A note on a forcing related to the S-space problem in the extension with a coherent Suslin tree

A note on a forcing related to the S-space problem in the extension with a coherent Suslin tree

Teruyuki Yorioka[1,*]

[1] Department of Mathematics, Shizuoka University, Ohya 836, Shizuoka, 422-8529, JAPAN.

Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words coherent Suslin tree, PFA(S), right-separated space, hereditarily separable space, S-space
MSC (2010) 03E35, 03E57, 54D20, 54D65

One of the main problems about $\mathsf{PFA}(S)$ is that whether a coherent Suslin tree forces that there are no S-spaces under $\mathsf{PFA}(S)$. We analyze a forcing notion related to this problem, and show that under $\mathsf{PFA}(S)$, S forces that every topology on ω_1 generated by a basis in the ground model is not an S-topology. This supplements the previous work due to Stevo Todorcevic [25].

1 Introduction

A regular space (X, τ) is called hereditarily separable if every subspace is separable, and is called hereditarily Lindelöf if every subspace is Lindelöf. Their properties look like dual notions in the sense that points are switched with open sets in their definitions. It was one of famous open problems in general topology whether they coincide. A regular space is called an S-space [(1)] if it is hereditarily separable but not hereditarily Lindelöf, and is called an L-space if it is hereditarily Lindelöf but not hereditarily separable. Stevo Todorcevic proved [22] that PFA implies that there are no S-spaces, and Justin Tatch Moore proved [10, 11] that there are L-spaces . Zoltán Szentmiklóssy proved that $\mathsf{MA}_{\aleph_1}$ implies [17] that there are no compact S-spaces . For the study of S and L spaces, see [22], and [1, 13, 16].

The P-ideal dichotomy is defined by Todorcevic [23]. The origin of the P-ideal dichotomy is an analysis of the problem whether every hereditarily separable regular space is Lindelöf (i.e. there are no S-spaces [24, §23]), and he proved (e.g. [23]) that PFA implies the P-ideal dichotomy and if the P-ideal dichotomy holds and $\mathfrak{p} > \aleph_1$, then there are no S-spaces [24, §23]. According to [13, §7], Todorcevic firstly proved that PFA implies no S-spaces directly, that is, he proved that for each right-separated [(2)] hereditarily separable regular space of order type ω_1, there is a proper forcing which adds an uncountable discrete subspace. It follows that PFA implies no S-spaces, because every S-space has a right-separated subspace of order type ω_1, and a right-separated regular space of order type ω_1 is an S-space iff it has no uncountable discrete subspace (e.g. [13, §3]).

In [25], Stevo Todorcevic introduced the forcing axiom $\mathsf{PFA}(S)$, which says that there exists a coherent Suslin tree S such that the forcing axiom holds for every proper forcing which preserves S to be Suslin, that is, for every proper forcing $\mathbb{P}$ which preserves S to be Suslin and $\aleph_1$-many dense subsets D_α, $\alpha \in \omega_1$, of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which intersects all D_α's. Since the preservation of a Suslin tree by the proper forcing is closed under countable support iteration (due to Tadatoshi Miyamoto [9]), it is consistent relative to some large cardinal assumption that $\mathsf{PFA}(S)$ holds.

* Corresponding author E-mail: styorio@ipc.shizuoka.ac.jp
Supported by FY 2010 Researcher Exchange Program between JSPS and NSERC, Grant-in-Aid for Scientific Research (C) 22540124 Japan Society for the Promotion of Science, and Leading Young Researchers of Shizuoka University 2011.

(1) Usually, an S-space is denoted by an 'S'-space. However, in this note, we always use S as a (particular) coherent Suslin tree. So we adopt notation an 'S'-space.

Sometime an S-space is defined as a hereditarily separable non-Lindelöf regular space. But our terminology allows us to consider e.g. compact S-space [1]. We note that every compact space is of course Lindelöf.

(2) A space is called right-separated if the set of points can be well-ordered such that every initial segment is open. We note that an uncountable right-separated space is not Lindelöf, and a non-hereditarily Lindelöf space has an uncountable right-separated subspace [13, §3].

The first appearance of such a forcing axiom is in the paper [8] due to Paul B. Larson and Todorcevic. In this paper, they introduced the weak version of $\mathsf{PFA}(S)$, called Souslin's Axiom (in which the properness is replaced by the cccness), and under this axiom, the coherent Suslin tree S, which is a witness of the axiom, forces a weak fragment of Martin's Axiom. In [25], it is also proved that under $\mathsf{PFA}(S)$, S forces the open graph dichotomy [(3)] and the P-ideal dichotomy. Namely, many consequences of PFA are satisfied in the extension with S under $\mathsf{PFA}(S)$. On the other hand, many people proved that some consequences from $\diamondsuit$ are satisfied in the extension with a Suslin tree (e.g. [12, Theorem 6.15.]). In particular, the pseudo-intersection number $\mathfrak{p}$ is $\aleph_1$ in the extension with a Suslin tree. In fact, the extension with S under $\mathsf{PFA}(S)$ is designed as a universe which satisfied some consequences of $\diamondsuit$ and PFA simultaneously. By the use of this model, Larson and Todorcevic proved that the affirmative answer to Katětov's problem is consistent [8].

It is not known whether under $\mathsf{PFA}(S)$, S (which is a witness of $\mathsf{PFA}(S)$) forces that there are no S-spaces. In [25], Todorcevic proved that there are no compact S-spaces in the extension with S under $\mathsf{PFA}(S)$. To do this, he develops the theory of compact countably tight spaces in the extension with S under $\mathsf{PFA}(S)$, and proved that under $\mathsf{PFA}(S)$, S forces that every non-Lindelöf subspace of a compact countably tight space has an uncountable discrete subspace [25, 8.6 Theorem]. In fact, he proved that for every S-name for a non-Lindelöf subspace of a compact countably tight space, there is a proper forcing which adds an S-name for an uncountable discrete subspace. In this note, we will show the following.

Theorem *Under* $\mathsf{PFA}(S)$*, S forces that every topology on ω_1 generated by a basis in the ground model is not an* S*-topology.*

Mary Ellen Rudin and Todorcevic respectively proved that the negation of Suslin Hypothesis (i.e. there exists a Suslin tree) implies the existence of S-spaces ([14, 15] and [22, §5]). Therefore under $\mathsf{PFA}(S)$, there are S-spaces. By the theorem, we notice that such spaces cannot generate an S-topology in the extension with S under $\mathsf{PFA}(S)$.

We will show the theorem in a more general form by investigating forcing notions which are so-called S-name versions of forcing notions in the proof of [22, 8.9.Theorem]. In the next section, we give necessary notation, some comments and a sufficient claim (**Main Claim**) to show the theorem. We will give a proof of **Main Claim** in §§3-4. We give a precise strategy of the proof in the next section.

2 Preliminaries

2.1 A coherent Suslin tree

In this work, a coherent Suslin tree is a Suslin tree $S \subseteq \omega^{<\omega_1}$ such that

- for any s and t in S, $s \leq_S t$ iff $s \subseteq t$,
- S is closed under taking initial segments,
- for any s and t in S, the set $\{\alpha \in \min\{\mathsf{lv}(s), \mathsf{lv}(t)\}; s(\alpha) \neq t(\alpha)\}$ is finite (here, $\mathsf{lv}(s)$ is the length of s, that is, the domain of s), and
- for any $s \in S$ and $t \in \omega^{\mathsf{lv}(s)}$, if the set $\{\alpha \in \mathsf{lv}(s); s(\alpha) \neq t(\alpha)\}$ is finite, then $t \in S$ also.

For a countable ordinal α, let S_α be the set of the α-th level nodes, that is, the set of all members of S of domain α, and let $S_{\leq\alpha} := \bigcup_{\beta\leq\alpha} S_\beta$. For $s \in S$, we let

$$S{\restriction}s := \{u \in S; s \leq_S u\}.$$

We note that $\diamondsuit$, or adding a Cohen real, builds a coherent Suslin tree [20, 21]. A coherent Suslin tree has a strong homogeneity, that is, it has canonical commutative isomorphisms. Let s and t be nodes in S with the same level. Then we define a function $\psi_{s,t}$ from $S{\restriction}s$ into $S{\restriction}t$ such that for each $v \in S{\restriction}s$,

$$\psi_{s,t}(v) := t \cup (v{\restriction}[\mathsf{lv}(s), \mathsf{lv}(v)))$$

(3) This is so called the open coloring axiom [22, §8].

(here, $v\lceil[\mathsf{lv}(s), \mathsf{lv}(v))$ is the function v restricted to the domain $[\mathsf{lv}(s), \mathsf{lv}(v))$). We note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S with the same level, then $\psi_{t,u} \circ \psi_{s,t} = \psi_{s,u}$. (On a coherent Suslin tree, see e.g. [2,7].)

In [9], Miyamoto introduced the following characterization of the preservation of a Suslin tree by proper forcing extensions.

Theorem 2.1 (Miyamoto, [9, (1.1) Proposition.]) *For a Suslin tree S and a proper forcing $\mathbb{P}$, $\mathbb{P}$ preserves S to be Suslin iff for any sufficiently large regular cardinal θ, any countable elementary substructure N of $H(\theta)$ which contains $\mathbb{P}$ and S as members and $p \in \mathbb{P} \cap N$, there exists $q \leq_{\mathbb{P}} p$ which is $(\mathbb{P}, N)$-generic such that for every $t \in S$ of level $\omega_1 \cap N$, the pair $\langle p, t\rangle$ is $(\mathbb{P} \times S, N)$-generic.*

Proof. We show here only the if case, which is the necessary implication in this note. Suppose that $p \in \mathbb{P}$ and $\dot{A}$ be a $\mathbb{P}$-name for a maximal antichain in S.

We take a sufficiently large regular cardinal θ and a countable elementary substructure N of $H(\theta)$ which contains $\mathbb{P}$, S, p and $\dot{A}$ as members, and let $q \leq_{\mathbb{P}} p$ be as in the assumption. Then the set

$$D := \left\{\langle r, s\rangle \in \mathbb{P} \times S; r \Vdash_{\mathbb{P}} \text{“ } s \in \dot{A} \text{ ”}\right\}$$

is predense in $\mathbb{P} \times S$ and is a member of the model N.

Then by our assumption, for every $t \in S_{\omega_1 \cap N}$, the pair $\langle q, t\rangle$ is $(\mathbb{P} \times S, N)$-generic. Therefore for every $t \in S_{\omega_1 \cap N}$, $D \cap N$ is predense below $\langle q, t\rangle$ in $\mathbb{P} \times S$. Thus for every $t \in S_{\omega_1 \cap N}$,

$$q \Vdash_{\mathbb{P}} \text{“ } \exists s <_S t (s \in \dot{A}) \text{ ”}.$$

This says that

$$q \Vdash_{\mathbb{P}} \text{“ } \dot{A} \subseteq S \cap \omega^{<\omega_1 \cap N} \text{ ”},$$

hence q forces that $\dot{A}$ is countable. This finishes the proof. □

2.2 A forcing notion $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$

Let S be a coherent Suslin tree and $\dot{\tau}$ an S-name for a right-separated hereditarily separable regular topology on ω_1 of order type ω_1. If there exists a proper forcing which preserves S to be Suslin and adds an S-name for an uncountable discrete subset of $(\omega_1, \dot{\tau})$, then under $\mathsf{PFA}(S)$, S (which is a witness of $\mathsf{PFA}(S)$) forces that there are no S-spaces.

To define such a forcing notion, we use a sequence $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ of S-names such that for each $\alpha \in \omega_1$,

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \text{ and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha + 1, \omega_1) = \emptyset \text{ ”}.$$

We can find it because $(\omega_1, \dot{\tau})$ is an S-name for a right-separated regular space. Let κ be the least regular cardinal such that

$$\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle \in H(\kappa).$$

We consider a forcing notion $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ which adds an S-name for an uncountable discrete subset of $(\omega_1, \dot{\tau})$ as follows (Proposition 2.3). This forcing is fairly a *naive* way to add a desired uncountable set by use of side-condition-method as in [22, 8.9.Theorem] (see also [3, 19]). This forcing is very similar to the one used in the proof [25, 8.6 Theorem].

Definition 2.2 Suppose that S is a coherent Suslin tree, $\dot{\tau}$ is an S-name for a right-separated hereditarily separable regular topology on ω_1 of order type ω_1, and a sequence $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ of S-names satisfies that for each $\alpha \in \omega_1$,

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \text{ and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha + 1, \omega_1) = \emptyset \text{ ”}.$$

$\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ consists of finite functions p such that

- $\mathrm{dom}(p)$ is a finite $\in$-chain of countable elementary submodels of the structure $H(\kappa)$ which contain S, $\dot{\tau}$ and $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ as members,
- for any $M \in \mathrm{dom}(p)$, $p(M) = \langle t^p_M, \alpha^p_M \rangle \in (S \setminus M) \times (\omega_1 \setminus M)$ (hence $t^p_M \notin M$ and $\alpha^p_M \notin M$),
- for any $M \in \mathrm{dom}(p)$ and $\beta \in \omega_1 \cap M$, t^p_M decides whether $\beta \in \dot{U}_{\alpha^p_M}$ or not,
- for any $M, M' \in \mathrm{dom}(p)$, if $M \in M'$, then $t^p_M, \alpha^p_M \in M'$, and
- for any $M, M' \in \mathrm{dom}(p)$, if $t^p_M <_S t^p_{M'}$, then

$$t^p_{M'} \Vdash_S \text{“ } \alpha^p_M \notin \dot{U}_{\alpha^p_{M'}} \text{ ”},$$

ordered by extensions.

The following proposition guarantees that under $\mathsf{PFA}(S)$, if $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ is proper and preserves S to be Suslin, then S forces that $(\omega_1, \dot{\tau})$ has an uncountable discrete subspace (remember that S has a strong homogeneity). The following proof is similar to ones in [22, 8.9.Theorem] and [25, 8.6 Theorem].

Proposition 2.3 *If the forcing notion $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ is proper and preserves S to be Suslin, then some member of $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ forces that there are $s \in S$ and an S-name $\dot{\Gamma}$ such that*

$$s \Vdash_S \text{“ } \left\{ \dot{U}_\alpha; \alpha \in \dot{\Gamma} \right\} \textit{ witnesses that } \dot{\Gamma} \textit{ is uncountable discrete ”}.$$

P r o o f. In this proof, we write $\mathbb{P}$ instead of $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$. We denote by $\dot{G}$ the canonical $\mathbb{P}$-name for a generic filter, and by $\dot{G}_S$ the canonical S-name for a generic filter. Then we define $\mathbb{P}$-name $\dot{D}$ for a subset of S and $\mathbb{P}$-name $\dot{\Gamma}$ for an S-name for subset of ω_1 as follows:

$$\Vdash_{\mathbb{P}} \text{“ } \dot{D} := \left\{ t^p_M; p \in \dot{G} \text{ and } M \in \mathrm{dom}(p) \right\} \text{ ”}$$

and

$$\Vdash_{\mathbb{P}} \text{“ } \Vdash_S \text{“ } \dot{\Gamma} := \left\{ \alpha^p_M; p \in \dot{G} \text{ and } t^p_M \in \dot{D} \cap \dot{G}_S \right\} \text{ ” ”}.$$

It follows from the definition of $\mathbb{P}$ that for every $s \in S$,

$$\Vdash_{\mathbb{P}} \text{“ } s \Vdash_S \text{“ } \left\{ \dot{U}_\alpha; \alpha \in \dot{\Gamma} \right\} \text{ witnesses that } \dot{\Gamma} \text{ is discrete ” ”}.$$

So it suffices to show that we will find $q \in \mathbb{P}$ and $s \in S$ such that

$$q \Vdash_{\mathbb{P}} \text{“ } s \Vdash_S \text{“ } \dot{\Gamma} \text{ is uncountable ” ”}.$$

Let M be a countable elementary submodel of $H(\theta)$, for a sufficiently large regular cardinal θ (that is, $\mathbb{P} \in M$), which contains S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ and $H(\kappa)$. We take $q \in \mathbb{P}$ such that $\mathrm{dom}(q) = \{M \cap H(\kappa)\}$, and write $q(M \cap H(\kappa)) = \langle t, \alpha \rangle$ (hence $t \notin M$). Then by our assumption that $\mathbb{P}$ is proper, we claim that q forces that $\dot{D}$ is uncountable.

To show this, we note that, since $\mathbb{P} \in M$,

$$\Vdash_{\mathbb{P}} \text{“ } \dot{D} \in M[\dot{G}] \text{ ”}.$$

Suppose that q doesn't force that $\dot{D}$ is uncountable. Then there exists an extension r of q in $\mathbb{P}$ which forces that $\dot{D}$ is countable. Then by the elementarity and the definition of $\dot{D}$,

$$r \Vdash_{\mathbb{P}} \text{“ } t \in \dot{D} \subseteq M[\dot{G}] \text{ ”}.$$

However by the properness of $\mathbb{P}$ (this is our assumption),

$$\Vdash_{\mathbb{P}} \text{“ } M[\dot{G}] \cap S = M \cap S \text{ ”},$$

which is a contradiction because $t \notin M$.

By our assumption that $\mathbb{P}$ is proper (here, in particular, preserves the uncountablity) and preserves S to be Suslin, there are a member q' of $\mathbb{P}$ and $s \in \mathbb{P}$ such that q' forces in $\mathbb{P}$ that $\dot{D}$ is dense and uncountable above s in S. This is what we want. $\square$

It is not known whether the assumption of the above proposition is true in general. The following is a sufficient condition that $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ is proper and preserves S to be Suslin.

Main Claim *Let S be a coherent Suslin tree, $\dot{\tau}$ an S-name for a right-separated hereditarily separable regular topology on ω_1 of order type ω_1. Suppose that $\dot{\tau}$ has the following condition:*

($\star$) *For any point $\delta \in \omega_1$, S-name $\dot{U}$ for an open neighborhood of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot{U}'$ for an open neighborhood of δ such that $t \Vdash_S$ “ $\dot{U}' \subseteq \dot{U}$ ” and for every $s \in F$,*

$$s \Vdash_S \text{“ } \psi_{t,s}(\dot{U}') \textit{ is open in } \dot{\tau} \text{ ”}.$$

Then for any sequence $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ of S-names such that for each $\alpha \in \omega_1$,

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \textit{ and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha+1, \omega_1) = \emptyset \text{ ”},$$

$\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ *is proper and preserves S to be Suslin.*

For an explanation of isomorphisms for forcings, see e.g. [4, Ch. VII. §7] or [5, IV.4]. For an argument of isomorphisms on a coherent Suslin tree, see e.g. [8, 26]. We note that in the condition ($\star$) for an $s \in F$, it holds that $s = \psi_{t,s}(t)$ and

$$s \Vdash_S \text{“ } \psi_{t,s}(\dot{U}') \text{ is open in } \psi_{t,s}(\dot{\tau}) \text{ ”},$$

but it may happen that

$$s \nVdash_S \text{“ } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{ ”}.$$

That is, $\dot{\tau}$ doesn't have the condition ($\star$) in general. By thinking of the isomorphisms of names, the following proposition holds.

Proposition 2.4 *If $\dot{\tau}$ generated by a basis which is in the ground model, then $\dot{\tau}$ satisfies the condition ($\star$).*

This guarantees that **Theorem** in the introduction follows from **Main Claim**.

To show **Main Claim**, we separate two parts: §3 and §4. In §3, we *don't* assume that $\dot{\tau}$ satisfies the condition ($\star$), that is, the argument in §3 applies to any $\dot{\tau}$. In fact, the scenario of this section is essentially same as in Todorcevic's one of [25, 8.6 Theorem]. In **Lemma 3.1** (§3), we give a sufficient condition to prove **Main Claim**.

We use the condition ($\star$) in §4 to show **Lemma 4.1** which implies **Main Claim**. In [25, 8.6 Theorem], to prove his theorem, he introduced the $\mathcal{U}$-sequentiality for a non-principal ultrafilter $\mathcal{U}$ on ω and used it to prove the similar statement to Lemma 4.1.

In both sections, our argument is concentrated only on how to prove that $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ preserves S to be Suslin, using Theorem 2.1. We notice that it follows from the same (or more simple) argument that $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle)$ is proper.

3 A sufficient lemma to show the properness and the preservation of S

In this section, we prove the following lemma.

Lemma 3.1 *Let S be a coherent Suslin tree, $\dot{\tau}$ an S-name for a right-separated hereditarily separable regular topology of order type ω_1, and $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ a sequence of S-names such that for each $\alpha \in \omega_1$,*

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \text{ and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha+1, \omega_1) = \emptyset \text{ ”}.$$

Suppose that $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ satisfies the following condition:

($\bullet$) *For any countable elementary substructure N of $H(\theta)$ which contains S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ and $H(\kappa)$ as members, $r \in \mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle)$ with $r \cap N = \emptyset$, $u \in S$ with $\mathsf{lv}\,(t^r_M) \le \mathsf{lv}\,(u)$ for all $M \in \mathrm{dom}(r)$, and S-name $\dot{X} \in N$ for an uncountable subset of ω_1, there are $\beta \in \omega_1 \cap N$ and $s \in S \cap N$ such that $s \le_S u$, $s \Vdash_S$ “ $\beta \in \dot{X}$ ”, and for every $M \in \mathrm{dom}(r)$,*

$$t^r_M \Vdash_S \text{“ } \beta \notin \dot{U}_{\alpha^r_M} \text{ ”}.$$

Then $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle)$ is proper and preserves S to be Suslin.

In this proof, we use ($\bullet$) only in the last part of the proof. That is, we don't need ($\bullet$) in the other parts, so we can apply the argument to the forcing notion $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle)$ for a general $\dot{\tau}$ and $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$. In the proof, we will mention the place where ($\bullet$) is used.

In §4, we show that any sequence $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ as in **Definition 2.2** satisfies ($\bullet$) whenever $\dot{\tau}$ satisfies the condition ($\star$), which finishes the proof of **Main Claim**. We say again that we *don't* assume the condition ($\star$) in this section.

Proof of Lemma 3.1. Let S be a coherent Suslin tree, $\dot{\tau}$ an S-name for a right-separated hereditarily separable regular topology on ω_1 of order type ω_1, and $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ a sequence of S-names such that for each $\alpha \in \omega_1$,

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \text{ and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha+1, \omega_1) = \emptyset \text{ ”}.$$

In this proof, we write $\mathbb{P}$ instead of $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle)$.

Let θ be a large enough regular cardinal, N a countable elementary submodel of $H(\theta)$ such that N contains S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$, $\mathbb{P}$ and $H(\kappa)$ as members, and $p_0 \in \mathbb{P} \cap N$.

It is true that for every $p \in \mathbb{P}$, if $N \cap H(\kappa) \in \mathrm{dom}(p)$, then p is $(N, \mathbb{P})$-generic. Such a conclusion is a general one for every forcing with models as side-conditions. At first, we should prove it. This proof is included in the following theorem (by ignoring the coordinate of the Suslin tree). So we omit the details of this proof.

For each $M \in \mathrm{dom}(p_0)$, we write $p_0(M) = \langle t^{p_0}_M, \alpha^{p_0}_M \rangle$. Let $N' := N \cap H(\kappa)$, which is a countable elementary submodel of $H(\kappa)$. We take (arbitrary) $\alpha^{p_1}_{N'} \in \omega_1 \setminus N$, and take $t^{p_1}_{N'} \in S \setminus N$ such that for every $M \in \mathrm{dom}(p_0)$, $t^{p_0}_M$ and $t^{p_1}_{N'} \restriction (\omega_1 \cap N)$ are incomparable in S, and $t^{p_1}_{N'}$ decides whether $\beta \in \dot{U}_{\alpha^{p_1}_{N'}}$ for every $\beta \in \omega_1 \cap N$ ($= \omega_1 \cap N'$). Then we define

$$p_1 := p_0 \cup \{ \langle N', \langle t^{p_1}_{N'}, \alpha^{p_1}_{N'} \rangle \rangle \},$$

which is a condition of $\mathbb{P}$ and moreover an extension of p_0 [(4)]. Let $s_1 \in S_{\omega_1 \cap N}$.

We show that $\langle p_1, s_1 \rangle$ is $(N, \mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle) \times S)$-generic whenever $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ satisfies ($\bullet$).

Let $\mathcal{D} \in N$ be a dense open subset of $\mathbb{P} \times S$. Let $r \le_{\mathbb{P}} p_1$ and $u \ge_S s_1$ be such that $\langle r, u \rangle \in \mathcal{D}$. By extending u if necessary, we may assume that for every $M \in \mathrm{dom}(r)$, $\mathsf{lv}\,(u) \ge \mathsf{lv}\,(t^r_M)$ holds (where we denote $r(M) = \langle t^r_M, \alpha^r_M \rangle$). By the coherency of S, we can take $\gamma \in \omega_1 \cap N$ such that for every $M \in \mathrm{dom}(r)$,

$$\{ \xi \in \mathsf{lv}\,(t^r_M) \cap \mathsf{lv}\,(s_1) ; t^r_M(\xi) \ne s_1(\xi) \} \subseteq \gamma.$$

(4) Actually, the following argument works whenever p_1 is an extension of p_0 in $\mathbb{P}$ such that $N' \in \mathrm{dom}(p_1)$.

We note that

$$\{\xi \in \mathsf{lv}\,(t^r_M) \cap \mathsf{lv}\,(s_1)\,; t^r_M(\xi) \neq s_1(\xi)\} = \{\xi \in \mathsf{lv}\,(t^r_M) \cap \omega_1 \cap N; t^r_M(\xi) \neq u(\xi)\}\,.$$

Let $\{M^r_i; i \in m\}$ be the $\in$-increasing enumeration of the set $\mathrm{dom}(r) \setminus N$.

For each $v \in S$, we define

$T^{-1}_v := \Big\{ \langle \alpha^q_M; M \in \mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N)\rangle\,;$

- $q \in \mathbb{P}$,
- q is an end-extension of $r \cap N$,
- $\langle q, v\rangle \in \mathcal{D}$,
- $|q| = |r|$, and letting $\{M^q_i; i \in m\}$ be the $\in$-increasing enumeration of the set $\mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N)$,
- for every $M \in \mathrm{dom}(q)$, $\mathsf{lv}\,(t^q_M) \leq \mathsf{lv}\,(v)$,
- for every $i \in m$, $t^q_{M^q_i}\restriction\gamma = t^r_{M^r_i}\restriction\gamma$ and

$$t^q_{M^q_i}\restriction\left[\gamma, \mathsf{lv}\left(t^q_{M^q_i}\right)\right) = v\restriction\left[\gamma, \mathsf{lv}\left(t^q_{M^q_i}\right)\right) \quad\text{iff}\quad t^r_{M^r_i}\restriction\left[\gamma, \mathsf{lv}\left(t^r_{M^r_i}\right)\right) = u\restriction\left[\gamma, \mathsf{lv}\left(t^r_{M^r_i}\right)\right) \Big\}.$$

We note that the set $\left\{T^{-1}_v; v \in S\right\}$ belongs to the model N, and for any $v, v' \in S$, if $v \leq_S v'$, then $T^{-1}_v \subseteq T^{-1}_{v'}$. Moreover, we note that $\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ is in T^{-1}_u. We consider each T^{-1}_v as a tree which consists of all initial segments of its members, where each sequence $\langle \alpha^q_M; M \in \mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N)\rangle$ is considered to be ordered by the usual order on ordinals. By induction on $i < m$, for each $v \in S$, we define the set

$$T^i_v := T^{i-1}_v \setminus \Big\{\sigma \in T^{i-1}_v; \exists \sigma' \in T^{i-1}_v \text{ such that } |\sigma'| = m - i - 1, \sigma' \subseteq \sigma \text{ and}$$
$$v \nVdash_S \text{“ } \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma' \frown \langle\beta\rangle \in T^{i-1}_t\right)\right\} \text{ is uncountable ”}\Big\}.$$

Then we note that for any $v, v' \in S$, if $v \leq_S v'$, then $T^{m-1}_v \subseteq T^{m-1}_{v'}$. Moreover, by the construction of T^i_v's, for every $v \in S$, $i < m$ and $\sigma \in T^{i-1}_v$ of height $m - i - 1$, if σ is still in T^i_v, then the set of successors of σ in T^i_v coincides with the set of successors of σ in T^{i-1}_v. It follows that for every $\sigma \in T^{m-1}_v$ of length $< m$,

$$\left\{\tau \in T^{m-1}_v; \sigma \subseteq \tau\right\} = \left\{\tau \in T^{m-|\sigma|-1}_v; \sigma \subseteq \tau\right\}.$$

Therefore it follows that for every $\sigma \in T^{m-1}_v$ which is not terminal,

$$v \Vdash_S \text{“ } \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma \frown \langle\beta\rangle \in T^{m-1}_t\right)\right\} \text{ is uncountable ”}.$$

We note that the set $\left\{T^{m-1}_v; v \in S\right\}$ also belongs to the model $N \cap H(\kappa) = N'$. So for each $M \in \mathrm{dom}(r \setminus N)$, $\left\{T^{m-1}_v; v \in S\right\} \in M$.

Fact 3.2 *$\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ is a cofinal path through T^{m-1}_u.*

Proof of Fact 3.2. Recall that $\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ belongs to the tree T^{-1}_u. By induction on $i < m$, we show that the initial segment of the sequence $\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ of length $m - i - 1$ belongs to the tree T^i_u.

Suppose that $\sigma \frown \langle\alpha\rangle$ is an initial segment of the sequence $\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ and $\sigma \frown \langle\alpha\rangle \in T^{i-1}_u$. We will show that $\sigma \in T^i_u$, that is,

$$u \Vdash_S \text{“ } \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma \frown \langle\beta\rangle \in T^{i-1}_t\right)\right\} \text{ is uncountable ”}.$$

Let $M \in \mathrm{dom}(r \backslash N)$ be such that $\sigma \in M$ and $\alpha \notin M$. Since $\mathsf{lv}(u) \geq \mathsf{lv}(t^r_M) \geq \omega_1 \cap M$, $\{w \in S \cap M; w <_S u\}$ forms a (S, M)-generic filter.

Suppose that

$$u \not\Vdash_S \text{“} \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown \langle\beta\rangle \in T_t^{i-1}\right)\right\} \text{ is uncountable”}.$$

Then some extension of u forces that $\left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown \langle\beta\rangle \in T_t^{i-1}\right)\right\}$ is countable. Since such an extension also generates a (M, S)-generic filter and the phrase “the set $\left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown \langle\beta\rangle \in T_t^{i-1}\right)\right\}$ is countable” can be described in $M[\dot{G}]$, there exists $w \in S \cap M$ such that $w \leq_S u$ and

$$w \Vdash_S \text{“} \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown \langle\beta\rangle \in T_t^{i-1}\right)\right\} \text{ is countable”}$$

[(5)]. Since S is $\aleph_0$-distributive, there are a countable set Z in N and $w' \in S \cap M$ such that $w \leq_S w' \leq_S u$ and

$$w' \Vdash \text{“} Z = \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown \langle\beta\rangle \in T_t^{i-1}\right)\right\} \text{”}.$$

This is a contradiction because $u \geq_S w'$ and

$$u \Vdash_S \text{“} \alpha \in \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown \langle\beta\rangle \in T_t^{i-1}\right)\right\} \setminus Z \text{”}.$$

⊣(Fact 3.2)

Therefore, the set

$$T' := \left\{v \in S; u{\restriction}\gamma \leq_S v \text{ and } T_v^{m-1} \text{ is of height } m\right\}$$

is not empty, in particular, contains u as a member. We note that T' is in N.

We have not used (•) yet. We will use it from now on. Since $u \in T' \in N$ and $\mathsf{lv}(u) \geq \omega_1 \cap N$, there exists $s_2 \in S \cap N$ such that $s_2 \leq_S u$ and $T_{s_2}^{m-1}$ has a cofinal branch of height m [(6)]. Let

$$a := \left\{i \in m; t^r_{M_i^r}{\restriction}\left[\gamma, \mathsf{lv}\left(t^r_{M_i^r}\right)\right) = u{\restriction}\left[\gamma, \mathsf{lv}\left(t^r_{M_i^r}\right)\right)\right\}.$$

If a is empty, then for any cofinal path of $T_{s_2}^{m-1}$ in N and its witness $p_2 \in \mathbb{P} \cap N$, $\langle p_2, s_2\rangle \in \mathcal{D} \cap N$ and by the choice of γ, $\langle r \cup p_2, u\rangle$ is a common extension of $\langle r, u\rangle$ and $\langle p_2, s_2\rangle$. Because then

- since $p_2 \in N$ and $N \cap H(\kappa) \in \mathrm{dom}(p_1) \subseteq \mathrm{dom}(r)$, the set $\mathrm{dom}(p_2) \cup \mathrm{dom}(r)$ forms an $\in$-chain, and
- for any $M \in \mathrm{dom}(p_2) \setminus \mathrm{dom}(r \cap N)$ and $M' \in \mathrm{dom}(r \setminus N)$, it is true that $\mathsf{lv}(t^{p_2}_M) \leq \mathsf{lv}(s_2) < \omega_1 \cap N \leq \mathsf{lv}(t^r_{M'})$, $t^{p_2}_M{\restriction}[\gamma, \mathsf{lv}(t^{p_2}_M)) \neq s_2{\restriction}[\gamma, \mathsf{lv}(t^{p_2}_M))$ (since a is empty and $s_2 <_S u$), $t^r_{M'}{\restriction}[\gamma, \omega_1 \cap N) = u{\restriction}[\gamma, \omega_1 \cap N)$ and $s_2 \leq_S u$, hence it holds that $t^{p_2}_M \not\leq_S t^r_{M'}$.

So the proof is finished. Therefore the interesting case is that a is not empty.

Suppose that a is not empty. For each $i \in m$, let

$$b_i := \left\{j \in a; t^r_{M_j^r}{\restriction}\gamma = t^r_{M_i^r}{\restriction}\gamma\right\}.$$

We note that for each $j \in b_i$, by the choice of γ, $t^r_{M_j^r}{\restriction}(\omega_1 \cap N) = t^r_{M_i^r}{\restriction}(\omega_1 \cap N)$.

(5) If $\mathsf{lv}(u) \geq \omega_1 \cap M$ and $A \in M \cap \mathcal{P}(S)$ contains u as a member, then there exists $w \in A \cap M$ with $w \leq_S u$. Indeed, the set $\{t \in S; (S{\restriction}t) \cap A = \emptyset \text{ or } t \in A\}$ is in M and dense in S. So there exists $w <_S u$ which belongs to this set (we should remember that the set $\{w \in S; w <_S u\}$ is an (S, M)-generic filter). Since $u \in A$, we have $w \in A$.

(6) Recall that if $\mathsf{lv}(u) \geq \omega_1 \cap N$ and $A \in N \cap \mathcal{P}(S)$ contains u as a member, then there exists $w \in A \cap N$ with $w \leq_S u$. Here, we consider the set $\left\{v \in S; T_v^{m-1} \text{ has a cofinal branch of height } m\right\}$. It belongs to the model N.

Let $\dot{X}_0$ be an S-name such that

$$\Vdash_S \text{“ } \dot{X}_0 := \left\{ \beta \in \omega_1 ; \exists t \in \dot{G} \Big(\langle \beta \rangle \in T_t^{m-1} \Big) \right\} \text{ ”.}$$

We note that $\dot{X}_0 \in N$ and since $T_{s_2}^{m-1}$ has a cofinal branch,

$$s_2 \Vdash_S \text{“ } \dot{X}_0 \text{ is uncountable ”.}$$

By applying the condition (•) to the tuple N, $r \restriction \left\{ M_j^r ; j \in b_0 \right\}$, u and $\dot{X}_0$, we get $\beta_0 \in \omega_1 \cap N$ and $s_2' \in S \cap N$ such that $s_2 \leq_S s_2' \leq_S u$, $s_2' \Vdash_S$ “ $\beta_0 \in \dot{X}_0$ ”, and for every $j \in b_0$,

$$t_{M_j^r}^r \Vdash_S \text{“ } \beta_0 \notin \dot{U}_{\alpha_{M_j^r}^r} \text{ ”.}$$

Next, let $\dot{X}_1$ be an S-name such that

$$\Vdash_S \text{“ } \dot{X}_1 := \left\{ \beta \in \omega_1 ; \exists t \in \dot{G} \Big(\langle \beta_0, \beta \rangle \in T_t^{m-1} \Big) \right\} \text{ ”.}$$

We note that $\dot{X}_1 \in N$ and since $T_{s_2'}^{m-1}$ also has a cofinal branch,

$$s_2' \Vdash_S \text{“ } \dot{X}_1 \text{ is uncountable ”.}$$

By applying the condition (•) to the tuple N, $r \restriction \left\{ M_j^r ; j \in b_1 \right\}$, u and $\dot{X}_1$, we get $\beta_1 \in \omega_1 \cap N$ and $s_2'' \in S \cap N$ such that $s_2' \leq_S s_2'' \leq_S u$, $s_2'' \Vdash_S$ “ $\beta_1 \in \dot{X}_1$ ”, and for every $j \in b_1$,

$$t_{M_j^r}^r \Vdash_S \text{“ } \beta_1 \notin \dot{U}_{\alpha_{M_j^r}^r} \text{ ”.}$$

By repeating this procedure, we can take $s_3 \in S \cap N$ and a cofinal branch $\langle \beta_i ; i \in m \rangle$ through $T_{s_3}^{m-1} \subseteq T_{s_3}^{-1}$ such that $s_2 \leq_S s_3 \leq_S s_1$ ($\leq_S u$) and for every $i \in m$ and $j \in b_i$,

$$t_{M_j^r}^r \Vdash_S \text{“ } \beta_i \notin \dot{U}_{\alpha_{M_j^r}^r} \text{ ”.}$$

Since $\langle \beta_i ; i \in m \rangle \in T_{s_3}^{m-1} \cap N \subseteq T_{s_3}^{-1} \cap N$, there exists $p_3 \in \mathbb{P} \cap N$ which is its witness. Then $\langle p_3, s_3 \rangle \in \mathcal{D} \cap N$. We note that for any $M \in \mathrm{dom}(p_3)$ and $M' \in \mathrm{dom}(r) \setminus N$, if $t_M^{p_3}$ and $t_{M'}^r$ are comparable in S (i.e. $t_M^{p_3} \leq_S t_{M'}^r$ holds), then it follows that $t_M^{p_3} \restriction [\gamma, \mathsf{lv}(t_M^{p_3})) = s_3 \restriction [\gamma, \mathsf{lv}(t_M^{p_3}))$ and $t_M^{p_3} \restriction \gamma = t_{M'}^r \restriction \gamma$. Therefore by the choice of p_3, $r \cup p_3$ satisfies the last condition of the definition of $\mathbb{P}(\dot{\tau}, \left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle)$. Thus $\langle r \cup p_3, u \rangle$ is a common extension of $\langle r, u \rangle$ and $\langle p_3, s_3 \rangle$, which finishes the proof. ⊣(Lemma 3.1)

4 Proof of Main Claim

We don't know whether there exists $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ as in **Definition 2.2** which satisfies (•) for each $\dot{\tau}$ as in **Definition 2.2** in general. The following lemma provides a sufficient condition for this.

Lemma 4.1 *Let S be a coherent Suslin tree, and $\dot{\tau}$ an S-name for a right-separated hereditarily separable regular topology on ω_1 of order type ω_1. Suppose that $\dot{\tau}$ satisfies the following condition:*

($\star$) *For any point $\delta \in \omega_1$, S-name $\dot{U}$ for an open neighborhood of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot{U}'$ for an open neighborhood of δ such that $t \Vdash_S$ “ $\dot{U}' \subseteq \dot{U}$ ” and for every $s \in F$,*

$$s \Vdash_S \text{“ } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{ ”.}$$

Then for any a sequence $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ of S-names such that for each $\alpha \in \omega_1$,

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \text{ and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha + 1, \omega_1) = \emptyset \text{ ”,}$$

it satisfies the condition (•) in Lemma 3.1.

Proof of Lemma 4.1. Suppose that N, r, u and $\dot{X}$ are as in the assumption of the condition ($\bullet$). Then we note that $u \notin N$, that is, $\mathsf{lv}(u) \geq \omega_1 \cap N$.

Since $\{w \in S \cap N; w <_S u\}$ forms a (S, N)-generic filter, S is $\aleph_0$-distributive and $(\omega_1, \dot{\tau})$ is an S-name for a hereditarily separable space, there are $t \in S \cap N$ and a countable set $Y \in N$ such that $t \leq_S u \restriction (\omega_1 \cap N)$, and

$$t \Vdash_S \text{“ } Y \subseteq \dot{X} \text{ and } \mathrm{cl}_{\dot{\tau}}(Y) = \mathrm{cl}_{\dot{\tau}}(\dot{X}) \text{ ”}.$$

We note that since $\dot{X}$ is an S-name for an uncountable subset of ω_1, it is true that

$$t \Vdash_S \text{“ } \mathrm{cl}_{\dot{\tau}}(Y) \text{ is uncountable ”}.$$

Let $\mathrm{dom}(r) = \{M_\zeta; \zeta < k\}$ and take $w_\zeta \in S$, $\zeta < k$, and $\delta \in \omega_1$ such that $(t \leq_S)\ u \leq_S w_0$, for each $\zeta < k$, $t^r_{M_\zeta} \leq_S w_\zeta$, all w_ζ are of the same level, $\delta > \max_{\zeta < k} \alpha^r_{M^r_\zeta}$, and

$$w_0 \Vdash_S \text{“ } \delta \in \mathrm{cl}_{\dot{\tau}}(Y) \text{ ”}.$$

Then by the property of $\dot{U}_\alpha$, we note that for each $\zeta < k$,

$$w_\zeta \Vdash_S \text{“ } \delta \notin \mathrm{cl}_{\dot{\tau}}(\dot{U}_{\alpha^r_{M_\zeta}}) \text{ ”}.$$

By induction on $\zeta < k$, using the condition ($\star$), we take an S-name $\dot{V}_\zeta$ such that

- $w_\zeta \Vdash_S$ “ $\delta \in \dot{V}_\zeta \in \dot{\tau}$ and $\mathrm{cl}_{\dot{\tau}}(\dot{U}_{\alpha^r_{M_\zeta}}) \cap \dot{V}_\zeta = \emptyset$ ”,
- for every $\zeta' \in k$, $w_{\zeta'} \Vdash_S$ “ $\psi_{w_\zeta, w_{\zeta'}}(\dot{V}_\zeta)$ is open in $\dot{\tau}$ ”, and
- $w_{\zeta+1} \Vdash_S$ “ $\dot{V}_{\zeta+1} \subseteq \psi_{w_\zeta, w_{\zeta+1}}(\dot{V}_\zeta)$ ”.

It follows from the last conditions that

$$\begin{aligned} w_{k-1} = \psi_{w_{k-2}, w_{k-1}}(w_{k-2}) \Vdash_S \text{“ } & \psi_{w_{k-2}, w_{k-1}}(\dot{V}_{k-2}) \\ & \subseteq \psi_{w_{k-2}, w_{k-1}}(\psi_{w_{k-3}, w_{k-2}}(\dot{V}_{k-3})) = \psi_{w_{k-3}, w_{k-1}}(\dot{V}_{k-3}) \text{ ”}, \end{aligned}$$

and so

$$w_{k-1} \Vdash_S \text{“ } \dot{V}_{k-1} \subseteq \psi_{w_{k-2}, w_{k-1}}(\dot{V}_{k-2}) \subseteq \psi_{w_{k-3}, w_{k-1}}(\dot{V}_{k-3}) \text{ ”}.$$

Therefore, by induction, for every $\zeta \in k$,

$$w_{k-1} \Vdash_S \text{“ } \dot{V}_{k-1} \subseteq \psi_{w_\zeta, w_{k-1}}(\dot{V}_\zeta) \text{ ”},$$

and hence it follows that

$$w_\zeta = \psi_{w_{k-1}, w_\zeta}(w_{k-1}) \Vdash_S \text{“ } \psi_{w_{k-1}, w_\zeta}(\dot{V}_{k-1}) \subseteq \psi_{w_{k-1}, w_\zeta}(\psi_{w_\zeta, w_{k-1}}(\dot{V}_\zeta)) = \dot{V}_\zeta \text{ ”}.$$

We take $\beta \in Y$ and $x \geq_S w_0$ such that $x \Vdash_S$ “ $\beta \in \psi_{w_{k-1}, w_0}(\dot{V}_{k-1})$ ”. Then for every $\zeta \in k$,

$$\psi_{w_0, w_\zeta}(x) \Vdash_S \text{“ } \beta \in \psi_{w_0, w_\zeta}(\psi_{w_{k-1}, w_0}(\dot{V}_{k-1})) = \psi_{w_{k-1}, w_\zeta}(\dot{V}_{k-1}) \subseteq \dot{V}_\zeta, \text{ hence } \beta \notin \dot{U}_{\alpha^r_{M_\zeta}} \text{ ”}.$$

Since it holds that $\beta \in Y \subseteq \omega_1 \cap N$, $\dot{X} \in N$, $t \leq_S u \leq_S w_0 \leq_S x$, $\mathsf{lv}(x) > \omega_1 \cap N$, and

$$t \Vdash_S \text{“ } Y \subseteq \dot{X} \text{ ”},$$

there exists $s \in S \cap N$ such that $s \leq_S x$ and

$$s \Vdash_S \text{“ } \beta \in \dot{X} \text{ ”}.$$

Then we note that $s \leq_S u$. Since $\beta \in Y \subseteq \omega_1 \cap N \subseteq M_\zeta$ and $t^r_{M_\zeta} \leq_S w_\zeta \leq_S \psi_{w_0, w_\zeta}(x)$, by the definition of conditions of $\mathbb{P}$, for every $\zeta < k$,

$$t^r_{M_\zeta} \Vdash_S \text{“ } \beta \notin \dot{U}_{\alpha^r_{M_\zeta}} \text{ ”},$$

which is what we want. ⊣(Lemma4.1)

Acknowledgements

This work has been done when the author was participating in a Researcher Exchange Program between JSPS and NSERC in 2010 hosted by Franklin D. Tall form April to August 2010. Then the author has taken a personal lecture from Franklin D. Tall about his paper [18], and has taken a lecture from Paul B. Larson about his preprint [6]. I would like to thank them. And I would like to thank Stevo Todorcevic for encouraging me about a study of $\mathsf{PFA}(S)$ and a useful comment of this note, thank to Dilip Raghavan for a comment of the introduction of this note.

I am thankful all the members of the Toronto set theory group, and to the staffs of the Fields Institute. A special thanks to Franklin D. Tall for his hospitality during the visit.

I have told the result of this note in Joint FWF–JSPS Seminar on Forcing in Set Theory, from January 23 to 28, 2012. Then I received some comments in the conference, especially Arthur Fischer. I would like to thank the organizers and participants of the conference.

Finally, I would like to thank the referee for careful reading and giving useful comments and suggestions.

References

[1] I. Juhász. *A survey of S- and L-spaces*. Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), pp. 675–688, Colloq. Math. Soc. János Bolyai, 23, North-Holland, Amsterdam-New York, 1980.

[2] B. König. *Trees, Games and Reflections*, Ph.D. thesis (2002) at the Ludwig-Maximilians-Universität München.

[3] P. Koszmider. *Models as side conditions*. Set theory (Cura?ao, 1995; Barcelona, 1996), 99–107, Kluwer Acad. Publ., Dordrecht, 1998.

[4] K. Kunen. *Set theory. An introduction to independence proofs*. Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1980.

[5] K. Kunen. *Set theory*. Studies in Logic (London), 34. College Publications, London, 2011.

[6] P. B. Larson. *Notes on Todorcevic's Erice lectures on forcing with a coherent Souslin tree*. Preprint, 2010.

[7] P. Larson and S. Todorcevic. *Chain conditions in maximal models*. Fund. Math. 168 (2001), no. 1, 77–104.

[8] P. Larson and S. Todorcevic. *Katětov's problem*. Trans. Amer. Math. Soc. 354 (2002), no. 5, 1783–1791.

[9] T. Miyamoto. *ω_1-Souslin trees under countable support iterations*. Fund. Math. 142 (1993), no. 3, 257–261.

[10] J. Moore. *A solution to the L space problem*. J. Amer. Math. Soc. 19 (2006), no. 3, 717–736.

[11] J. Moore. *An L space with a d-separable square*. Topology Appl. 155 (2008), no. 4, 304–307.

[12] J. Moore, M. Hrušák and M. Džamonja. *Parametrized $\diamondsuit$ principles*. Transactions of American Mathematical Society, 356 (2004), no. 6, 2281–2306.

[13] J. Roitman. *Basic S and L*. Handbook of set-theoretic topology, 295–326, North-Holland, Amsterdam, 1984.

[14] M. E. Rudin. *A normal hereditarily separable non-Lindelöf space*. Illinois J. Math. 16 (1972), 621–626.

[15] M. E. Rudin. *A non-normal hereditarily-separable space*. Illinois J. Math. 18 (1974), 481–483.

[16] M. E. Rudin. *S and L spaces*. Surveys in general topology, pp. 431–444, Academic Press, New York-London-Toronto, Ont., 1980.

[17] Z. Szentmiklóssy. *S-spaces and L-spaces under Martin's axiom*. Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), pp. 1139–1145, Colloq. Math. Soc. János Bolyai, 23, North-Holland, Amsterdam-New York, 1980.

[18] F. D. Tall. *PFA(S)[S]: more mutually consistent topological consequences of PFA and $V = L$*, Canad. J. Math. 64 (2012), no. 5, 1182–1200.

[19] S. Todorcevic. *A note on the proper forcing axiom*. Axiomatic set theory (Boulder, Colo., 1983), 209–218, Contemp. Math., 31, Amer. Math. Soc., Providence, RI, 1984.

[20] S. Todorcevic. *Trees and linearly ordered sets*. Handbook of set-theoretic topology, 235–293, North-Holland, Amsterdam, 1984.

[21] S. Todorcevic. *Partitioning pairs of countable ordinals*. Acta Math. 159 (1987), no. 3-4, 261–294.

[22] S. Todorcevic. *Partition Problems in Topology*. volume 84 of *Contemporary mathematics*. American Mathematical Society, Providence, Rhode Island, 1989.

[23] S. Todorcevic. *A dichotomy for P-ideals of countable sets*. Fund. Math. 166 (2000), no. 3, 251–267.

[24] S. Todorcevic. *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic 17 (2011), no. 1, 1–72.

[25] S. Todorcevic. *Forcing with a coherent Suslin tree*. Preprint, 2011.

[26] T. Yorioka. *Uniformizing ladder system colorings and the rectangle refining property*. Proc. Amer. Math. Soc. 138 (2010), no. 8, 2961–2971.