A correction to “A non-implication between fragments of Martin’s Axiom related to a property which comes from Aronszajn trees”

A correction to "A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees"

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Abstract

In the paper *A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees* [1], **Proposition 2.7** is *not* true. To avoid this error and correct **Proposition 2.7**, the definition of the property $\mathsf{R}_{1,\aleph_1}$ is changed. In [1], all proofs of lemmas and theorems but **Lemma 6.9** are valid about this definition without changing the proofs. We give a new statement and a new proof of **Lemma 6.9**.

In the paper *A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees* [1], **Proposition 2.7** is *not* true. For example, T is an Aronszajn tree, t_1 and t_3 are incomparable node of T in a model N, t_2 is a node of T such that $t_2 \notin N$ and $t_1 <_T t_2$, $\sigma := \{t_2, t_3\}$ (which is in $a(T)$) and I be an uncountable subset of $a(T)$ which forms a Δ-system with root $\{t_1, t_3\}$. Then $\sigma \cap N = \{t_1\} \subseteq \{t_1, t_3\}$, but every element of I is incompatible with σ in $a(T)$.

To avoid this error and correct **Proposition 2.7**, the definition of the property $\mathsf{R}_{1,\aleph_1}$ is changed as follows.

Theorem 2.6. *A forcing notion $\mathbb{Q}$ in FSCO has the property $\mathsf{R}_{1,\aleph_1}$ if for any regular cardinal κ larger than $\aleph_1$, countable elementary submodel N of $H(\kappa)$ which has the set $\{\mathbb{Q}\}$, $I \in [\mathbb{Q}]^{\aleph_1} \cap N$ and $\sigma \in \mathbb{Q} \setminus N$, if I forms a Δ-system with root (exactly) $\sigma \cap N$, then there exists $I' \in [I]^{\aleph_1} \cap N$ such that every member of I' is compatible with σ in $\mathbb{Q}$.*

Similarly, we should also change **Proposition 2.8** and **Proposition 2.10.2** as follows.

Proposition 2.8. *The property $\mathsf{R}_{1,\aleph_1}$ is closed under finite support products in the following sense.*

If $\{\mathbb{Q}_\xi; \xi \in \Sigma\}$ is a set of forcing notions in FSCO with the property $\mathsf{R}_{1,\aleph_1}$, κ is a large enough regular cardinal, N is a countable elementary submodel of

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$H(\kappa)$ which has the set $\{\{\mathbb{Q}_\xi; \xi \in \Sigma\}\}$, I is an uncountable subset of the finite support product $\prod_{\xi\in\Sigma} \mathbb{Q}_\xi$ in N, $\vec{\sigma} \in \prod_{\xi\in\Sigma} \mathbb{Q}_\xi \setminus N$, I forms a Δ-system with root (exactly) $\vec{\sigma} \cap N$, that is,

- *the set $\{\mathrm{supp}(\vec{\tau}); \tau \in I\}$ forms a Δ-system with root (exactly) $\mathrm{supp}(\vec{\sigma}) \cap N$, where $\mathrm{supp}(\vec{\tau}) := \{\xi \in \Sigma; \vec{\tau}(\xi) \neq \emptyset\}$,*
- *for each $\xi \in \mathrm{supp}(\vec{\sigma}) \cap N$, the set $\{\vec{\tau}(\xi); \tau \in I\}$ forms a Δ-system with root (exactly) $\vec{\sigma}(\xi) \cap N$,*

then there exists $I' \in [I]^{\aleph_1} \cap N$ such that every element of I' is compatible with $\vec{\sigma}$ in $\prod_{\xi\in\Sigma} \mathbb{Q}_\xi$.

Proposition 2.10.2. *Let $\mathbb{Q}$ be a forcing notion in FSCO with the property $\mathsf{R}_{1,\aleph_1}$. Suppose that κ is a regular cardinal larger than $\aleph_1$, N is a countable elementary submodel of $H(\kappa)$ which has the set $\{\mathbb{Q}\}$, $\langle I_i; i \in n\rangle$ is a finite sequence of members of the set $[\mathbb{Q}]^{\aleph_1} \cap N$, and $\sigma \in \mathbb{Q} \setminus N$ such that the union $\bigcup_{i\in n} I_i$ forms a Δ-system with root (exactly) $\sigma \cap N$.*

Then there exists $\langle \tau_i; i \in n\rangle \in \prod_{i\in n} I_i$ such that there exists a common extension of σ and the τ_i in $\mathbb{Q}$.

The new definition of the property $\mathsf{R}_{1,\aleph_1}$ is less restrictive. All examples in the paper [1] has this property. In [1], all proofs of lemmas and theorems but **Lemma 6.9** are valid about this definition without changing the proofs. For example, in the proof of **Proposition 2.7**, we have only to check for an uncountable subset I of $a(\mathbb{P})$ in a countable elementary submodel N of $H(\kappa)$ and $\sigma \in a(\mathbb{P}) \setminus N$ such that I forms a Δ-system with root $\sigma \cap N$. The proof of this proposition is completely same to the one in [1]. The proofs of **Theorems 5.3** and **5.4** are adopted fot this new definition. Because the property $\mathsf{R}_{1,\aleph_1}$ are applied for uncountable sets which form Δ-systems with root exact "$\tau \cap N$" in the proofs of **Theorems 5.3** and **5.4** in [1]. We apply the new **Proposition 2.10.2** to these Δ-systems.

We have to change only the statement and the proof of **Lemma 6.9** as follows.

Lemma 6.9. *Suppose that $\mathbb{Q}$ is a forcing notion in FSCO with the property $\mathsf{R}_{1,\aleph_1}$, I is an uncountable subset of $\mathbb{Q}$ such that*

- *I forms a Δ-system with root ϵ, and*
- *for every σ and τ in I, either $\max(\sigma \setminus \epsilon) < \min(\tau \setminus \epsilon)$ or $\max(\tau \setminus \epsilon) < \min(\sigma \setminus \epsilon)$,*

$\vec{M} = \langle M_\alpha; \alpha \in \omega_1\rangle$ is a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{\mathbb{Q}, I\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha\rangle \in M_\alpha$, and $S \subseteq \omega_1 \setminus \{0\}$ is stationary.

Then $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is (T, S)-preserving.

Proof. Let $\mathbb{Q}$, I, $\vec{M}$, S be as in the assumption of the statement of the lemma, and T, θ, N as in the statement of the definition of the (T,S)-preservation, (moreover we suppose $\vec{M} \in N$, to calculate levels of conditions in $\mathbb{Q}$) and $\langle h, f\rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S) \cap N$. Suppose that $\omega_1 \cap N \notin S$, because if $\omega_1 \cap N \in S$, then the condition $\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N\rangle\}, f\rangle$ is as desired.

Let

$$\delta := \sup\{F(\omega_1 \cap N) + 1; F \in ({}^{\omega_1}\omega_1) \cap N\}.$$

Since N is countable, δ is a countable ordinal. We will show that the condition $\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$ of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is our desired one.

By Lemma 6.6 (in the original paper [1]), $\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$ is $(N, \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S))$-generic. Suppose that $x \in T$ of height $\omega_1 \cap N$ such that for any subset $A \in N$ of T, if $x \in A$, then there is $y \in A$ such that $y <_T x$. Let $\dot{A} \in N$ be a $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$-name for a subset of T. We will show that

$$\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle \Vdash_{\mathbb{Q}} \text{“ } x \notin \dot{A} \text{ or } \exists y \in \dot{A}\,(y <_T x) \text{ ”}.$$

Let $\langle h', f'\rangle \leq_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$, and assume that

$$\langle h', f'\rangle \nVdash_{\mathbb{Q}} \text{“ } x \notin \dot{A} \text{ ”}.$$

By strengthening $\langle h', f'\rangle$ if necessary, we may assume that

$$\langle h', f'\rangle \Vdash_{\mathbb{Q}} \text{“ } x \in \dot{A} \text{ ”}.$$

We note that $\langle h' \restriction N, f' \restriction N\rangle$ is in N (because $\omega_1 \cap N \in \mathrm{dom}(h')$) and for every $\sigma \in \mathrm{dom}(f') \setminus N$, $\min(\sigma \setminus \epsilon) > \delta$ by the definition of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$. Let

$$L := \{f'(\sigma); \sigma \in \mathrm{dom}(f') \ \&\ f'(\sigma) \in \omega_1 \cap N\},$$

which is a finite subset of N, hence is in N. For each $\alpha \in L$, let

$$\tau_\alpha := \bigcup\left((f')^{-1}[\{\alpha\}]\right).$$

Then $\langle \tau_\alpha; \alpha \in L\rangle$ is a condition of the product ${}^L\mathbb{Q}$ and for each $\alpha \in L$, τ_α is an extension of all members of $(f')^{-1}[\{\alpha\}]$ in $\mathbb{Q}$. The sequence $\langle \tau_\alpha; \alpha \in L\rangle$ does not belong to N, however we notice that the sequence $\langle \tau_\alpha \cap N; \alpha \in L\rangle$ belongs to N. We define a function F with the domain

$$\{t \in T; \mathsf{ht}_T(t) > \max(\mathrm{dom}(h' \restriction N))\}$$

such that for each $t \in T$ of height larger than $\max(\mathrm{dom}(h' \restriction N))$,

$$\begin{aligned} F(t) := \sup\Big\{\beta \in \omega_1;\ \ & \text{there exists } \langle k, g\rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S) \text{ such that} \\ & \bullet\, \min(\mathrm{dom}(k)) = \mathsf{ht}_T(t), \\ & \bullet\, k(\mathsf{ht}_T(t)) = \beta, \\ & \bullet\, \langle (h' \restriction N) \cup k, (f' \restriction N) \cup g\rangle \text{ is a condition of } \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S), \\ & \bullet\, \langle (h' \restriction N) \cup k, (f' \restriction N) \cup g\rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{“ } t \in \dot{A} \text{ ”, and} \\ & \bullet\, \text{for all } \alpha \in L,\ \min\left(\bigcup\left(g^{-1}[\{\alpha\}] \setminus \epsilon\right)\right) \geq \beta \qquad \Big\}. \end{aligned}$$

Then F belongs to N. Let

$$B := \{t \in T; \mathsf{ht}_T(t) > \max(\mathrm{dom}(h' \restriction N)) \ \&\ F(t) = \omega_1\},$$

which is also in N. We define a function F' with the domain

$$[\max(\mathrm{dom}(h' \restriction N)) + 1, \omega_1)$$

such that for a countable ordinal β larger than $\max(\mathrm{dom}(h' \restriction N))$,

$$F'(\beta) := \sup\left\{F(t) + 1; t \in T \setminus B \ \&\ \mathsf{ht}_T(t) \in \left(\max(\mathrm{dom}(h' \restriction N)), \beta\right]\right\}.$$

This F' is a function from ω_1 into ω_1 and also in N. Hence $F'(\omega_1 \cap N) < \delta$ by the definition of δ. Since letting $k = h' \setminus (h' \restriction N)$ and $g = f' \setminus (f' \restriction N)$, $k(\mathsf{ht}_T(x)) = h'(\omega_1 \cap N) = \delta$, $\langle (h' \restriction N) \cup k, (f' \restriction N) \cup g\rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)}$ " $x \in \dot{A}$ " and $\min\left(\bigcup\left(g^{-1}[\{\alpha\}] \setminus \epsilon\right)\right) \geq \delta$, $F(x) \geq \delta$ holds. Therefore x have to belong to B. Thus by our assumption, there exists $y \in B$ such that $y <_T x$.

Since $F(y) = \omega_1$ and both F and y belong to N, there exists an uncountable subset $\{\langle k_\xi, g_\xi\rangle; \xi \in \omega_1\}$ of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ such that for each ξ and η in ω_1 with $\xi < \eta$,

- $\langle (h' \restriction N) \cup k_\xi, (f' \restriction N) \cup g_\xi\rangle$ is a condition of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$,
- $\langle (h' \restriction N) \cup k_\xi, (f' \restriction N) \cup g_\xi\rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)}$ " $y \in \dot{A}$ ",
- for all $\alpha \in L$,

$$\begin{aligned}\max(\tau_\alpha \cap N) &< \min\left(\bigcup\left({g_\xi}^{-1}[\{\alpha\}] \setminus \epsilon\right)\right)\\ &< \max\left(\bigcup\left({g_\xi}^{-1}[\{\alpha\}] \setminus \epsilon\right)\right) < \min\left(\bigcup\left({g_\eta}^{-1}[\{\alpha\}] \setminus \epsilon\right)\right).\end{aligned}$$

For each $\xi \in \omega_1$ and $\alpha \in L$, let

$$\mu_{\xi,\alpha} := \bigcup\left((f' \restriction N)^{-1}[\{\alpha\}] \cup {g_\eta}^{-1}[\{\alpha\}]\right).$$

Then for every $\alpha \in L$, since

$$\tau_\alpha \cap N = \bigcup\left((f' \restriction N)^{-1}[\{\alpha\}]\right)$$

(because of the assumption of I), the set $\{\mu_{\xi,\alpha}; \xi \in \omega_1\}$ forms a Δ-system with root $\tau_\alpha \cap N$. So by the property $\mathsf{R}_{1,\aleph_1}$ of ${}^L\mathbb{Q}$ of Proposition 2.8, there exists $J'' \in [\omega_1]^{\aleph_1} \cap N$ such that every member of the set $\{\langle \mu_{\xi,\alpha}; \alpha \in L\rangle; \xi \in J''\}$ is compatible with $\langle \tau_\alpha; \alpha \in L\rangle$ in ${}^L\mathbb{Q}$. Therefore when we take any $\xi \in J'' \cap N$, for every $\alpha \in L$, $\mu_{\xi,\alpha} \cup \tau_\alpha$ is an extension of all members of $(f')^{-1}[\{\alpha\}] \cup {g_\xi}^{-1}[\{\alpha\}]$ in $\mathbb{Q}$, so $\langle h' \cup k_\xi, f' \cup g_\xi\rangle$ is a common extension of $\langle h', f'\rangle$ and $\langle k_\xi, g_\xi\rangle$ in $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$. Moreover it follows that

$$\langle h' \cup k_\xi, f' \cup g_\xi\rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{" } y \in \dot{A} \text{ "}.$$

□

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References

[1] T. Yorioka. *A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees*. Ann. Pure Appl. Logic 161 (2010), no. 4, 469–487.