Calculation Note for N=3-extended Supersymmetric Schwarzian and Liouville Theories

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Abstract

We give proofs for the important formulae required in the article [1].

Contents

1 Introduction

For the $N = 3$ superconformal diffeomorphism the Kiirillov-Kostant 2-form was given by (5.1) in [1] as

$$2\Omega_{(b,c)} = \int dxd^3\theta\Big\{-d\Big[2y\Big(\Delta^{\frac{1}{2}}b(f,\varphi)+c\mathcal{S}(f,\varphi;x,\theta)\Big)\Big]-cyD_{\theta1}D_{\theta2}D_{\theta3}y\Big\}.$$

We show that the top component of the anomalous term takes the form

$$[y\epsilon_{ijk}D_iD_jD_ky]_{\theta^3} = d[\cdots]+\partial_x[\cdots].$$

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The boundary term $d[\cdots]$ is explicitly calculated so that we get we get the formula (5.2) in [1]

$$\int dxd^3\theta\ yD_{\theta 1}D_{\theta 2}D_{\theta 3}y = d\gamma.$$

We also calculate the top component of the $N=3$ super-Schwarzian derivative $\mathcal{S}(f,\varphi;x,\theta)$.

2 Various formulae

$$f = h + \theta_i\psi_i + \frac{1}{2}\epsilon_{ijk}\theta_i\theta_j t_k + \theta_1\theta_2\theta_3\omega,$$

$$\varphi_k = \eta_k + \theta_k\rho + \frac{1}{2}\epsilon_{ijk}\theta_i\theta_j\tau + \theta_k(\theta\cdot\tau) + \theta_1\theta_2\theta_3 r_k,$$

$$\varphi_k\varphi_l = \eta_k\eta_l + \theta_k\eta_l - \theta_l\eta_k + \frac{1}{2}\eta_k\epsilon_{lij}\theta_i\theta_j\tau - \frac{1}{2}\tau_l\epsilon_{kij}\theta_i\theta_j\tau + \eta_k\theta_l(\theta\cdot\tau) - \eta_l\theta_k(\theta\cdot\tau) + \theta_k\theta_l\rho^2$$
$$+\theta_1\theta_2\theta_3(-\eta_k r_l + \eta_l r_k + 2\epsilon_{kli}\rho\tau_i),$$

$$\varphi_k\varphi_l\varphi_m = \eta_k\eta_l\eta_m + \eta_k\eta_l\theta_m\rho + (\theta_k\rho\eta_l - \theta_l\rho\eta_k)\eta_m +$$
$$+\Big(\frac{1}{2}\eta_k\eta_l\epsilon_{mij}\theta_i\theta_j\tau + \frac{1}{2}\eta_m\eta_k\epsilon_{lij}\theta_i\theta_j\tau + \frac{1}{2}\eta_l\eta_m\epsilon_{kij}\theta_i\theta_j\tau\Big)$$
$$+\Big(\eta_k\eta_l\theta_m(\theta\cdot\tau + \eta_m\eta_k\theta_l(\theta\cdot\tau) + \eta_l\eta_m\theta_k(\theta\cdot\tau\Big) + (\theta_k\theta_l\eta_m + \theta_l\theta_m\eta_k + \theta_m\theta_k\eta_l)\rho^2$$
$$+\theta_1\theta_2\theta_3\Big(\eta_k\eta_l r_m + \epsilon_{klm}\rho^3 + \rho(\theta\cdot\tau)(2\theta_m\theta_k\eta_l + 2\theta_l\theta_m\eta_k) + 2\rho\tau\eta_k\delta_{lm}$$
$$+(-\eta_k r_l + \eta_l r_k + 2\epsilon_{kli}\tau_i\rho)\eta_m\Big),$$

$$\epsilon_{klm}\varphi_k\varphi_l\varphi_m = \epsilon_{klm}\eta_k\eta_l\eta_m + 3\theta_k\eta_l\eta_m\rho + 3\Big(-(\theta\cdot\eta)^2\tau + \epsilon_{klm}\theta_k\theta_l\eta_m\rho^2 + \epsilon_{klm}\theta_k\eta_l\eta_m(\theta\cdot\eta)\Big)$$
$$+3\theta_1\theta_2\theta_3\Big(\epsilon_{klm}\eta_k\eta_l r_m - 4(\eta\cdot\tau)\rho + 2\rho^3\Big),$$

$$D_l f = \psi_l + (\theta_l\partial_x h + \epsilon_{lij}\theta_i t_j) + (\theta_l\theta_i\partial_x\psi_i + \frac{1}{2}\epsilon_{lij}\theta_i\theta_j\omega) + \theta_1\theta_2\theta_3\partial_x t_l,$$

$$D_l\varphi_k = \delta_{lk}\rho + \Big(\theta_l\partial_x\eta_k - \epsilon_{lkj}\theta_j\tau + \delta_{lk}(\theta\cdot\tau) - \theta_k\tau_l\Big) + (\theta_l\theta_k\partial_x\rho + \frac{1}{2}\epsilon_{lij}\theta_i\theta_j r_k)$$
$$+\theta_1\theta_2\theta_3(\delta_{lk}\partial_x\tau + \epsilon_{lkj}\partial_x\tau_j),$$

$$\varphi_k D_l\varphi_k = \eta_l\rho + \theta_l\Big(-\theta_l(\eta\cdot\partial\eta) + \rho^2\Big) - \epsilon_{lij}\theta_i\eta_j\tau + \eta_l(\theta\cdot\tau) + \tau_l(\theta\cdot\eta)$$
$$+\theta_l\Big((\theta\cdot\eta)\partial_x\rho - (\theta\cdot\partial_x\eta)\rho + 2(\theta\cdot\tau)\rho\Big) + \frac{1}{2}\epsilon_{lij}\theta_i\theta_j\Big((\eta\cdot r) - \tau\rho\Big)$$
$$+\theta_1\theta_2\theta_3\Big[-\eta_l\partial_x\tau - \tau\partial_x\eta_l + 2r_l\rho + 4\tau\tau_l + \epsilon_{lij}(-\eta_i\partial_x\tau_j + \partial_x\eta_i\tau_j - \tau_i\tau_j)\Big],$$

$$\epsilon_{pnl}D_nD_l\varphi_k = -2\delta_{pk}\tau - 2\epsilon_{pkl}\tau_l$$
$$+\Big(-2\epsilon_{pkl}\theta_l\partial_x\rho - 2\theta_p r_k\Big)$$
$$+\Big(\epsilon_{pnl}\theta_n\theta_l\partial_x^2\eta_k + 2\theta_k\theta_p\partial_x\tau - \epsilon_{pkl}\theta_l(\theta\cdot\partial_x\tau) + \epsilon_{pml}\theta_k\theta_m\partial_x\tau_l + \epsilon_{kml}\theta_p\theta_m\partial_x\tau_l\Big)$$
$$+\epsilon_{pml}\theta_m\theta_l\theta_k\partial_x^2\rho,$$

$$(\epsilon_{pnl}D_nD_l\varphi_k)D_p\varphi_k = -6\Big[\rho\tau + \Big(2(\theta\cdot\tau)\tau + \frac{1}{2}\epsilon_{pnl}\theta_p\tau_n\tau_l\Big)\Big)$$
$$+\Big(\frac{1}{2}\epsilon_{pnl}\theta_p\theta_n r_l\tau + (\theta\cdot r)(\theta\cdot\tau) + \epsilon_{pnl}\theta_p\theta_n\tau_l\partial_x\rho - \frac{1}{2}\epsilon_{pnl}\theta_p\theta_n(\partial_x\tau_l)\rho\Big)$$
$$-\frac{1}{6}\epsilon_{pnl}\theta_p\theta_n\theta_l\Big(3\tau\partial_x\tau + 3(\tau\cdot\partial_x\tau) - 2(\partial_x\rho)^2 + \rho\partial_x^2\rho - (r\cdot r)\Big)\Big].$$

Constraints due to the superconformal conditions $D_l f = \varphi_k D_l\varphi_k$:

$$\partial_x h = -(\eta\cdot\partial_x\eta) + \rho^2, \qquad \psi_l = \eta_l\rho,$$

$$t_i = -\eta_i \tau - \epsilon_{ijk}\eta_j\tau_k, \qquad \omega = (\eta \cdot r) - \tau\rho,$$
$$\tau_i = \partial_x \eta_i, \qquad 0 = \rho r_i + \tau\tau_i + \frac{1}{2}\epsilon_{ijk}\tau_j\tau_k.$$

Calculation of $y = \frac{df + (\varphi \cdot d\varphi)}{\Delta}$:

$$\begin{aligned}
&df + (\varphi \cdot d\varphi) \\
&\quad = dh + (\eta \cdot d\eta) + 2\rho(\theta \cdot d\eta) - \epsilon_{ijk}\theta_i\theta_j d\eta_k \tau + 2(\theta \cdot d\eta)(\theta \cdot \tau) \\
&\qquad + \theta_1\theta_2\theta_3\Big(2(d\eta \cdot r) + 2\rho d\tau - 4d\rho\tau\Big), \\
&\Delta = \partial_x f + (\varphi \cdot \partial_x \varphi) \qquad\qquad (\text{ Note that } \quad D_l\varphi_k D_l\varphi_m = \delta_{km}\Delta \ ) \\
&\quad = \rho^2 + 2\rho(\theta \cdot \tau) - \epsilon_{ijk}\theta_i\theta_j\tau_k\tau + 2(\theta \cdot \tau)^2 \\
&\qquad + \theta_1\theta_2\theta_3[2(r \cdot \tau) + 2\rho\partial_x\tau - 4\partial\rho\tau], \\
&\frac{1}{\Delta} = \frac{1}{\rho^2} - \frac{2}{\rho^3}(\theta \cdot \partial\eta) + \frac{1}{\rho^4}\epsilon_{ijk}\theta_i\theta_j\partial_x\eta_k\tau + \frac{2}{\rho^4}(\theta \cdot \partial_x\eta)^2 \\
&\qquad - \theta_1\theta_2\theta_3\frac{1}{\rho^4}\Big(2(r \cdot \partial_x\eta) + 2\rho\partial_x\tau - 4\partial_x\rho\tau\Big), \\
&\Delta^{\frac{1}{2}} = \rho + \theta\tau - \frac{1}{2\rho}\epsilon_{ijk}\theta_i\theta_j\tau_k\tau + \frac{1}{2\rho}(\theta \cdot \tau)^2 + \theta_1\theta_2\theta_3\Big[\frac{1}{\rho}(r \cdot \tau) + \partial_x\tau - 2\frac{\partial_x\rho}{\rho}\tau - \frac{1}{2\rho^2}(\theta \cdot \tau)^3\Big] \\
&y = \frac{1}{\rho^2}(dh + (\eta \cdot d\eta)) \\
&\qquad - \frac{2}{\rho^3}(dh + (\eta \cdot d\eta))(\theta \cdot \partial\eta) + \frac{2}{\rho}(\theta \cdot d\eta) \\
&\qquad + (dh + (\eta \cdot d\eta))\frac{1}{\rho^4}\Big(\epsilon_{ijk}\theta_i\theta_j\partial\eta_k\tau + 2(\theta \cdot \partial_x\eta)^2\Big) \\
&\qquad\qquad\qquad - \frac{2}{\rho^2}(\theta \cdot d\eta)(\theta \cdot \partial\eta) - \frac{1}{\rho^2}\epsilon_{ijk}\theta_i\theta_j d\eta_k\tau \\
&\qquad + \theta_1\theta_2\theta_3\Big[- \frac{1}{\rho^4}(dh + (\eta \cdot d\eta))\Big(2(r \cdot \partial_x\eta) + 2\rho\partial_x\tau - 4\partial_x\rho\tau\Big) \\
&\qquad\qquad\qquad + \frac{1}{\rho^2}\Big(2(d\eta \cdot r) + 2\rho d\tau - 4d\rho\tau\Big)\Big].
\end{aligned}$$

Using

$$\epsilon_{ijk}D_iD_jD_k = \epsilon_{ijk}[\partial_i\partial_j\partial_k + 3\theta_i\partial_j\partial_k\partial_x + 3\theta_i\theta_j\partial_k\partial_x^2 + \theta_i\theta_j\theta_k\partial_x^3],$$

we get

$$\begin{aligned}
&[\epsilon_{ijk}D_iD_jD_k y]_{\theta^0} \\
&\quad = \frac{6}{\rho^4}(dh + (\eta \cdot d\eta))\Big(2(r \cdot \partial_x\eta) + 2\rho\partial_x\tau - 4\partial_x\rho\tau\Big) \\
&\qquad - \frac{6}{\rho^2}\Big(2(d\eta \cdot r) + 2\rho d\tau - 4d\rho\tau\Big), \\
&[\epsilon_{ijk}D_iD_jD_k y]_{\theta^1} \\
&\quad = 12\partial_x\Big[\frac{1}{\rho^4}(dh + (\eta \cdot d\eta))\Big(- (\theta \cdot \partial_x\eta)\tau + \epsilon_{ijk}\theta_i\partial_x\eta_j\partial_x\eta_k\Big) \\
&\qquad\qquad + \frac{1}{\rho^2}\Big((\theta \cdot d\eta)\tau - \epsilon_{ijk}\theta_i d\eta_j\partial_x\eta_k\Big)\Big], \\
&[\epsilon_{ijk}D_iD_jD_k y]_{\theta^2} \\
&\quad = 6\epsilon_{ijk}\theta_i\theta_j\partial_x^2\Big[- \frac{1}{\rho^3}(dh + (\eta \cdot d\eta))\partial_x\eta_k + \frac{1}{\rho}d\eta_k\Big], \\
&[\epsilon_{ijk}D_iD_jD_k y]_{\theta^3} \\
&\quad = \epsilon_{ijk}\theta_i\theta_j\theta_k\partial_x^3\Big[\frac{1}{\rho^2}(dh + (\eta \cdot d\eta))\Big].
\end{aligned}$$

Then we have

$$
\begin{aligned}
&[y\epsilon_{ijk}D_iD_jD_ky]_{\theta^3} \\
&\quad = [y]_{\theta^0} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^3} + [y]_{\theta^1} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^2} \\
&\qquad +[y]_{\theta^2} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^1} + [y]_{\theta^3} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^0},
\end{aligned}
$$

with

$$
\begin{aligned}
&[y]_{\theta^0} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^3} \\
&\quad = \theta_1\theta_2\theta_3\Big\{6\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x^3\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big\}, \\
&[y]_{\theta^1} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^2} \\
&\quad = \theta_1\theta_2\theta_3\Big\{24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\frac{\partial_x\eta_k}{\rho}\partial_x^2\Big(\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\frac{\partial_x\eta_k}{\rho}\Big) \\
&\qquad +24\Big(\frac{d\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big) - 48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big)\Big\}, \\
&[y]_{\theta^2} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^1} \\
&\quad = \theta_1\theta_2\theta_3\Big\{48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} \\
&\qquad +12\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big] \\
&\qquad\quad \times\Big[-4\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\tau}{\rho}\partial_x(\frac{d\eta_m}{\rho}\frac{\partial_x\eta_m}{\rho}) - 4\epsilon_{lmn}\partial_x(\frac{d\eta_l}{\rho}\frac{\tau}{\rho})\frac{\partial\eta_m}{\rho}\frac{\partial\eta_n}{\rho} \\
&\qquad\quad +2\Big(\frac{\partial_x\eta}{\rho}\frac{\tau}{\rho}\cdot\partial_x(\frac{d\eta}{\rho}\frac{\tau}{\rho})\Big) - 2\Big(\partial_x(\frac{\partial_x\eta}{\rho}\frac{\tau}{\rho})\cdot\frac{d\eta}{\rho}\frac{\tau}{\rho}\Big) \\
&\qquad\quad -8\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\Big] \\
&\qquad +12\Big[2\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\tau}{\rho} + 2\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big) \\
&\qquad\quad +2\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{d\eta}{\rho})\Big) - 2\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\partial_x(\frac{\partial_x\eta}{\rho})\cdot\frac{d\eta}{\rho}\Big) \\
&\qquad\quad +4\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{d\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big]\Big\}, \\
&[y]_{\theta^3} \times [\epsilon_{ijk}D_iD_jD_ky]_{\theta^0} \\
&\quad = \theta_1\theta_2\theta_3\Big\{48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[\Big(\frac{r}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\frac{d\eta}{\rho}\cdot\frac{r}{\rho}\Big) + (\frac{\partial_x\tau}{\rho} - 2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})(\frac{d\tau}{\rho} - 2\frac{d\rho}{\rho}\frac{\tau}{\rho}) \\
&\qquad\qquad +\Big(\frac{r}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)(\frac{d\tau}{\rho} - 2\frac{d\rho}{\rho}\frac{\tau}{\rho}) - \Big(\frac{d\eta}{\rho}\cdot\frac{r}{\rho}\Big)(\frac{\partial_x\tau}{\rho} - 2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})\Big] \\
&\qquad -24\Big[\Big(\frac{d\eta}{\rho}\cdot\frac{r}{\rho}\Big) + \frac{d\tau}{\rho} - 2\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big]^2\Big\} \\
&\quad = \theta_1\theta_2\theta_3\Big\{48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[(\frac{\partial_x\tau}{\rho} - 2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})(\frac{d\tau}{\rho} - 2\frac{d\rho}{\rho}\frac{\tau}{\rho}) \\
&\qquad\qquad -\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}(\frac{\partial_x\tau}{\rho} - 2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}) \\
&\qquad\qquad -\frac{1}{2}\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}(\frac{d\tau}{\rho} - 2\frac{d\rho}{\rho}\frac{\tau}{\rho}) \\
&\qquad\qquad +\frac{1}{2}\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{d\eta_n}{\rho}(\frac{\partial_x\tau}{\rho} - 2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})\Big] \\
&\qquad -24\Big[2\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\eta_1}{\rho}\frac{\partial_x\eta_2}{\rho}\frac{\partial_x\eta_3}{\rho} + (\frac{d\tau}{\rho} - 2\frac{d\rho}{\rho}\frac{\tau}{\rho})^2
\end{aligned}
$$

$$+2\Big(\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}-\frac{1}{2}\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{d\eta_n}{\rho}\Big)(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})\Big]\Big\}.$$

3 Calculation of $[y\epsilon_{ijk}D_iD_jD_ky]_{\theta^3}$

Using the result for this quantity in the previous we find

$$\int dx^2d\theta_1d\theta_2d\theta_3[y\epsilon_{ijk}D_iD_jD_ky]_{\theta^3}=\int dx^2(L_{\tau^0}+L_{\tau^1}+L_{\tau^2}),$$

in which the Lagrangian density is expanded in terms of the fermionic field component τ.

$$\begin{aligned}
L_{\tau^0}=&\,6\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x^3\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\\
&+48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)\\
&+24\Big(\frac{d\eta}{\rho}\cdot\partial_x^2\frac{d\eta}{\rho}\Big)-48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big)\\
&-96\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\\
&+24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\\
&-24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\partial_x(\frac{d\eta}{\rho})\cdot\frac{\partial_x\eta}{\rho}\Big)+24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big),
\end{aligned}$$

$$\begin{aligned}
L_{\tau^1}=&\,48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&+12\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\\
&\qquad\times\Big[-4\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\tau}{\rho}\partial_x(\frac{d\eta_m}{\rho}\frac{\partial_x\eta_m}{\rho})-4\epsilon_{lmn}\partial_x(\frac{d\eta_l}{\rho}\frac{\tau}{\rho})\frac{\partial\eta_m}{\rho}\frac{\partial\eta_n}{\rho}\\
&+48\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{d\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&-24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big)\\
&+24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{d\eta_n}{\rho}\Big(\partial_x(\frac{\tau}{\rho})-\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}\Big)\\
&-48\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\eta_1}{\rho}\frac{\partial_x\eta_2}{\rho}\frac{\partial_x\eta_3}{\rho}\\
&+24\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{d\eta_n}{\rho}\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
L_{\tau^2}=&\,24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[\Big(\frac{\partial_x\eta}{\rho}\frac{\tau}{\rho}\cdot\partial_x(\frac{d\eta}{\rho}\frac{\tau}{\rho})\Big)-\Big(\partial_x(\frac{\partial_x\eta}{\rho}\frac{\tau}{\rho})\cdot\frac{d\eta}{\rho}\frac{\tau}{\rho}\Big)\Big]\\
&+24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\tau}{\rho}\\
&+48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[(\frac{\partial_x\tau}{\rho}-2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})-\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}(\frac{\partial_x\tau}{\rho}-2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})\Big]\\
&-24(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})^2-48\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{d\tau}{\rho}\\
=&\,24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\tau}{\rho}
\end{aligned}$$

$$+48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big](\frac{\partial_x\tau}{\rho}-2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})$$
$$-24(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})^2-48\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{d\tau}{\rho}.$$

These $L_{\tau^0}, L_{\tau^1}, L_{\tau^2}$ are calculated in Sections **3.1** , **3.2**, and **3.3** respectively. Each of them is shown to take a form $d[\cdots]+\partial_x[\cdots]$.

3.1 L_{τ^0}

The Lagrangian density to $O(\tau^0)$ has been given just above as

$$\begin{aligned}
L_{\tau^0}&=6\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x^3\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\\
&+48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)\\
&+24\Big(\frac{d\eta}{\rho}\cdot\partial_x^2\frac{d\eta}{\rho}\Big)-48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big)\\
&-96\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\\
&+24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\\
&-24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\partial_x(\frac{d\eta}{\rho})\cdot\frac{\partial_x\eta}{\rho}\Big)+24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big).
\end{aligned}$$

First of all we calculate the the first term in L_{τ^0}, which may be written as $-6A\partial_x A$ with

$$\begin{aligned}
A&=\partial_x\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\\
&=\frac{d\rho^2}{\rho^2}-\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big).
\end{aligned}$$

$A\partial_x A$ may be put in the expansion form in η

$$A\partial_x A=[A\partial_x A]_{\eta^0}+[A\partial_x A]_{\eta^2}+[A\partial_x A]_{\eta^4},$$

in which

$$\begin{aligned}
[A\partial_x A]_{\eta^0}&=(\frac{d\rho^2}{\rho^2}-\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2})\partial_x(\frac{d\rho^2}{\rho^2}-\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2})\\
&=d\Big[\log\rho^2\partial_x d(\log\rho^2)+\partial_x(\frac{1}{\rho^2})\partial_x(\frac{1}{\rho^2})dh\rho^2\Big]\quad\cdots\cdots\quad d-boundary\\
&\quad\underbrace{-\partial_x(\frac{1}{\rho^2})\partial_x(\frac{1}{\rho^2})dhd(\eta\cdot\partial_x\eta)}_{O(\eta^2)},\\
[A\partial_x A]_{\eta^2}&=2(\frac{d\rho^2}{\rho^2}-\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2})\partial_x\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big],\\
[A\partial_x A]_{\eta^4}&=\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big]\\
&\quad\times\partial_x\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big].
\end{aligned}$$

Using this we expand L_{τ^0} in η as

$$L_{\tau^0}=L_{\tau^0,\eta^0}+L_{\tau^0,\eta^2}+L_{\tau^0,\eta^4}+L_{\tau^0,\eta^6},$$

in which

$$
\begin{aligned}
L_{\tau^0,\eta^0} &= -6[A\partial_x A]_{\eta^0},\\
L_{\tau^0,\eta^2} &= -6[A\partial_x A]_{\eta^2}\\
&\quad +48\frac{dh}{\rho^2}\partial_x(\frac{dh}{\rho^2})\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)\\
&\quad +24\Big(\frac{d\eta}{\rho}\cdot\partial_x^2\frac{d\eta}{\rho}\Big)-48\frac{dh}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big),\\
L_{\tau^0,\eta^4} &= -6[A\partial_x A]_{\eta^4}\\
&\quad +48\frac{dh}{\rho^2}\partial_x\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)+48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\partial_x(\frac{dh}{\rho^2})\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)\\
&\quad -48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big)\\
&\quad -96\frac{dh}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\\
&\quad +24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)\\
&\quad -24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\partial_x(\frac{d\eta}{\rho})\cdot\frac{\partial_x\eta}{\rho}\Big)+24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big),\\
L_{\tau^0,\eta^6} &= 48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\partial_x\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)\\
&\quad -96\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big).
\end{aligned}
$$

3.1.1 L_{τ^0,η^0}

Summary for L_{τ^0,η^0}

$$
\begin{aligned}
L_{\tau^0,\eta^0} &= -6d\Big[\log\rho^2\partial_x d(\log\rho^2)+\partial_x(\frac{1}{\rho^2})\partial_x(\frac{1}{\rho^2})dh\rho^2\Big]\\
&\quad +\underbrace{6\partial_x(\frac{1}{\rho^2})\partial_x(\frac{1}{\rho^2})dhd(\eta\cdot\partial_x\eta)}_{O(\eta^2)}
\end{aligned}
$$

3.1.2 L_{τ^0,η^2}

Terms with dh

$$
\begin{aligned}
&= 12\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\partial_x\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big]\\
&\quad +48\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big[\Big(\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})+\partial_x(\frac{\partial_x\rho}{\rho})\frac{\eta}{\rho}+\frac{\partial_x\rho}{\rho}\partial_x(\frac{\eta}{\rho})\Big)\Big]\\
&\quad \underbrace{-48\frac{dh}{\rho^2}\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{O(\eta^4)}\\
&\quad -48\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x^2(d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho})\Big)\\
&= -12\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\partial_x(\frac{d\rho^2}{\rho^2})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad (1)
\end{aligned}
$$

$$-12\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \tag{2}$$

$$+24(\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2})\partial_x\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big) \tag{3}$$

$$+48\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \tag{4}$$

$$+\partial_x(\frac{\partial_x\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \tag{5}$$

$$+\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \tag{6}$$

$$+\frac{\partial_x\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big] \tag{7}$$

$$\underbrace{-48\frac{dh}{\rho^2}\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{O(\eta^4)}$$

$$-48\frac{dh}{\rho^2}\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2 d(\frac{\eta}{\rho})\Big) \tag{8}$$

$$+\frac{\partial_x\rho}{\rho}\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \tag{9}$$

$$+2\frac{\partial_x\rho}{\rho}\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \tag{10}$$

$$+\partial_x^2(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \tag{11}$$

$$+\frac{d\rho}{\rho}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \tag{12}$$

$$+\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2 d(\frac{\eta}{\rho})\Big)\Big] \tag{13}$$

Terms with dh, which comes from L_{τ^0,η^0}

$$= 6\frac{\partial_x\rho^2}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\Big[d\Big(\frac{\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)+\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big]$$

$$= -d\Big[6\frac{\partial_x\rho^2}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big] \quad \cdots\cdots \quad d-boundary$$

$$+12\frac{\partial_x\rho^2}{\rho^2}d(\frac{\partial_x\rho^2}{\rho^2})\frac{dh}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \tag{14}$$

$$+12\frac{\partial_x\rho^2}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big). \tag{15}$$

The terms (1) $\sim$ (15) are appropriately combined to get cancelled each other.

$$(1)+(10)+(14) = -48\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\partial_x(\frac{d\rho^2}{\rho^2})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big), \tag{A}$$

$(2)+(6)+(9)=0$,

$(5)+(7)+(15)$

$$= -24\partial_x(\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2})\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+24\frac{d\partial_x h}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)$$

$$= -24\partial_x(\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2})\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \tag{B}$$

$$\underbrace{-\,24\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{O(\eta^4)},$$

$$(4)+(12)=24\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big), \qquad (C)$$

$$(11)=48\frac{d\partial_x h}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)$$

$$-48\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+48\frac{dh}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)$$

$$=\underbrace{48\frac{d\rho^2}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)}_{\text{without } dh}-\underbrace{48\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)}_{O(\eta^4)}$$

$$-48\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+48\frac{dh}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big), \qquad (D)$$

$$(3)+(13)=d\Big[48\frac{dh}{\rho^2}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big] \quad \cdots\cdots\cdots \quad d-boundary$$

$$-48\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad (E)$$

$$+48\frac{dh}{\rho^2}d(\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad (F)$$

$$+24\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot d\partial_x(\frac{\eta}{\rho})\Big), \qquad (G)$$

$$(8)=d\Big[48\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big]$$

$$-48\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)+48\frac{dh}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)$$

$$=d\Big[48\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big]$$

$$-48\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)$$

$$-48\frac{d\partial_x h}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)$$

$$+48\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)$$

$$-48\frac{dh}{\rho^2}\Big(\partial_x^2 d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad (\text{the last term }=-(8))$$

$$=d\Big[24\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big] \quad \cdots\cdots\cdots \quad d-boundary$$

$$-24\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad (H)$$

$$\underbrace{-\,24\frac{d\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{\text{without } dh}+\underbrace{24\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{O(\eta^4)}$$

$$+24\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big). \qquad (I)$$

Further cancellations occur among the terms $(A) \sim (I)$.

$$(A) + (D) + (F) = -48\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big),$$
$$(B) + (E) = -24\partial_x\Big\{\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big\}\frac{d\rho^2}{\rho^2},$$
$$(C) + (H) = 0,$$
$$(G) + (I) = 0.$$

Summary 1 for L_{τ^0,η^2}

$$\begin{aligned}
&[\text{The terms with } dh]_{\eta^2}\\
&\quad = -d\Big[6\frac{\partial_x\rho^2}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big]\\
&\qquad +d\Big[48\frac{dh}{\rho^2}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big]\\
&\qquad +d\Big[24\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big]\\
&\qquad \underbrace{-48\frac{dh}{\rho^2}\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{O(\eta^4)}\\
&\qquad \underbrace{-24\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{O(\eta^4)}\\
&\qquad \underbrace{+48\frac{d\rho^2}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)}_{\text{without } dh}\underbrace{-48\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)}_{O(\eta^4)}\\
&\qquad \underbrace{-24\frac{d\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{\text{without } dh}\underbrace{+24\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{O(\eta^4)}
\end{aligned}$$

Terms without dh

$$\begin{aligned}
&= -12\frac{d\rho^2}{\rho^2}\partial_x\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big]\\
&\quad +24\Big(\frac{d\eta}{\rho}\cdot\partial_x^2\frac{d\eta}{\rho}\Big)\\
&= 12\frac{d\rho^2}{\rho^2}\partial_x\Big[\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big]\\
&\quad -24\frac{d\rho^2}{\rho^2}\Big[\Big(\partial_x^2(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\\
&\qquad\qquad +\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x d(\frac{\eta}{\rho})\Big)\Big]\\
&\quad -24\Big(\partial_x d(\frac{\eta}{\rho})+\partial_x(\frac{d\rho}{\rho}\frac{\eta}{\rho})\cdot\partial_x d(\frac{\eta}{\rho})+\partial_x(\frac{d\rho}{\rho}\frac{\eta}{\rho})\Big)\\
&= 12\frac{d\rho^2}{\rho^2}\partial_x\Big[\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big] \qquad (1)
\end{aligned}$$

$$-24\frac{d\rho^2}{\rho^2}\Big[\Big(\partial_x^2(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big) \tag{2}$$

$$+\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x d(\frac{\eta}{\rho})\Big)\Big] \tag{3}$$

$$-24\Big[\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x d(\frac{\eta}{\rho})\Big) \tag{4}$$

$$+\frac{1}{2}\partial_x(\frac{d\rho^2}{\rho^2})\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \tag{5}$$

$$-\partial_x(\frac{d\rho^2}{\rho^2})\Big(\partial_x d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \tag{6}$$

$$-\frac{d\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big]. \tag{7}$$

Terms without dh, which come from L_{τ^0,η^2}

$$= 24\frac{d\rho^2}{\rho^2}\partial_x(\frac{d\rho^2}{\rho^2})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) - 24\frac{d\rho^2}{\rho^2}\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big). \tag{8}$$

The terms without (1) $\sim$ (7) are appropriately combined to get canceled each other.

$$(1)+(5) = -24\partial_x(\frac{d\rho^2}{\rho^2})\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big),$$

$$(2)+(6) = d\Big\{24\frac{d\rho^2}{\rho^2}\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big\} \quad \cdots\cdots\cdots \quad d-boundary$$

$$-24\frac{d\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big),$$

$$(3)+(7) = 48\frac{d\rho^2}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big),$$

$$(4) = d\Big\{-24\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x d(\frac{\eta}{\rho})\Big)\Big\}. \quad \cdots\cdots\cdots \quad d-boundary$$

These are summed with (8) to give

Summary 2 for L_{τ^0,η^2}

[The terms without dh]$_{\eta^2}$

$$= d\Big\{24\frac{d\rho^2}{\rho^2}\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big\} - d\Big\{24\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x d(\frac{\eta}{\rho})\Big)\Big\}$$

3.1.3 L_{τ^0,η^4}

Terms with dh

$$= 48\frac{dh}{\rho^2}\partial_x\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big) + 48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\partial_x(\frac{dh}{\rho^2})\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)$$

$$-96\frac{dh}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)$$

$$= 48\frac{dh}{\rho^2}\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big) + \Big(\frac{\eta}{\rho}\cdot\partial_x d(\frac{\eta}{\rho})\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)$$

$$+48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)(\frac{d\partial_x h}{\rho^2} - \frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2})\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)$$

$$-96\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho}) + \frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho}) + \frac{d\rho}{\rho}\frac{\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big)$$

$$= 48\frac{dh}{\rho^2}\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)+\Big(\frac{\eta}{\rho}\cdot\partial_x d(\frac{\eta}{\rho})\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big) \quad (1)$$

$$+\underbrace{48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{d\rho^2}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{\text{without } dh}$$

$$-\underbrace{48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{O(\eta^6)}$$

$$-48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big) \quad (2)$$

$$-96\frac{dh}{\rho^2}\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big) \quad (3)$$

$$+\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big) \quad (4)$$

$$+\frac{d\rho}{\rho}\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big). \quad (5)$$

From $Summary\ 1 for\ L_{\tau^0,\eta^2}$ we have

the terms with dh

$$= -48\frac{dh}{\rho^2}\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)$$

$$= -48\frac{dh}{\rho^2}\Big[d\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \quad (6)$$

$$+\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big]\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big). \quad (7)$$

The terms (1) $\sim$ (6) are combined to get canceled each other.

$(1 + (3) + (6) = 0,$

$(2) + (4) = 0,$

$(5) + (7) = 0.$

$Summary\ 1\ for\ L_{\tau^0,\eta^4}$

[The terms with $dh]_{\eta^4}$

$$= \underbrace{48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{d\rho^2}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{\text{without } dh}-\underbrace{48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{O(\eta^6)}$$

Terms without dh

$$= -6\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big]$$

$$\times\partial_x\Big[-\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+2\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big] \quad (1)$$

$$-48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x^2(\frac{d\eta}{\rho})\Big) \quad (2)$$

$$+24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big) \quad (3)$$

$$-24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\partial_x(\frac{d\eta}{\rho})\cdot\frac{\partial_x\eta}{\rho}\Big) \tag{4}$$

$$+24\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big). \tag{5}$$

Calculate each line to find

$$(1) = -6\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)^2\partial_x(\frac{d\rho^2}{\rho^2}) \qquad < 1 >$$

$$+6\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\cdot 2\Big(\partial_x^2(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big) \qquad < 2 >$$

$$+6\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\cdot 2\Big(\partial_x(\frac{\eta}{\rho})\cdot d\partial_x(\frac{\eta}{\rho})\Big) \qquad < 3 >$$

$$+12\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\partial_x(\frac{d\rho^2}{\rho^2})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad < 4 >$$

$$+12\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 5 >$$

$$-24\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\partial_x\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big), \qquad < 6 >$$

$$(2) = -48\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x^2(d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho})\Big)$$

$$= -48\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot d\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 7 >$$

$$+\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 8 >$$

$$+\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x^2(\frac{d\rho}{\rho}) \qquad < 9 >$$

$$+\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\frac{d\rho}{\rho} \qquad < 10 >$$

$$+\frac{\partial_x\rho}{\rho}\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 11 >$$

$$+2\frac{\partial_x\rho}{\rho}\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big), \qquad < 12 >$$

$$(3) = 24\Big(d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho}\Big)\times\Big(\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\partial_x\eta}{\rho}\Big)$$

$$= 24\Big[\Big(d(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)+2\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big]$$

$$\times\Big[\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)+\frac{\partial_x^2\rho}{\rho}\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big)+2(\frac{\partial_x\rho}{\rho})^2\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big]$$

$$= 24\Big(d(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 13 >$$

$$+24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad < 14 >$$

$$+24(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad < 15 >$$

$$+24\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 16 >$$

$$+48(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad < 17 >$$

$$+48\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad < 18 >$$

$$+48\frac{d\rho}{\rho}\partial_x(\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <19>$$

$$+48\frac{d\rho}{\rho}(\frac{\partial_x\rho}{\rho})^2\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <20>$$

$$+48\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <21>$$

$$+96\frac{d\rho}{\rho}(\frac{\partial_x\rho}{\rho})^2\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big), \qquad <22>$$

$$\begin{aligned}
(4) = &-24\Big(d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\Big)\\
&\times\Big(\partial_x d(\frac{\eta}{\rho})+\partial_x(\frac{d\rho}{\rho})\frac{\eta}{\rho}+\frac{d\rho}{\rho}\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\Big)\\
= &-24\Big[\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)+\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big]\\
&\times\Big[\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)+\frac{\partial_x\rho}{\rho}\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)+\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\\
= &-24\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <23>\\
&-24\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <24>\\
&+24\partial_x(\frac{d\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <25>\\
&+24\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <26>\\
&-24\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <27>\\
&-24\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <28>\\
&-24\frac{d\rho}{\rho}\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)^2 \qquad <29>\\
&-24\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <30>\\
&-24(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <31>\\
&+24\frac{\partial_x\rho}{\rho}\partial_x(\frac{d\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <32>\\
&+24(\frac{\partial_x\rho}{\rho})^2\frac{d\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big), \qquad <33>
\end{aligned}$$

$$\begin{aligned}
(5) = &\,24\Big(d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})+\frac{\partial_x\rho}{\rho}\frac{\eta}{\rho}\Big)\\
&\times\Big(d(\frac{\eta}{\rho})+\frac{d\rho}{\rho}\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})+\partial_x(\frac{\partial_x\rho}{\rho})\frac{\eta}{\rho}\Big)+\frac{\partial_x\rho}{\rho}\partial_x(\frac{\eta}{\rho})\Big)\\
= &\,24\Big[\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)+\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big]\\
&\times\Big[\Big(d(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)+\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\\
&\quad+\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big)+\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big]\\
= &\,24\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <34>
\end{aligned}$$

$$+24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big) \qquad <35>$$

$$-24\frac{d\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <36>$$

$$-24\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <37>$$

$$+24\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(d(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <38>$$

$$+24(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <39>$$

$$-24\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <40>$$

$$-24\frac{d\rho}{\rho}(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <41>$$

$$+24\frac{d\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <42>$$

$$+24\frac{d\rho}{\rho}\partial_x(\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <43>$$

$$+24\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big). \qquad <44>$$

$O(\eta^4)$ contribution from [The terms with $dh]_{\eta^2}$ in L_{τ^0,η^2} :

$$= -\underbrace{24\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{O(\eta^4)}$$

$$-\underbrace{48\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\partial_x(\frac{d\rho}{\rho})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)}_{O(\eta^4)}$$

$$+\underbrace{24\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\partial_x d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)}_{O(\eta^4)}$$

$$= -24d\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <45>$$

$$-24\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\partial_x(\frac{d\rho^2}{\rho^2})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <46>$$

$$-24\Big(\frac{\eta}{\rho}\cdot d\partial_x(\frac{\eta}{\rho})\Big)\partial_x(\frac{d\rho^2}{\rho^2})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <47>$$

$$-24\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\partial_x(\frac{d\rho^2}{\rho^2})\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad <48>$$

$$+24\Big(d(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <49>$$

$$+24\Big(\frac{\eta}{\rho}\cdot d\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big) \qquad <50>$$

$$+24\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big). \qquad <51>$$

$O(\eta^4)$ contribution from [The terms with $dh]_{\eta^4}$

$$= -48\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)$$

$$= -48\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <52>$$

$$-48\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{\partial_x\rho}{\rho}) \qquad <53>$$

$$-48\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad <54>$$

$$-48\frac{d\rho^2}{\rho^2}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)(\frac{\partial_x\rho}{\rho})^2\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big). \qquad <55>$$

The terms $<1>\sim<55>$ are combined without repetition to get canceled against each other.

$$<1>+<29>+<45>+<48>= d\Big\{-12\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho 2}{\rho 2}\Big(\frac{\eta}{\rho}\cdot\partial(\frac{\eta}{\rho})\Big)^2\Big\}, \quad \cdots\cdots \quad d-boundary$$

$$<2>+<42>=0,$$

$$<3>+<27>+<51>=0,$$

$$<4>+<25>+<46>=0,$$

$$<5>+<36>=0,$$

$$<6>+<34>+<49>=0,$$

$$<7>+<13>+<23>+<50>=0,$$

$$\{<8>+<24>+<30>\}+\{<16>+<38>\}$$

$$= -24\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot d\partial_x^2(\frac{\eta}{\rho})\Big)+24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)$$

$$+\{(16)+(38)\}$$

$$= 24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) \qquad (A)$$

$$-24d(\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big) \qquad (B)$$

$$+d\Big\{24\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{d\rho}{\rho})\Big)\Big\}, \quad \cdots\cdots\cdots \quad d-boundary$$

$$<9>+<47>$$

$$= 24\Big(\partial_x(\frac{\eta}{\rho})\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{d\rho^2}{\rho^2})+24\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{d\rho^2}{\rho^2})$$

$$= d\Big\{12\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho})\Big)^2\partial_x(\frac{d\rho^2}{\rho^2})\Big\}$$

$$-24\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho})\Big)\Big(d\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{d\rho^2}{\rho^2})$$

$$+24\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{d\rho^2}{\rho^2})$$

$$= d\Big\{12\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho})\Big)^2\partial_x(\frac{d\rho^2}{\rho^2})\Big\}$$

$$-(<9>+<47>)$$

$$-24\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x^2(\frac{d\rho^2}{\rho^2})$$

$$+24\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{d\rho^2}{\rho^2})$$

$$= d\Big\{6\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho})\Big)^2\partial_x(\frac{d\rho^2}{\rho^2})\Big\} \quad \cdots\cdots\cdots \quad d-boundary$$

$$-12\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x^2(\frac{d\rho^2}{\rho^2}) \qquad (C)$$

$$+12\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\partial_x(\frac{d\rho^2}{\rho^2}), \qquad (D)$$

$$<10>+<18>+<52>=0,$$

$$<11>+<21>+<54>=0,$$

$$<26>+<44>=0,$$

$$<28>+<37>+<40>=-24\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho}\partial_x\Big[\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big], \qquad (E)$$

$$\begin{aligned}
&<12>+\{<19>+<53>\}+\{<32>+<43>\}\\
&\quad=-96\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\frac{\partial_x\rho}{\rho}\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)-48\frac{d\rho}{\rho}\partial_x(\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\\
&\qquad+24\partial_x(\frac{\partial_x\rho}{\rho}\frac{d\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\\
&\quad=-48\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\frac{\partial_x\rho}{\rho}\partial_x(\frac{d\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) && (F)\\
&\qquad-24\partial_x(\frac{d\rho}{\rho}\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big), && (G)
\end{aligned}$$

$$\begin{aligned}
&<14>+<35>\\
&\quad=d\Big\{-24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big\} \quad \cdots\cdots\cdots \quad d-boundary\\
&\qquad+24d\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big) && (H)\\
&\qquad-24\partial_x(\frac{\partial_x\rho}{\rho})\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big)\Big(\frac{\eta}{\rho}\cdot d\partial_x(\frac{\eta}{\rho})\Big), && (I)
\end{aligned}$$

$$\begin{aligned}
&<15>+<17>+<31>+<39>\\
&\quad=d\Big\{-24(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big\} \quad \cdots\cdots\cdots \quad d-boundary\\
&\qquad+48\frac{\partial_x\rho}{\rho}d(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big), && (J)
\end{aligned}$$

$$<20>+<33>+<41>=0,$$

$$<22>+<55>=0.$$

For the terms $(A)\sim(J)$ further combinations are taken without repetition to find cancellation

$(A)+(I)=0,$
$(B)+(D)=0,$
$(C)+(H)=0,$
$(E)+(G)=0,$
$(F)+(J)=0.$

Summary 2 for L_{τ^0,η^4}

$$
\begin{aligned}
&[\text{The terms without } dh]_{\eta^4}\\
&\quad = d\Big\{-12\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho 2}{\rho 2}\Big(\frac{\eta}{\rho}\cdot\partial(\frac{\eta}{\rho})\Big)^2\Big\}\\
&\qquad + d\Big\{24\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{d\rho}{\rho})\Big)\Big\}\\
&\qquad + d\Big\{6\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho})\Big)^2\partial_x(\frac{d\rho^2}{\rho^2})\Big\}\\
&\qquad + d\Big\{-24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big\}\\
&\qquad + d\Big\{-24(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big\}
\end{aligned}
$$

3.1.4 L_{τ^0,η^6}

$$
\begin{aligned}
L_{\tau^0,\eta^6} = {} & 48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\partial_x\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)\\
& -96\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\frac{\partial_x\eta}{\rho}\cdot\frac{\partial_x^2\eta}{\rho}\Big). \qquad (1)
\end{aligned}
$$

The term to $O(\eta^6)$ coming from [The terms with $dh]_{\eta^4}$ in L_{τ^0,η^4}

$$
= -\underbrace{48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{d(\eta\cdot\partial_x\eta)}{\rho^2}\Big(\frac{\partial_x\eta}{\rho}\cdot\partial_x(\frac{\partial_x\eta}{\rho})\Big)}_{O(\eta^6)}, \qquad (2)
$$

$(1)+(2)=0.$

3.2 L_{τ^1}

The Lagrangian density to $O(\tau^1)$ has been given in the beginning of this Section by

$$
\begin{aligned}
L_{\tau^1} = {} & 48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\partial_x\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} \qquad (1)\\
& +12\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\\
& \qquad\times\Big[-4\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\tau}{\rho}\partial_x(\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho})-4\epsilon_{lmn}\partial_x(\frac{d\eta_l}{\rho}\frac{\tau}{\rho})\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho} \qquad (2)\\
& +48\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{d\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} \qquad (3)\\
& -24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big) \qquad (4)\\
& +24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{d\eta_n}{\rho}\Big(\partial_x(\frac{\tau}{\rho})-\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}\Big) \qquad (5)\\
& -48\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\eta_1}{\rho}\frac{\partial_x\eta_2}{\rho}\frac{\partial_x\eta_3}{\rho} \qquad (6)\\
& +24\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{d\eta_n}{\rho}\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big)\Big]. \qquad (7)
\end{aligned}
$$

The terms (1) $\sim$ (4) are combined to get canceled against each other.

$$(1) = 48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\frac{d\rho^2}{\rho^2}\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}, \qquad (a)$$

$$\begin{aligned}
(2) &= 48\partial_x\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&\quad +48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[\epsilon_{lmn}\partial_x(\frac{\partial_x\eta_l}{\rho})\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} - \epsilon_{lmn}\partial_x(\frac{d\eta_l}{\rho})\frac{\partial\eta_m}{\rho}\frac{\partial\eta_n}{\rho}\frac{\tau}{\rho}\Big]\\
&= 24\partial_x\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} && (b)\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big](-\frac{\partial_x\rho^2}{\rho^2})\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} && (c)\\
&\quad +24\Big[\frac{d\rho^2}{\rho^2} - 2\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} && (d)\\
&\quad +48\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[\frac{1}{2}\epsilon_{lmn}\frac{d\eta_m}{\rho}\partial_x\Big(\frac{\partial_x\eta_n}{\rho}\frac{\partial_x\eta_l}{\rho}\Big)\frac{\tau}{\rho} - \epsilon_{lmn}\partial_x(\frac{d\eta_l}{\rho})\frac{\partial\eta_m}{\rho}\frac{\partial\eta_n}{\rho}\frac{\tau}{\rho}\Big], && (e)
\end{aligned}$$

$$\begin{aligned}
(3) &= 24\epsilon_{lmn}\frac{d\eta_l}{\rho}d\Big(\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\Big)\frac{\tau}{\rho} + 48\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\frac{d\rho}{\rho}\\
&= -d\Big\{24\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\} \quad \cdots\cdots\cdots \quad d-boundary\\
&\quad -24\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}d(\frac{\tau}{\rho}) + 72\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\frac{d\rho}{\rho}, && (f)
\end{aligned}$$

$$\begin{aligned}
(4) &= d\Big\{24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\}\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big](-\frac{d\rho^2}{\rho^2})\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&\quad -24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big)\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\frac{d\rho}{\rho}\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]3\epsilon_{lmn}d(\frac{\partial_x\eta_l}{\rho})\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&= d\Big\{24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\} \quad \cdots\cdots\cdots \quad d-boundary\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big](-\frac{d\rho^2}{\rho^2})\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} && (g)\\
&\quad -24\Big[\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} && (h)\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\frac{d\rho}{\rho} && (i)\\
&\quad +24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\Big[3\epsilon_{lmn}\partial_x(\frac{d\eta_l}{\rho})\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho} && (j)\\
&\qquad\qquad +3\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho} && (k)\\
&\qquad\qquad -3\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\frac{d\rho}{\rho}\Big]. && (l)
\end{aligned}$$

The terms $(a) \sim (l)$ are combined with (5) $\sim$ (7) to get simplified as

$$(a) + (g) + (i) + (l) = 0,$$

$$(b) + (c) + (e) + (j) + (k) + (5) = \partial_x\Big\{24\Big[\frac{dh}{\rho^2} + \Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\} = 0,$$

$$
\begin{aligned}
(d)+(f)+(7) &= -48\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{d\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&= 48\frac{d\eta_k}{\rho}(\epsilon_{kln}\frac{\partial_x\eta_1}{\rho}\frac{\partial_x\eta_2}{\rho}\frac{\partial_x\eta_3}{\rho})\epsilon_{lmn}\frac{d\eta_m}{\rho}\frac{\tau}{\rho},\\
(h)+(6) &= -32\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho},\\
\{(d)&+(f)+(7)\}+\{(h)+(6)\}=0.
\end{aligned}
$$

Summary for L_{τ^1}

$$
L_{\tau^1} = -d\Big\{24\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\}+d\Big\{24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\}
$$

3.3 L_{τ^2}

The Lagrangian density to $O(\tau^2)$ has been given in the beginning of this Section by

$$
\begin{aligned}
L_{\tau^2} &= 24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\tau}{\rho}\\
&\quad+48\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big](\frac{\partial_x\tau}{\rho}-2\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho})(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})\\
&\quad-24(\frac{d\tau}{\rho}-2\frac{d\rho}{\rho}\frac{\tau}{\rho})^2-48\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{d\tau}{\rho}.
\end{aligned}
$$

We take expansion in η

$$
L_{\tau^2} = L_{\tau^2,\eta^0}+L_{\tau^2,\eta^2},
$$

with

$$
\begin{aligned}
L_{\tau^2,\eta^0} &= 48\frac{dh}{\rho^2}\Big(\partial_x(\frac{\tau}{\rho})-\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}\Big)\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big) &&(1)\\
&\quad-24\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big)^2, &&(2)\\
L_{\tau^2,\eta^2} &= 24\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{\partial_x\tau}{\rho} &&(3)\\
&\quad+48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big(\partial_x(\frac{\tau}{\rho})-\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}\Big)\Big(d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big) &&(4)\\
&\quad-48\Big(\frac{d\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)\frac{\tau}{\rho}\frac{d\tau}{\rho}. &&(5)
\end{aligned}
$$

The terms $(1),(2),(4)$ are calculated as follows.

$$
\begin{aligned}
(1) &= 48\frac{dh}{\rho^2}\Big[\partial_x(\frac{\tau}{\rho})d(\frac{\tau}{\rho})-\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big]\\
&= 24\Big\{-d\Big[\frac{dh}{\rho^2}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big]+\frac{dh}{\rho^2}\frac{d\rho^2}{\rho^2}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}-\frac{dh}{\rho^2}d\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big\}\\
&\quad+24\Big\{\partial_x\Big[\frac{dh}{\rho^2}\frac{\tau}{\rho}d(\frac{\tau}{\rho})\Big]+\frac{dh}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{\tau}{\rho}d(\frac{\tau}{\rho})-\frac{dh}{\rho^2}\frac{\tau}{\rho}d\partial_x(\frac{\tau}{\rho})\\
&\qquad-\frac{d\partial_x h}{\rho^2}\frac{\tau}{\rho}d(\frac{\tau}{\rho})\Big\}\\
&\quad+48\frac{dh}{\rho^2}\Big[-\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}d(\frac{\tau}{\rho})-\frac{d\rho}{\rho}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big]
\end{aligned}
$$

$$= -24d\Big[\frac{dh}{\rho^2}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big] - 24\frac{d\partial_x h}{\rho^2}\frac{\tau}{\rho}d(\frac{\tau}{\rho}) + \partial - boundary$$

$$= -24d\Big[\frac{dh}{\rho^2}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big] \quad \cdots\cdots \quad d - boundary$$

$$-24\frac{d\rho^2}{\rho^2}\frac{\tau}{\rho}d(\frac{\tau}{\rho}) \qquad < a >$$

$$\underbrace{-24\frac{1}{\rho^2}d\Big(\eta\cdot\partial_x\eta\Big)\frac{\tau}{\rho}d(\frac{\tau}{\rho})}_{O(\eta^2)},$$

$$(2) = -24\Big[d(\frac{\tau}{\rho})\cdot d(\frac{\tau}{\rho}) - 2d(\frac{\tau}{\rho})\frac{d\rho}{\rho}\frac{\tau}{\rho}\Big]$$

$$= -24d\Big[(\frac{\tau}{\rho})d(\frac{\tau}{\rho})\Big] \quad \cdots\cdots \quad d - boundary$$

$$+48d(\frac{\tau}{\rho})\frac{d\rho}{\rho}\frac{\tau}{\rho}, \qquad < b >$$

$$(4) = 48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big[\partial_x(\frac{\tau}{\rho})d(\frac{\tau}{\rho}) - \frac{d\rho}{\rho}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big] \qquad < c >$$

$$-48\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\partial_x\rho}{\rho}\frac{\tau}{\rho}d(\frac{\tau}{\rho}). \qquad < d >$$

Contribution to $L_{\tau^2.\eta^2}$ from the calculation of $L_{\tau^2.\eta^0}$

$< L_{\tau^2,\eta^2} >_{from\ 0(\eta^0)}$

$$= 24\Big[\frac{1}{\rho^2}\Big(d\eta\cdot\partial_x\eta\Big) + \Big(\frac{\eta}{\rho}\cdot\frac{d\partial_x\eta}{\rho}\Big)\Big]\frac{\tau}{\rho}d(\frac{\tau}{\rho})$$

$$= 24\Big[\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big) + \Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{d\eta}{\rho})\Big)\Big]\frac{\tau}{\rho}d(\frac{\tau}{\rho})$$

$$= 24\Big[2\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\frac{\tau}{\rho}d(\frac{\tau}{\rho})$$

$$-\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\partial_x(\frac{\tau}{\rho})d(\frac{\tau}{\rho})$$

$$-\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}d\partial_x(\frac{\tau}{\rho})\Big]$$

$$= 24\Big[2\Big(\frac{d\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\frac{\tau}{\rho}d(\frac{\tau}{\rho}) \qquad < e >$$

$$-\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\partial_x(\frac{\tau}{\rho})d(\frac{\tau}{\rho}) \qquad < f >$$

$$-\Big(\frac{d\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\partial_x(\frac{\tau}{\rho}) \qquad < g >$$

$$-\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{d\rho^2}{\rho^2}\frac{\tau}{\rho}\partial_x(\frac{\tau}{\rho}) \qquad < h >$$

$$+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)d(\frac{\tau}{\rho})\partial_x(\frac{\tau}{\rho})\Big] \qquad < i >$$

$$+d\Big[24\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\partial_x(\frac{\tau}{\rho})\Big]. \quad \cdots\cdots\cdots \quad d - boundary$$

In the sum $< a > + \cdots + < i >$ we find cancellations among the terms

$< a > + < b >= 0$,

$< c > + < f > + < h > + < i >= 0$,

$(3)+ < g >= 0$,

$(5)+<d>+<e>0.$

Summary for L_{τ^2}

$$L_{\tau^2}=-d\Big[24\frac{dh}{\rho^2}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}\Big]-d\Big[24(\frac{\tau}{\rho})d(\frac{\tau}{\rho})\Big]+d\Big[24\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\partial_x(\frac{\tau}{\rho})\Big]$$

This is the end of the long calculations of

$$\int dx^2d\theta_1d\theta_2d\theta_3[y\epsilon_{ijk}D_iD_jD_ky]_{\theta^3}=\int dx^2(L_{\tau^0}+L_{\tau^1}+L_{\tau^2}).$$

We have shown that there remains only $d-$ boundary terms as

$$\begin{aligned}
&\int dx^2d\theta_1d\theta_2d\theta_3[y\epsilon_{ijk}D_iD_jD_ky]_{\theta^3}\\
&=\frac{1}{6}d\int dx\Big[-6\Big\{\log\rho^2\partial_xd(\log\rho^2)+\partial_x(\frac{1}{\rho^2})\partial_x(\frac{1}{\rho^2})dh\rho^2)\Big\}\\
&\quad+\Big\{-6\frac{\partial_x\rho^2}{\rho^2}\frac{\partial_x\rho^2}{\rho^2}\frac{dh}{\rho^2}\Big(\frac{\eta}{\rho}\cdot\frac{\partial_x\eta}{\rho}\Big)+48\frac{dh}{\rho^2}\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\\
&\qquad+24\frac{dh}{\rho^2}\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_x^2(\frac{\eta}{\rho})\Big)\Big\}\\
&\quad+\Big\{-24\frac{d\rho^2}{\rho^2}\Big(\partial_x^2(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)+24\Big(\partial_x(\frac{\eta}{\rho})\cdot\partial_xd(\frac{\eta}{\rho})\Big)\Big\}\\
&\quad+\Big\{-12\frac{\partial_x\rho^2}{\rho^2}\frac{d\rho2}{\rho2}\Big(\frac{\eta}{\rho}\cdot\partial(\frac{\eta}{\rho})\Big)^2+24\frac{\partial_x\rho}{\rho}\Big(\frac{\eta}{\rho}\cdot d(\frac{\eta}{\rho})\Big(\frac{\eta}{\rho}\cdot\partial_x^2(\frac{d\rho}{\rho})\Big)\\
&\qquad+6\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)^2\partial_x(\frac{d\rho^2}{\rho^2})\Big\}\\
&\qquad\quad-24\partial_x(\frac{\partial_x\rho}{\rho})\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big(\partial_x(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big\}\\
&\qquad\quad-24(\frac{\partial_x\rho}{\rho})^2\Big(d(\frac{\eta}{\rho})\cdot\frac{\eta}{\rho}\Big)\Big)\Big(\frac{\eta}{\rho}\cdot\partial_x(\frac{\eta}{\rho})\Big)\Big\}\\
&\quad+\Big\{24\epsilon_{lmn}\frac{d\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\\
&\qquad+24\Big[\frac{dh}{\rho^2}+\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\Big]\epsilon_{lmn}\frac{\partial_x\eta_l}{\rho}\frac{\partial_x\eta_m}{\rho}\frac{\partial_x\eta_n}{\rho}\frac{\tau}{\rho}\Big\}\\
&\quad-24\Big\{\frac{dh}{\rho^2}\partial_x(\frac{\tau}{\rho})\frac{\tau}{\rho}+(\frac{\tau}{\rho})d(\frac{\tau}{\rho})-\Big(\frac{\eta}{\rho}\cdot\frac{d\eta}{\rho}\Big)\frac{\tau}{\rho}\partial_x(\frac{\tau}{\rho})\Big\}\Big].
\end{aligned}$$

4 Calculation of $\mathcal{S}(f,\varphi;x,\theta)$ and $[\Delta^{\frac{1}{2}}\epsilon_{klm}\varphi_k\varphi_l\varphi_m]_{\theta^3}$

The $N=3$ super-Schwarzian derivative is given by

$$\mathcal{S}(f,\varphi;x,\theta)=\frac{2}{\Delta}(\epsilon_{pnl}D_nD_l\varphi_k)D_p\varphi_k.$$

It is expanded in components as

$$\mathcal{S}(f,\varphi;x,\theta)=2\Big[\frac{\tau}{\rho}+\frac{1}{2\rho^2}\epsilon_{pnl}\theta_p\tau_n\tau_l \\ +\Big(-\frac{1}{2\rho^3}(\theta\cdot\tau)^2\tau+\frac{1}{\rho^2}\epsilon_{pnl}\theta_p\theta_n\tau_l\partial_x\rho-\frac{1}{2\rho}\epsilon_{pnl}\theta_p\theta_n\partial_x\tau_l\Big) \\ -\frac{1}{6\rho^2}\epsilon_{pnl}\theta_p\theta_n\theta_l\Big(\tau\partial_x\tau+(\tau\cdot\partial_x\tau)-2(\partial_x\rho)^2+\rho\partial_x^2\rho\Big)\Big].$$

Here use was of made of the constraints.

The $N=3$ super-Schwarzian theory contains a quantity $\Delta^{\frac{1}{2}}\epsilon_{klm}\varphi_k\varphi_l\varphi_m$ in addition to the above Schwarzian derivative. The top component is calculated as

$$[\Delta^{\frac{1}{2}}\epsilon_{klm}\varphi_k\varphi_l\varphi_m]_{\theta^3}=\rho^4-3\rho^2(\eta\cdot\tau)+\frac{3}{2}(\eta\cdot\tau)^2 \\ +\frac{1}{2}\epsilon_{klm}\eta_k\eta_l(\rho r_m)+\frac{1}{6}\epsilon_{klm}\eta_k\eta_l\eta_l\Big(\frac{1}{\rho}(r\cdot\tau)+\partial_x\tau-2\frac{\partial_x\rho}{\rho}\tau-\frac{1}{2\rho^2}(\theta\cdot\tau)^3\Big).$$

By using the constraints it becomes

$$[\Delta^{\frac{1}{2}}\epsilon_{klm}\varphi_k\varphi_l\varphi_m]_{\theta^3}=\rho^4-3\rho^2(\eta\cdot\tau)+2(\eta\cdot\tau)^2+\frac{1}{6}\epsilon_{klm}\eta_k\eta_l\eta_l\Big(-2\frac{\partial_x\rho}{\rho}\tau\Big).$$

References

[1] S. Aoyama, "$N=3$-extended supersymmetric Schwarzian and Liouville theory"