}_{M}\left(P_{d_{2}}\right)$. Therefore, it is enough to show that

$$
\begin{equation*}
\operatorname{Rej}_{M}\left(P_{d_{2}}\right)=R_{M} \oplus I_{M} \tag{4.1}
\end{equation*}
$$

By Lemma 4.1 we have

$$
\operatorname{Rej}_{M}\left(P_{d_{2}}\right)=\operatorname{Rej}_{P_{M} \oplus R_{M} \oplus I_{M}}\left(P_{d_{2}}\right)=\operatorname{Rej}_{P_{M}}\left(P_{d_{2}}\right) \oplus \operatorname{Rej}_{R_{M}}\left(P_{d_{2}}\right) \oplus \operatorname{Rej}_{I_{M}}\left(P_{d_{2}}\right)
$$

By Theorem 1.1(3) we have $\operatorname{Hom}_{A}\left(R_{M}, P_{d_{2}}\right)=0$ and $\operatorname{Hom}_{A}\left(I_{M}, P_{d_{2}}\right)=0$, which shows that

$$
\operatorname{Rej}_{R_{M}}\left(P_{d_{2}}\right)=R_{M} \quad \text { and } \quad \operatorname{Rej}_{I_{M}}\left(P_{d_{2}}\right)=I_{M}
$$

If a preprojective indecomposable module $P_{i}$ is a direct summand of $M$, then it follows from $(i-1, i)=\underline{\operatorname{dim}} P_{i} \leq \underline{\operatorname{dim}} M=\left(d_{1}, d_{2}\right)$ that $i \leq d_{2}$ (see Remark 1.2(2)). Therefore, we have $P_{M}=\bigoplus_{i=1}^{d_{2}} P_{i}^{\left(a_{i}\right)}$ for some $a_{i} \geq 0$ (we identify $P_{i}$ with $F\left(P_{i}\right)$ ), and then $\operatorname{Rej}_{P_{M}}\left(P_{d_{2}}\right)=$ $\bigoplus_{i=1}^{d_{2}}\left(\operatorname{Rej}_{P_{i}}\left(P_{d_{2}}\right)\right)^{\left(a_{i}\right)}$. Now if $i \leq d_{2}$, then by Remark 1.2(1) we have a monomorphism $P_{i} \rightarrow P_{d_{2}}$, which shows that $\operatorname{Rej}_{P_{i}}\left(P_{d_{2}}\right)=0$ for all $i \leq d_{2}$, and therefore

$$
\operatorname{Rej}_{P_{M}}\left(P_{d_{2}}\right)=0 .
$$

Hence the equality (4.1) holds.
Proposition 4.3 (Calculation of $\left.I_{M}\right)$. If $\left\{g_{1}, \ldots, g_{b}\right\}$ is a basis of $\operatorname{Hom}_{A}\left(I_{d_{1}}, R_{M} \oplus\right.$ $I_{M}$ ), then we have

$$
\sum_{i=1}^{b} \operatorname{Im} g_{i}=I_{M}
$$

Proof. By assumption it is obvious that $\sum_{i=1}^{b} \operatorname{Im} g_{i}=\operatorname{Tr}_{R_{M} \oplus I_{M}}\left(I_{d_{1}}\right)$. Therefore it is enough to show that

$$
\begin{equation*}
\operatorname{Tr}_{R_{M} \oplus I_{M}}\left(I_{d_{1}}\right)=I_{M} \tag{4.2}
\end{equation*}
$$

By Lemma 4.1 we have

$$
\operatorname{Tr}_{R_{M} \oplus I_{M}}\left(I_{d_{1}}\right)=\operatorname{Tr}_{R_{M}}\left(I_{d_{1}}\right) \oplus \operatorname{Tr}_{I_{M}}\left(I_{d_{1}}\right) .
$$

By Theorem 1.1(3) we have $\operatorname{Hom}_{A}\left(I_{d_{1}}, R_{M}\right)=0$, which shows that

$$
\operatorname{Tr}_{R_{M}}\left(I_{d_{1}}\right)=0 .
$$

If a preinjective indecomposable module $I_{i}$ is a direct summand of $M$, then it follows from $(i, i-1)=\underline{\operatorname{dim}} I_{i} \leq \underline{\operatorname{dim}} M=\left(d_{1}, d_{2}\right)$ that $i \leq d_{1}$. Therefore we have $I_{M}=\bigoplus_{i=1}^{d_{1}} I_{i}^{\left(b_{i}\right)}$ for some $b_{i} \geq 0$ (we identify $I_{i}$ with $F\left(I_{i}\right)$ ), and then $\operatorname{Tr}_{I_{M}}\left(I_{d_{1}}\right)=\bigoplus_{i=1}^{d_{1}}\left(\operatorname{Tr}_{I_{i}}\left(I_{d_{1}}\right)\right)^{\left(b_{i}\right)}$. Now if $i \leq d_{1}$, then we have an epimorphism $I_{d_{1}} \rightarrow I_{i}$, which shows that $\operatorname{Tr}_{I_{i}}\left(I_{d_{1}}\right)=I_{i}$ for all $i \leq d_{1}$, and therefore

$$
\operatorname{Tr}_{I_{M}}\left(I_{d_{1}}\right)=I_{M} .
$$

Hence the equality (4.2) holds.
By Propositions 4.2 and 4.3 we have the following.
Proposition 4.4 (Calculation of $R_{M}$ ). Let $\left\{f_{1}, \ldots, f_{a}\right\}$ a basis of $\operatorname{Hom}_{A}\left(M, P_{d_{2}}\right)$ and $\left\{g_{1}, \ldots, g_{b}\right\}$ a basis of $\operatorname{Hom}_{A}\left(I_{d_{1}}, \bigcap_{i=1}^{a} \operatorname{Ker} f_{i}\right)$. Then we have

$$
R_{M} \cong\left(\bigcap_{i=1}^{a} \operatorname{Ker} f_{i}\right) /\left(\sum_{i=1}^{b} \operatorname{Im} g_{i}\right)
$$

By this isomorphism we identify $R_{M}$ with the right hand side. Since $R_{M}=\left(R_{M}(\alpha), R_{M}(\beta)\right)$ is the direct sum of regular indecomposable modules, both $R_{M}(\alpha)$ and $R_{M}(\beta)$ are square matrices, say of size $d$. Put $R(\infty):=\operatorname{Tr}_{R_{M}}\left(R_{d}(\infty)\right)$. Note that $\operatorname{Tr}_{R_{M}}\left(R_{d}(\infty)\right)=\operatorname{Tr}_{R_{M}}\left(\bigoplus_{n=1}^{d} R_{n}(\infty)\right)$ because there exists an epimorphism $R_{n}(\infty) \rightarrow R_{m}(\infty)$ for $n \geq m$. Then $R_{M}=R(\infty) \oplus R^{\prime}$ for some $A$-submodule $R^{\prime}=\left(X^{\prime}, Y^{\prime}\right)$ of $R_{M}$ such that $R^{\prime}$ has no direct summand of the form $R_{n}(\infty)$ for any $n$ by Theorem 1.1(3)(iii). [Decompose $R_{M}$ into indecomposables of the form $R_{n}(\lambda)$ with $n \geq 1, \lambda \in \mathbb{P}^{1}(\mathbb{k})$. Then $R^{\prime}$ is given by the direct sum of those direct summands of the form $R_{n}(\lambda)$ with $\lambda \neq \infty$ because $R(\infty)$ is given by the direct sum of summands of the form $R_{n}(\infty)$.] Since the matrix $X^{\prime}$ is invertible, we have

$$
R^{\prime} \cong\left(E_{l},\left(X^{\prime}\right)^{-1} Y^{\prime}\right)
$$

for some $l \leq d$. Therefore, the set $\Lambda$ of eigenvalues of $\left(X^{\prime}\right)^{-1} Y^{\prime}$ is finite.
Then by Propositions 4.2, 4.3 and 4.4, we obtain the following.
Theorem 4.5. Set

$$
S_{M}:=\left\{P_{i}, I_{j}, R_{k}(\lambda) \mid 1 \leq i \leq d_{2}, 1 \leq j \leq d_{1}, 1 \leq k \leq d, \lambda \in \Lambda \cup\{\infty\}\right\}
$$

Then this gives a solution to the problem (II) for the Kronecker algebra.
Remark 4.6. Note that if $R(\infty)=0$, then we can replace $S_{M}$ by

$$
\left\{P_{i}, I_{j}, R_{k}(\lambda) \mid 1 \leq i \leq d_{2}, 1 \leq j \leq d_{1}, 1 \leq k \leq d, \lambda \in \Lambda\right\}
$$

## 5. Examples for the Kronecker algebra

(1) For a preprojective module $M=P_{3}=\left(\left[\begin{array}{ll}1 & 0 \\ 0 & 1 \\ 0 & 0\end{array}\right],\left[\begin{array}{ll}0 & 0 \\ 1 & 0 \\ 0 & 1\end{array}\right]\right)$ with $\underline{\operatorname{dim}} M=(2,3)$, we will compute $p_{n}(M)(n \in \mathbb{N})$ and then we will give $\boldsymbol{d}_{M}\left(P_{n}\right)(n \in \mathbb{N})$. By Definition 3.1 we have $p_{1}(M)=0$,

$$
\begin{aligned}
& p_{2}(M)=\operatorname{rank}[M(\beta) \mid M(\alpha)]=\operatorname{rank}\left[\begin{array}{cc|cc}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{array}\right]=3, \\
& p_{3}(M)=\operatorname{rank}\left[\begin{array}{c|c|c}
M(\beta) & M(\alpha) & 0 \\
\hline 0 & M(\beta) & M(\alpha)
\end{array}\right]=\operatorname{rank}\left[\begin{array}{cc|cc|c}
0 & 0 & 1 & 0 & \\
1 & 0 & 0 & 1 & \\
0 & 1 & 0 & 0 & \\
\hline & & 0 & 0 & 1 \\
& 1 & 0 & 0 \\
& & 0 & 1 & 0 \\
\hline
\end{array}\right]=6, \\
& p_{4}(M)=\operatorname{rank}\left[\begin{array}{c|c|c|c}
M(\beta) & M(\alpha) & 0 & 0 \\
\hline 0 & M(\beta) & M(\alpha) & 0 \\
\hline 0 & 0 & M(\beta) & M(\alpha)
\end{array}\right]=\operatorname{rank}\left[\begin{array}{ccccc|c} 
& 1 & 0 & 0 & 1 & 0 \\
& & \\
1 & 0 & 0 & 1 & \\
& 0 & 1 & 0 & 0 & \\
\hline & & & 0 & 0 & 1
\end{array}\right)
\end{aligned}
$$

and

$$
p_{5}(M)=\operatorname{rank}\left[\begin{array}{c|c|c|c|c}
M(\beta) & M(\alpha) & 0 & 0 & 0 \\
\hline 0 & M(\beta) & M(\alpha) & 0 & 0 \\
\hline 0 & 0 & M(\beta) & M(\alpha) & 0 \\
\hline 0 & 0 & 0 & M(\beta) & M(\alpha)
\end{array}\right]
$$

$=\operatorname{rank}\left[\begin{array}{cc|cc|cc|c|c}0 & 0 & 1 & 0 & & & & \\ 1 & 0 & 0 & 1 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ \\ \hline & & 0 & 0 & 1 & 0 & & \\ & & 1 & 0 & 0 & 1 & & \\ & & 0 & 1 & 0 & 0 & & \\ & & & \\ \hline & & & 0 & 0 & 1 & 0 & \\ & & & 1 & 0 & 0 & 1 & \\ & & & 0 & 1 & 0 & 0 & \\ \hline & & & & & 0 & 0 & 1 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 \\ & & 0 & 0\end{array}\right]=10$.

Similarly, we have $p_{n}(M)=2 n$ for $n \geq 3$. Hence by Theorem 3.3 we have

$$
\begin{aligned}
& \boldsymbol{d}_{M}\left(P_{1}\right)=3-p_{2}(M)=0, \\
& \boldsymbol{d}_{M}\left(P_{2}\right)=2 p_{2}(M)-p_{1}(M)-p_{3}(M)=6-0-6=0, \\
& \boldsymbol{d}_{M}\left(P_{3}\right)=2 p_{3}(M)-p_{2}(M)-p_{4}(M)=12-3-8=1,
\end{aligned}
$$

and for $n \geq 4$,

$$
\boldsymbol{d}_{M}\left(P_{n}\right)=2 p_{n}(M)-p_{n-1}(M)-p_{n+1}(M)=2 \cdot 2 n-2(n-1)-2(n+1)=0 .
$$

Thus we can confirm $\boldsymbol{d}_{M}\left(P_{3}\right)=1$ and $\boldsymbol{d}_{M}\left(P_{n}\right)=0$ for $n \neq 3$.
(2) For a module $M=\left(\left[\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right], 0_{2,2}\right)=P_{1} \oplus R_{1}(1) \oplus I_{1}$ with $\underline{\operatorname{dim}} M=(2,2)$, we will compute $\operatorname{Rej}_{M}\left(P_{2}\right)$ and $\operatorname{Tr}_{\operatorname{Rej}_{M}\left(P_{2}\right)}\left(I_{2}\right)$. Recall that $P_{2}=\left(\left[\begin{array}{l}1 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 1\end{array}\right]\right)$. If $(X, Y) \in$ $\operatorname{Hom}_{A}\left(M, P_{2}\right)$, then we have $X=0_{1,2}, Y=\left[\begin{array}{ll}a & 0 \\ b & 0\end{array}\right]$ for some $a, b \in \mathbb{k}$, and we can take $\left\{f_{1}=\left(0_{1,2},\left[\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right]\right), f_{2}=\left(0_{1,2},\left[\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right]\right)\right\}$ as a basis of $\operatorname{Hom}_{A}\left(M, P_{2}\right)$. Hence we have

$$
\operatorname{Rej}_{M}\left(P_{2}\right)=\operatorname{Ker} f_{1} \cap \operatorname{Ker} f_{2}=\left(\left[\begin{array}{ll}
1 & 0
\end{array}\right], 0_{1,2}\right)=R_{1}(1) \oplus I_{1}
$$

with $\underline{\operatorname{dim}} \operatorname{Rej}_{M}\left(P_{2}\right)=(2,1)$ and have $M / \operatorname{Rej}_{M}\left(P_{2}\right) \cong P_{1}$. Moreover, recall that $I_{2}=$ $\left(\left[\begin{array}{ll}1 & 0\end{array}\right],\left[\begin{array}{ll}0 & 1\end{array}\right]\right)$. If $(X, Y) \in \operatorname{Hom}_{A}\left(I_{2}, \operatorname{Rej}_{M}\left(P_{2}\right)\right)$, then we have $X=\left[\begin{array}{ll}0 & 0 \\ c & d\end{array}\right], Y=0_{1,1}$ for some $c, d \in \mathbb{k}$, and we can also take $\left\{g_{1}=\left(\left[\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right], 0_{1,1}\right), g_{2}=\left(\left[\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right], 0_{1,1}\right)\right\}$ as a basis of $\operatorname{Hom}_{A}\left(I_{2}, \operatorname{Rej}_{M}\left(P_{2}\right)\right)$. Therefore, we have

$$
\operatorname{Tr}_{\operatorname{Rej}_{M}\left(P_{2}\right)}\left(I_{2}\right)=\operatorname{Im} g_{1}+\operatorname{Im} g_{2}=\left(\mathrm{J}_{0,1}, \mathrm{~J}_{0,1}\right)=I_{1}
$$

with $\operatorname{dim} \operatorname{Tr}_{\operatorname{Rej}_{M}\left(P_{2}\right)}\left(I_{2}\right)=(1,0)$, and $\operatorname{Rej}_{M}\left(P_{2}\right) / \operatorname{Tr}_{\operatorname{Rej}_{M}\left(P_{2}\right)}\left(I_{2}\right) \cong R_{1}(1)$. Thus we can confirm the process to get $S_{M}=\left\{P_{1}, P_{2}, I_{1}, I_{2}, R_{1}(1)\right\}$ in Section 4.

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[^0]:    ${ }^{1}$ a terminology used in [7]

[^1]:    ${ }^{2}$ Here $s(\mu)$ and $t(\nu)$ stand for the source of $\mu$ and the target of $\nu$ and compositions are written from the right to the left.

[^2]:    ${ }^{3} 1 \neq g \in G, x \in C_{0}$ implies $g x \neq x$

