On presentations and a derived equivalence classification of orbit categories

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Contents

Introduction

In this paper, we investigated problems on orbit categories. The first one is to give presentations of Grothendieck constructions that are generalizations of orbit categories, and second one deals with a derived equivalence classification of algebras that have the form of orbit categories.

Presentations of Grothendieck constructions. Throughout chapter 1 I is a small category, $\Bbbk$ is a commutative ring, and $\Bbbk$-**Cat** denotes the 2-category of all $\Bbbk$-categories, $\Bbbk$-functors between them and natural transformations between $\Bbbk$-functors.

The Grothendieck construction is a way to form a single category $\mathrm{Gr}(X)$ from a diagram X of small categories indexed by a small category I, which first appeared in [8, §8 of Exposé VI]. As is exposed by Tamaki [19] this construction has been used as a useful tool in homotopy theory (e.g., [20]) or topological combinatorics (e.g., [21]). This can be also regarded as a generalization of orbit category construction from a category with a group action.

In [5] we defined a notion of derived equivalences of (colax) functors from I to $\Bbbk$-**Cat**, and in [6] we have shown that if (colax) functors $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ are derived equivalent, then so are their Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. An easy example of a derived equivalent pair of functors is given by using diagonal functors: For a category $\mathcal{C}$ define the *diagonal* functor $\Delta(\mathcal{C})\colon I \to \Bbbk\text{-}\mathbf{Cat}$ to be a functor sending all objects of I to $\mathcal{C}$ and all morphisms in I to the identity functor of $\mathcal{C}$. Then if categories $\mathcal{C}$ and $\mathcal{C}'$ are derived equivalent, then so are their diagonal functors $\Delta(\mathcal{C})$ and $\Delta(\mathcal{C}')$. Therefore, to compute examples of derived equivalent pairs using this result, it will be useful to present Grothendieck constructions of functors by quivers with relations. We already have computations in two special cases. First for a $\Bbbk$-algebra A, which we regard as a $\Bbbk$-category with a single object, we noted in [6] that if I is a semigroup G, a poset S, or the free category $\mathbb{P}Q$ of a quiver Q, then the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ is isomorphic to the semigroup algebra AG, the incidence algebra AS, or the path-algebra AQ, respectively. Second in [4] we gave a quiver presentation of the orbit category $\mathcal{C}/G$ for each $\Bbbk$-category $\mathcal{C}$ with an action of a semigroup G in the case that $\Bbbk$ is a field, which can be seen as a computation of a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ of each functor $X \colon G \to \Bbbk\text{-}\mathbf{Cat}$.

In chapter 1 we generalize these two results in the following way:

(1) We compute the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ for each $\Bbbk$-algebra A and each small category I.

(2) We give a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ for each functor $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ and each small category I when $\Bbbk$ is a field.

Derived equivalence classification of generalized multifold extensions of piecewise hereditary algebras of tree type. Throughout chapter 2 we fix an algebraically closed field $\Bbbk$, and assume that all algebras are basic and finite-dimensional $\Bbbk$-algebras and that all categories are $\Bbbk$-categories.

The classification of algebras under derived equivalences seems to be first explicitly investigated by Rickard in [16], which gave the derived equivalence classification of Brauer tree algebras (implicitly there exists an earlier work [7] by Assem–Happel giving the classification of gentle tree algebras). After that the first named author gave the classification of representation-finite self-injective algebras (see also [1] and Membrillo-Hernández [14] for type A_n). The technique used there (a covering technique for derived equivalences developed in [1]) is applicable also for representation-infinite algebras; it requires that the algebras in consideration have the form of orbit categories (usually of repetitive categories of some algebras having no oriented cycles in their ordinary quivers). In fact, it was applied in [3] to give the classification of twisted multifold extensions of piecewise hereditary algebras of tree type by giving a complete invariant. Here an algebra is called a *twisted multifold extension* of an algebra A if it has the form

$$T^n_\psi(A) := \hat{A}/\langle\hat{\psi}\nu_A^n\rangle \tag{0.1}$$

for some positive integer n and some automorphism ψ of A, where $\hat{A}$ is the repetitive algebra of A, ν_A is the Nakayama automorphism of $\hat{A}$ and $\hat{\psi}$ is the automorphism of $\hat{A}$ naturally induced from ψ (see Definition 2.1 and Lemma 2.2 for details); and an algebra A is called a *piecewise hereditary algebra of tree type* if A is an algebra derived equivalent to a hereditary algebra whose ordinary quiver is an oriented tree. In chapter 2 we extend this classification to a wider class of algebras. To state this class of algebras we introduce the following terminologies. For an integer n we say that an automorphism ϕ of $\hat{A}$ has a *jump n* if $\phi(A^{[0]}) = A^{[n]}$. An algebra of the form

$$\hat{A}/\langle\phi\rangle$$

for some automorphism ϕ of $\hat{A}$ with jump n for some positive integer n is called a *generalized multifold extension* of A. Since obviously $\hat{\psi}\nu_A^n$ is an automorphism of $\hat{A}$ with jump n in the formula (0.1), twisted multifold extensions are generalized multifold extensions. We are now able to state our purpose. In chapter 2 we will give the derived equivalence classification of generalized multifold extensions of piecewise hereditary algebras of tree type by giving a complete invariant. Note that most algebras in this class are wild and that the tame part of the class has a big intersection with the class of self-injective algebras of Euclidean type studied by Skowroński in [17] (see Remark 2.7).

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Chapter I.

Presentations of Grothendieck constructions

In section 1 we give necessary definitions and recall the fact that all categories can be presented by quivers and relations. Sections 2 and 3 are devoted to the computation (1) and a quiver presentation (2) above, respectively. Finally in section 4 we give some examples.

1. Preliminaries

Throughout this paper $Q = (Q_0, Q_1, t, h)$ is a quiver, where $t(\alpha) \in Q_0$ is the *tail* and $h(\alpha) \in Q_0$ is the *head* of each arrow α of Q. For each path μ of Q, the tail and the head of μ is denoted by $t(\mu)$ and $h(\mu)$, respectively. For each non-negative integer n the set of all paths of Q of length at least n is denoted by $Q_{\geq n}$. In particular $Q_{\geq 0}$ denotes the set of all paths of Q.

A category $\mathcal{C}$ is called a $\Bbbk$-*category* if for each $x, y \in \mathcal{C}$, $\mathcal{C}(x, y)$ is a $\Bbbk$-module and the compositions are $\Bbbk$-bilinear.

Definition 1.1. Let Q be a quiver.

(1) The *free* category $\mathbb{P}Q$ of Q is the category whose underlying quiver is $(Q_0, Q_{\geq 0}, t, h)$ with the usual composition of paths.
(2) The *path* $\Bbbk$-category of Q is the $\Bbbk$-linearization of $\mathbb{P}Q$ and is denoted by $\Bbbk Q$.

Definition 1.2. Let $\mathcal{C}$ be a category. We set

$$\mathrm{Rel}(\mathcal{C}) := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \mathcal{C}(i, j) \times \mathcal{C}(i, j),$$

elements of which are called *relations* of $\mathcal{C}$. Let $R \subseteq \mathrm{Rel}(\mathcal{C})$. For each $i, j \in \mathcal{C}_0$ we set

$$R(i, j) := R \cap (\mathcal{C}(i, j) \times \mathcal{C}(i, j)).$$

(1) The smallest congruence relation

$$R^c := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \{(dac, dbc) \mid c \in \mathcal{C}(-, i), d \in \mathcal{C}(j, -), (a, b) \in R(i, j)\}$$

containing R is called the *congruence relation* generated by R.

(2) For each $i, j \in \mathcal{C}_0$ we set

$$R^{-1}(i,j) := \{(g,f) \in \mathcal{C}(i,j) \times \mathcal{C}(i,j) \mid (f,g) \in R(i,j)\}$$
$$1_{\mathcal{C}(i,j)} := \{(f,f) \mid f \in \mathcal{C}(i,j)\}$$
$$S(i,j) := R(i,j) \cup R^{-1}(i,j) \cup 1_{\mathcal{C}(i,j)}$$
$$S(i,j)^1 := S(i,j)$$
$$S(i,j)^n := \{(h,f) \mid \exists g \in \mathcal{C}(i,j), (g,f) \in S(i,j), (h,g) \in S(i,j)^{n-1}\} \quad (\text{for all } n \geq 2)$$
$$S(i,j)^\infty := \bigcup_{n \geq 1} S(i,j)^n, \text{ and set}$$
$$R^e := \bigcup_{(i,j) \in \mathcal{C}_0 \times \mathcal{C}_0} S(i,j)^\infty.$$

R^e is called the *equivalence relation* generated by R.

(3) We set $R^\# := (R^c)^e$ (cf. [10]).

Remark 1.3. In the statement (2) above, $S(i,j)^\infty$ is the smallest equivalence relation on $\mathcal{C}(i,j)$ containing $R(i,j)$ for each $i, j \in \mathcal{C}_0$.

Definition 1.4. Let $\mathcal{C}$ be a category and $R \subseteq \mathrm{Rel}(\mathcal{C})$. Then a category $\mathcal{C}/R^\#$ is defined as follows:

(i) $(\mathcal{C}/R^\#)_0 := \mathcal{C}_0$.

(ii) For $i, j \in (\mathcal{C}/R^\#)_0$, $(\mathcal{C}/R^\#)(i,j) := \mathcal{C}(i,j)/R^\#(i,j)$.
For each $f \in (\mathcal{C}/R^\#)(i,j)$, we set $\overline{f}$ the equivalence class of f in $R^\#$.

(iii) For $i, j, k \in (\mathcal{C}/R^\#)_0$ and $\overline{f} \in (\mathcal{C}/R^\#)(i,j)$, $\overline{g} \in (\mathcal{C}/R^\#)(j,k)$, $\overline{g} \circ \overline{f} := \overline{g \circ f}$.

(iv) A functor $F : \mathcal{C} \to \mathcal{C}/R^\#$ is defined as follows:
  (a) For $i \in \mathcal{C}_0$, $F(i) = i$.
  (b) For $i, j \in \mathcal{C}(i,j)$ and $f \in \mathcal{C}(i,j)$, $F(f) = \overline{f}$.

Remark 1.5. In definition1.4, $R^\#$ is a congruence relation, therefore the composition in (iii) is well-defined.

The following is well known (cf. [13]).

Proposition 1.6. *Let $\mathcal{C}$ be a category, and $R \subseteq \mathrm{Rel}(\mathcal{C})$. Then the category $\mathcal{C}/R^\#$ and the functor $F : \mathcal{C} \to \mathcal{C}/R^\#$ defined above satisfy the following conditions.*

(i) *For each $i, j \in \mathcal{C}_0$ and each $(f, f') \in R(i,j)$ we have $Ff = Ff'$.*

(ii) *If a functor $G : \mathcal{C} \to \mathcal{D}$ satisfies $Gf = Gf'$ for all $f, f' \in \mathcal{C}(i,j)$ and all $i, j \in \mathcal{C}_0$ with $(f, f') \in R(i,j)$, then there exists a unique functor $G' : \mathcal{C}/R^\# \to \mathcal{D}$ such that $G' \circ F = G$.*

Definition 1.7. Let Q be a quiver and $R \subseteq \mathrm{Rel}(\mathbb{P}Q)$. We set

$$\langle Q \mid R \rangle := \mathbb{P}Q/R^\#.$$

The following is straightforward.

Proposition 1.8. *Let $\mathcal{C}$ be a category, Q the underlying quiver of $\mathcal{C}$, and set*

$$R := \{(e_i, \mathbb{1}_i), (\mu, [\mu]) \mid i \in Q_0, \mu \in Q_{\geq 2}\} \subseteq \mathrm{Rel}(\mathbb{P}Q),$$

where e_i is the path of length 0 at each vertex $i \in Q_0$, and $[\mu] := \alpha_n \circ \cdots \circ \alpha_1$ (the composite in $\mathcal{C}$) for all paths $\mu = \alpha_n \ldots \alpha_1 \in Q_{\geq 2}$ with $\alpha_1, \ldots, \alpha_n \in Q_1$. Then

$$\mathcal{C} \cong \langle Q \mid R \rangle.$$

By this statement, an arbitrary category is presented by a quiver and relations. Throughout the rest of this paper I is a small category with a presentation $I = \langle Q \mid R \rangle$.

2. Grothendieck constructions of Diagonal functors

Definition 2.1. Let $X : I \to \Bbbk\text{-}\mathbf{Cat}$ be a functor. Then a category $\mathrm{Gr}(X)$, called the *Grothendieck construction* of X, is defined as follows:

(i) $(\mathrm{Gr}(X))_0 := \bigcup_{i \in I_0} \{(i, x) \mid x \in X(i)_0\}$

(ii) For $(i, x), (j, y) \in (\mathrm{Gr}(X))_0$

$$\mathrm{Gr}(X)((i, x), (j, y)) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y)$$

(iii) For $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)((i, x), (j, y))$ and $g = (g_b)_{b \in I(j,k)} \in \mathrm{Gr}(X)((j, y), (k, z))$

$$g \circ f := \left(\sum_{\substack{c = ba \\ a \in I(i,j) \\ b \in I(j,k)}} g_b X(b) f_a \right)_{c \in I(i,k)}$$

Definition 2.2. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$. Then the *diagonal functor* $\Delta(\mathcal{C})$ of $\mathcal{C}$ is a functor from I to $\Bbbk\text{-}\mathbf{Cat}$ sending each arrow $a \colon i \to j$ in I to $\mathbb{1}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ in $\Bbbk\text{-}\mathbf{Cat}$.

In this section, we fix a $\Bbbk$-algebra A which we regard as a $\Bbbk$-category with a single object $*$ and with $A(*, *) = A$. The *quiver algebra* AQ of Q over A is the A-linearization of $\mathbb{P}Q$, namely $AQ := A \otimes_{\Bbbk} \Bbbk Q$.

Definition 2.3. The ideal of AQ generated by the elements $g - h$ with $(g, h) \in R$ is denoted by $\langle R \rangle_A$:

$$\langle R \rangle_A := AQ\{g - h \mid (g, h) \in R\}AQ.$$

The purpose of this section is to prove the following theorem which computes the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of $\Delta(A) : I \to \Bbbk\text{-}\mathbf{Cat}$.

Theorem 2.4. $\mathrm{Gr}(\Delta(A)) \cong AQ/\langle R \rangle_A$.

To prove this theorem, we use the following two lemmas.

Lemma 2.5. *Let S be a set, $E \subseteq S \times S$ an equivalence relation on S. Then*

$$(\bigoplus_{x \in S} Ax)/(\sum_{(g,h)\in E} A(g-h)) \cong \bigoplus_{\overline{x}\in S/E} A\overline{x}$$

Proof. Let $\varepsilon : \bigoplus_{x\in S} Ax \to \bigoplus_{\overline{x}\in S/E} A\overline{x}$ be a homomorphism of A-modules defined by $x \mapsto \overline{x}$ $(x \in S)$. Then the sequence

$$0 \to \sum_{(g,h)\in E} A(g-h) \hookrightarrow \bigoplus_{x\in S} Ax \xrightarrow{\varepsilon} \bigoplus_{\overline{x}\in S/E} A\overline{x} \to 0$$

is exact. Indeed, since ε is obviously a surjection by definition, it is enough to show that $\operatorname{Ker}\varepsilon = \sum_{(g,h)\in E} A(g-h)$.

For each $(g, h) \in E$ we have

$$\varepsilon(g-h) = \overline{g-h} = \overline{g} - \overline{h} = 0,$$

and hence $\sum_{(g,h)\in E} A(g-h) \subseteq \operatorname{Ker}\varepsilon$.

To prove the reverse inclusion, let $\sum_{x\in S} a_x x \in \operatorname{Ker}\varepsilon$ $(a_x \in A)$. Then since

$$0 = \varepsilon\left(\sum_{x\in S} a_x x\right) = \sum_{x\in S} a_x \overline{x} = \sum_{\overline{x}\in S/E}\sum_{x'\in\overline{x}} a_{x'}\overline{x},$$

we have $\sum_{x'\in\overline{x}} a_{x'} = 0$ for each $\overline{x} \in S/E$, and hence for each $x \in S$ we have

$$a_x = -\sum_{x'\in\overline{x}\setminus\{x\}} a_{x'}$$

and

$$\sum_{x'\in\overline{x}} a_{x'}x' = a_x x + \sum_{x'\in\overline{x}\setminus\{x\}} a_{x'}x' = \sum_{x'\in\overline{x}\setminus\{x\}} a_{x'}(x'-x).$$

Let L be a complete set of representatives in S/E. Then we have

$$\sum_{x\in S} a_x x = \sum_{x\in L}\sum_{(x,x')\in E\setminus\{(x,x)\}} a_{x'}(x'-x).$$

Hence $\operatorname{Ker}\varepsilon \subseteq \sum_{(g,h)\in E} A(g-h)$ and we have $\operatorname{Ker}\varepsilon = \sum_{(g,h)\in E} A(g-h)$. $\square$

We will give an explicit form of $\langle R\rangle_A$ as follows.

Lemma 2.6. *For each $i, j \in Q_0$,*

$$\langle R\rangle_A(i,j) = \sum_{(g,h)\in R^{\#}(i,j)} A(g-h)$$

Proof. Define J by setting $J(i,j) := \sum_{(g,h)\in R^{\#}(i,j)} A(g-h)$ for all $i,j \in Q_0$. First, we prove that J is an ideal of AQ. It is obvious that $J(i,j)$ is closed under addition. Let $a \in AQ(i',i)$, $b \in AQ(j,j')$, $c \in J(i,j)$. Then there exist $a_\alpha, b_\beta, c_{g,h} \in A$ such that

$$\begin{aligned} a &= \sum_{\alpha\in\mathbb{P}Q(i'i)} a_\alpha \alpha \\ b &= \sum_{\beta\in\mathbb{P}Q(j,j')} b_\beta \beta \\ c &= \sum_{(g,h)\in R^{\#}(i,j)} c_{g,h}(g-h) \end{aligned}$$

and

$$\begin{aligned} bca &= \left(\sum_{\beta\in\mathbb{P}Q(j,j')} b_\beta\beta\right)\left(\sum_{(g,h)\in R^{\#}(i,j)} c_{g,h}(g-h)\right)\left(\sum_{\alpha\in\mathbb{P}Q(i'i)} a_\alpha\alpha\right) \\ &= \sum_{\beta\in\mathbb{P}Q(j,j')}\sum_{(g,h)\in R^{\#}(i,j)}\sum_{\alpha\in\mathbb{P}Q(i',i)} b_\beta c_{g,h} a_\alpha(\beta g\alpha - \beta h\alpha), \end{aligned}$$

where we have $(\beta g\alpha, \beta h\alpha) \in R^{\#}$ for all $(g,h) \in R^{\#}(i,j)$. Hence $bca \in J(i',j')$ as desired.

Next, we prove that $\langle R\rangle_A(i,j) = J(i,j)$. Since $R \subseteq R^{\#}$, for each $(g,h) \in R(i,j)$ we have

$$g - h \in J(i,j).$$

Hence $\langle R\rangle_A(i,j) \subseteq J(i,j)$ because J is an ideal of AQ. Further for each $(g,h) \in R^c(i,j)$, there exist $(g',h') \in R(i',j')$, $e \in \mathbb{P}Q(i,i')$ and $f \in \mathbb{P}Q(j',j)$ such that

$$(g,h) = (fg'e, fh'e).$$

Then

$$g - h = fg'e - fh'e = f(g'-h')e \in \langle R\rangle_A(i,j).$$

Hence also for each $(g,h) \in R^{\#}(i,j)$ we have $g-h \in \langle R\rangle_A(i,j)$ because $\langle R\rangle_A$ is an additive subgroup of AQ. Therefore $J(i,j) \subseteq \langle R\rangle_A(i,j)$, and hence $\langle R\rangle_A(i,j) = J(i,j)$. $\square$

Proof of Theorem2.4

The object classes and the morphism spaces of $\mathrm{Gr}(\Delta(A))$ and $AQ/\langle R\rangle_A$ are given as follows.

$\mathrm{Gr}(\Delta(A))$:

(i) $\mathrm{Gr}(\Delta(A))_0 = \{(i,*) \mid i \in Q_0\}$.

(ii) For $(i,*),(j,*) \in \mathrm{Gr}(\Delta(A))_0$

$$\begin{aligned}\mathrm{Gr}(\Delta(A))((i,*),(j,*)) &= \bigoplus_{a\in I(i,j)} \Delta(A)(j)(\Delta(A)(a)(*),*)\\ &= \bigoplus_{a\in I(i,j)} A(*,*) = A^{(I(i,j))}\end{aligned}$$

$AQ/\langle R\rangle_A$:

(i) $(AQ/\langle R\rangle_A)_0 = Q_0$.
(ii) For $i,j \in (AQ/\langle R\rangle_A)_0$

$$\begin{aligned}(AQ/\langle R\rangle_A)(i,j) &= (\bigoplus_{a\in \mathbb{P}Q(i,j)} Aa)/\langle R\rangle_A(i,j)\\ &= (\bigoplus_{a\in \mathbb{P}Q(i,j)} Aa)/\sum_{(g,h)\in R^{\#}(i,j)} A(g-h)\\ &= \bigoplus_{a\in I(i,j)} Aa\end{aligned}$$

by Lemma 2.6 and the last equality is given by the isomorphism in Lemma 2.5. We define a functor $F : \mathrm{Gr}(\Delta(A)) \to AQ/\langle R\rangle_A$ by

$$(i,*) \mapsto i$$

$$(f_a)_{a\in I(i,j)} \mapsto \sum_{a\in I(i,j)} f_a a$$

for each $(f_a)_{a\in I(i,j)} : (i,*) \to (j,*)$ in $\mathrm{Gr}(\Delta(A))$. We check that F is well-defined as a $\mathbb{k}$-linear functor. For each $(i,*) \in \mathrm{Gr}(\Delta(A))_0$ we have

$$\begin{aligned}F(\mathbb{1}_{(i,*)}) &= F((\delta_{1_i a})_{a\in I(i,i)})\\ &= \sum_{a\in I(i,i)} \delta_{1_i a} a\\ &= 1_i\end{aligned}$$

For each $f \in \mathrm{Gr}(\Delta(A))((i,*,),(j,*))$ and $g \in \mathrm{Gr}(\Delta(A))((j,*,),(k,*))$, there exist $f_a, g_b \in A$ $(a \in I(i,j), b \in I(j,k))$ such that

$$\begin{aligned}f &= (f_a)_{a\in I(i,j)}\\ g &= (g_b)_{b\in I(j,k)}.\end{aligned}$$

Then

$$
\begin{aligned}
F(g \circ f) &= F\left(\left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} g_b f_a\right)_{c\in I(i,k)}\right)\\
&= \sum_{c\in I(i,k)}\left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} g_b f_a\right) c\\
F(g)F(f) &= \left(\sum_{b\in I(j,k)} g_b b\right)\left(\sum_{a\in I(i,j)} f_a a\right)\\
&= \sum_{c\in I(i,k)}\left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} g_b f_a\right) c\\
&= F(g\circ f).
\end{aligned}
$$

Hence F is a functor. Obviously F is $\Bbbk$-linear. It is clear that F is bijective on objects and that for each $i, j \in Q_0$, F induces an isomorphism

$$\mathrm{Gr}(\Delta(A))((i,*),(j,*)) \to (AQ/\langle R\rangle_A)(i,j)$$

by the definition of F. Therefore $\mathrm{Gr}(\Delta(A)) \cong AQ/\langle R\rangle_A$. □

Remark 2.7. Theorem 2.4 can be written in the form

$$\mathrm{Gr}(\Delta(A)) \cong A \otimes_{\Bbbk} (\Bbbk Q/\langle R\rangle_{\Bbbk}).$$

3. The quiver presentation of Grothendieck constructions

In this section we give a quiver presentation of the Grothendieck construction of an arbitrary functor $I \to \Bbbk$-**Cat**. Throughout this section we assume that $\Bbbk$ is a field.

Theorem 3.1. *Let $X : I \to \Bbbk$-**Cat** be a functor, and for each $i \in I$ set $X(i) = \Bbbk Q^{(i)}/\langle R^{(i)}\rangle$ with $\Phi^{(i)} \colon \Bbbk Q^{(i)} \to X(i)$ the canonical morphism, where $R^{(i)} \subseteq \Bbbk Q^{(i)}$, $\langle R^{(i)}\rangle \cap \{e_x \mid x \in Q(i)_0\} = \emptyset$. Then the Grothendieck construction of X is presented by the quiver with relations (Q, R') defined as follows.*

Quiver: $Q' = (Q'_0, Q'_1, t', h')$, where

(i) $Q'_0 := \bigcup_{i \in I} \{ {}_i x \mid x \in Q_0^{(i)} \}$.

(ii) $Q'_1 := \bigcup_{i \in I} \{ \{ {}_i\alpha \mid \alpha \in Q_1^{(i)} \} \cup \{ (a, {}_i x) : {}_i x \to {}_j(ax) \mid a : i \to j \in Q_1, x \in Q_0^{(i)}, ax \neq 0 \} \}$,
where we set $ax := X(\overline{a})(x)$.

(iii) *For* $\alpha \in Q_1^{(i)}$, $t'({}_i\alpha) = t^{(i)}(\alpha)$ *and* $h'({}_i\alpha) = h^{(i)}(\alpha)$.

(iv) *For* $a : i \to j \in Q_1, x \in Q_0^{(i)}$, $t'(a, {}_i x) = {}_i x$ *and* $h'(a, {}_i x) = {}_j(ax)$.

Relations: $R' := R'_1 \cup R'_2 \cup R'_3$, *where*

(i) $R'_1 := \{ \sigma^{(i)}(\mu) \mid i \in Q_0, \mu \in R^{(i)} \}$,
where we set $\sigma^{(i)} : \Bbbk Q^{(i)} \hookrightarrow \Bbbk Q'$.

(ii) $R'_2 := \{ \pi(g, {}_i x) - \pi(h, {}_i x) \mid i, j \in Q_0, (g, h) \in R(i, j), x \in Q_0^{(i)} \}$,
where for each path a *in* Q *we set*

$$\pi(a, {}_i x) := (a_n, {}_{i_{n-1}}(a_{n-1}a_{n-2} \dots a_1 x)) \dots (a_2, {}_{i_1}(a_1 x))(a_1, {}_i x)$$

if $a = a_n \dots a_2 a_1$ *for some arrows* $a_1, \dots, a_n$ *in* Q, *and*

$$\pi(a, {}_i x) := e_{{}_i x}$$

if $a = e_i$ *for some* $i \in Q_0$.

(iii) $R'_3 := \{ (a, {}_i y){}_i\alpha - {}_j(a\alpha)(a, {}_i x) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)} \}$, *where we take* $a\alpha : ax \to ay$ *so that* $\Phi^{(j)}(a\alpha) \in X(\overline{a})\Phi^{(i)}(\alpha)$:

$$\begin{tikzcd}
\alpha \in \Bbbk Q^{(i)} \arrow[r, "\Phi^{(i)}"] & X(i) \arrow[d, "X(\overline{a})"] \\
a\alpha \in \Bbbk Q^{(j)} \arrow[r, "\Phi^{(j)}"] & X(j).
\end{tikzcd}$$

Note that the ideal $\langle R' \rangle$ *is independent of the choice of* $a\alpha$ *because* $R'_1 \subseteq R'$.

Proof. We define a $\Bbbk$-functor $\Phi : \Bbbk Q' \to \mathrm{Gr}(X)$ as follows:

(i) for ${}_i x \in Q'_0$, $\Phi({}_i x) = (i, x)$;

(ii) for ${}_i\alpha : {}_i x \to {}_i y \in Q_1^{(i)}$, $\Phi({}_i\alpha) = (\delta_{\mathbf{1}_i a} \Phi^{(i)}(\alpha))_{a \in I(i,i)}$;

(iii) for $(a, {}_i x) : {}_i x \to {}_j(ax) \in Q'_1$, $\Phi((a, {}_i x)) = (\delta_{\overline{a}b} \mathbb{1}_{X(\overline{a})(x)})_{b \in I(i,j)}$;

(iv) for $\alpha_n \alpha_{n-1} \dots \alpha_1 \in \mathbb{P}Q'$ $(\alpha_1, \dots, \alpha_n \in Q')$

$$\Phi(\alpha_n \alpha_{n-1} \dots \alpha_1) := \Phi(\alpha_n)\Phi(\alpha_{n-1}) \dots \Phi(\alpha_1); \text{ and}$$

(v) for $f := \sum_{\alpha \in \mathbb{P}Q'({}_i x, {}_j y)} f_\alpha \alpha \in \Bbbk Q'({}_i x, {}_j y)$ $(f_\alpha \in \Bbbk)$

$$\Phi(f) := \sum_{\alpha \in \mathbb{P}Q'({}_i x, {}_j y)} f_\alpha \Phi(\alpha).$$

Claim 1. Φ *is well-defined as a* $\Bbbk$*-functor, and is bijective on objects.*

Indeed, this is clear by noting that for each ${}_ix \in Q'_0$ we have

$$\begin{aligned}\Phi(e_{{}_ix}) &= (\delta_{1_i,a}\Phi^{(i)}(e_x))_{a\in I(i,i)}\\ &= \mathbb{1}_{(i,x)}.\end{aligned}$$

Claim 2. $\Phi(R') = 0$.

Indeed, for each $i \in Q_0$, $\alpha, \beta \in Q_1^{(i)}$ we have

$$\begin{aligned}\Phi({}_i\beta_i\alpha) &= \Phi({}_i\beta)\Phi({}_i\alpha)\\ &= (\delta_{1_i,b}\Phi^{(i)}(\beta))_{b\in I(i,i)}(\delta_{1_i,a}\Phi^{(i)}(\alpha))_{a\in I(i,i)}\\ &= \left(\sum_{\substack{c=ba\\ a\in I(i,i)\\ b\in I(i,i)}} \delta_{1_i,b}\Phi^{(i)}(\beta)X(b)(\delta_{1_i,a}\Phi^{(i)}(\alpha))\right)_{c\in I(i,i)}\\ &= (\delta_{1_i,c}\Phi^{(i)}(\beta\alpha))_{c\in I(i,i)},\end{aligned}$$

which shows that $\Phi(\sigma^{(i)}(\mu)) = (\delta_{1_i,c}\Phi^{(i)}(\mu))_{c\in I(i,i)}$ for each $\mu \in \mathbb{P}Q^{(i)}$, and that for each $\mu \in R^{(i)}$,

$$\Phi(\sigma^{(i)}(\mu)) = (\delta_{1_i,a}\Phi^{(i)}(\mu))_{a\in I(i,i)} = (\delta_{1_i,a}0)_{a\in I(i,i)} = 0.$$

Thus $\Phi(R'_1) = 0$.

For each $g_1 : i \to j, g_2 : j \to k \in Q_1$, ${}_ix \in Q'$,

$$\begin{aligned}\Phi(\pi(g_2g_1, {}_ix)) &= \Phi((g_2, {}_j(g_1x)))\Phi((g_1, {}_ix))\\ &= (\delta_{\overline{g_2},b}\mathbb{1}_{X(\overline{g_2})(g_1x)})_{b\in I(j,k)}(\delta_{\overline{g_1},a}\mathbb{1}_{X(\overline{g_1})(x)})_{a\in I(i,j)}\\ &= \left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} \delta_{\overline{g_2},b}\mathbb{1}_{X(\overline{g_2})(g_1x)}X(b)(\delta_{\overline{g_1},a}\mathbb{1}_{X(\overline{g_1})(x)})\right)_{c\in I(i,k)}\\ &= (\delta_{\overline{g_2g_1},c}\mathbb{1}_{X(\overline{g_2})(g_1x)}\mathbb{1}_{X(\overline{g_1})(x)})_{c\in I(i,k)}\\ &= (\delta_{\overline{g_2g_1},c}\mathbb{1}_{X(\overline{g_2g_1})(x)})_{c\in I(i,k)},\end{aligned}$$

which shows that $\Phi(\pi(g, {}_ix)) = (\delta_{\overline{g},b}\mathbb{1}_{X(\overline{g})(x)})_{b\in I(i,j)}$ for each $g \in \mathbb{P}Q$. Therefore

$$\begin{aligned}\Phi(\pi(g, {}_ix) - \pi(h, {}_ix)) &= \Phi(\pi(g, {}_ix)) - \Phi(\pi(h, {}_ix))\\ &= (\delta_{\overline{g},b}\mathbb{1}_{X(\overline{g})(x)})_{b\in I(i,j)} - (\delta_{\overline{h},a}\mathbb{1}_{X(\overline{h})(x)})_{a\in I(i,j)}\\ &= 0\end{aligned}$$

because $\overline{g} = \overline{h}$ for each $(g, h) \in R(i, j)$. Thus $\Phi(R'_2) = 0$.

For $a : i \to j \in Q_1$, $\alpha : x \to y \in Q_1^{(i)}$

$$\begin{aligned}
\Phi((a, {}_iy)_i\alpha) &= \Phi((a, {}_iy))\Phi({}_i\alpha) \\
&= (\delta_{\overline{a},c}\mathbb{1}_{X(\overline{a})(y)})_{c\in I(i,j)}(\delta_{1_i,b}\Phi^{(i)}(\alpha))_{b\in I(i,i)} \\
&= \left(\sum_{\substack{d=cb \\ b\in I(i,i) \\ c\in I(i,j)}} \delta_{\overline{a},c}\mathbb{1}_{X(\overline{a})(y)}X(c)(\delta_{1_i,b}\Phi^{(i)}(\alpha))\right)_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}\mathbb{1}_{X(\overline{a})(y)}X(\overline{a})(\Phi^{(i)}(\alpha)))_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}X(\overline{a})(\Phi^{(i)}(\alpha)))_{d\in I(i,j)},
\end{aligned}$$

$$\begin{aligned}
\Phi({}_j(a\alpha)(a, {}_ix)) &= \Phi({}_j(a\alpha))\Phi((a, {}_ix)) \\
&= (\delta_{1_j,c}\Phi^{(j)}(a\alpha))_{c\in I(j,j)}(\delta_{\overline{a},b}\mathbb{1}_{X(\overline{a})(x)})_{b\in I(i,j)} \\
&= \left(\sum_{\substack{d=cb \\ b\in I(i,j) \\ c\in I(j,j)}} \delta_{1_j,c}\Phi^{(j)}(a\alpha)X(c)(\delta_{\overline{a},b}\mathbb{1}_{X(\overline{a})(x)})\right)_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}\Phi^{(j)}(a\alpha)X(1_j)(\mathbb{1}_{X(\overline{a})(x)}))_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}\Phi^{(j)}(a\alpha))_{d\in I(i,j)}.
\end{aligned}$$

Since $X(\overline{a})(\Phi^{(i)}(\alpha)) = \Phi^{(j)}(a\alpha)$ by the choice of $a\alpha$, we have

$$\Phi((a, {}_iy)_i\alpha) = \Phi({}_j(a\alpha)(a, {}_ix)).$$

Hence $\Phi(R'_3) = 0$, and finally $\Phi(R') = 0$.

By the claim above we see that Φ induces a functor $\overline{\Phi} : \Bbbk Q'/\langle R'\rangle \to \mathrm{Gr}(X)$. We prove that $\overline{\Phi}$ is an isomorphism. To this end, we first consider a basis of $(\Bbbk Q'/\langle R'\rangle)({}_ix, {}_jy)$ for each ${}_ix$, ${}_jy \in Q'_0$.

Claim 3. *For each $(g, h) \in R^{\#}(i, j)$ and $x \in Q^{(i)}$, $\overline{\pi(g, {}_ix)} = \overline{\pi(h, {}_ix)}$.*

Indeed, there exist some $(a, b) \in R(i', j')$, $c \in \mathbb{P}Q(i, i')$ and $d \in \mathbb{P}Q(j', j)$ such that

$$(g, h) = (dac, dbc).$$

Then

$$\begin{aligned}
\pi(g, {}_ix) - \pi(h, {}_ix) &= \pi(dac, {}_ix) - \pi(dbc, {}_ix) \\
&= \pi(d, {}_{j'}(acx))\pi(a, {}_{i'}(cx))\pi(c, {}_ix) - \pi(d, {}_{j'}(bcx))\pi(b, {}_{i'}(cx))\pi(c, {}_ix) \\
&= \pi(d, {}_{j'}(acx))(\pi(a, {}_{i'}(cx)) - \pi(b, {}_{i'}(cx)))\pi(c, {}_ix).
\end{aligned}$$

Therefore since $\pi(a, {}_{i'}(cx)) - \pi(b, {}_{i'}(cx)) \in R'$, we have $\pi(g, {}_ix) - \pi(h, {}_ix) \in R'$. Hence $\overline{\pi(g, {}_ix)} = \overline{\pi(h, {}_ix)}$.

For each $a : i \to j$ in I we define a functor $\tilde{X}(a) \colon \Bbbk Q^{(i)} \to \Bbbk Q^{(j)}$ as follows:

- For each $x \in Q_0^{(i)}$, $\tilde{X}(a)(x) := X(\overline{a})(x)$.
- For each arrow $\alpha : x \to y$ in $Q^{(i)}$, $\tilde{X}(a)(\alpha) := a\alpha$.
- For each path $\mu := \alpha_n \dots \alpha_1$ $(n \geq 2)$ in $Q^{(i)}$, $\tilde{X}(a)(\mu) := \tilde{X}(a)(\alpha_n) \dots \tilde{X}(a)(\alpha_1)$.

Claim 4. *For each ${}_ix, {}_jy \in Q'_0$ and $\mu \in \mathbb{P}Q'({}_ix, {}_jy)$, there exist some $a \in I(i,j)$ and $\nu \in \Bbbk Q^{(j)}({}_j(ax), {}_jy)$ such that $\overline{\mu} = \overline{\nu\pi(a, {}_ix)}$.*

Indeed, since $(b, {}_kv){}_k\alpha - {}_l(b\alpha)(b, {}_ku) \in R'$ for each $b : k \to l$ in Q_1 and $\alpha : u \to v$ in $Q_1^{(k)}$, we have

$$\overline{(b, {}_kv){}_k\alpha} = \overline{{}_l(b\alpha)(b, {}_ku)},$$

which implies

$$\overline{(b, {}_kv)\sigma^{(k)}(\lambda)} = \overline{\sigma^{(l)}\tilde{X}(b)(\lambda)(b, {}_ku)}$$

for each $\lambda \in \Bbbk Q^{(k)}({}_ku, {}_kv)$. By using this formula in the path μ we can move factors of the form $\overline{(b, {}_kv)}$ to the right, and finally we have

$$\overline{\mu} = \overline{\nu(a_t, x_t) \cdots (a_1, x_1)}$$

for some $0 \leq t \in \mathbb{Z}$, $\nu \in \Bbbk Q^{(j)}$, $x_1, \cdots, x_t \in Q'_0$, $a_1, \cdots, a_t \in Q_1$, where $(a_t, x_t) \cdots (a_1, x_1)$ is a path of length t in Q', and hence we have $(a_t, x_t) \cdots (a_1, x_1) = \pi(a, x_1)$ $(a := a_t \cdots a_1)$. Hence we have $\nu \in \Bbbk Q^{(j)}({}_j(ax), {}_jy)$ and $\overline{\mu} = \overline{\nu\pi(a, {}_ix)}$.

Claim 5. $\mathcal{M} := \{\overline{\alpha\pi(a, {}_ix)} | a \in I(i,j), \alpha \in \mathcal{M}_j(ax, y)\}$ *is a basis of* $(\Bbbk Q'/\langle R' \rangle)({}_ix, {}_jy)$*, where* $\mathcal{M}_j(ax, y)$ *is a basis of* $(\Bbbk Q^{(j)}/\langle R^{(j)} \rangle)(ax, y)$.

Indeed, assume $\displaystyle\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}\overline{\alpha\pi(a,{}_ix)}=0$. Then

$$
\begin{aligned}
&\overline{\Phi}\left(\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}\overline{\alpha\pi(a,{}_ix)}\right)\\
&=\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}\overline{\Phi}(\overline{\alpha})\overline{\Phi}(\overline{\pi(a,{}_ix)})\\
&=\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\delta_{1_j,c}\Phi^{(j)}(\alpha))_{c\in I(j,j)}(\delta_{a,b}\mathbb{1}_{X(a)(x)})_{b\in I(i,j)}\\
&=\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\sum_{\substack{d=cb\\ b\in I(i,j)\\ c\in I(j,j)}}\delta_{1_j,c}\Phi^{(j)}(\alpha)X(c)(\delta_{a,b}\mathbb{1}_{X(a)(x)}))_{d\in I(i,j)}\\
&=\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\delta_{a,d}\Phi^{(j)}(\alpha)X(1_j)(\mathbb{1}_{X(a)(x)}))_{d\in I(i,j)}\\
&=\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\delta_{a,d}\Phi^{(j)}(\alpha))_{d\in I(i,j)}\\
&=\left(\Phi^{(j)}\left(\sum_{\alpha\in\mathbb{P}Q^{(j)}(ax,y)} k_{d,\alpha}\alpha\right)\right)_{d\in I(i,j)}\\
&=0
\end{aligned}
$$

Since $\alpha\in\mathcal{M}_j(ax,y)$, we have $k_{d,\alpha}=0$. Therefore $\mathcal{M}$ is a basis of $(\Bbbk Q'/\langle R'\rangle)({}_ix,{}_jy)$.

Here we define $\sigma_a\colon X(j)(X(a)(x),y)\hookrightarrow\displaystyle\bigoplus_{a\in I(i,j)}X(j)(X(a)(x),y)$ by $\mu\mapsto(\delta_{b,a}\mu)_{b\in I(i,j)}$ for each $\mu\in X(j)(X(a)(x),y)$. Then a basis of $\mathrm{Gr}(X)((i,x),(j,y))$ is written by $\displaystyle\bigcup_{a\in I(i,j)}\sigma_a(\Phi^{(j)}(\mathcal{M}_j(ax,y)))$, and for each $\overline{\alpha\pi(a,{}_ix)}\in\mathcal{M}$ we have

$$
\begin{aligned}
\overline{\Phi}(\overline{\alpha\pi(a,{}_ix)}) &= (\delta_{a,d}\Phi^{(j)}(\alpha))_{d\in I(i,j)}\\
&= \sigma_a\Phi^{(j)}(\alpha).
\end{aligned}
$$

Hence $\overline{\Phi}$ induces an isomorphism $(\Bbbk Q'/\langle R'\rangle)({}_ix,{}_jy)\xrightarrow{\sim}\mathrm{Gr}(X)((i,x),(j,y))$.

Therefore $\overline{\Phi}$ is an isomorphism. $\square$

Remark 3.2. The description of the proof of Claim 5 in the proof of Theorem 8.1 in [4] is not complete. This corresponds to Claim 4 above, and the formula (8.4) in [4] should be replaced by a linear combination

$$\overline{\eta} = \sum t_{y,\alpha_s,\dots} \overline{e_y \alpha_s \dots \alpha_1 (g_t, x_t) \dots (g_1, x_1)}$$

with $t_{y,\alpha_s,\dots} \in \Bbbk$. Correspondingly, we must remove "$\overline{\eta} =$" in the last formula in Claim 5 there. The earlier version arXiv:0807.4706v6 of the paper records the correct proof.

4. Examples

In this section, we illustrate Theorems 2.4 and 3.1 by some examples.

Example 4.1. Let Q be the quiver

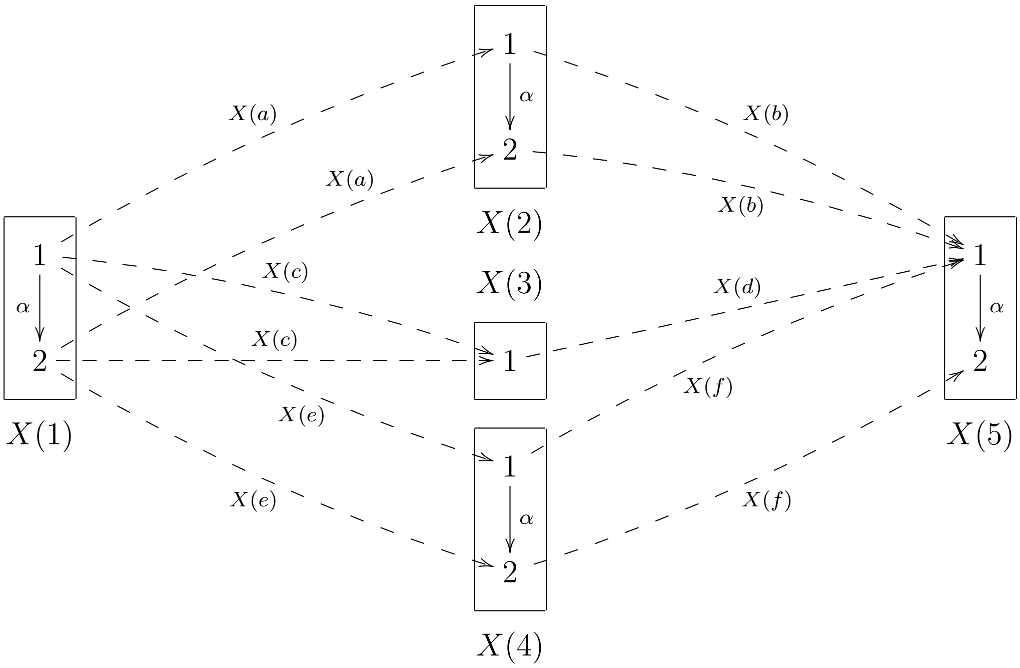
$$\begin{tikzcd}
 & 2 \arrow[dr, "b"] & \\
1 \arrow[ur, "a"] \arrow[r, "c"] \arrow[dr, "e"'] & 3 \arrow[r, "d"] & 5 \\
 & 4 \arrow[ur, "f"'] &
\end{tikzcd}$$

and let $R = \{(ba, dc)\}$. Then the category $I := \langle Q \mid R \rangle$ is not given as a semigroup, as a poset or as the free category of a quiver. For any algebra A consider the diagonal functor $\Delta(A)\colon I \to \Bbbk$-**Cat**. Then by Theorem 2.4 the category $\mathrm{Gr}(\Delta(A))$ is given by

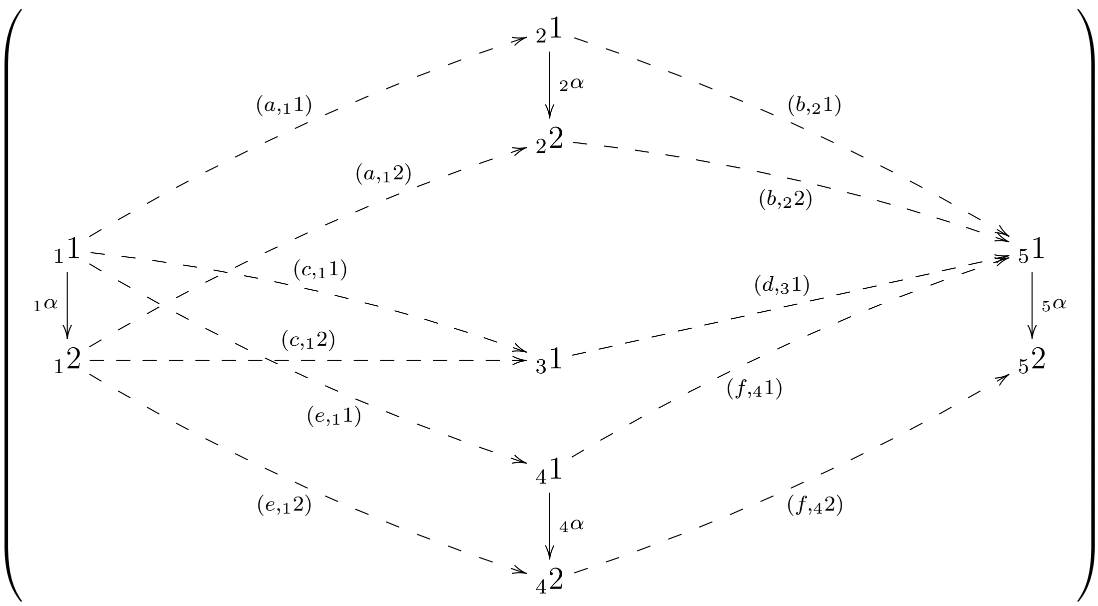
$$AQ/\langle ba - dc \rangle.$$

Remark 4.2. Let Q and Q' be quivers having neither double arrows nor loops, and let $f\colon Q_0 \to Q'_0$ be a map (a *vertex map* between Q and Q'). If $Q(x, y) \neq \emptyset$ ($x, y \in Q_0$) implies $Q'(f(x), f(y)) \neq \emptyset$ or $f(x) = f(y)$, then f induces a $\Bbbk$-functor $\hat{f}\colon \Bbbk P \to \Bbbk P'$ defined by the following correspondence: For each $x \in Q_0$, $\hat{f}(e_x) := e_{f(x)}$, and for each arrow $a\colon x \to y$ in Q, $f(a)$ is the unique arrow $f(x) \to f(y)$ (resp. $e_{f(x)}$) if $f(x) \neq f(y)$ (resp. if $f(x) = f(y)$).

Example 4.3. Let $I = \langle Q \mid R \rangle$ be as in the previous example. Define a functor $X : I \to \Bbbk$-**Cat** by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by broken arrows between them:



Then by Theorem 3.1 $\mathrm{Gr}(X)$ is presented by the quiver

$Q' =$ 

with relations

$$\begin{aligned} R' &= \{\pi(ba,{}_11) - \pi(dc,{}_11), \pi(ba,{}_12) - \pi(dc,{}_12)\} \\ &\quad \cup \{(a,{}_iy)_i\alpha - {}_j(a\alpha)(a,{}_ix) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\}, \end{aligned}$$

where the new arrows are presented by broken arrows.

Example 4.4 (Semigroup case)**.** Define a category $I = \langle Q \mid R \rangle$ by setting

$$Q = \left(1 \circlearrowright g \right), \quad R = \{(g^2, g^3)\}.$$

Then I can be regarded as a semigroup with the presentation $\langle g \mid g^2 = g^3\rangle$. We define a functor $X : G \to \Bbbk\text{-}\mathbf{Cat}$ as follows. Let $Q^{(1)}$ be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \,.$$

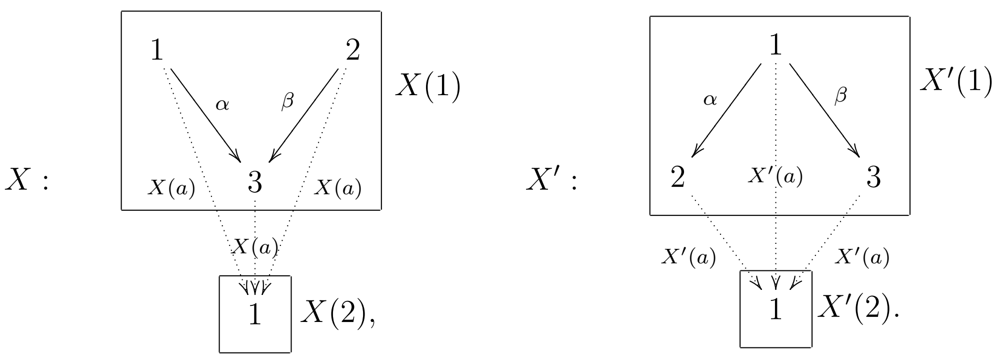
and set $X(1) := \Bbbk Q^{(1)}$, and define an endofunctor $X(g)$ of $X(1)$ as the $\Bbbk$-functor induced by the vertex map $X(g)(1) = 2, X(g)(2) = 3, X(g)(3) = 3$. Then by Theorem 3.1 $\mathrm{Gr}(X)$ is presented by the quiver

$$Q' = \left(\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, dotted, "(g,1)"'] & 2 \arrow[r, shift left, "\beta"] \arrow[r, shift right, dotted, "(g,2)"'] & 3 \arrow[loop right, dotted, "(g,3)"] \end{tikzcd} \right)$$

with relations

$$\begin{aligned} R' = \{&(g,3)(g,2)(g,1) - (g,2)(g,1), (g,3)(g,3)(g,2) - (g,3)(g,2),\\ &(g,3)(g,3)(g,3) - (g,3)(g,3), (g,2)\alpha - \beta(g,1), (g,3)\beta - (g,2)\}. \end{aligned}$$

Example 4.5. Let $Q = (\,1 \xrightarrow{a} 2\,)$ and $I := \mathbb{P}Q$. Define functors $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by dotted arrows between them:



Then by Theorem 3.1 $\mathrm{Gr}(X)$ is given by the following quiver with no relations

$$\left(\begin{tikzcd} {}_1 1 \arrow[dr, "{}_1\alpha"] \arrow[ddr, dotted, "{(a,{}_1 1)}"'] & & {}_1 2 \arrow[dl, "{}_1\beta"'] \arrow[ddl, dotted, "{(a,{}_1 2)}"] \\ & {}_1 3 \arrow[d, dotted, "{(a,{}_1 3)}"'] & \\ & {}_2 1 & \end{tikzcd} , \left\{ \begin{matrix} (a, {}_1 3)_1\alpha - (a, {}_1 1), \\ (a, {}_1 3)_1\beta - (a, {}_1 2) \end{matrix} \right\} \right) \cong \left(\begin{tikzcd} {}_1 1 \arrow[dr, "{}_1\alpha"] & & {}_1 2 \arrow[dl, "{}_1\beta"'] \\ & {}_1 3 \arrow[d, "{(a,{}_1 3)}"'] & \\ & {}_2 1 & \end{tikzcd} \right),$$

and $\mathrm{Gr}(X')$ is given by the following quiver with a commutativity relation

$$
\left(\begin{tikzcd}
 & {}_1 1 \arrow[dl, "{}_1\alpha"'] \arrow[dr, "{}_1\beta"] \arrow[dd, dotted, "{(a,{}_1 1)}"'] & \\
{}_1 2 \arrow[dr, dotted, "{(a,{}_1 2)}"'] & & {}_1 3 \arrow[dl, dotted, "{(a,{}_1 3)}"] \\
 & {}_2 1 &
\end{tikzcd}, \left\{ \begin{matrix} (a, {}_1 2)_1\alpha - (a, {}_1 1), \\ (a, {}_1 3)_1\beta - (a, {}_1 1) \end{matrix} \right\} \right) \cong \left(\begin{tikzcd}
 & {}_1 1 \arrow[dl, "{}_1\alpha"'] \arrow[dr, "{}_1\beta"] & \\
{}_1 2 \arrow[dr, dotted, "{(a,{}_1 2)}"'] \arrow[rr, phantom, "\circlearrowright"] & & {}_1 3 \arrow[dl, dotted, "{(a,{}_1 3)}"] \\
 & {}_2 1 &
\end{tikzcd} \right).
$$

By using the main theorem in [6] derived equivalences between $X(1)$ and $X'(1)$ and between $X(2)$ and $X'(2)$ are glued together to have a derived equivalence between $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.

Chapter II.

Derived equivalence classification of generalized multifold extensions of piecewise hereditary algebras of tree type

After preparations in section 1 we first reduce the problem to the case of hereditary tree algebras in section 2. Then we investigate scalar multiples in the repetitive category of a hereditary tree algebras in section 3, which is a central part of the proof of the main result. In section 4 we show that any generalized multifold extension of a piecewise hereditary algebra of tree type is derived equivalent to a twisted multifold extension of the same type, which immediately yields the desired classification result.

1. Preliminaries

For a category R we denote by R_0 and R_1 the class of objects and morphisms of R, respectively. A category R is said to be *locally bounded* if it satisfies the following:

- Distinct objects of R are not isomorphic;
- $R(x, x)$ is a local algebra for all $x \in R_0$;
- $R(x, y)$ is finite-dimensional for all $x, y \in R_0$; and
- The set $\{y \in R_0 \mid R(x, y) \neq 0 \text{ or } R(y, x) \neq 0\}$ is finite for all $x \in R_0$.

A category is called *finite* if it has only a finite number of objects.

A pair (A, E) of an algebra A and a complete set $E := \{e_1, \dots, e_n\}$ of orthogonal primitive idempotents of A can be identified with a locally bounded and finite category R by the following correspondences. Such a pair (A, E) defines a category $R_{(A,E)} := R$ as follows: $R_0 := E$, $R(x, y) := yAx$ for all $x, y \in E$, and the composition of R is defined by the multiplication of A. Then the category R is locally bounded and finite. Conversely, a locally bounded and finite category R defines such a pair (A_R, E_R) as follows: $A_R := \bigoplus_{x,y \in R_0} R(x, y)$ with the usual matrix multiplication (regard each element of A as a matrix indexed by R_0), and $E_R := \{(\mathbb{1}_x \delta_{(i,j),(x,x)})_{i,j \in R_0} \mid x \in R_0\}$. We always regard an algebra A as a locally bounded and finite category by fixing a complete set A_0 of orthogonal primitive idempotents of A.

For a locally bounded category A, we denote by $\operatorname{Mod} A$ the category of all (right) A-modules (= contravariant functors from A to the category $\operatorname{Mod} k$ of k-vector spaces); by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely presented objects; and by $\operatorname{prj} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely generated projective objects. $\mathcal{K}^{\mathrm{b}}(\mathcal{A})$ denotes the bounded homotopy category of an additive category $\mathcal{A}$.

2. Repetitive categories

Definition 2.1. Let A be a locally bounded category.

(1) The *repetitive category* $\hat{A}$ of A is a $\Bbbk$-category defined as follows ($\hat{A}$ turns out to be locally bounded again):

- $\hat{A}_0 := A_0 \times \mathbb{Z} = \{x^{[i]} := (x, i) \mid x \in A_0, i \in \mathbb{Z}\}$.
- $\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} \mid f \in A(x, y)\} & \text{if } j = i, \\ \{\phi^{[i]} \mid \phi \in DA(y, x)\} & \text{if } j = i + 1, \\ 0 & \text{otherwise,} \end{cases}$ for all $x^{[i]}, y^{[j]} \in \hat{A}_0$.
- For each $x^{[i]}, y^{[j]}, z^{[k]} \in \hat{A}_0$ the composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is given as follows.
  - (i) If $i = j, j = k$, then this is the composition of A $A(y, z) \times A(x, y) \to A(x, z)$.
  - (ii) If $i = j, j + 1 = k$, then this is given by the right A-module structure of DA: $DA(z, y) \times A(x, y) \to DA(z, x)$.
  - (iii) If $i + 1 = j, j = k$, then this is given by the left A-module structure of DA: $A(y, z) \times DA(y, x) \to DA(z, x)$.
  - (iv) Otherwise, the composition is zero .

(2) We define an automorphism ν_A of $\hat{A}$, called the *Nakayama automorphism* of $\hat{A}$, by $\nu_A(x^{[i]}) := x^{[i+1]}$, $\nu_A(f^{[i]}) := f^{[i+1]}$, $\nu_A(\phi^{[i]}) := \phi^{[i+1]}$ for all $i \in \mathbb{Z}, x \in A_0, f \in A_1, \phi \in \bigcup_{x,y \in A_0} DA(y, x)$.

(3) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $\hat{A}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow \hat{A}, x \mapsto x^{[n]}$, the embedding functor.

We cite the following from [3, Lemma 2.3].

Lemma 2.2. *Let $\psi\colon A \to B$ be an isomorphism of locally bounded categories. Denote by $\psi^y_x\colon A(y, x) \to B(\psi y, \psi x)$ the isomorphism defined by ψ for all $x, y \in A$. Define $\hat{\psi}\colon \hat{A} \to \hat{B}$ as follows.*

- *For each $x^{[i]} \in \hat{A}, \hat{\psi}(x^{[i]}) := (\psi x)^{[i]}$;*
- *For each $f^{[i]} \in \hat{A}(x^{[i]}, y^{[i]}), \hat{\psi}(f^{[i]}) := (\psi f)^{[i]}$; and*
- *For each $\phi^{[i]} \in \hat{A}(x^{[i]}, y^{[i+1]}), \hat{\psi}(\phi^{[i]}) := (D((\psi^y_x)^{-1})(\phi))^{[i]} = (\phi \circ (\psi^y_x)^{-1})^{[i]}$.*

Then

1. *$\hat{\psi}$ is an isomorphism.*
2. *Given an isomorphism $\rho\colon \hat{A} \to \hat{B}$, the following are equivalent.*
   - (a) *$\rho = \hat{\psi}$;*
   - (b) *ρ satisfies the following.*
     - (i) *$\rho\nu_A = \nu_B\rho$;*
     - (ii) *$\rho(A^{[0]}) = A^{[0]}$;*

(iii) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \mathbf{1}^{[0]}}\downarrow & & \downarrow{\scriptstyle \mathbf{1}^{[0]}} \\ A^{[0]} & \xrightarrow[\rho]{} & B^{[0]} \end{array}$$

is commutative; and

(iv) $\rho(\phi^{[0]}) = (\phi \circ (\psi_x^y)^{-1})^{[0]}$ *for all* $x, y \in A$ *and all* $\phi \in DA(y, x)$.

Let R be a locally bounded category with the Jacobson radical J and with the ordinary quiver Q. Then by definition of Q there is a bijection $f\colon Q_0 \to R_0, x \mapsto f_x$ and injections $\bar{a}_{y,x}\colon Q_1(x, y) \to J(f_x, f_y)/J^2(f_x, f_y)$ such that $\bar{a}_{y,x}(Q_1(x, y))$ forms a basis of $J(f_x, f_y)/J^2(f_x, f_y)$, where $Q_1(x, y)$ is the set of arrows from x to y in Q for all $x, y \in Q_0$. For each $\alpha \in Q_1(x, y)$ choose $a_{y,x}(\alpha) \in J(f_x, f_y)$ such that $a(\alpha) + J^2(f_x, f_y) = \bar{a}_{y,x}(\alpha)$. Then the pair (f, a) of the bijection f and the family a of injections $a_{y,x}\colon Q_1(x, y) \to J(f_x, f_y)$ $(x, y \in Q_0)$ uniquely extends to a full functor $\Phi\colon \Bbbk Q \to R$, which is called a *display functor* for R.

A path μ from y to x in a quiver with relations (Q, I) is called *maximal* if $\mu \notin I$ but $\alpha\mu, \mu\beta \in I$ for all arrows $\alpha, \beta \in Q_1$. For a k-vector space V with a basis $\{v_1, \dots, v_n\}$ we denote by $\{v_1^*, \dots, v_n^*\}$ the basis of DV dual to the basis $\{v_1, \dots, v_n\}$. In particular if $\dim_k V = 1$, $v^* \in DV$ is defined for all $v \in V \backslash \{0\}$.

An algebra is called a *tree algebra* if its ordinary quiver is an oriented tree.

Lemma 2.3. *Let A be a tree algebra and $\Phi : \Bbbk Q \to A$ a display functor with $I := \mathrm{Ker}\,\Phi$. Then*

(1) Φ *uniquely induces the display functor* $\hat{\Phi} : \Bbbk\hat{Q} \to \hat{A}$ *for* $\hat{A}$, *where*

(i) $\hat{Q} = (\hat{Q}_0, \hat{Q}_1, \hat{s}, \hat{t})$ *is defined as follows:*

- $\hat{Q}_0 := Q_0 \times \mathbb{Z} = \{x^{[i]} := (x, i) \mid x \in Q_0, i \in \mathbb{Z}\}$,
- $Q_1 \times \mathbb{Z} := \{\alpha^{[i]} := (\alpha, i) \mid \alpha \in Q_1, i \in \mathbb{Z}\}$,
  $\hat{Q}_1 := (Q_1 \times \mathbb{Z}) \sqcup \{\mu^{*[i]} \mid \mu$ *is a maximal path in* $(Q, I), i \in \mathbb{Z}\}$,
- $\hat{s}(\alpha^{[i]}) := s(\alpha)^{[i]}$, $\hat{t}(\alpha^{[i]}) := t(\alpha)^{[i]}$ *for all* $\alpha^{[i]} \in Q_1 \times \mathbb{Z}$, *and if* μ *is a maximal path from* y *to* x *in* (Q, I) *then,* $\hat{s}(\mu^{*[i]}) := x^{[i]}$, $\hat{t}(\mu^{*[i]}) := y^{[i+1]}$.

(ii) $\hat{\Phi}$ *is defined by* $\hat{\Phi}(x^{[i]}) := (\Phi x)^{[i]}$, $\hat{\Phi}(\alpha^{[i]}) := (\Phi\alpha)^{[i]}$, *and* $\hat{\Phi}(\mu^{*[i]}) := (\Phi(\mu)^*)^{[i]}$ *for all* $i \in \mathbb{Z}$, $x \in Q_0$, $\alpha \in Q_1$ *and maximal paths* μ *in* (Q, I).

(2) *We define an automorphism* ν_Q *of* $\hat{Q}$ *by* $\nu_Q(x^{[i]}) := x^{[i+1]}$, $\nu_Q(\alpha^{[i]}) := \alpha^{[i+1]}$, $\nu_Q(\mu^{*[i]}) := \mu^{*[i+1]}$ *for all* $i \in \mathbb{Z}, x \in Q_0, \alpha \in Q_1$, *and maximal paths* μ *in* (Q, I).

(3) $\operatorname{Ker}\hat{\Phi}$ *is equal to the ideal* $\hat{I}$ *defined by the full commutativity relations on* $\hat{Q}$ *and the zero relations* $\mu = 0$ *for those paths* μ *of* $\hat{Q}$ *for which there is no path* $\hat{t}(\mu) \rightsquigarrow \nu_Q(\hat{s}(\mu))$. *(Therefore note that if a path* $\alpha_n \cdots \alpha_1$ *is in* I, *then* $\alpha_n^{[i]} \cdots \alpha_1^{[i]}$ *is in* $\hat{I}$ *for all* $i \in \mathbb{Z}$.*)*

Let R be a locally bounded category. A morphism $f : x \to y$ in R_1 is called a *maximal nonzero morphism* if $f \neq 0$ and $fg = 0, hf = 0$ for all $g \in \operatorname{rad} R(z, x), h \in \operatorname{rad} R(y, z), z \in R_0$.

Lemma 2.4. *Let* A *be an algebra and* $x^{[i]}, y^{[j]} \in \hat{A}_0$. *Then there exists a maximal nonzero morphism in* $\hat{A}(x^{[i]}, y^{[j]})$ *if and only if* $y^{[j]} = \nu_A(x^{[i]})$.

Proof. This follows from the fact that $\hat{A}(\text{-}, x^{[i+1]}) \cong D\hat{A}(x^{[i]}, \text{-})$ for all $i \in \mathbb{Z}, x \in A_0$. $\square$

Lemma 2.5. *Let* A *be an algebra. Then the actions of* $\phi\nu_A$ *and* $\nu_A\phi$ *coincide on the objects of* $\hat{A}$ *for all* $\phi \in \operatorname{Aut}(\hat{A})$.

Proof. Let $x^{[i]} \in \hat{A}_0$. Then there is a maximal nonzero morphism in $\hat{A}(x^{[i]}, \nu_A(x^{[i]}))$ by Lemma 2.4. Since ϕ is an automorphism of $\hat{A}$, there is a maximal nonzero morphism in $\hat{A}(\phi(x^{[i]}), \phi(\nu_A(x^{[i]})))$. Hence $\phi(\nu_A(x^{[i]})) = \nu_A(\phi(x^{[i]}))$ by the same lemma. $\square$

The following is immediate by the lemma above.

Proposition 2.6. *Let* A *be an algebra,* n *an integer, and* ϕ *an automorphism of* $\hat{A}$. *Then the following are equivalent:*

(1) ϕ *is an automorphism with jump* n;
(2) $\phi(A^i) = A^{[i+n]}$ *for some integer* i;
(3) $\phi(A^j) = A^{[j+n]}$ *for all integers* j; *and*
(4) $\phi = \sigma\nu_A^n$ *for some automorphism* σ *of* $\hat{A}$ *with jump* 0.

Remark 2.7. Let A be an algebra.

(1) In Skowroński [17, 18] an automorphism ϕ of $\hat{A}$ is called *rigid* if $\phi(A^{[j]}) = A^{[j]}$ for all $j \in \mathbb{Z}$. Hence ϕ is rigid if and only if it is an automorphism with jump 0 by the proposition above. Therefore for an integer n, ϕ is an automorphism with jump n if and only if $\phi = \sigma\nu_A^n$ for some rigid automorphism σ of $\hat{A}$.
(2) Noting this fact we see by [18, Theorem 4.7] that the class of self-injective algebras of Euclidean type contains a lot of generalized multifold extensions of piecewise hereditary algebras of tree type.

In the sequel, we always assume that n is a positive integer when we consider a morphism with jump n.

3. Derived equivalences and tilting subcategories

Let R be a locally bounded category and $\phi \in \mathrm{Aut}(R)$. Then ϕ induces an equivalence ${}^{\phi}(\text{-}) : \mathrm{mod}\, R \to \mathrm{mod}\, R$ defined by ${}^{\phi}M := M \circ \phi^{-1} : R \to \mathrm{mod}\, k$ for all $M \in \mathrm{mod}\, R$. In particular for $R(\text{-}, x)$ with $x \in R$, we have ${}^{\phi}(R(\text{-}, x)) = R(\phi^{-1}(\text{-}), x) \cong R(\text{-}, \phi x)$, and the last isomorphism is given by ϕ itself. Thus the identification ${}^{\phi}(R(\text{-}, x)) = R(\text{-}, \phi x)$ depends on ϕ, and the subset $\{R(\text{-}, x) \mid x \in R\}$ of $\mathrm{prj}\, R$ is not $\langle {}^{\phi}(\text{-}) \rangle$-stable in a strict sense. This makes it difficult to give explicitly a complete set of representatives of isoclasses of indecomposable objects of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, R)$ which is $\langle \mathcal{K}^{\mathrm{b}}({}^{\phi}(\text{-})) \rangle$-stable. To avoid this difficulty we used in [2] the formal additive hull $\mathrm{add}\, R$ ([9, 2.1 Example 8]) of R defined below instead of $\mathrm{prj}\, R$.

Definition 3.1. Let R be a locally bounded category. The *formal additive hull* $\mathrm{add}\, R$ of R is a category defined as follows.

- $(\mathrm{add}\, R)_0 := \{\bigoplus_{i=1}^{n} x_i := (x_1, \dots, x_n) \mid n \in \mathbb{N},\ x_1, \dots, x_n \in R_0\}$;
- For each $x = \bigoplus_{i=1}^{m} x_i, y = \bigoplus_{j=1}^{m} y_i \in (\mathrm{add}\, R)_0$,

$(\mathrm{add}\, R)(x, y) := \{(\mu_{j,i})_{j,i} \mid \mu_{j,i} \in R(x_i, y_j)$ for all $i = 1, \dots, m, j = 1, \dots, n\}$; and

- The composition is given by the matrix multiplication.

We regard that R is contained in $\mathrm{add}\, R$ by the embedding $(f\colon x \to y) \mapsto ((f)\colon (x) \to (y))$ for all f in R.

Remark 3.2. Let R and ϕ be as above.

(1) Define a functor $\eta_R\colon\ \mathrm{add}\, R \to \mathrm{prj}\, R$ by $(x_1, \dots, x_n) \mapsto R(\text{-}, x_1) \oplus \cdots \oplus R(\text{-}, x_n)$ and $(\mu_{ji})_{j,i} \mapsto (R(\text{-}, \mu_{ji}))_{j,i}$. Then η_R is an equivalence, called the *Yoneda* equivalence.

(2) Let $F\colon R \to S$ be a functor of locally bounded categories. Then F naturally induces functors $\mathrm{add}\, F\colon\ \mathrm{add}\, R \to \mathrm{add}\, S$ and $\tilde{F} := \mathcal{K}^{\mathrm{b}}(\mathrm{add}\, F)\colon \mathcal{K}^{\mathrm{b}}(\mathrm{add}\, R) \to \mathcal{K}^{\mathrm{b}}(\mathrm{add}\, S)$, which are isomorphisms if F is an isomorphism. Namely, $\mathrm{add}\, F$ is defined by $(x_1, \dots, x_n) \mapsto (Fx_1, \dots, Fx_n)$ and $(\mu_{ji}) \mapsto (F\mu_{ji})$ for all objects $(x_1, \dots, x_n)$ and all morphisms (μ_{ji}) in $\mathrm{add}\, R$; and $\tilde{F}$ is defined by $\mathrm{add}\, F$ componentwise. Further if $G\colon S \to T$ is a functor of locally bounded categories, then we have $(GF)\tilde{} = \tilde{G}\tilde{F}$.

(3) The automorphism ϕ acts on $\mathcal{K}^{\mathrm{b}}(\mathrm{add}\, R)$ as $\tilde{\phi}$, and ${}^{\phi}\mathcal{K}^{\mathrm{b}}(\eta_R)(X^{\cdot}) \cong \mathcal{K}^{\mathrm{b}}(\eta_R)(\tilde{\phi}(X^{\cdot}))$ for all $X^{\cdot} \in \mathcal{K}^{\mathrm{b}}(\mathrm{add}\, R)$.

We cite the following from [2, Proposition 5.1] which follows from Keller [11] (Cf. Rickard [15], [1, Proposition 1.1]).

Proposition 3.3. *Let R and S be locally bounded categories. Then the following are equivalent:*

(1) *There is a triangle equivalence $\mathcal{D}(\mathrm{Mod}\, S) \to \mathcal{D}(\mathrm{Mod}\, R)$; and*

(2) *There is a full subcategory E of $\mathcal{K}^{\mathrm{b}}(\mathrm{add}\, R)$ such that*

(a) $\mathcal{K}^{\mathrm{b}}(\operatorname{add} R)(T, U[n]) = 0$ *for all* $T, U \in E$ *and all* $n \neq 0$;
(b) *R is contained in the smallest full triangulated subcategory of* $\mathcal{K}^{\mathrm{b}}(\operatorname{add} R)$ *containing E that is closed under direct summands and isomorphisms; and*
(c) *E is isomorphic to S.*

Definition 3.4. We say that locally bounded categories R and S are *derived equivalent* if one of the equivalent conditions above holds. In (2) the triple (R, E, S) is called a *tilting triple* and $E \subseteq \mathcal{K}^{\mathrm{b}}(\operatorname{add} R)$ is called a *tilting subcategory* for R.

Theorem 1.5 in [1] is interpreted as follows.

Theorem 3.5. *If (A, E, B) is a tilting triple of locally bounded categories with an isomorphism $\psi\colon E \to B$, then $(\hat{A}, \hat{E}, \hat{B})$ is also a tilting triple with the isomorphism $\hat{\psi}\colon \hat{E} \to \hat{B}$, where $\hat{E}$ is isomorphic to (and identified with) the full subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{add} \hat{A})$ consisting of the $(\mathbb{1}^{[n]})\tilde{}\,(T)$ with $T \in E$, $n \in \mathbb{Z}$.*

For a group G acting on a category S we say that a subclass E of the objects of S is *G-stable* (resp. *G-stable up to isomorphisms*) if $gx \in E$ (resp. if gx is isomorphic to some object in E) for all $g \in G$ and $x \in E$.

Proposition 3.6. *Let (A, E, B) be a tilting triple of locally bounded categories with an isomorphism $\psi\colon E \to B$, g an automorphism of $\hat{A}$ and h an automorphism of $\hat{B}$. Then $\hat{A}/\langle g\rangle$ is derived equivalent to $\hat{B}/\langle h\rangle$ if*

(1) *g is of infinite order and $\langle g\rangle$ acts freely on $\hat{A}$;*
(2) *$\hat{E}$ is $\langle \tilde{g}\rangle$-stable; and*
(3) *The following diagram commutes:*

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\hat{\psi}} & \hat{B} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle h} \\ \hat{E} & \xrightarrow[\hat{\psi}]{} & \hat{B}. \end{array}$$

Remark 3.7. Let E be a tilting subcategory for a locally bounded category R and G a group acting on R. If E is G-stable up to isomorphisms, then there exists a tilting subcategory E' for R such that $E \cong E'$ and E' is G-stable (see [1, Remark 3.2] and [2, Lemma 5.3.3 and Remark 5.3(2)]).

4. Reduction to hereditary tree algebras

Let Q be a quiver. We denote by $\bar{Q}$ the underlying graph of Q, and call Q *finite* if both Q_0 and Q_1 are finite sets. Each automorphism of Q is regarded as an automorphism of $\bar{Q}$ preserving the orientation of Q, thus $\operatorname{Aut}(Q)$ can be regarded as a subgroup of $\operatorname{Aut}(\bar{Q})$. Suppose now that Q is a finite oriented tree. Then it is also known that $\mathrm{Aut}(Q) \leq \mathrm{Aut}_0(\bar{Q}) := \{f \in \mathrm{Aut}(\bar{Q}) \mid f(x) = x \text{ for some } x \in Q_0\}$. We say that Q is an *admissibly oriented* tree if $\mathrm{Aut}(Q) = \mathrm{Aut}_0(\bar{Q})$. We quote the following from [3, Lemma 4.1]:

Lemma 4.1. *For any finite tree T there exists an admissibly oriented tree Q with a unique source such that $\bar{Q} = T$.*

We cite the following from [3, Lemma 5.4].

Lemma 4.2. *Let A be a piecewise hereditary algebra of type Q for an admissibly oriented tree Q. Then there is a tilting triple (A, E, kQ) such that E is $\langle\tilde{\phi}\rangle$-stable up to isomorphisms for all $\phi \in \mathrm{Aut}(A)$.*

By the following proposition we can reduce the derived equivalence classification of generalized multifold extensions of *piecewise hereditary* algebras of tree type to the corresponding problem of generalized multifold extensions of *hereditary* tree algebras. The second statement also enables us to compare the generalized multifold extension and a twisted version corresponding to it using the repetitive category of the common hereditary algebra.

Proposition 4.3. *Let A be a piecewise hereditary algebra of tree type $\bar{Q}$ for an admissibly oriented tree Q, and n a positive integer. Then we have the following:*

(1) *For any $\phi \in \mathrm{Aut}(\hat{A})$ with jump n, there exists some $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi\rangle$; and*
(2) *If we set $\phi' := \nu_A^n\hat{\phi}_0 \in \mathrm{Aut}(\hat{A})$, where $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}\phi\mathbb{1}^{[0]}$, then there exists some $\psi' \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi'\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi'\rangle$, and that the actions of ψ and ψ' coincide on the objects of $\widehat{\Bbbk Q}$.*

Proof. (1) We set $\phi_i := (\mathbb{1}^{[i]})^{-1}\nu^{-n}\phi\mathbb{1}^{[i]} \in \mathrm{Aut}(A)$ for all $i \in \mathbb{Z}$. By Lemma 4.2, there exists a tilting triple $(A, E, \Bbbk Q)$ with an isomorphism $\zeta\colon E \to \Bbbk Q$ such that E is $\langle\tilde{\eta}\rangle$-stable up to isomorphisms for all $\eta \in \mathrm{Aut}(A)$. In particular, E is $\langle\tilde{\phi}_i\rangle$-stable up to isomorphisms for all $i \in \mathbb{Z}$. Then $(\hat{A}, \hat{E}, \widehat{\Bbbk Q})$ is a tilting triple with the isomorphism $\hat{\zeta}$ by Theorem 3.5 and the following holds.

Claim 6. *$\hat{E}$ is $\langle\tilde{\phi}\rangle$-stable up to isomorphisms.*

Indeed, for each $T \in E_0$ and $i \in \mathbb{Z}$, we have

$$\begin{aligned} \tilde{\phi}(\mathbb{1}^{[i]})\tilde{}(T) &= (\nu^n\nu^{-n}\phi\mathbb{1}^{[i]})\tilde{}(T) \\ &= (\nu^n\mathbb{1}^{[i]}\phi_i)\tilde{}(T) \\ &= (\mathbb{1}^{[i+n]})\tilde{}\,\tilde{\phi}_i(T). \end{aligned} \tag{4.1}$$

Since E is $\langle\tilde{\phi}_i\rangle$-stable up to isomorphisms, we have $\tilde{\phi}_i(T) \cong T'$ for some $T' \in E$, and hence $\tilde{\phi}((\mathbb{1}^{[i]})\tilde{\ }(T)) \cong (\mathbb{1}^{[i+n]})\tilde{\ }(T') \in \hat{E}$, as desired.

By Remark 3.7, we have a $\langle\tilde{\phi}\rangle$-stable tilting subcategory $\hat{E}'$ and an isomorphism $\theta\colon \hat{E}' \xrightarrow{\sim} \hat{E}$. Therefore by Proposition 3.6 $\hat{A}/\langle\phi\rangle$ and $\hat{E}'/\langle\tilde{\phi}\rangle$ are derived equivalent. If we set $\psi := (\hat{\zeta}\theta)\tilde{\phi}(\hat{\zeta}\theta)^{-1}$, then (4.1) shows that ψ is an automorphism with jump n, and that $\hat{E}'/\langle\tilde{\phi}\rangle \cong \widehat{\Bbbk Q}/\langle\psi\rangle$. Hence $\hat{A}/\langle\phi\rangle$ and $\widehat{\Bbbk Q}/\langle\psi\rangle$ are derived equivalent.

(2) Note that ϕ' is also an automorphism with jump n. By the same argument we see that $\hat{E}$ is also $\langle\tilde{\phi}'\rangle$-stable up to isomorphisms; there exists a $\langle\tilde{\phi}'\rangle$-stable tilting subcategory $\hat{E}''$ and an isomorphism $\theta'\colon \hat{E}'' \xrightarrow{\sim} \hat{E}$; and $\hat{A}/\langle\phi'\rangle$ and $\hat{E}''/\langle\tilde{\phi}'\rangle$ are derived equivalent. Set $\psi' := (\hat{\zeta}\theta')\tilde{\phi}'(\hat{\zeta}\theta')^{-1}$, then ψ' is an automorphism with jump n, $\hat{E}''/\langle\tilde{\phi}'\rangle \cong \widehat{\Bbbk Q}/\langle\psi'\rangle$, and $\hat{A}/\langle\phi'\rangle$ and $\widehat{\Bbbk Q}/\langle\psi'\rangle$ are derived equivalent. Now for $i = 0$ the equality (4.1) shows that $\tilde{\phi}(\mathbb{1}^{[0]})\tilde{\ }(T) = (\mathbb{1}^{[n]})\tilde{\ }\tilde{\phi}_0(T)$ for all $T \in E_0$. Since $\phi'_0 = \phi_0$, the same calculation shows that $\tilde{\phi}'(\mathbb{1}^{[0]})\tilde{\ }(T) = (\mathbb{1}^{[n]})\tilde{\ }\tilde{\phi}_0(T)$ for all $T \in E_0$. Thus the actions of $\tilde{\phi}$ and $\tilde{\phi}'$ coincide on the objects of $E^{[0]}$, which shows that the actions of ψ and ψ' coincide on the objects of $\Bbbk Q^{[0]}$. Hence by Lemma 2.5 their actions coincide on the objects of $\widehat{\Bbbk Q}$. Indeed, $\psi(x^{[i]}) = \psi\nu^i(x^{[0]}) = \nu^i\psi(x^{[0]}) = \nu^i\psi'(x^{[0]}) = \psi'\nu^i(x^{[0]}) = \psi'(x^{[i]})$ for all $x \in Q_0$ and $i \in \mathbb{Z}$. $\square$

5. Hereditary tree algebras

Remark 5.1. Let Q be an oriented tree.

(1) We may identify $\widehat{\Bbbk Q} = \Bbbk\hat{Q}/\hat{I}$ as stated in Lemma 2.3, and we denote by $\overline{\mu}$ the morphism $\mu + \hat{I}$ in $\widehat{\Bbbk Q}$ for each morphism μ in $\Bbbk\hat{Q}$.

(2) Let $x, y \in \hat{Q}_0$. Since $\hat{I}$ contains full commutativity relations, we have $\dim_{\Bbbk} \widehat{\Bbbk Q}(x, y) \leq 1$, and in particular $\hat{Q}$ has no double arrows.

(3) Let $\alpha\colon x \to y$ be in $\hat{Q}_1$ and $\phi \in \operatorname{Aut}(\widehat{\Bbbk Q})$. Then there exists a unique arrow $\phi x \to \phi y$ in $\hat{Q}$, which we denote by $(\hat{\pi}\phi)(\alpha)$, and we have $\phi(\overline{\alpha}) = \phi_\alpha\overline{(\hat{\pi}\phi)(\alpha)} \in \widehat{\Bbbk Q}(\phi x, \phi y)$ for a unique $\phi_\alpha \in \Bbbk^\times := \Bbbk \setminus \{0\}$. This defines an automorphism $\hat{\pi}\phi$ of $\hat{Q}$, and thus a group homomorphism $\hat{\pi} : \operatorname{Aut}(\widehat{\Bbbk Q}) \to \operatorname{Aut}(\hat{Q})$.

(4) Similarly, let $\alpha\colon x \to y$ be in Q_1 and $\psi \in \operatorname{Aut}(\Bbbk Q)$. Then there exists a unique arrow $\psi x \to \psi y$ in Q, which we denote by $(\pi\psi)(\alpha)$. This defines an automorphism $\pi\psi$ of Q, and thus a group homomorphism $\pi : \operatorname{Aut}(\Bbbk Q) \to \operatorname{Aut}(Q)$.

We cite the following from [3, Proposition 7.4].

Proposition 5.2. *Let R be a locally bounded category, and g, h automorphisms of R acting freely on R. If there exists a map $\rho\colon R_0 \to \Bbbk^\times$ such that $\rho(y)g(f) = h(f)\rho(x)$ for all morphisms $f\colon x \to y$ in R, then $R/\langle g\rangle \cong R/\langle h\rangle$.* $\square$

Definition 5.3. (1) For a quiver $Q = (Q_0, Q_1, s, t)$ we set $Q[Q_1^{-1}]$ to be the quiver

$$Q[Q_1^{-1}] := (Q_0, Q_1 \sqcup \{\alpha^{-1} \mid \alpha \in Q_1\}, s', t'),$$

where $s'|_{Q_1} := s$, $t'|_{Q_1} := t$, $s'(\alpha^{-1}) := t(\alpha)$ and $t'(\alpha^{-1}) := s(\alpha)$ for all $\alpha \in Q_1$. A *walk* in Q is a path in $Q[Q_1^{-1}]$.

(2) Suppose that Q is a finite oriented tree. Then for each $x, y \in Q_0$ there exists a unique shortest walk from x to y in Q, which we denote by $w(x, y)$. If $w(x, y) = \alpha_n^{\varepsilon_n} \cdots \alpha_1^{\varepsilon_1}$ for some $\alpha_1, \cdots, \alpha_n \in Q_1$ and $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$, then we define a subquiver $W(x, y)$ of Q by $W(x, y) := (W(x, y)_0, W(x, y)_1, s', t')$, where $W(x, y)_0 := \{s(\alpha_i), t(\alpha_i) \mid i = 1, \dots, n\}$, $W(x, y)_1 := \{\alpha_1, \dots, \alpha_n\}$, and s', t' are restrictions of s, t to $W(x, y)_1$, respectively. Since Q is an oriented tree, $w(x, y)$ is uniquely recovered by $W(x, y)$. Therefore we can identify $w(x, y)$ with $W(x, y)$, and define a *sink* and a *source* of $w(x, y)$ as those in $W(x, y)$.

The following is a central part of the proof of the main result.

Proposition 5.4. *Let Q be a finite oriented tree and ϕ, ψ automorphisms of $\widehat{\Bbbk Q}$ acting freely on $\widehat{\Bbbk Q}$. If the actions of ϕ and ψ coincide on the objects of $\widehat{\Bbbk Q}$, then there exists a map $\rho\colon (\hat{Q}_0 =)\ \widehat{\Bbbk Q}_0 \to \Bbbk^\times$ such that $\rho(y)\psi(f) = \phi(f)\rho(x)$ for all morphisms $f\colon x \to y$ in $\widehat{\Bbbk Q}$. Hence in particular, $\widehat{\Bbbk Q}/\langle\phi\rangle$ is isomorphic to $\widehat{\Bbbk Q}/\langle\psi\rangle$.*

Proof. Assume that the actions of $\phi, \psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ coincides on the objects of $\widehat{\Bbbk Q}$. Then ϕ and ψ induce the same quiver automorphism $q = \hat{\pi}\phi = \hat{\pi}\psi$ of $\hat{Q}$, and there exist $(\phi_\alpha)_{\alpha \in \hat{Q}_1}, (\psi_\alpha)_{\alpha \in \hat{Q}_1} \in (k^\times)^{\hat{Q}_1}$ such that for each $\alpha \in \hat{Q}_1$ we have

$$\phi(\overline{\alpha}) = \phi_\alpha \overline{q(\alpha)}, \quad \psi(\overline{\alpha}) = \psi_\alpha \overline{q(\alpha)}.$$

For each path $\lambda = \alpha_n \cdots \alpha_1$ in $\hat{Q}$ with $\alpha_1, \dots, \alpha_n \in \hat{Q}_1$ we set $\phi_\lambda := \phi_{\alpha_n} \cdots \phi_{\alpha_1}$. Then we have

$$\phi(\overline{\lambda}) = \phi_\lambda \overline{q(\lambda)},$$

where $q(\lambda) := q(\alpha_n) \cdots q(\alpha_1)$ because $\phi(\overline{\alpha_n}) \cdots \phi(\overline{\alpha_1}) = \phi_{\alpha_n} \cdots \phi_{\alpha_1} \overline{q(\alpha_n) \cdots q(\alpha_1)}$.

To show the statement we may assume that $\psi_\alpha = 1$ for all $\alpha \in \hat{Q}_1$. Since for each $x, y \in \hat{Q}_0$ the morphism space $\widehat{\Bbbk Q}(x, y)$ is at most 1-dimensional and has a basis of the form $\overline{\mu}$ for some path μ, it is enough to show that there exists a map $\rho : \hat{Q}_0 \to \Bbbk^\times$ satisfying the following condition:

$$\rho(v^{[j]}) = \phi_\beta \rho(u^{[i]}) \quad \text{for all } \beta : u^{[i]} \to v^{[j]} \text{ in } \hat{Q}_1. \tag{5.1}$$

We define a map ρ as follows:
Fix a maximal path $\mu\colon y \rightsquigarrow x$ in Q. Then x is a sink and y is a source in Q. We can write μ as $\mu = \alpha_l \cdots \alpha_1$ for some $\alpha_1, \dots, \alpha_l \in Q_1$. First

we set $\rho(x^{[0]}) := 1$. By induction on $0 \le i \in \mathbb{Z}$ we define $\rho(x^{[i]})$ and $\rho(x^{[-i]})$ by the following formulas:

$$\rho(x^{[i+1]}) := \phi_{\mu^{[i+1]}}\phi_{\mu^{*[i]}}\rho(x^{[i]}), \tag{5.2}$$

$$\rho(x^{[i-1]}) := \phi^{-1}_{\mu^{*[i-1]}}\phi^{-1}_{\mu^{[i]}}\rho(x^{[i]}). \tag{5.3}$$

Now for each $i \in \mathbb{Z}$ and $u \in Q_0$ if $w(u,x) = \beta_m^{\varepsilon_m}\cdots\beta_1^{\varepsilon_1}$ for some $\beta_1,\ldots,\beta_m \in Q_1$ and $\varepsilon_1,\ldots,\varepsilon_m \in \{1,-1\}$, then we set

$$\rho(u^{[i]}) := \phi^{-\varepsilon_1}_{\beta_1^{[i]}}\cdots\phi^{-\varepsilon_m}_{\beta_m^{[i]}}\rho(x^{[i]}). \tag{5.4}$$

We have to verify the condition (5.1).

Case 1. $\beta = \alpha^{[i]}\colon u^{[i]} \to v^{[i]}$ for some $i \in \mathbb{Z}$, and $\alpha\colon u \to v$ in Q_1. Since Q is an oriented tree, we have either $w(u,x) = w(v,x)\alpha$ or $w(v,x) = w(u,x)\alpha^{-1}$. In either case we have $\rho(v^{[i]}) = \phi_{\alpha^{[i]}}\rho(u^{[i]})$ by the formula (5.4).

Case 2. Otherwise, we have $\beta = \lambda^{*[i]} : u^{[i]} \to v^{[i+1]}$ for some maximal path $\lambda\colon v \rightsquigarrow u$ in Q and $i \in \mathbb{Z}$. In this case the condition (5.1) has the following form:

$$\rho(v^{[i+1]}) = \phi_{\lambda^{*[i]}}\rho(u^{[i]}). \tag{5.5}$$

Two paths are said to be *parallel* if they have the same source and the same target. We prepare the following for the proof.

Claim 7. *If ζ and η are parallel paths in $\hat{Q}$, then we have $\phi_\zeta = \phi_\eta$.*

Indeed, since $\zeta - \eta \in \hat{I}$, we have $\phi(\overline{\zeta}) = \phi(\overline{\eta})$, which shows

$$\phi_\zeta\overline{q(\zeta)} = \phi_\eta\overline{q(\eta)}.$$

Here we have $\overline{q(\zeta)} = \psi(\overline{\zeta}) = \psi(\overline{\eta}) = \overline{q(\eta)}$, and $\psi(\overline{\zeta}) \ne 0$ because $\overline{\zeta} \ne 0$. Hence $\phi_\zeta = \phi_\eta$, as required.

We now set $d(a,b)$ to be the number of sinks in $w(a,b)$ for all $a,b \in Q_0$. By induction on $d(y,v)$ we show that the condition (5.5) holds. Note that both v and y (resp. u and x) are sources (resp. sinks) in Q.

Assume $d(y,v) = 0$. Then $y = v$ because these are sources in Q. By formulas (5.4) and (5.2) we have

$$\rho(v^{[i+1]}) = \rho(y^{[i+1]}) = \phi^{-1}_{\alpha_1^{[i+1]}}\cdots\phi^{-1}_{\alpha_l^{[i+1]}}\rho(x^{[i+1]}) = \phi_{\mu^{*[i]}}\rho(x^{[i]}).$$

If $u = x$, then $\lambda = \mu$ and hence $\phi_{\mu^{*[i]}}\rho(x^{[i]}) = \phi_{\lambda^{*[i]}}\rho(u^{[i]})$. Thus (5.5) holds.

If $u \ne x$, then $\phi_{\mu^{*[i]}}\phi_{\mu^{[i]}} = \phi_{\lambda^{*[i]}}\phi_{\lambda^{[i]}}$ by Claim 7. Since Q is an oriented tree, we have $w(u,x) = \mu\lambda^{-1}$, and $\rho(u^{[i]}) = \phi_{\lambda^{[i]}}\phi^{-1}_{\mu^{[i]}}\rho(x^{[i]})$. Therefore

$$\rho(v^{[i+1]}) = \phi_{\mu^{*[i]}}\rho(x^{[i]}) = \phi_{\lambda^{*[i]}}\phi_{\lambda^{[i]}}\phi^{-1}_{\mu^{[i]}}\rho(x^{[i]}) = \phi_{\lambda^{*[i]}}\rho(u^{[i]}),$$

and (5.5) holds.

Assume $d(y,v) \ge 1$. Then we can write $w(y,v) = \nu_1^{-1}\nu_2\cdots\nu_{m-1}^{-1}\nu_m$ for some paths $\nu_1,\ldots,\nu_m$ of length at least 1 and $m \ge 2$. Set $z_1 :=$

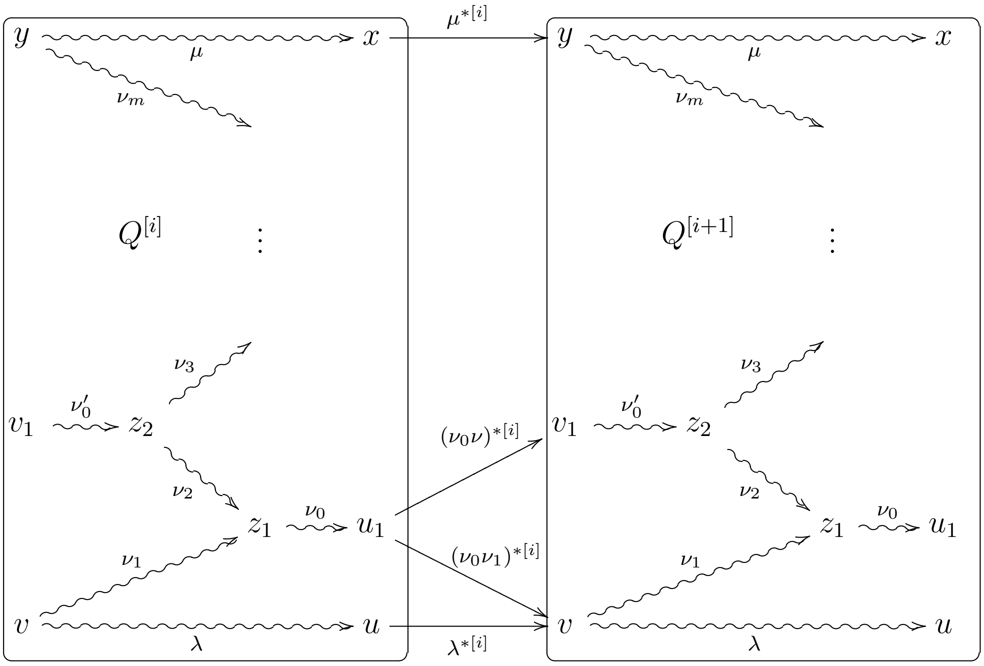


FIGURE 5.1

$t(\nu_2), z_2 := s(\nu_2)$. Then z_1 is a sink and z_2 is a source in $w(y, v)$. Since Q is a tree, there exists a unique maximal path of the form $\nu_0\nu_2\nu_0' \colon v_1 \rightsquigarrow u_1$ in Q for some paths ν_0, ν_0'. We set $\nu := \nu_2\nu_0'$. (See Figure 5.1, where we omitted the notations $[i], [i+1]$ for paths in $Q^{[i]}, Q^{[i+1]}$, respectively.) Since $d(v_1, y) = d(v, y) - 1$, we have

$$\rho(v_1^{[i+1]}) = \phi_{(\nu_0\nu)^{*[i]}}\rho(u_1^{[i]}) \tag{5.6}$$

by induction hypothesis. Since the paths $\nu^{[i+1]}(\nu_0\nu)^{*[i]}$ and $\nu_1^{[i+1]}(\nu_0\nu_1)^{*[i]}$ are parallel, we have

$$\phi_{\nu^{[i+1]}}\phi_{(\nu_0\nu)^{*[i]}} = \phi_{\nu_1^{[i+1]}}\phi_{(\nu_0\nu_1)^{*[i]}} \tag{5.7}$$

by Claim 1. Further by the result of Case 1 we have

$$\rho(v^{[i+1]}) = \phi^{-1}_{\nu_1^{[i+1]}}\phi_{\nu^{[i+1]}}\rho(v_1^{[i+1]}). \tag{5.8}$$

It follows from (5.6), (5.7) and (5.8) that

$$\rho(v^{[i+1]}) = \phi_{(\nu_0\nu_1)^{*[i]}}\rho(u_1^{[i]}).$$

(If $u_1 = u$, then $\nu_0\nu_1 = \lambda$ and this already gives (5.5).) Again by the result of Case 1 we have

$$\rho(u_1^{[i]}) = \phi_{(\nu_0\nu_1)^{[i]}}\phi^{-1}_{\lambda^{[i]}}\rho(u^{[i]}).$$

Since the paths $\lambda^{*[i]}\lambda^{[i]}$ and $(\nu_0\nu_1)^{*[i]}(\nu_0\nu_1)^{[i]}$ are parallel, we have

$$\phi_{\lambda^{*[i]}}\phi_{\lambda^{[i]}} = \phi(\nu_0\nu_1)^{*[i]}\phi(\nu_0\nu_1)^{[i]}$$

by Claim 1. The last three equalities give (5.5). $\square$

6. Main result

Theorem 6.1. *Let A be a piecewise hereditary algebra of tree type and ϕ an automorphism of $\hat{A}$ with jump n. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are derived equivalent, where we set $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}\phi\mathbb{1}^{[0]}$.*

Proof. Let T be the tree type of A. Then by Lemma 4.1 there exists an admissibly oriented tree Q with $\bar{Q} = T$. We set $\phi' := \nu^n_A\hat{\phi}_0$ $(= \hat{\phi}_0\nu^n_A)$. Then $T^n_{\phi_0}(A) = \hat{A}/\langle\phi'\rangle$. By Proposition 4.3(2) there exist some $\psi, \psi' \in \mathrm{Aut}(\widehat{\Bbbk Q})$ both with jump n such that $\hat{A}/\langle\phi\rangle$ (resp. $\hat{A}/\langle\phi'\rangle$) is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi\rangle$ (resp. $\widehat{\Bbbk Q}/\langle\psi'\rangle$), and the actions of ψ and ψ' coincide on the objects of $\widehat{\Bbbk Q}$. Then by Proposition 5.4 we have $\widehat{\Bbbk Q}/\langle\psi\rangle \cong \widehat{\Bbbk Q}/\langle\psi'\rangle$. Hence $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are derived equivalent. □

Definition 6.2. Let Λ be a generalized n-fold extension of a piecewise hereditary algebra A of tree type T, say $\Lambda = \hat{A}/\langle\phi\rangle$ for some $\phi \in \mathrm{Aut}(A)$ with jump n. Further let Q be an admissibly oriented tree with $\bar{Q} = T$. Then by Proposition 4.3 there exists $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi\rangle$. We define the (*derived equivalence*) *type* $\mathrm{type}(\Lambda)$ of Λ to be the triple $(T, n, \overline{\pi}(\psi_0))$, where $\psi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}_{\Bbbk Q}\psi\mathbb{1}^{[0]}$ and $\overline{\pi}(\psi_0)$ is the conjugacy class of $\pi(\psi_0)$ in $\mathrm{Aut}(T)$. $\mathrm{type}(\Lambda)$ is uniquely determined by Λ.

By Theorem 6.1, we can extend the main theorem in [3] as follows.

Theorem 6.3. *Let Λ, Λ' be generalized multifold extensions of piecewise hereditary algebras of tree type. Then the following are equivalent:*

(i) Λ *and* Λ' *are derived equivalent.*
(ii) Λ *and* Λ' *are stably equivalent.*
(iii) $\mathrm{type}(\Lambda) = \mathrm{type}(\Lambda')$.

Finally we pose a question concerning a refinement of Theorem 6.1.

Question. Under the setting of Theorem 6.1, when are the algebras $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ isomorphic?

In the forthcoming paper [12] we will give an affirmative answer.

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