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Some consequences from Proper Forcing Axiom together with large continuum and the negation of Martin's Axiom

By Teruyuki Yorioka

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Abstract. Recently, David Asperó and Miguel Angel Mota discovered a new method of iterated forcing using models as side conditions. The side condition method with models was introduced by Stevo Todorčević in the 1980s. The Asperó–Mota iteration enables us to force some Π_2-statements over $H(\aleph_2)$ with the continuum greater than $\aleph_2$. In this article, by using the Asperó–Mota iteration, we prove that it is consistent that $\mho$ fails, there are no weak club guessing ladder systems, $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = 2^{\aleph_0} > \aleph_2$ and $\mathrm{MA}_{\aleph_1}$ fails.

1. Introduction.

Since Cohen's discovery of the method of forcing, many mathematical statements have been shown to be consistent with ZFC. The method of the iterated forcing has been used in many of these consistency results. This method was first introduced by Solovay and Tennenbaum in their solution of Suslin's problem [**20**], that is, they proved that it is consistent that Suslin's Hypothesis holds (every connected ccc ordered topology is separable). Their iteration used finite support, and forcing conditions with the countable chain condition, and there are known limits to this approach (e.g. [**1**]). Shelah introduced the notion of proper partial orders, and proved that the properness is preserved by countable support iterations [**18**]. This technique has been used to prove the consistency of many Π_2 statements over $H(\aleph_2)$. Todorčević introduced the method of forcing by use of countable elementary submodels of $H(\kappa)$ for some regular cardinal κ, so called the side condition method [**21**], [**22**], [**23**]. This method introduces many proper partial orders and is widely applicable. The first example of this method combining with countable support iterations is to show that it is consistent that every hereditarily separable regular space is hereditarily Linderöf (this shows the S-space problem which was one of longstanding open problems in general topology).

It has been known that countable support iterations cannot be used to prove the consistency result with the continuum greater than $\aleph_2$. In [**3**], Asperó and Mota introduced a new method of the forcing iteration. Their forcing iteration is defined with the side condition method. Their iteration can be used to show some consistency results

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with the continuum greater that $\aleph_2$, which cannot be shown by finite support iterations. One of this applications is the following problem due to Moore.

Moore formulated the axiom $\mho$ in [**16**] to show that his solution of the five element basis problem for the uncountable linear orders in [**15**] needs the Mapping Reflection Principle in some sense. Moore pointed out that $\mho$ can be forced by a countable support iteration (see also [**3**], [**17**]). In [**16**], Moore asked whether $\mho$ follows from the assumption that the continuum is greater than $\aleph_2$. In [**3**], Asperó and Mota introduced the finitely proper forcing notions and the forcing axiom $\mathrm{PFA}^\star(\omega_1)$ for finitely proper forcings of size $\aleph_1$. Since finite properness is a stronger condition than properness, $\mathrm{PFA}^\star(\omega_1)$ follows from the proper forcing axiom. And it is proved that $\mathrm{PFA}^\star(\omega_1)$ implies the negation of $\mho$. They proved by use of their new iteration that it is consistent that $\mathrm{PFA}^\star(\omega_1)$ holds and the continuum is greater than $\aleph_2$. This answers Moore's question negatively.

$\mathrm{PFA}^\star(\omega_1)$ implies many Π_2-statements over $H(\aleph_2)$. One is Martin's Axiom for $\aleph_1$ many dense sets, denoted by $\mathrm{MA}_{\aleph_1}$, which has been introduced by Solovay and Tennenbaum. $\mathrm{MA}_{\aleph_1}$ implies that Suslin's Hypothesis, that is, every connected linearly ordered set which has the countable chain condition is separable. As said above, $\mathrm{PFA}^\star(\omega_1)$ implies the failure of $\mho$. As other examples, $\mathrm{PFA}^\star(\omega_1)$ implies that there are no weak club guessing sequences (see Section 2 for the definition). In [**4**], Asperó and Mota introduced the class Υ of forcing notions, which is a somewhat large class of posets with the $\aleph_2$-chain condition, and the forcing axiom $\mathrm{MA}_{<2^{\aleph_0}}(\Upsilon)$. By use of their iteration, it is proved that it is consistent that $\mathrm{MA}_{<2^{\aleph_0}}(\Upsilon)$ holds and $2^{\aleph_0} > \aleph_2$. $\mathrm{MA}_{<2^{\aleph_0}}(\Upsilon)$ implies both $\mathrm{PFA}^\star(\omega_1)$ and Martin's Axiom. Martin's Axiom implies that all cardinal invariants of the reals are equal to the continuum. In this paper, we deal with two small cardinal invariants: the pseudo-intersection number $\mathfrak{p}$ and the additivity of the measure $\mathrm{add}(\mathcal{N})$ (see Section 2 for the definitions). It should be pointed out that almost all cardinal invariants of the reals are not smaller than $\mathfrak{p}$ or $\mathrm{add}(\mathcal{N})$.

Their iteration seems to be the large possibility to introduce lots of new consistency results. To make it clear, it should be proved some preservation properties of the Asperó–Mota iteration. The aim of this paper is to give a preservation theorem of the Asperó–Mota iteration for some ccc structures, and to introduce the new consistency result using this property. More precisely, a preservation theorem of the Asperó–Mota iteration for certain destrtuctible gaps is given, and it is shown that it is consistent that $\mho$ fails no weak club guessing sequences, both $\mathfrak{p}$ and $\mathrm{add}(\mathcal{N})$ are equal to the continuum, and there exists a destructible gap.

This paper consists of the following sections. In Section 2, it is given the definitions of $\mho$, a weak club guessing sequence, $\mathfrak{t}$, $\mathrm{add}(\mathcal{N})$ and a special type of destructible gap in $\mathcal{P}(\omega)/\mathrm{fin}$, which is introduced by Todorčević [**24**]. Such a gap is one of ccc structures like a Suslin tree. In Section 3, we define $\mathbb{P}_\kappa$ by use of the Asperó–Mota iteration, and show that $\mathbb{P}_\kappa$ doesn't collapse any cardinals and forces that $\mho$ fails, there are no weak club guessing ladder systems and $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = 2^{\aleph_0} = \kappa$ (Theorem 3.5). In Section 4, we show that the forcing notion $\mathbb{P}_\kappa$ may not force $\mathrm{MA}_{\aleph_1}$. So it is consistent that $\mho$ fails, there are no weak club guessing ladder systems, $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = 2^{\aleph_0} > \aleph_2$, and $\mathrm{MA}_{\aleph_1}$ fails (Theorem 4.4). To show this, we use a special type of destructible gap in $\mathcal{P}(\omega)/\mathrm{fin}$, which is introduced by Todorčević [**24**]. The argument in Section 4 can be applied to other ccc structures like Suslin trees.

Notation and terminology in this article are quite standard in Set Theory, e.g. [**12**], [**13**], [**14**]. In this article, basics of forcing theory and elementary substructures are required, which can be found for example in [**12**], [**13**], [**14**]. The forcing notion in this article is an application of Asperó–Mota's sophisticated technique of [**3**]. But we don't assume any knowledge of the Asperó–Mota iteration. All of the proofs are fairly self-contained.

2. Preliminaries.

Throughout this article, let Lim denote the class of limit ordinals.

2.1. $\mho$.

$\mho$ is the statement that there is a sequence $\langle f_\xi; \xi \in \omega_1\rangle$ such that for each $\xi \in \omega_1$, f_ξ is a continuous function from ξ into ω and for every club C on ω_1, there exists $\xi \in \omega_1$ such that $f_\xi[C \cap \xi] = \omega$ [**16**]. Such a sequence $\langle f_\xi; \xi \in \omega_1\rangle$ is called a $\mho$-sequence. Let $\xi \in (\omega_1 \cap \mathrm{Lim}) \setminus \omega$ and f a continuous function from ξ into ω. Then for each $i \in \xi \cap \mathrm{Lim}$, the value of $f(i)$ is eventually equal to the values $f(j)$ for $j < i$, and so the set $\{i \in \xi; f(i+1) \neq f(i)\}$ is of order type $\leq \omega$. So there exists $B \subseteq \xi$ such that B is of order type $\leq \omega$ and for each $i \in \xi$, the value $f(i)$ is decided by the cardinality of the set $B \cap i$. We notice that there exists $k \in \omega$ such that for cofinally many $i \in \xi$, $f(i) \neq k$.

To simplify terminology, in this article we don't mention domains and ranges of continuous functions when they are obvious from the context.

2.2. Weak club guessing ladder systems.

A sequence $\langle C_\xi; \xi \in \omega_1 \cap \mathrm{Lim}\rangle$ is called a ladder system if for any $\xi \in \omega_1 \cap \mathrm{Lim}$, C_ξ is a cofinal subset of ξ and is of order type ω. A ladder system $\langle C_\xi; \xi \in \omega_1 \cap \mathrm{Lim}\rangle$ is called weak club guessing if for any club E on ω_1, there exists $\xi \in \omega_1 \cap \mathrm{Lim}$ such that $C_\xi \cap E$ is cofinal (i.e infinite) in ξ. In [**3**], Asperó and Mota pointed out that the random forcing preserves the statement that there are no weak club guessing ladder systems (in [**3**], they said that they learned this from Michael Hrusak). Shelah proved that a weak club guessing sequence cannot be destroyed by ω-proper forcings (see e.g. [**11**, Proposition 5.2]). Asperó and Mota noted that $\mathrm{PFA}^\star(\omega_1)$ implies that there are no weak club guessing ladder systems [**3**].

In this article, let $\mathcal{PF}$ denote the set of all finite partial functions that can be extended to a strictly increasing and continuous functions f from ω_1 into ω_1. A strictly increasing and continuous function on ω_1 is the increasing enumeration of some club subset of ω_1.

2.3. Cardinal invariants.

A family X of infinite subsets of ω has the (strong) finite intersection property if every finite subfamily of X has an infinite intersection. The pseudo-intersection number $\mathfrak{p}$ is the minimal size of a family X of infinite subsets of ω which has the strong finite intersection property such that no infinite subset of ω is contained mod-finite in each member of X (see e.g. [**10**, 6.22 Definition]). Bell proved that if $\lambda < \mathfrak{p}$, then for any

σ-centered forcing $\mathbb{P}$ and any family $\mathcal{D}$ of λ-many dense subsets of $\mathbb{P}$, there exists a $\mathcal{D}$-generic filter over $\mathbb{P}$ [**9**] (see also [**10**, 7.12 Theorem] and [**26**, Theorem 3.1]). So the statement $\mathfrak{p} = 2^{\aleph_0}$ can be considered as a weak fragment of Martin's Axiom.

A slalom is a function in the set $\prod_{n\in\omega}\left([\omega]^{\leq n+1}\setminus\{\emptyset\}\right)$. For a function f in ω^ω and a slalom φ, we say that φ captures f if for all but finitely many $n \in \omega$, $f(n) \in \varphi(n)$. Bartoszyński proved that the additivity of the null ideal is equal to the smallest size of a set F of functions in ω^ω such that for every slalom φ, there exists a member of F which is not captured by φ [**5**]. In this article, let $\mathcal{S}$ denote the set of all finite initial segments of slaloms. Given a set F of functions in ω^ω, the following is a well known forcing notion adding a slalom which captures all functions in the set F (see e.g. [**7**, Section 3.1]):

$$\mathbf{LOC}(F) := \left\{\langle p(0), p(1)\rangle \in \mathcal{S}\times [F]^{<\aleph_0}; |p(1)| \leq |p(0)|\right\},$$

and for each $p = \langle p(0), p(1)\rangle$ and $q = \langle q(0), q(1)\rangle$ in $\mathbf{LOC}(F)$, $q \leq_{\mathbf{LOC}(F)} p$ if and only if $q(0)$ end-extends $p(0)$, $q(1) \supseteq p(1)$ and for each $f \in p(1)$ and $n \in \mathrm{dom}(q(0))\setminus\mathrm{dom}(p(0))$, $f(n) \in q(0)(n)$. We note that for any $\mathbf{LOC}(F)$-generic filter G, the set $\bigcup_{p\in G} p(0)$ is a slalom which captures all functions in F. This is called a localization forcing.

We note that almost all cardinal invariants of the reals are larger than or equal to $\mathfrak{p}$ or $\mathrm{add}(\mathcal{N})$. See e.g. [**6**], [**10**].

2.4. Gaps in $\mathcal{P}(\omega)/\mathrm{fin}$.

A pregap in $\mathcal{P}(\omega)/\mathrm{fin}$ is a pair $(\mathcal{A},\mathcal{B})$ of subsets of $\mathcal{P}(\omega)$ such that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the set $a \cap b$ is finite (denoted by $a \perp b$). For subsets a and b of ω, we say that a is almost contained in b (denoted by $a \subseteq^* b$) if the set $a \setminus l$ is a subset of b for some $l \in \omega$. For a pregap $(\mathcal{A},\mathcal{B})$, if both $\mathcal{A}$ and $\mathcal{B}$ are well-ordered by $\subseteq^*$and these order types are κ and λ respectively, then we say that the pregap $(\mathcal{A},\mathcal{B})$ has type (κ,λ) or that $(\mathcal{A},\mathcal{B})$ is a (κ,λ)-pregap. For a pregap $(\mathcal{A},\mathcal{B})$ and $c \in \mathcal{P}(\omega)$, we say that c separates $(\mathcal{A},\mathcal{B})$ if $a \subseteq^* c$ and $c \perp b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If a pregap is not separated, we say that it is a gap. Moreover if a gap has type (κ,λ), it is called a (κ,λ)-gap. An (ω_1,ω_1)-pregap is called destructible if it can be destroyed by a forcing extension preserving cardinals. A destructible gap is an (ω_1,ω_1)-gap which is destructible. (For information on gaps and related notions, see e.g. [**19**].)

For an (ω_1,ω_1)-pregap $(\mathcal{A},\mathcal{B}) = \langle a_\xi, b_\xi; \xi\in\omega_1\rangle$ with $a_\xi \cap b_\xi$ empty for every $\xi \in \omega_1$, we say here that ξ and η in ω_1 are compatible if

$$(a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi) = \emptyset.$$

Then, by the characterization due to Kunen and Todorčević, we notice that an (ω_1,ω_1)-pregap is a gap if and only if it has no uncountable pairwise compatible subsets of ω_1, and that it is a destructible gap if and only if it has neither uncountable pairwise compatible subsets of ω_1 nor uncountable pairwise incompatible subsets of ω_1. Therefore Aronszajn trees and (ω_1,ω_1)-gaps have analogous characterizations, and so do Suslin trees and destructible gaps (see [**2**], also [**29**], [**30**]). In fact, we have many analogies with respect to their existence (see e.g. [**27**], [**28**]). It is known and easy to see that $\mathrm{MA}_{\aleph_1}$ implies that there are no Suslin trees and no destructible gaps.

For an (ω_1,ω_1)-pregap $(\mathcal{A},\mathcal{B}) = \langle a_\xi, b_\xi; \xi\in\omega_1\rangle$ with $a_\xi \cap b_\xi$ empty for every $\xi \in \omega_1$, we define the transitive relation $\lhd_{(\mathcal{A},\mathcal{B})}$ on ω_1 such that for ξ and η in ω_1,

$$\xi \lhd_{(\mathcal{A},\mathcal{B})} \eta :\iff \xi < \eta \ \&\ a_\xi \subseteq a_\eta \ \&\ b_\xi \subseteq b_\eta.$$

In this article, we use the following.

DEFINITION 2.1 (Todorčević [**24**]). *An (ω_1,ω_1)-pregap $(\mathcal{A},\mathcal{B}) = \langle a_\xi, b_\xi; \xi \in \omega_1\rangle$ (with $a_\xi \cap b_\xi$ empty for every $\xi \in \omega_1$) satisfies the property* (t) *if for any uncountable subset I of ω_1, there exist ξ and η in I with $\xi < \eta$ such that*

$$a_\xi \subseteq a_\eta \text{ and } b_\xi \subseteq b_\eta.$$

That is, $(\mathcal{A},\mathcal{B})$ satisfies the property (t) *if and only if every $\lhd_{(\mathcal{A},\mathcal{B})}$-incomparable subset of ω_1 is countable.*

A T-gap is an (ω_1,ω_1)-gap which satisfies the property (t).

By the definition, we note that a T-gap is a destructible gap. In [**24**], Todorčević proved that it is consistent that there exists a T-gap, and it is consistent that there exists a destructible gap but there are no T-gaps. For example, $\diamondsuit$ implies the existence of a T-gap. In the following proof, we use a Cohen real. This is the easiest proof as far as the author knows. This proof is in [**25**, Theorem 9.3].

LEMMA 2.2 (Todorčević). *It is consistent that there exists a T-gap.*

PROOF. Let $\langle a_\xi, b_\xi; \xi \in \omega_1\rangle$ be an (ω_1,ω_1)-gap such that for each $\xi \in \omega_1$, $a_\xi \cap b_\xi = \emptyset$. (This always exists as shown by Hausdorff.) We write $\mathbb{C} = (2^{<\omega}, \supseteq)$ as Cohen forcing, and here we consider that $\Vdash_{\mathbb{C}}$“ $\dot{c} = \left\{n \in \omega; \left(\bigcup \dot{G}\right)(n) = 1\right\}$ ”. We will show that $\langle a_\xi \cap \dot{c}, b_\xi \cap \dot{c}; \xi \in \omega_1\rangle$ is a $\mathbb{C}$-name for a T-gap.

First, we will show that in the extension with $\mathbb{C}$, $\langle a_\xi \cap \dot{c}, b_\xi \cap \dot{c}; \xi \in \omega_1\rangle$ still forms a gap. Let $\dot{I}$ be a $\mathbb{C}$-name for an uncountable subset of ω_1 and let $p \in \mathbb{C}$. We can take a sequence $\langle p_i, \xi_i; i \in \omega_1\rangle$ such that

- $p_i \leq_{\mathbb{C}} p$ and $\xi_i \in \omega_1$,
- $p_i \Vdash_{\mathbb{C}}$“ $\xi_i \in \dot{I}$ ”, and
- if $i < j < \omega_1$, then $\xi_i < \xi_j$.

Since $\mathbb{C}$ is countable, there exists $p' \in \mathbb{C}$ such that the set $\{i \in \omega_1; p_i = p'\}$ is uncountable. Since the sequence

$$\langle a_{\xi_i} \setminus \mathrm{dom}(p'), b_{\xi_i} \setminus \mathrm{dom}(p'); i \in \omega_1 \ \&\ p_i = p'\rangle$$

is also a gap, there are i and j in ω_1 and $l \in \omega$ such that $i < j$, $p_i = p_j = p'$ and

$$l \in \left((a_{\xi_i} \setminus \mathrm{dom}(p')) \cap \left(b_{\xi_j} \setminus \mathrm{dom}(p')\right)\right) \cup \left(\left(a_{\xi_j} \setminus \mathrm{dom}(p')\right) \cap (b_{\xi_i} \setminus \mathrm{dom}(p'))\right).$$

Let $q \in 2^{l+1}$ be such that $q \supseteq p'$ and for each $n \in (l+1) \setminus \mathrm{dom}(p')$, $q(n) = 1$. Then we note that

$$q \Vdash_{\mathbb{C}} \text{“} \big((a_{\xi_i} \cap \dot{c}) \cap \big(b_{\xi_j} \cap \dot{c}\big)\big) \cup \big(\big(a_{\xi_j} \cap \dot{c}\big) \cap (b_{\xi_i} \cap \dot{c})\big) \neq \emptyset \text{”}.$$

It follows that

$$\Vdash_{\mathbb{C}} \text{“} \dot{I} \text{ is not pairwise compatible with respect to } \langle a_\xi \cap \dot{c}, b_\xi \cap \dot{c}; \xi \in \omega_1 \rangle \text{”}.$$

Secondly, we will show that in the extension with $\mathbb{C}$, $\langle a_\xi \cap \dot{c}, b_\xi \cap \dot{c}; \xi \in \omega_1 \rangle$ satisfies the property (t). Let $\dot{I}$ be a $\mathbb{C}$-name for an uncountable subset of ω_1 and let $p \in \mathbb{C}$. Then as seen above, we can take a sequence $\langle p_i, \xi_i; i \in \omega_1 \rangle$ such that

- $p_i \leq_{\mathbb{C}} p$ and $\xi_i \in \omega_1$,
- $p_i \Vdash_{\mathbb{C}}$“$\xi_i \in \dot{I}$”, and
- if $i < j < \omega_1$, then $\xi_i < \xi_j$.

Since $\mathbb{C}$ is countable, there exists $p' \in \mathbb{C}$ such that the set $\{i \in \omega_1; p_i = p'\}$ is uncountable. Take i and j in ω_1 such that $i < j$, $p_i = p_j = p'$ and

$$a_{\xi_i} \cap \mathrm{dom}(p') = a_{\xi_j} \cap \mathrm{dom}(p') \text{ and } b_{\xi_i} \cap \mathrm{dom}(p') = b_{\xi_j} \cap \mathrm{dom}(p').$$

Then there exists $l \in \omega$ with $l \geq \mathrm{dom}(p')$ such that

$$a_{\xi_i} \setminus l \subseteq a_{\xi_j} \text{ and } b_{\xi_i} \setminus l \subseteq b_{\xi_j}.$$

Let $q \in 2^l$ be such that $q \supseteq p'$ and for each $n \in l \setminus \mathrm{dom}(p')$, $q(n) = 0$. Then we note that

$$q \Vdash_{\mathbb{C}} \text{“} a_{\xi_i} \cap \dot{c} \subseteq a_{\xi_j} \cap \dot{c} \text{ and } b_{\xi_i} \cap \dot{c} \subseteq b_{\xi_j} \cap \dot{c} \text{”},$$

which finishes the proof. □

3. The definition and basics of $\mathbb{P}_\kappa$.

We define the forcing notion $\mathbb{P}_\kappa$ which is used in this article. The present forcing construction takes place in the framework from [**3**], [**4**], although the notation we are using here differs somewhat from the one in [**3**], [**4**].

Throughout this article, suppose that

- κ is an uncountable regular cardinal such that $2^{<\kappa} = \kappa$,
- Φ is a surjection from κ to $H(\kappa)$ such that for every $x \in H(\kappa)$, $\Phi^{-1}[\{x\}]$ is unbounded in κ,
- $\langle \theta_\alpha; \alpha \in \kappa + 1 \rangle$ is a sequence of regular cardinals which is increasing fast enough, that is, $H(\kappa) \in H(\theta_0)$ and for each $\alpha \in \kappa$, $\langle H(\theta_\beta), \triangle_\beta; \beta < \alpha \rangle$ belongs to $H(\theta_\alpha)$ [1], where $\triangle_\beta$ is a well-ordering of $H(\theta_\beta)$,

[1]For example, in [**3**], it is defined that $\theta_0 = (2^\kappa)^+$ and $\theta_\alpha := \left(2^{\sup\{\theta_\beta; \beta \in \alpha\}}\right)^+$ for each $\alpha \in \kappa$.

- for each $\alpha \in \kappa + 1$, $\mathcal{M}^*_\alpha$ is the set of countable elementary substructures of $H(\theta_\alpha)$ which contain the set

$$\Big\{\omega_1, H(\kappa), \Phi, \big\langle H(\theta_\beta), \triangle_\beta, \mathcal{M}^*_\beta; \beta \in \alpha\big\rangle\Big\}$$

as a member,

- for each $\alpha \in \kappa + 1$, $\mathcal{M}_\alpha := \{N \cap H(\kappa); N \in \mathcal{M}^*_\alpha\}$.

We note that for each $\alpha < \beta < \kappa + 1$ and $M^* \in \mathcal{M}^*_\beta$, if $\alpha \in M^*$, then $M^* \cap H(\theta_\alpha)$ belongs to $\mathcal{M}^*_\alpha$, and in this case, $M^* \cap H(\kappa)$ belongs to both $\mathcal{M}_\beta$ and $\mathcal{M}_\alpha$. Therefore for every $\alpha \in \kappa + 1$, $\mathcal{M}_\alpha \subseteq \mathcal{M}_0$.

For each $M \in \mathcal{M}_0$, let $\overline{M}$ denote the transitive collapse of M, and let Ψ_M denote the transitive collapsing map from M onto $\overline{M}$. We always consider the members of $\bigcup_{\alpha\in\kappa+1} \mathcal{M}_\alpha$ as substructures of the structure $\Big\langle H(\kappa), \in, \omega_1, \Phi\Big\rangle$. (This situation is the same as in [**22**, Section 4].) So when M and M' in $\mathcal{M}_\alpha$ have the same transitive collapse in this sense, the composition ${\Psi_{M'}}^{-1} \circ \Psi_M$ is an isomorphism from the structure $\Big\langle M, \in, \omega_1, \Phi\restriction M\Big\rangle$ onto the structure $\Big\langle M', \in, \omega_1, \Phi\restriction M'\Big\rangle$. For each $M \in \mathcal{M}_0$, since M is countable and ω_1 is of uncountable cofinality, it is true that $\sup(\omega_1 \cap M) < \omega_1$, and moreover $\omega_1 \cap M$ is a countable ordinal. And if M and M' in $\mathcal{M}_0$ are isomorphic in the above sense, then $\omega_1 \cap M = \omega_1 \cap M'$.

For each $\alpha \in (\kappa+1) \setminus \{0\}$, we will define the forcing notion $\mathbb{P}_\alpha$ as a subset of the set

$$\left(\bigcup_{\mathcal{N}\in[\mathcal{M}_0]^{<\aleph_0}} {}^{\mathcal{N}}(\alpha+1)\right) \times \prod_{[1,\alpha)}^{\text{finite support}} H(\kappa).$$

We note that the set $[\mathcal{M}_0]^{<\aleph_0}$ is a subset of $H(\kappa)$. Therefore for each $\alpha \in \kappa \setminus \{0\}$, each $\mathbb{P}_\alpha$ is a subset of $H(\kappa)$. As seen below, for each $\alpha \in \kappa$, $\mathbb{P}_{\alpha+1}$ is defined from the set

$$\Big\{\omega_1, H(\kappa), \Phi, \big\langle H(\theta_\beta), \triangle_\beta, \mathcal{M}^*_\beta; \beta \in \alpha+2\big\rangle\Big\},$$

and for each $\alpha \in (\kappa+1) \cap \mathrm{Lim}$, $\mathbb{P}_\alpha$ is defined from the set

$$\Big\{\omega_1, H(\kappa), \Phi, \big\langle H(\theta_\beta), \triangle_\beta, \mathcal{M}^*_\beta; \beta \in \alpha\big\rangle\Big\}.$$

Therefore, for each $\alpha \in \kappa$, every element of $\mathcal{M}^*_{\alpha+2}$ contains $\mathbb{P}_{\alpha+1}$ as a member, and for each $\alpha \in (\kappa+1) \cap \mathrm{Lim}$, every element of $\mathcal{M}^*_\alpha$ contains $\mathbb{P}_\alpha$ as a member. We will define the conditions p of $\mathbb{P}_\alpha$ such that $\mathrm{dom}(p(0))$ is a finite system of members of $\mathcal{M}_0$ as in [**22**, Section 4]. $p(0)$ works for properness of $\mathbb{P}_\alpha$ and the $\left(2^{\aleph_0}\right)^+$-chain condition of $\mathbb{P}_\alpha$. To simplify notation, for each $p \in \mathbb{P}_\alpha$ and $M \in \mathrm{dom}(p(0))$, we write

$$p(0, M) := p(0)(M).$$

To define $\mathbb{P}_\alpha$, we introduce following notation. For each $\alpha < \beta < \kappa + 1$ and $p \in \mathbb{P}_\beta$, $p{\downarrow}\alpha$ is defined as the function with domain α such that

- for each $\gamma \in \alpha \setminus \{0\}$, $(p{\downarrow}\alpha)(\gamma) := p(\gamma)$, and
- $(p{\downarrow}\alpha)(0)$ is the function with domain $\mathrm{dom}(p(0))$ such that for each $M \in \mathrm{dom}(p(0))$, $(p{\downarrow}\alpha)(0, M) := \min\{p(0, M), \alpha\}$.

For a function f and a set B, we denote $f{\restriction}B$ as the usual restricted function, that is, the function f restricted to the domain $\mathrm{dom}(f) \cap B$. For each $\alpha \in (\kappa + 1) \setminus \{0\}$ and $p \in \mathbb{P}_\alpha$, we define

$$\mathrm{supp}(p) := \Big\{\gamma \in \alpha \setminus \{0\}; p(\gamma) \neq \emptyset\Big\}.$$

DEFINITION 3.1. *The forcing notion $\mathbb{P}_\alpha$ is defined by induction on $\alpha \in \kappa + 1$ as follows.*

Basic stage: *$\mathbb{P}_1$ consists of functions p with domain 1 $(= \{0\})$ such that $p(0)$ is a function with $\mathrm{ran}(p(0)) = \{1\}$ such that*

(0-1) *$\mathrm{dom}(p(0))$ is a finite subset of $\mathcal{M}_0$,*

(0-2) *for each $M, M' \in \mathrm{dom}(p(0))$, if $\omega_1 \cap M = \omega_1 \cap M'$, then $\overline{M} = \overline{M'}$,*

(0-3) *for each $M, M' \in \mathrm{dom}(p(0))$, if $\omega_1 \cap M' < \omega_1 \cap M$, then*

- *$\overline{M'} \in M$, and*
- *there exists $M'' \in \mathrm{dom}(p(0))$ such that $\overline{M''} = \overline{M}$ and $M' \in M''$* [2],

and

(0-4) *for each $M, M' \in \mathrm{dom}(p(0))$, if $\omega_1 \cap M = \omega_1 \cap M'$, then the function $\Big({\Psi_{M'}}^{-1} \circ \Psi_M\Big){\restriction}(\kappa \cap M \cap M')$ is identity* [3],

and the order on $\mathbb{P}_1$ is defined so that for each $p, q \in \mathbb{P}_1$, $q \leq_{\mathbb{P}_1} p$ if and only if $\mathrm{dom}(q(0)) \supseteq \mathrm{dom}(p(0))$.

Successor stages: *Let $\alpha \in \kappa \setminus \{0\}$ and suppose that $\mathbb{P}_\alpha$ is defined. To define $\mathbb{P}_{\alpha+1}$, we consider the following cases.*

Case 1. *$\Phi(\alpha)$ is a sequence $\Big\langle \dot{f}^\alpha_\xi; \xi \in \omega_1 \Big\rangle$ such that for each $\xi \in \omega_1$, $\dot{f}^\alpha_\xi$ is a $\mathbb{P}_\alpha$-name for a continuous function from ξ into ω.*

Then $\mathbb{P}_{\alpha+1}$ consists of the functions p with domain $\alpha + 1$ such that

1. *$p{\downarrow}\alpha \in \mathbb{P}_\alpha$,*
2. *$p(0)$ is a function into $(\alpha + 2) \setminus \{0\}$,*
3. *$p(\alpha)$ is a pair $\langle p(\alpha, 0), p(\alpha, 1)\rangle$ such that*

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[2]This property is a basic property of the $\aleph_2$-pic version of the side condition method. See e.g. [**22**, Section 4].

[3]This property comes from the Asperó–Mota iteration [**3**]. As said in [**3**], this property is used only when the Asperó–Mota iteration of length a limit ordinal of uncountable cofinality is proper.

(α-1) $p(\alpha,0)$ *is a member of* $\mathcal{PF}$ *(here, recall that* $\mathcal{PF}$ *is the set of all finite partial functions that can be extended to a strictly increasing and continuous functions* f *from* ω_1 *into* ω_1 *Section* 2.2*),*

(α-2) $p(\alpha,1)$ *is a finite partial function from* ω_1 *into* ω*,*

(α-3) *for each* $\xi \in \mathrm{dom}(p(\alpha,1))$*,*

$$p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“ for cofinally many } i \in \xi,\ \dot{f}^\alpha_\xi(i) \neq p(\alpha,1)(\xi), \text{ and } p(\alpha,1)(\xi) \notin \dot{f}^\alpha_\xi\Big[\mathrm{ran}(p(\alpha,0))\Big] \text{ ”},$$

and

(α-4) *for each* $M \in \mathrm{dom}(p(0)) \cap \mathcal{M}_{\alpha+1}$*, if* $p(0,M) = \alpha+1$*, then*

- $p(\alpha,0){\restriction}(\omega_1 \cap M)$ *is a partial function from* $\omega_1 \cap M$ *into* $\omega_1 \cap M$*,*
- $\omega_1 \cap M \in \mathrm{dom}(p(\alpha,0))$ *and* $p(\alpha,0)(\omega_1 \cap M) = \omega_1 \cap M$*,*
- *for each* $\xi \in \mathrm{dom}(p(\alpha,1))$ *with* $\omega_1 \cap M < \xi$*,*

$$p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{f}^\alpha_\xi(\omega_1 \cap M) \neq p(\alpha,1)(\xi) \text{ ”},$$

  and

- *if* $\omega_1 \cap M \in \mathrm{dom}(p(\alpha,1))$*, then for any* $x \in [M]^{<\aleph_0}$*,*

$$\begin{gathered} p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“there are } q \in \dot{G}_{\mathbb{P}_\alpha},\ M' \in \mathrm{dom}(q(0)) \cap \mathcal{M}_{\alpha+1} \cap M \\ \text{such that } x \in M', q(0,M') = \alpha, \text{ and } \dot{f}^\alpha_{\omega_1\cap M}(\omega_1 \cap M') \neq p(\alpha,1)(\omega_1 \cap M) \text{ ”}, \end{gathered}$$

and the order on $\mathbb{P}_{\alpha+1}$ *is defined so that for each* $p,q \in \mathbb{P}_{\alpha+1}$*,* $q \leq_{\mathbb{P}_{\alpha+1}} p$ *if and only if*

- $q{\downarrow}\alpha \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$*,*
- *for each* $M \in \mathrm{dom}(p(0))$*,* $q(0,M) \geq p(0,M)$*,*
- $q(\alpha,0) \supseteq p(\alpha,0)$*, and*
- $q(\alpha,1) \supseteq p(\alpha,1)$*.*

Case 2. $\Phi(\alpha)$ *is a sequence* $\left\langle \dot{C}^\alpha_\xi ; \xi \in \omega_1 \right\rangle$ *such that for each* $\xi \in \omega_1 \cap \mathrm{Lim}$*,* $\dot{C}^\alpha_\xi$ *is a* $\mathbb{P}_\alpha$*-name for a cofinal subset of* ξ *of order type* ω*, and for each* $\xi \in \omega_1 \setminus \mathrm{Lim}$*,* $\dot{C}^\alpha_\xi = \{\xi - 1\}$*.*

Then $\mathbb{P}_{\alpha+1}$ *consists of the functions* p *with domain* $\alpha+1$ *such that*

1. $p{\downarrow}\alpha \in \mathbb{P}_\alpha$*,*
2. $p(0)$ *is a function into* $(\alpha+2) \setminus \{0\}$*,*
3. $p(\alpha)$ *is a pair* $\langle p(\alpha,0), p(\alpha,1)\rangle$ *such that*

   (α-1) $p(\alpha,0)$ *is a member of* $\mathcal{PF}$*,*

(α-2) *$p(\alpha,1)$ is a regressive finite partial function from $\omega_1 \cap \mathrm{Lim}$ into ω_1,*

(α-3) *for each $\xi \in \mathrm{dom}(p(\alpha,1))$,*

$$p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} \mathrm{ran}(p(\alpha,0)) \cap \dot{C}^\alpha_\xi \subseteq p(\alpha,1)(\xi) \text{”},$$

and

(α-4) *for each $M \in \mathrm{dom}(p(0))$, if there exists $M' \in \mathrm{dom}(p(0)) \cap \mathcal{M}_{\alpha+1}$ such that $\omega_1 \cap M' = \omega_1 \cap M$ and $p(0,M') = \alpha+1$, then*

- *$p(\alpha,0) \restriction (\omega_1 \cap M)$ is a partial function from $\omega_1 \cap M$ into $\omega_1 \cap M$,*
- *$\omega_1 \cap M \in \mathrm{dom}(p(\alpha,0))$ and $p(\alpha,0)(\omega_1 \cap M) = \omega_1 \cap M$, and*
- *for each $\xi \in \mathrm{dom}(p(\alpha,1))$,*

$$p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\omega_1 \cap M \notin \dot{C}^\alpha_\xi \cap p(\alpha,1)(\xi) \text{”},$$

and the order on $\mathbb{P}_{\alpha+1}$ is defined so that for each $p, q \in \mathbb{P}_{\alpha+1}$, $q \leq_{\mathbb{P}_{\alpha+1}} p$ if and only if

- *$q{\downarrow}\alpha \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$,*
- *for each $M \in \mathrm{dom}(p(0))$, $q(0,M) \geq p(0,M)$,*
- *$q(\alpha,0) \supseteq p(\alpha,0)$, and*
- *$q(\alpha,1) \supseteq p(\alpha,1)$.*

Case 3. *$\Phi(\alpha)$ is a sequence $\left\langle \dot{x}^\alpha_\xi ; \xi \in \nu \right\rangle$ of $\mathbb{P}_\alpha$-names for infinite subsets of ω, for some $\nu \in \kappa$, such that*

$$\Vdash_{\mathbb{P}_\alpha} \text{“} \left\{ \dot{x}^\alpha_\xi ; \xi \in \nu \right\} \text{ satisfies the finite intersection property”}.$$

Then $\mathbb{P}_{\alpha+1}$ consists of the functions p with domain $\alpha+1$ such that

1. *$p{\downarrow}\alpha \in \mathbb{P}_\alpha$,*
2. *$p(0)$ is a function into $(\alpha+2) \setminus \{0\}$, and*
3. *$p(\alpha)$ is a pair $\langle p(\alpha,0), p(\alpha,1) \rangle$ such that $p(\alpha,0)$ is a finite subset of ω and $p(\alpha,1)$is a finite subset of ν,*

and the order on $\mathbb{P}_{\alpha+1}$ is defined so that for each $p, q \in \mathbb{P}_{\alpha+1}$, $q \leq_{\mathbb{P}_{\alpha+1}} p$ if and only if

- *$q{\downarrow}\alpha \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$,*
- *for each $M \in \mathrm{dom}(p(0))$, $q(0,M) \geq p(0,M)$,*
- *$q(\alpha,0)$ end-extends $p(\alpha,0)$,*
- *$q(\alpha,1) \supseteq p(\alpha,1)$, and*

- *for each $\xi \in p(\alpha, 1)$,*

$$q{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}q(\alpha,0) \setminus p(\alpha,0) \subseteq \dot{x}^\alpha_\xi \text{”}.$$

Case 4. $\Phi(\alpha)$ *is a sequence* $\left\langle \dot{g}^\alpha_\xi; \xi \in \nu \right\rangle$ *of $\mathbb{P}_\alpha$-names for functions in ω^ω, for some $\nu \in \kappa$.* *Then $\mathbb{P}_{\alpha+1}$ consists of the functions p with domain $\alpha + 1$ such that*

1. $p{\downarrow}\alpha \in \mathbb{P}_\alpha$,
2. $p(0)$ *is a function into* $(\alpha + 2) \setminus \{0\}$,
3. $p(\alpha)$ *is a pair* $\langle p(\alpha, 0), p(\alpha, 1)\rangle$ *such that* $p(\alpha, 0) \in \mathcal{S}$ *(recall that $\mathcal{S}$ is the set of all finite initial segments of slaloms* Section 2.3) *and* $p(\alpha, 1)$*is a finite subset of ν, and*
4. $p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha}$ “$\forall k \geq |p(\alpha,0)|$, $\left|\left\{\dot{g}^\alpha_\xi(k); \xi \in p(\alpha,1)\right\}\right| \leq |p(\alpha,0)|$ ”,

and the order on $\mathbb{P}_{\alpha+1}$ is defined so that for each $p, q \in \mathbb{P}_{\alpha+1}$, $q \leq_{\mathbb{P}_{\alpha+1}} p$ if and only if

- $q{\downarrow}\alpha \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$,
- *for each $M \in \mathrm{dom}(p(0))$, $q(0, M) \geq p(0, M)$,*
- *$q(\alpha, 0)$ end-extends $p(\alpha, 0)$,*
- *$q(\alpha, 1) \supseteq p(\alpha, 1)$, and*
- *for each $\xi \in p(\alpha, 1)$,*

$$q{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“for every } n \in |q(\alpha,0)| \setminus |p(\alpha,0)|,\ \dot{g}^\alpha_\xi(n) \in q(\alpha,0)(n) \text{”}.$$

Case 5. *Otherwise.* *Then $\mathbb{P}_{\alpha+1}$ consists of the functions p with domain $\alpha + 1$ such that*

1. $p{\downarrow}\alpha \in \mathbb{P}_\alpha$,
2. $p(0)$ *is a function into* $(\alpha + 2) \setminus \{0\}$, *and*
3. $p(\alpha) = \langle \emptyset, \emptyset \rangle$,

and the order on $\mathbb{P}_{\alpha+1}$ is defined so that for each $p, q \in \mathbb{P}_{\alpha+1}$, $q \leq_{\mathbb{P}_{\alpha+1}} p$ if and only if

- $q{\downarrow}\alpha \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$, *and*
- *for each $M \in \mathrm{dom}(p(0))$, $q(0, M) \geq p(0, M)$.*

Limit stages: *Let $\beta \leq \kappa$ be a limit ordinal and suppose that $\mathbb{P}_\alpha$ is defined for every $\alpha \in \beta$. Then $\mathbb{P}_\beta$ consists of the functions p with domain β such that*

1. *there exists $\alpha \in \beta \setminus \{0\}$ such that $p{\downarrow}\alpha \in \mathbb{P}_\alpha$ and for every $\gamma \in [\alpha, \beta)$, $p(\gamma) = \langle \emptyset, \emptyset \rangle$, and*

2. *$p(0)$ is a function into $(\beta+1)\setminus\{0\}$,*

and the order on $\mathbb{P}_\beta$ is defined so that for each $p, q \in \mathbb{P}_\beta$, $q \leq_{\mathbb{P}_\beta} p$ if and only if

- *there exists $\alpha \in \beta \setminus \{0\}$ such that for every $\gamma \in [\alpha, \beta)$, $q(\gamma) = p(\gamma) = \langle \emptyset, \emptyset \rangle$ holds, and $q{\downarrow}\alpha \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$, and*
- *for each $M \in \mathrm{dom}(p(0))$, $q(0, M) \geq p(0, M)$.*

We have a couple of comments about the definition of $\mathbb{P}_\alpha$.

At limit stages, $\mathbb{P}_\beta$ is defined as a direct limit. So for each condition $p \in \mathbb{P}_\alpha$, $\mathrm{supp}(p)$ is finite. Moreover, we will see that if $\alpha < \beta < \kappa+1$, then $\mathbb{P}_\alpha$ completely embeds into $\mathbb{P}_\beta$ (Proposition 3.2). Thus $\mathbb{P}_\alpha$ can be considered as a finite support iteration with models as side conditions.

As said before the definition of $\mathbb{P}_\alpha$, for each $\alpha \in \kappa \setminus \{0\}$, $\mathbb{P}_\alpha$ is a subset of $H(\kappa)$. However, $\mathbb{P}_\alpha$ is *not* a member of $H(\kappa)$, because $H(\kappa)$ doesn't have the sets $\mathcal{M}_\alpha$'s. So we cannot set-force over any member of $\mathrm{dom}(p(0))$ for any $p \in \mathbb{P}_\alpha$. However, for each $\alpha \in \kappa$, every member of $\mathcal{M}^*_{\alpha+1}$ contains $\mathbb{P}_\alpha$ as a member. So we can consider the forcing relation $\Vdash_{\mathbb{P}_\alpha}$ in a model in $\mathcal{M}^*_{\alpha+1}$.

In our definition, for each condition p in $\mathbb{P}_\alpha$, $\mathrm{dom}(p(0))$ plays the role of a side condition of models as in e.g. [**22**, Section 4]. But in [**22**, Section 4], the models coming from side conditions in a forcing condition are not required to satisfy the requirement (0-4) in Definition 3.1. The requirement (0-4) in Definition 3.1 will be used in the proof of properness of $\mathbb{P}_\alpha$ for limit ordinals $\alpha \in (\kappa+1) \cap \mathrm{Lim}$ of uncountable cofinality. Asperó–Mota's Φ-symmetric system is a set of the form $\mathrm{dom}(p(0))$ with the property that

(0-5) for every $M_0, M_1 \in \mathrm{dom}(p(0))$ and $M' \in \mathrm{dom}(p(0)) \cap M_0$, if $\omega_1 \cap M_0 = \omega_1 \cap M_1$, then

$$\left(\Psi_{M_1}{}^{-1} \circ \Psi_{M_0}\right)(M') \in \mathrm{dom}(p(0)).$$

We note that the set of conditions of $\mathbb{P}_\alpha$ with the property (0-5) above is dense in $\mathbb{P}_\alpha$.

In [**3**], a forcing-condition of the Asperó–Mota iteration is a pair (p, Δ) such that p is a working part and Δ is a side-condition part. Δ consists of finitely many pairs (N, γ) such that N is a model (a member of $\mathcal{M}_0$) and γ is an ordinal. Then γ indicates the stage where the condition is N-generic. They call such a γ a marker. Markers have two roles. One is for guaranteeing the complete embeddability between intermediate stages, and the other one is for the genericity of the model N. In [**3**], the set $\mathrm{dom}(\Delta)$ forms a Φ-symmetric system, and each model in $\mathrm{dom}(\Delta)$ has its own marker. In our definition, for each $p \in \mathbb{P}_\alpha$, $\mathrm{dom}(p(0))$ plays the same role as the sets $\mathrm{dom}(\Delta)$ in [**3**], $p(0)$ assigns markers to the models in $\mathrm{dom}(p(0))$, and $p \restriction \left(\mathrm{dom}(p) \setminus \{0\}\right)$ is a working part. In the version of this article, if we ignore Case 1, then we can define our construction in such a way that for each $p \in \mathbb{P}_\alpha$, a marker of the model $M \in \mathrm{dom}(p(0))$ depends only on $\omega_1 \cap M$, and then the proofs of Lemma 3.3 and Lemma 3.4 in all the cases but Case 1 works well. But it may happen that $p(0)$ has to assign different markers to models M

and M' in $\mathrm{dom}(p(0))$ with $\omega_1 \cap M = \omega_1 \cap M'$ in the proofs of Lemma 3.3 and Lemma 3.4 when we deal with Case 1.

We should take care of the definitions of the successor stages $\mathbb{P}_{\alpha+1}$. As seen below, $\mathbb{P}_\alpha$ has the $\aleph_2$-chain condition under CH, so since $\omega_1 < \kappa$, $\mathbb{P}_\alpha$-names for a subset of ω_1 can be considered as members of $H(\kappa)$, by thinking of their nice-names. So a $\mathbb{P}_\alpha$-name for a function on some countable ordinal can be considered as a member of $H(\kappa)$, that is, a sequence of $\aleph_1$-many $\mathbb{P}_\alpha$-names for continuous functions can be considered as a member of $H(\kappa)$. Therefore this definition (in particular Case 1 and Case 2) makes sense.

For each $\mathrm{i} \in \{1, 2, 3, 4\}$, we write

$$\mathrm{Ci} := \Big\{\alpha \in \kappa; \Phi(\alpha) \text{ is of the form in Case i}\Big\}.$$

We have the following.

Observation. For each $\alpha \in (\kappa + 1) \setminus \{0\}$, $p \in \mathbb{P}_\alpha$, $\beta \in \mathrm{supp}(p) \cap (\mathrm{C1} \cup \mathrm{C2})$ and $M \in \mathrm{dom}(p(0))$, if there exists a model M' in the set $\mathrm{dom}(p(0)) \cap \mathcal{M}_{\beta+1}$ such that $\omega_1 \cap M' = \omega_1 \cap M$ and $\beta+1 \le p(0, M')$, then $p(\beta, 0) \restriction (\omega_1 \cap M)$ is a partial function from $\omega_1 \cap M$ into $\omega_1 \cap M$, $\omega_1 \cap M \in \mathrm{dom}(p(\beta, 0))$, and $p(\beta, 0)(\omega_1 \cap M) = \omega_1 \cap M$.

This observation follows immediately from the definition and is the key of proofs of the chain condition of $\mathbb{P}_\alpha$ (Lemma 3.3) and properness of $\mathbb{P}_\alpha$ (Lemma 3.4). This is one of the important roles of the markers in the Asperó–Mota iteration.

We note that a marker may not be increased freely. For example, suppose that $p \in \mathbb{P}_{\alpha+1}$, $\alpha \in \mathrm{C1}$, $M \in \mathrm{dom}(p(0)) \cap \mathcal{M}_{\alpha+2}$, $p(0, M) < \alpha + 1$, and there exists $\xi \in \mathrm{dom}(p(\alpha, 0)) \cap M$ such that $p(\alpha, 0)(\xi) \ge \omega_1 \cap M$ (that is, $p(\alpha, 0) \restriction (\omega_1 \cap M)$ is not a partial function from $\omega_1 \cap M$ into $\omega_1 \cap M$). In this case, if $q \in \mathbb{P}_{\alpha+1}$ is an extension of p in $\mathbb{P}_{\alpha+1}$, then $q(0, M)$ has to be *less than* $\alpha + 1$. Because if $q(0, M)$ was equal to $\alpha + 1$, then q would *not* satisfy the requirement (α-4) in Definition 3.1 Case 1, hence q would *not* be a condition of $\mathbb{P}_{\alpha+1}$.

At the non-trivial stage $\alpha \in \mathrm{C1}$, it follows from the definition of $\mathbb{P}_{\alpha+1}$ and the proof of Theorem 3.5 that

$$\begin{aligned} \Vdash_{\mathbb{P}_{\alpha+1}} \text{“} & \textstyle\bigcup_{p \in \dot{G}} p(\alpha, 0) \text{ is a strictly increasing continuous map from } {\omega_1}^{\mathbf{V}} \text{ into } {\omega_1}^{\mathbf{V}}, \\ & \textstyle\bigcup_{p \in \dot{G}} p(\alpha, 1) \text{ is a function from } {\omega_1}^{\mathbf{V}} \text{ into } \omega, \text{ and} \\ & \text{for each } \xi \in {\omega_1}^{\mathbf{V}},\ \Big(\textstyle\bigcup_{p \in \dot{G}} p(\alpha, 1)\Big)(\xi) \notin \dot{f}^{\alpha}_{\xi}\Big[\mathrm{ran}\Big(\textstyle\bigcup_{p \in \dot{G}} p(\alpha, 0)\Big) \cap \xi\Big] \text{”}, \end{aligned}$$

where ${\omega_1}^{\mathbf{V}}$ represents the first uncountable cardinal in the ground model in the extension (see e.g. [**8**, Section 3]). It is proved that $\mathbb{P}_{\alpha+1}$ is proper (Lemmas 3.4). So $\mathbb{P}_{\alpha+1}$ doesn't collapse ω_1, hence by the genericity argument,

$$\Vdash_{\mathbb{P}_{\alpha+1}} \text{“ } \mathrm{ran}\Big(\textstyle\bigcup_{p \in \dot{G}} p(\alpha, 0)\Big) \text{ is club on } \omega_1 \text{ ”}.$$

Therefore, $\mathbb{P}_{\alpha+1}$ forces that $\Big\langle \dot{f}^{\alpha}_{\xi}; \xi \in \omega_1 \Big\rangle$ is not a $\mho$-sequence.

At stage C2, the situation is similar to C1. At this stage,

$$\Vdash_{\mathbb{P}_{\alpha+1}} \text{“} \bigcup_{p\in\dot G} p(\alpha,0) \text{ is a strictly increasing continuous map from } {\omega_1}^{\mathbf{V}} \text{ into } {\omega_1}^{\mathbf{V}},$$
$$\bigcup_{p\in\dot G} p(\alpha,1) \text{ is a regressive function from } {\omega_1}^{\mathbf{V}} \cap \mathrm{Lim} \text{ into } {\omega_1}^{\mathbf{V}}, \text{ and}$$
$$\text{for each } \xi \in {\omega_1}^{\mathbf{V}} \cap \mathrm{Lim},\ \mathrm{ran}\left(\bigcup_{p\in\dot G} p(\alpha,0)\right) \cap \dot C^{\alpha}_{\xi} \subseteq \left(\bigcup_{p\in\dot G} p(\alpha,1)\right)(\xi)\text{”}.$$

So $\mathbb{P}_{\alpha+1}$ forces that $\left\langle \dot C^{\alpha}_{\xi}; \xi \in \omega_1 \cap \mathrm{Lim}\right\rangle$ is not weak club guessing for the club $\mathrm{ran}\left(\bigcup_{p\in\dot G} p(\alpha,0)\right)$.

We note that at stage C3 of the above definition, the iterands generically add a pseudo-intersection of the family $\left\{\dot x^{\alpha}_{\xi}; \xi \in \nu\right\}$, and at stage C4 of the above definition, the iterands generically add a slalom which captures all $\dot g^{\alpha}_{\xi}$'s.

Proposition 3.2. *For each $\alpha, \beta \in (\kappa+1)\setminus\{0\}$ with $\alpha<\beta$, the canonical mapping sending each $\mathbb{P}_\alpha$-condition p to the $\mathbb{P}_\beta$-condition $p \cup \left\{ \langle \gamma, \langle \emptyset, \emptyset\rangle\rangle ; \gamma \in [\alpha,\beta)\right\}$ is a complete embedding from $\mathbb{P}_\alpha$ into $\mathbb{P}_\beta$.*

Proof. Suppose that $q \in \mathbb{P}_\alpha$ and $p \in \mathbb{P}_\beta$ are such that $q \leq_{\mathbb{P}_\alpha} p{\downarrow}\alpha$. We define q' as the function with domain α such that

- for each $\gamma \in \alpha \setminus \{0\}$, $q'(\gamma) := q(\gamma)$, and
- $q'(0)$ is the function with domain $\mathrm{dom}(q(0))$ such that for each $M \in \mathrm{dom}(q(0))$, $q'(0,M) := \max\{q(0,M), p(0,M)\}$.

Then $q' ^\frown \big(p{\restriction}[\alpha,\beta)\big)$ is a condition of $\mathbb{P}_\beta$, and is a common extension of the condition $q \cup \left\{ \langle \gamma, \langle \emptyset, \emptyset\rangle\rangle ; \gamma \in [\alpha,\beta)\right\}$ and p in $\mathbb{P}_\beta$. □

Lemma 3.3. *For every $\alpha \in (\kappa+1)\setminus\{0\}$, $\mathbb{P}_\alpha$ has the $\left(2^{\aleph_0}\right)^+$-chain condition. In fact, every subset of $\mathbb{P}_\alpha$ of size $\left(2^{\aleph_0}\right)^+$ has a pairwise compatible subset of size $\left(2^{\aleph_0}\right)^+$.*

Proof. This proof is similar to the one in Todorčević's paper [**22**, p. 720]. Before starting the proof, we review the compatibility condition of the forcing at stages in C3 and C4.

At the iterands α falling under Case 3, for each p and q in $\mathbb{P}_{\alpha+1}$, if $p{\downarrow}\alpha$ and $q{\downarrow}\alpha$ are compatible in $\mathbb{P}_\alpha$ and r is their common extension and $p(\alpha,0) = q(\alpha,0)$, then by letting $r' \in \mathbb{P}_\alpha$ be such that $r'(\gamma) := r(\gamma)$ for all $\gamma \in \alpha\setminus\{0\}$, $\mathrm{dom}(r'(0)) := \mathrm{dom}(r(0))$ and for each $M \in \mathrm{dom}(r'(0))$,

$$r'(0,M) := \begin{cases} \alpha+1 & \text{if } M \in \mathrm{dom}(p(0)) \cup \mathrm{dom}(q(0)) \text{ and} \\ & \quad \max\{p(0,M), q(0,M)\} = \alpha+1, \\ r(0,M) & \text{otherwise,} \end{cases}$$

$r' ^\frown \Big\langle \big\langle p(\alpha,0), p(\alpha,1) \cup q(\alpha,1)\big\rangle \Big\rangle$ is a common extension of p and q in $\mathbb{P}_{\alpha+1}$.

At iterands α falling under Case 4, for each p and q in $\mathbb{P}_{\alpha+1}$, if

- $p{\downarrow}\alpha$ and $q{\downarrow}\alpha$ are compatible in $\mathbb{P}_\alpha$ and r is their common extension,

- $p(\alpha,0) = q(\alpha,0)$,
- $p(\alpha,1)$ and $q(\alpha,1)$ have the same size, and
- the length of $p(\alpha,0)$ is not shorter than $2 \cdot |p(\alpha,1)|$,

then by letting $r' \in \mathbb{P}_\alpha$ be as in the previous case, $r' \frown \left\langle \langle p(\alpha,0), p(\alpha,1) \cup q(\alpha,1) \rangle \right\rangle$ is a common extension of p and q in $\mathbb{P}_{\alpha+1}$. For each $p \in \mathbb{P}_{\alpha+1}$, there are $r \in \mathbb{P}_\alpha$ and $\sigma \in \mathcal{S}$ such that $r \leq_{\mathbb{P}_\alpha} p \downarrow \alpha$, $p(\alpha,0) \subseteq \sigma$, $r \frown \langle \sigma, p(\alpha,1) \rangle \in \mathbb{P}_{\alpha+1}$ and the length of σ is not shorter than $2 \cdot |p(\alpha,1)|$. Then $r \frown \langle \sigma, p(\alpha,1) \rangle$ is an extension of p in $\mathbb{P}_{\alpha+1}$ for any such σ.

Suppose that $\alpha \in (\kappa+1) \setminus \{0\}$ and $\left\{p_i; i \in \left(2^{\aleph_0}\right)^+\right\}$ is a set of $\left(2^{\aleph_0}\right)^+$-many conditions in $\mathbb{P}_\alpha$. By extending each p_i if necessary, we may assume that for each $\beta \in \mathrm{supp}(p_i) \cap \mathrm{C4}$, the length of $p_i(\beta,0)$ is not shorter than $2 \cdot |p_i(\beta,1)|$ [4]. By shrinking the set if necessary, we may assume that

(•) for each $i, j \in \left(2^{\aleph_0}\right)^+$,

$$\left\{\overline{M}; M \in \mathrm{dom}(p_i(0))\right\} = \left\{\overline{M}; M \in \mathrm{dom}(p_j(0))\right\},$$

- the set $\left\{\mathrm{supp}(p_i); i \in \left(2^{\aleph_0}\right)^+\right\}$ forms a Δ-system with root s,

(•) the set $\left\{(\bigcup \mathrm{dom}(p_i(0))) \cap \kappa; i \in \left(2^{\aleph_0}\right)^+\right\}$ forms a Δ-system with root K (which is a countable subset of κ),

(•) for each $i \in \left(2^{\aleph_0}\right)^+$, $(\mathrm{supp}(p_i) \setminus s) \cap K = \emptyset$,

(•) for each $i, j \in \left(2^{\aleph_0}\right)^+$, $M \in \mathrm{dom}(p_i(0))$ and $M' \in \mathrm{dom}(p_j(0))$, if $\overline{M} = \overline{M'}$, then $M \cap \kappa$ and $M' \cap \kappa$ are order isomorphic and the corresponding isomorphism fixes $\kappa \cap M \cap M'$ (which is a subset of K) [5],

- for each $\gamma \in s \cap (\mathrm{C1} \cup \mathrm{C2})$, all $p_i(\gamma)$'s are the same,
- for each $\gamma \in s \cap \mathrm{C3}$, all $p_i(\gamma,0)$'s are the same, and
- for each $\gamma \in s \cap \mathrm{C4}$,
  - all $p_i(\gamma,0)$'s are the same, and
  - all $p_i(\gamma,1)$'s have the same size.

Then we note that for each distinct i and j, p_i and p_j are compatible in $\mathbb{P}_\alpha$. Too see this, let q be the function from α such that

[4]This can be done by reverse-induction on $\mathrm{supp}(p_i) \cap \mathrm{C4}$, i.e. by the induction on $\max(\mathrm{supp}(p_i) \cap \mathrm{C4})$.

[5]In [**3**], Asperó and Mota point out that the corresponding isomorphism between M and M' fixes $\kappa \cap M \cap M'$ if and only if for every two consecutive ordinals ξ_0 and ξ_1, the order types of the sets $\{\mu \in \kappa \cap M; \xi_0 < \mu < \xi_1\}$ and $\{\mu \in \kappa \cap M'; \xi_0 < \mu < \xi_1\}$ are the same (these order types are countable ordinals).

- $q(0)$ is a function with domain $\mathrm{dom}(p_i(0)) \cup \mathrm{dom}(p_j(0))$, such that for each $M \in \mathrm{dom}(p_i(0)) \cup \mathrm{dom}(p_j(0))$,

$$q(0, M) := \begin{cases} p_i(0, M), & \text{if } M \in \mathrm{dom}(p_i(0)) \setminus \mathrm{dom}(p_j(0)), \\ p_j(0, M), & \text{if } M \in \mathrm{dom}(p_j(0)) \setminus \mathrm{dom}(p_i(0)), \\ \max\{p_i(0, M), p_j(0, M)\}, & \text{if } M \in \mathrm{dom}(p_i(0)) \cap \mathrm{dom}(p_j(0)), \end{cases}$$

- $q{\restriction}[2, \alpha)$ is a function with support $\mathrm{supp}(p_i) \cup \mathrm{supp}(p_j)$ such that
  - for each $\gamma \in s \cap (\mathrm{C3} \cup \mathrm{C4})$, $q(\gamma, 0) := p_i(\gamma, 0)$ and

$$q(\gamma, 1) := p_i(\gamma, 1) \cup p_j(\gamma, 1),$$

  - for each $\gamma \in \mathrm{supp}(p_i) \setminus (s \cap (\mathrm{C3} \cup \mathrm{C4}))$, $q(\gamma) := p_i(\gamma)$, and
  - for each $\gamma \in \mathrm{supp}(p_j) \setminus \mathrm{supp}(p_i)$, $q(\gamma) := p_j(\gamma)$.

Such a q is a canonical amalgamation of p_i and p_j. Then by the above items **(•)**, $q(0)$ satisfies the requirements in *Basic stage* in Definition 3.1. Now both p_i and p_j satisfy OBSERVATION. So q does as well. Thus q satisfies the requirement (α-4) in Definition 3.1 Case 1 and Case 2. Thus q is a condition of $\mathbb{P}_\alpha$, and is a common extension of p_i and p_j. $\square$

For $\alpha \in (\kappa + 1) \setminus \{0\}$, $p \in \mathbb{P}_\alpha$ and a countable elementary submodel N of $H(\theta)$ for some large enough θ with $p \in N$, we define the condition p^{+N} such that

- $p^{+N}(0) := p(0) \cup \Big\{ \langle N \cap H(\kappa)\ , \alpha \rangle \Big\}$,
- $\mathrm{supp}(p^{+N}) := \mathrm{supp}(p)$ (since $p \in N$, $\mathrm{supp}(p) \subseteq N$),
- for each $\beta \in \mathrm{supp}(p^{+N}) \cap (\mathrm{C1} \cup \mathrm{C2})$,

$$p^{+N}(\beta, 0) := p(\beta, 0) \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}$$

  and $p^{+N}(\beta, 1) := p(\beta, 1)$, and
- for each $\beta \in \mathrm{supp}(p^{+N}) \setminus (\mathrm{C1} \cup \mathrm{C2})$, $p^{+N}(\beta) := p(\beta)$.

We note that if $\alpha \in \kappa + 1$, $p \in \mathbb{P}_\alpha$ and $N^* \in \bigcup_{\gamma \in \kappa+1} \mathcal{M}^*_\gamma$ with $\{\mathbb{P}_\alpha, p\} \in N^*$ (then $\mathrm{supp}(p) \subseteq N^*$), then $p^{+N^*}(\beta, 0)$ is in $\mathcal{PF}$ for every β in $\mathrm{supp}(p) \cap (\mathrm{C1} \cup \mathrm{C2})$, hence p^{+N^*} is still a condition of $\mathbb{P}_\alpha$, because for every $f \in N^*$ which is a strictly increasing and continuous function on ω_1,

$$f(\omega_1 \cap N^*) = \omega_1 \cap N^*.$$

We should notice here that p^{+N^*} is the weakest extension p' of p satisfying that

- $N^* \cap H(\kappa) \in \mathrm{dom}(p')$, and

- for every $\beta \in \mathrm{supp}(p') \cap (\mathrm{C1} \cup \mathrm{C2}) \cap N$, $\omega_1 \cap N^* \in \mathrm{dom}(p'(\beta, 0))$ and

$$p'(\beta, 0)(\omega_1 \cap N^*) = \omega_1 \cap N^*.$$

The following lemma says that p^{+N^*} is a typical $(N^*, \mathbb{P}_\alpha)$-generic condition. We remember that each member of $\mathcal{M}^*_{\alpha+1}$ contains $\mathbb{P}_\alpha$ as a member.

LEMMA 3.4. 1. *For every* $\alpha \in (\kappa+1) \setminus (\mathrm{Lim} \cup \{0\})$, $N^* \in \mathcal{M}^*_{\alpha+1}$ *and* $p \in \mathbb{P}_\alpha$, *if* $N^* \cap H(\kappa)$ *is in* $\mathrm{dom}(p(0))$ *and* $p(0, N^* \cap H(\kappa)) = \alpha$, *then* p *is* $(N^*, \mathbb{P}_\alpha)$*-generic.*

2. *For every* $\alpha \in (\kappa+1) \cap \mathrm{Lim}$, $N^* \in \mathcal{M}^*_\alpha$ *and* $p \in \mathbb{P}_\alpha$, *if* $N^* \cap H(\kappa)$ *is in* $\mathrm{dom}(p(0))$ *and* $p(0, N^* \cap H(\kappa)) = \alpha$, *then* p *is* $(N^*, \mathbb{P}_\alpha)$*-generic.*

Therefore, $\mathbb{P}_\alpha$ *is proper.*

PROOF. This is proved by induction on α.

Basic stage: This proof is included in the proof of properness of the forcing notion in [**22**, Lemma 4].

Suppose that $N^* \in \mathcal{M}^*_2$, $p \in \mathbb{P}_1$ satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(p(0))$ (and then $p(0, N^* \cap H(\kappa)) = 1$), $\mathcal{D} \in N^*$ is an open dense subset of $\mathbb{P}_1$, and $q \leq_{\mathbb{P}_1} p$. It suffices to find $r \in \mathcal{D} \cap N^*$ which is compatible with q. By extending q if necessary, we may assume that $q \in \mathcal{D}$. Moreover, by extending q if necessary again, we may assume that for each $M \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M = \omega_1 \cap N^*$ and each $M' \in \mathrm{dom}(q(0)) \cap M$, the set

$$\left({\Psi_{N^* \cap H(\kappa)}}^{-1} \circ \Psi_M\right)(M')$$

is a member of $\mathrm{dom}(q(0))$.

We consider the set

$$\begin{aligned}\mathcal{E} := \Big\{ r \in \mathcal{D}; \quad & - \left\{\overline{M}; M \in \mathrm{dom}(r(0))\right\} \text{ end-extends the set} \\ & \qquad \vec{t} := \left\{\overline{M}; M \in \mathrm{dom}(q(0)) \text{ and } \omega_1 \cap M < \omega_1 \cap N^*\right\}, \\ & \text{and} \\ & - \mathrm{dom}(r(0)) \text{ includes } \mathrm{dom}(q(0)) \cap N^* \Big\}.\end{aligned}$$

Even if q may not be in N^*, we note that $\mathcal{E}$ is in N^*, because the sets $\vec{t}$ and $\mathrm{dom}(q(0)) \cap N^*$ are members of N^*. So by the elementarity of N^*, we can find $r \in \mathcal{E} \cap N^*$. We note then that $\mathrm{dom}(r(0)) \subseteq N^*$. We define the function $q \wedge r$ from 1 $(= \{0\})$ such that

$$\begin{aligned}\mathrm{dom}((q \wedge r)(0)) := {} & \mathrm{dom}(q(0)) \cup \mathrm{dom}(r(0)) \cup \\ & \Big\{ \left({\Psi_M}^{-1} \circ \Psi_{N^* \cap H(\kappa)}\right)(M'); M \in \mathrm{dom}(q(0)) \text{ with } \omega_1 \cap M = \omega_1 \cap N^* \\ & \qquad \& \ M' \in \mathrm{dom}(r(0)) \text{ with } \overline{M'} \notin \vec{t} \Big\}\end{aligned}$$

(and for each $M \in \mathrm{dom}((q \wedge r)(0))$, $(q \wedge r)(0, M) = 1$). $q \wedge r$ is a canonical amalgamation of q and r. We can check that $(q \wedge r)(0)$ satisfies the requirement in *Basic stage* in Definition 3.1. We check only the non-trivial situation: For any $M \in \mathrm{dom}(r(0))$ so that

$\overline{M} \notin \vec{t}$, and any $M' \in \mathrm{dom}(q(0)) \setminus \mathrm{dom}(r(0))$ with $\omega_1 \cap M' < \omega_1 \cap N^*$, there exists $K_1 \in \mathrm{dom}((q \wedge r)(0))$ such that $\omega_1 \cap K_1 = \omega_1 \cap M$ and $M' \in K_1$.

We note that for such M and M', $\omega_1 \cap M' < \omega_1 \cap M$. By the requirement for $\mathrm{dom}(q(0))$ in *Basic stage* in Definition 3.1, there exists $M'' \in \mathrm{dom}(q(0))$ such that $\omega_1 \cap M'' = \omega_1 \cap N^*$ and $M' \in M''$. So by our assumption on q, the set

$$\left(\Psi_{N^* \cap H(\kappa)}{}^{-1} \circ \Psi_{M''}\right)(M')$$

is in $\mathrm{dom}(q(0)) \cap N^*$, which is included in $\mathrm{dom}(r(0))$. By the requirement for $\mathrm{dom}(r(0))$ in *Basic stage* in Definition 3.1, there exists $K_0 \in \mathrm{dom}(r(0))$ such that $\omega_1 \cap K_0 = \omega_1 \cap M$ and

$$\left(\Psi_{N^* \cap H(\kappa)}{}^{-1} \circ \Psi_{M''}\right)(M') \in K_0.$$

Then

$$M' \in \left(\Psi_{M''}{}^{-1} \circ \Psi_{N^* \cap H(\kappa)}\right)(K_0) \in \mathrm{dom}((q \wedge r)(0)).$$

Therefore $q \wedge r$ is a common extension of q and r, hence q and r are compatible in $\mathbb{P}_1$.

Stages in C1: Suppose that $\alpha \in \mathrm{C1}$, $N^* \in \mathcal{M}^*_{\alpha+2}$, $p \in \mathbb{P}_{\alpha+1}$ satisfies that $N^* \cap H(\kappa)$ is in $\mathrm{dom}(p(0))$ and $p(0, N^* \cap H(\kappa)) = \alpha + 1$, $\mathcal{D} \in N^*$ is an open dense subset of $\mathbb{P}_{\alpha+1}$ and $q \leq_{\mathbb{P}_{\alpha+1}} p$ with $q \in \mathcal{D}$. Then $N^* \cap H(\theta_{\alpha+1})$ belongs to $\mathcal{M}^*_{\alpha+1}$ and $N^* \cap H(\kappa)$ is in both $\mathcal{M}_{\alpha+1}$ and $\mathcal{M}_{\alpha+2}$. Since Φ is in N^* and $\alpha \in N^*$, it follows that

$$\left\langle \dot{f}^\alpha_\xi ; \xi \in \omega_1 \right\rangle = \Phi(\alpha) \in N^* \cap H(\kappa),$$

hence $\left\langle \dot{f}^\alpha_\xi ; \xi \in \omega_1 \right\rangle$ is a member of $N^* \cap H(\kappa)$. We consider the non-trivial case: Suppose that $\omega_1 \cap N^* \in \mathrm{dom}(q(\alpha, 1))$. If not, the proof would be much simpler.

By extending $q \downarrow \alpha$ if necessary, we may assume that for each ξ in the set $\mathrm{dom}(q(\alpha,1)) \setminus (N^* \cup \{\omega_1 \cap N^*\})$, there exists a union e_ξ of finitely many intervals in $\xi \cap N^*$ (possibly empty) such that

$$q{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} e_\xi = \left\{ i \in \xi \cap N^* ; \dot{f}^\alpha_\xi(i) = q(\alpha,1)(\xi) \right\} \text{”}.$$

Since

$$q{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} q(\alpha,0)(\xi \cap N^*) = q(\alpha,0)(\omega_1 \cap N^*) = \omega_1 \cap N^* \notin \left\{ i \in \xi ; \dot{f}^\alpha_\xi(i) = q(\alpha,1)(\xi) \right\} \\ \text{and } \dot{f}^\alpha_\xi \text{ is continuous”},$$

we note that $\sup(e_\xi) < \omega_1 \cap N^*$. Since q is an extension of p in $\mathbb{P}_{\alpha+1}$, it holds that

$$q(0, N^* \cap H(\kappa)) \geq p(0, N^* \cap H(\kappa)) = \alpha + 1,$$

for every $\gamma \in \mathrm{supp}(q) \cap (\mathrm{C1} \cup \mathrm{C2})$,

$$q(\gamma,0)\cap N^* = q(\gamma,0)\restriction N^*,$$

and it also holds that $\omega_1 \cap N^* \in \mathrm{dom}(q(\gamma,0))$ and

$$q(\gamma,0)(\omega_1\cap N^*) = \omega_1 \cap N^*.$$

We define the set

$$\begin{aligned}\mathcal{D}' := \Big\{ f \in {}^{\alpha+1}H(\aleph_1);\ &\text{there exists } r' \in \mathcal{D} \text{ such that}\\ &- \left\{\overline{M}; M \in \mathrm{dom}(r'(0))\right\} \text{ end-extends the set}\\ &\qquad \left\{\overline{M}; M \in \mathrm{dom}(q(0)) \text{ and } \omega_1\cap M < \omega_1 \cap N^*\right\},\\ &- \text{for every } M \in \mathrm{dom}(r'(0)), \text{ if}\\ &\qquad \max\left\{\omega_1\cap M'; M' \in \mathrm{dom}(q(0)) \text{ and } \omega_1\cap M' < \omega_1\cap N^*\right\} < \omega_1\cap M,\\ &\text{then for every } \gamma \in \mathrm{supp}(q)\cap(\mathrm{C1}\cup\mathrm{C2}),\ \{q(\gamma,0)\cap N^*, q(\gamma,1)\cap N^*\} \in M,\\ &- \mathrm{dom}(r'(0)) \text{ includes } \mathrm{dom}(q(0))\cap N^*, \text{ and}\\ &- f(0) = \left\{\overline{M}; M \in \mathrm{dom}(r'(0))\right\} \text{ and } f\restriction[1,\alpha+1) = r'\restriction[1,\alpha+1)\ \Big\}.\end{aligned}$$

We note that $\mathcal{D}'$ is in $N^*\cap H(\kappa)$. By the requirement (α-4) in Definition 3.1 Case 1, there are an extension $q' \in \mathbb{P}_\alpha$ of $q{\downarrow}\alpha$ and $M \in \mathrm{dom}(q'(0)) \cap \mathcal{M}_{\alpha+1}\cap N^*$ such that

- the model M contains the set

$$\begin{aligned}&\{\mathcal{D}'\}\cup\{e_\xi; \xi \in \mathrm{dom}(q(\alpha,1))\setminus(N^*\cup\{\omega_1\cap N^*\})\}\\ &\qquad \cup\{q(\gamma,0)\cap N^*, q(\gamma,1)\cap N^*; \gamma\in\mathrm{supp}(q)\cap(\mathrm{C1}\cup\mathrm{C2})\},\end{aligned}$$

- $q'(0,M) = \alpha$, and
- $q' \Vdash_{\mathbb{P}_\alpha}$"$\dot{f}^\alpha_{\omega_1\cap N^*}(\omega_1\cap M) \neq q(\alpha,1)(\omega_1\cap N^*)$".

Let $M^* \in \mathcal{M}^*_{\alpha+1}\cap N^*$ be such that $M = M^*\cap H(\kappa)$. By the inductive hypothesis, q' is $(M^*,\mathbb{P}_\alpha)$-generic. We take an extension q'' of q' in $\mathbb{P}_\alpha$ and a union $e_{\omega_1\cap N^*}$ of finitely many intervals in $\omega_1\cap M^*$ (possibly empty) such that

$$q'' \Vdash_{\mathbb{P}_\alpha}\text{“}e_{\omega_1\cap N^*} = \left\{i \in \omega_1\cap M^*; f^\alpha_{\omega_1\cap N^*}(i) = q(\alpha,1)(\omega_1\cap N^*)\right\}\text{”}.$$

We note again that $\sup(e_{\omega_1\cap N^*}) < \omega_1\cap M^*$. Then $(q'')^\frown\Big\langle \langle\emptyset,\emptyset\rangle \Big\rangle$ is still a condition of $\mathbb{P}_{\alpha+1}$ and compatible with q in $\mathbb{P}_{\alpha+1}$. By extending $\mathrm{dom}(q''(0))$ if necessary, we may assume that for each $M \in \mathrm{dom}(q''(0))$ with $\omega_1\cap M = \omega_1\cap M^*$ and each $M' \in \mathrm{dom}(q''(0))\cap M$, the set

$$\left({\Psi_{M^*\cap H(\kappa)}}^{-1}\circ\Psi_M\right)(M')$$

is a member of $\mathrm{dom}(q''(0))$.

We define the set

$$\mathcal{E} := \Big\{ r \in \mathbb{P}_\alpha;\ \textit{either}\ \text{there exists}\ \langle u_0, u_1 \rangle\ \text{such that}$$

- $\left\langle \{ \overline{M}; M \in \mathrm{dom}(r(0)) \} \right\rangle {}^\frown (r{\restriction}[1,\alpha]) {}^\frown \Big\langle \langle u_0, u_1 \rangle \Big\rangle \in \mathcal{D}'$,
- $u_0 \supseteq q(\alpha, 0) \cap N^*$,
- $u_1 \supseteq q(\alpha, 1) \cap N^*$, and
- $(*)$ for each $\xi \in \mathrm{dom}(q(\alpha,1)) \setminus N^*$, $\mathrm{ran}(u_0) \cap e_\xi = \emptyset$,

or the condition r is not compatible with any condition of $\mathbb{P}_\alpha$ which satisfies the above property $\Big\}$.

Though q doesn't belong to M^*, we note that $\mathcal{E}$ is a member of M^*. Since $\mathcal{E}$ is predense in $\mathbb{P}_\alpha$ and q'' is $(M^*, \mathbb{P}_\alpha)$-generic, there exists $r \in \mathcal{E} \cap M^*$ which is compatible with q'' in $\mathbb{P}_\alpha$. Since q'' satisfies the *either* case, so does r. Let $\langle u_0, u_1 \rangle \in M^*$ be a witness that $r \in \mathcal{E}$ in M^* $(\subseteq N^*)$, and let $r' \in \mathcal{D} \cap N^*$ be a witness that $\left\langle \{ \overline{M}; M \in \mathrm{dom}(r(0)) \} \right\rangle {}^\frown (r{\restriction}[1,\alpha)) {}^\frown \Big\langle \langle u_0, u_1 \rangle \Big\rangle$ is in $\mathcal{D}'$.

We claim that q, $q'' {}^\frown \Big\langle \langle \emptyset, \emptyset \rangle \Big\rangle$ and r' are compatible in $\mathbb{P}_{\alpha+1}$. By the definitions of $\mathcal{D}'$ and $\mathcal{E}$ and the facts that $q'' \leq_{\mathbb{P}_\alpha} q{\downarrow}\alpha$ and q'' is compatible with r in $\mathbb{P}_\alpha$, it follows that $q{\downarrow}\alpha$, q'' and $r'{\downarrow}\alpha$ are compatible in $\mathbb{P}_\alpha$. The point of this claim is that a common extension of q'' and $r'{\downarrow}\alpha$ forces that an amalgamation of $q(\alpha)$ and $\langle u_0, u_1 \rangle$ still satisfies the requirement (α-3) in Definition 3.1 Case 1 at stage $\alpha + 1$, because of the requirement $(*)$ above. Therefore q, $q'' {}^\frown \Big\langle \langle \emptyset, \emptyset \rangle \Big\rangle$ and r' are compatible in $\mathbb{P}_{\alpha+1}$, which finishes the proof in this case.

Stages in C2: The proof of this case is almost the same as in the previous one. But this is somewhat simpler. Suppose that $\alpha \in \mathrm{C2}$, $N^* \in \mathcal{M}_{\alpha+2}$, $p \in \mathbb{P}_{\alpha+1}$ satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(p(0))$ and $p(0, N^* \cap H(\kappa)) = \alpha + 1$, $\mathcal{D} \in N^*$ is an open dense subset of $\mathbb{P}_{\alpha+1}$ and $q \leq_{\mathbb{P}_{\alpha+1}} p$ with $q \in \mathcal{D}$. By extending $q{\downarrow}\alpha$ if necessary, we may assume that for each ξ in the set $\mathrm{dom}(q(\alpha,1)) \setminus (N^* \cup \{\omega_1 \cap N^*\})$, there exists a finite subset e_ξ of $\omega_1 \cap N^*$ such that

$$q{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\, e_\xi = \dot{C}^\alpha_\xi \cap N^* \,\text{”}.$$

We define the set

$$\mathcal{E}' := \Big\{ r \in \mathbb{P}_\alpha;\ \textit{either}\ \text{there exists}\ M^* \in \mathcal{M}^*_{\alpha+1}\ \text{which contains the set}$$

$$\{\mathcal{D}\} \cup \{ e_\xi; \xi \in \mathrm{dom}(q(\alpha,1)) \setminus (N^* \cup \{\omega_1 \cap N^*\}) \}$$

such that there exists $K \in \mathrm{dom}(r(0))$ so that $\overline{K} = \overline{M^* \cap H(\kappa)}$ and $r(0, K) = \alpha$,

$$\left. \textit{or} \text{ there are no extensions of } r \text{ which satisfy the above property } \right\}.$$

We note that $\mathcal{E}' \in N^*$ and $\mathcal{E}'$ is dense in $\mathbb{P}_\alpha$. By the inductive hypothesis, $q\downarrow\alpha$ is $(N^*, \mathbb{P}_\alpha)$-generic. So there exists $q_0' \in \mathcal{E}' \cap N^*$ which is compatible with $q\downarrow\alpha$. Since $q\downarrow\alpha$ satisfies the *either* case in the definition $\mathcal{E}'$, so does q_0'. Let $M^* \in \mathcal{M}^*_{\alpha+1} \cap N^*$ witness that q_0' satisfies the *either* case in the definition of $\mathcal{E}'$, and q' a common extension of q_0' and $q\downarrow\alpha$ in $\mathbb{P}_\alpha$. By the inductive hypothesis again, q' is $(M^*, \mathbb{P}_\alpha)$-generic.

We take an extension q'' of q' in $\mathbb{P}_\alpha$ and a finite subset $e_{\omega_1 \cap N^*}$ of $\omega_1 \cap M^*$ such that

$$q'' \Vdash_{\mathbb{P}_\alpha} \text{“} e_{\omega_1\cap N^*} = \dot{C}^\alpha_{\omega_1 \cap N^*} \cap M^* \text{”}.$$

The rest of the proof of this case is similar to the previous one (in this case, we don't need to take r' as above).

Stages in C3. Suppose that $\alpha \in \mathrm{C3}$, $N^* \in \mathcal{M}^*_{\alpha+2}$, $p \in \mathbb{P}_{\alpha+1}$ satisfies that $N^* \cap H(\kappa)$ is in $\mathrm{dom}(p(0))$ and $p(0, N^* \cap H(\kappa)) = \alpha + 1$, $\mathcal{D} \in N^*$ is an open dense subset of $\mathbb{P}_{\alpha+1}$, and $q \in \mathcal{D}$ is such that $q \leq_{\mathbb{P}_{\alpha+1}} p$. Then the set

$$\begin{aligned} \mathcal{E} := \Big\{ s \in \mathbb{P}_\alpha; & \textit{either} \text{ there exists } r \in \mathcal{D} \text{ such that} \\ & \qquad s \leq_{\mathbb{P}_\alpha} r\downarrow\alpha \text{ and } r(\alpha, 0) = q(\alpha, 0) \\ & \textit{or} \text{ there are no extensions of } s \text{ which satisfy the above property} \Big\} \end{aligned}$$

belongs to N^*, and is dense open in $\mathbb{P}_\alpha$. Since $p\downarrow\alpha$ is $(N^*, \mathbb{P}_\alpha)$-generic (by the inductive hypothesis) and $q\downarrow\alpha$ is an extension of $p\downarrow\alpha$ in $\mathbb{P}_\alpha$, there exists $s \in \mathcal{E} \cap N^*$ such that s is compatible with $q\downarrow\alpha$ in $\mathbb{P}_\alpha$. Since $q\downarrow\alpha$ satisfies the *either* case in the definition of $\mathcal{E}$, there exists $r \in N^*$ witnessing that s satisfies the *either* case in the definition of $\mathcal{E}$. Then r and q are compatible in $\mathbb{P}_{\alpha+1}$, which finishes the proof in this case.

Stages in C4. This proof is quite similar to the previous one. Suppose that $\alpha \in \mathrm{C4}$, $N^* \in \mathcal{M}^*_{\alpha+2}$, $p \in \mathbb{P}_{\alpha+1}$ satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(p(0))$ and $p(0, N^* \cap H(\kappa)) = \alpha+1$, $\mathcal{D} \in N^*$ is an open dense subset of $\mathbb{P}_{\alpha+1}$, and $q \in \mathcal{D}$ is such that $q \leq_{\mathbb{P}_{\alpha+1}} p$. By extending only $q\downarrow\alpha$ and $q(\alpha, 0)$ if necessary, we may assume that the length of $q(\alpha, 0)$ is not shorter than $2 \cdot |q(\alpha, 1)|$. To finish the proof, it suffices to argue as in the previous paragraph for the set

$$\begin{aligned} \mathcal{E} := \Big\{ s \in \mathbb{P}_\alpha; & \textit{either} \text{ there exists } r \in \mathcal{D} \text{ such that} \\ & \qquad s \leq_{\mathbb{P}_\alpha} r\downarrow\alpha, \ r(\alpha, 0) = q(\alpha, 0) \text{ and } |r(\alpha, 1)| = |q(\alpha, 1)| \\ & \textit{or} \text{ there are no extensions of } s \text{ which satisfy the above property} \Big\}. \end{aligned}$$

The rest of the proof for this case is the same as for the previous case.

Limit stages: Suppose that α is a limit ordinal, $N^* \in \mathcal{M}^*_\alpha$ (then $\mathbb{P}_\alpha \in N^*$), $p \in \mathbb{P}_\alpha$ which satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(p(0))$ and $p(0, N^* \cap H(\kappa)) = \alpha$, $\mathcal{D} \in N^*$ is an open dense subset of $\mathbb{P}_\alpha$, and $q \in \mathcal{D}$ such that $q \leq_{\mathbb{P}_\alpha} p$. We consider two cases: α is of countable cofinality and is of uncountable cofinality. The former case is straight forward, because then $\alpha \cap N^*$ is cofinal in α and hence we can take $\beta \in \alpha \cap N^*$ such that $\max(\mathrm{supp}(q)) < \beta$ holds. But we may not take such a β in the latter case, that is, it may happen that $\mathrm{supp}(q)$ is not bounded by $\sup(\alpha \cap N^*)$. So we need more argument for the latter case than for the former case.

Suppose that α is of uncountable cofinality. By the requirement for $\mathrm{dom}(q(0))$ in *Basic stage* in Definition 3.1, for each $M' \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M' < \omega_1 \cap N^*$, there exists $M \in \mathrm{dom}(q(0))$ such that $\omega_1 \cap M = \omega_1 \cap N^*$ and $M' \in M$, and then by the requirement (0-4) in Definition 3.1 [6],

$$\begin{aligned}\sup(M' \cap N^* \cap \alpha) &= \sup\left(\left({\Psi_{N^* \cap H(\kappa)}}^{-1} \circ \Psi_M\right)(M') \cap N^* \cap \alpha\right) \\ &\leq \sup\left(\left({\Psi_{N^* \cap H(\kappa)}}^{-1} \circ \Psi_M\right)(M') \cap \alpha\right).\end{aligned}$$

Since N^* thinks that the set $\left({\Psi_{N^* \cap H(\kappa)}}^{-1} \circ \Psi_M\right)(M')$ is countable and α is of uncountable cofinality,

$$\sup\left(\left({\Psi_{N^* \cap H(\kappa)}}^{-1} \circ \Psi_M\right)(M') \cap \alpha\right) \in N^* \cap \alpha.$$

So there exists $\beta \in \alpha \cap N^*$ such that

- $\max(\mathrm{supp}(q) \cap \sup(\alpha \cap N^*)) < \beta$ and
- for every $M' \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M' < \omega_1 \cap N^*$,
$$\sup(M' \cap N^* \cap \alpha) < \beta.$$

By the second requirement of β, we note that

- for every $M' \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M' < \omega_1 \cap N^*$ and $\gamma \in [\beta, \alpha) \cap N^*$, $M' \notin \mathcal{M}_\gamma$.

Let

$\mathcal{E} := \Big\{ r \in \mathbb{P}_\beta$; *either* there exists $r' \in \mathcal{D}$ such that

1. $r \leq_{\mathbb{P}_\beta} r' \downarrow \beta$,
2. $\left\{\overline{M}; M \in \mathrm{dom}(r'(0))\right\}$ end-extends the set
$$\left\{\overline{M}; M \in \mathrm{dom}(q(0)) \text{ and } \omega_1 \cap M < \omega_1 \cap N^*\right\},$$
3. $\mathrm{dom}(r'(0))$ includes $\mathrm{dom}(q(0)) \cap N^*$,
4. for each $M \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M < \omega_1 \cap N^*$, there exists $M' \in \mathrm{dom}(q(0))$ such that $\omega_1 \cap M' = \omega_1 \cap M$ and $q(0, M') = \alpha$ if and only if there exists $M' \in \mathrm{dom}(r'(0))$ such that $\omega_1 \cap M' = \omega_1 \cap M$ and $r'(0, M') = \alpha$,

or there are no extensions of r which satisfy the above property $\Big\}$.

[6]This is the only place where the requirement (0-4) is used.

We note that $\mathcal{E}$ also belongs to the model N^* and is dense open in $\mathbb{P}_\beta$. By the inductive hypothesis, $p{\downarrow}\beta$ is $(N^*, \mathbb{P}_\beta)$-generic and $q{\downarrow}\beta \leq_{\mathbb{P}_\beta} p{\downarrow}\beta$. So there exists $r \in \mathcal{E} \cap N^*$ which is compatible with $q{\downarrow}\beta$ in $\mathbb{P}_\beta$. Since $q{\downarrow}\beta$ satisfies the *either* case in the definition of $\mathcal{E}$, so does r. Then there exists $r' \in \mathcal{D} \cap N^*$ which witnesses that $r \in \mathcal{E}$.

We will see that r' is compatible with q in $\mathbb{P}_\alpha$. Now we note that $q \downarrow \beta$ and $r' \downarrow \beta$ are compatible in $\mathbb{P}_\beta$, because $q \downarrow \beta$ and r are compatible in $\mathbb{P}_\beta$ and $r \leq_{\mathbb{P}_\alpha} r' \downarrow \beta$. Since $r' \in N^*$, both $\mathrm{supp}(r')$ and $\mathrm{dom}(r'(0))$ are included in N^*. So it holds that

$$\mathrm{ran}(r'(0)) \subseteq (\alpha + 1) \cap N^* = (\alpha \cap N^*) \cup \{\alpha\},$$

that is, for any $M \in \mathrm{dom}(r'(0))$, $r'(0, M)$ is either less than $\sup(\alpha \cap N^*)$ or equal to α. Thus by the above observation, every M in $\mathrm{dom}(q(0))$ does not influence the coordinate $r'(\gamma)$ for any $\gamma \in (\mathrm{supp}(r') \cap (\mathrm{C1} \cup \mathrm{C2})) \setminus N^*$ (which is less than $\sup(\alpha \cap N^*)$) even if $q(0, M) \geq \beta$. Combining this fact and $\max(\mathrm{supp}(q) \cap \sup(\alpha \cap N^*)) < \beta$, we note that the canonical amalgamation of $r'{\downarrow}\sup(\alpha \cap N^*)$ and $q{\downarrow}\sup(\alpha \cap N^*)$ satisfies Observation, and hence $r'{\downarrow}\sup(\alpha \cap N^*)$ is compatible with $q{\downarrow}\sup(\alpha \cap N^*)$ in $\mathbb{P}_{\sup(\alpha \cap N^*)}$. Moreover, for any $\gamma \in \mathrm{supp}(q) \setminus N^*$ and $M \in \mathrm{dom}(r'(0))$, since $M \subseteq N^*$ holds M does not belong to $\mathcal{M}_{\gamma+1}$ (because $\gamma + 1 \notin M$). Thus every M in $\mathrm{dom}(r'(0))$ does not influence the coordinate $q(\gamma)$ for any $\gamma \in (\mathrm{supp}(q) \cap (\mathrm{C1} \cup \mathrm{C2}))) \setminus N^*$ even if $r'(0, M) = \alpha$. So the canonical amalgamation of q and r' satisfies Observation, and hence r' is compatible with q in $\mathbb{P}_\alpha$.

Suppose that α is of countable cofinality. Then since $\alpha \cap N$ is cofinal in α and both $\mathrm{supp}(q)$ and $\mathrm{ran}(q(0))$ are finite, we can take $\beta \in \alpha \cap N^*$ such that

- $\max(\mathrm{supp}(q)) < \beta$ and
- for every $M' \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M' < \omega_1 \cap N^*$, $q(0, M') \in \beta \cup \{\alpha\}$.

As in the case that α is of uncountable cofinality, we define $\mathcal{E}$ and take $r \in \mathcal{E} \cap N^*$ and $r' \in \mathcal{D} \cap N^*$ as above. Then by a similar observation, we note that r' is compatible with q in $\mathbb{P}_\alpha$. □

Therefore, as a corollary, we conclude the following.

Theorem 3.5. *Supposing that κ is an uncountable regular cardinal with $2^{<\kappa} = \kappa$ and* CH *holds, $\mathbb{P}_\kappa$ doesn't destroy any cardinal and forces that* ℧ *fails, there are no weak club-guessing ladder systems on ω_1, and $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = 2^{\aleph_0} = \kappa$ holds.*

Proof. By Lemma 3.3 and CH, $\mathbb{P}_\kappa$ has the $\aleph_2$-chain condition. By Lemma 3.4, $\mathbb{P}_\kappa$ is proper. Therefore $\mathbb{P}_\kappa$ preserves all cardinals.

In the non-trivial stages $\alpha + 1$, $\mathbb{P}_{\alpha+1}$ adds a real, so $\mathbb{P}_\kappa$ forces that $2^{\aleph_0} \geq \kappa$. And since the number of $\mathbb{P}_\kappa$-nice names for reals is equal to the cardinality

$$\left| \bigcup_{X \in [\kappa]^{\leq \aleph_1}} {}^X 2 \right|^{\aleph_0},$$

which is equal to $\kappa^{\aleph_1} = \kappa$, $\mathbb{P}_\kappa$ forces that $2^{\aleph_0} \leq \kappa$. Putting these inequalities together, we notice that $\mathbb{P}_\kappa$ forces that $2^{\aleph_0} = \kappa$.

In the rest of the proof, we show that $\mathbb{P}_\kappa$ forces the failure of $\mho$. The other cases can be proved more easily. Let $\left\langle \dot{f}_\xi; \xi \in \omega_1 \right\rangle$ be a sequence of $\mathbb{P}_\kappa$-names for continuous functions. By the $\aleph_2$-cc of $\mathbb{P}_\kappa$ and Proposition 3.2, there exists $\beta \in \kappa$ such that for every $\alpha \in [\beta, \kappa+1)$, $\left\langle \dot{f}_\xi; \xi \in \omega_1 \right\rangle$ can be considered as a sequence of $\mathbb{P}_\alpha$-nice names for continuous functions. Therefore by the property of Φ, there exists $\alpha \in [\beta, \kappa)$ such that $\Phi(\alpha)$ is the sequence $\left\langle \dot{f}_\xi; \xi \in \omega_1 \right\rangle$. Therefore, the bookkeeping argument works well for stages in C1. Similarly, Φ also works as a bookkeeping function for the other stages.

The only non-trivial density argument is that when $\alpha \in \mathrm{C1}$,

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\operatorname{dom}\left(\bigcup_{p \in \dot{G}} \operatorname{dom}(p(\alpha, 1))\right) = \omega_1\text{”}.$$

It suffices to show that for each $\delta \in \omega_1$, the set

$$\{r \in \mathbb{P}_{\alpha+1}; \delta \in \operatorname{dom}(r(\alpha, 1))\}$$

is dense in $\mathbb{P}_{\alpha+1}$.

Let $p \in \mathbb{P}_{\alpha+1}$. A non-trivial case is that there exists $N^* \in \mathcal{M}^*_{\alpha+1}$ such that $N^* \cap H(\kappa) \in \operatorname{dom}(p(0))$, $p(0, N^* \cap H(\kappa)) = \alpha + 1$ and $\omega_1 \cap N^* \notin \operatorname{dom}(p(\alpha, 1))$. For each $M \in \operatorname{dom}(p(0)) \cap \mathcal{M}_{\alpha+1}$ with $\omega_1 \cap M = \omega_1 \cap N^*$ and $p(0, M) = \alpha + 1$, let $\left\{x^M_n; n \in \omega\right\}$ enumerate the set $[M]^{<\aleph_0}$, and for each $n \in \omega$, we take a $\mathbb{P}_\alpha$-name $\dot{K}^M_n$ such that

$$\begin{aligned} &p{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“ for some } q \in \dot{G},\ \left\{x^M_i; i \leq n\right\} \in \dot{K}^M_n \in \operatorname{dom}(q(0)) \cap \mathcal{M}_{\alpha+1} \cap M \\ &\text{and } q(0, \dot{K}^M_n) = \alpha\text{”}. \end{aligned}$$

This can be done because, since p is $(M^*, \mathbb{P}_\alpha)$-generic (where $M^* \in \mathcal{M}^*_{\alpha+1}$ is such that $M = M^* \cap H(\kappa)$) by Lemma 3.4, for each $n \in \omega$, the set

$$\Big\{q \in \mathbb{P}_\alpha;\ \text{there exists } K \in \operatorname{dom}(q(0)) \cap \mathcal{M}_{\alpha+1} \cap M \text{ such that } \left\{x^M_i; i \leq n\right\} \in K \text{ and } q(0, K) = \alpha\Big\}$$

is dense below $p{\downarrow}\alpha$ in $\mathbb{P}_\alpha$. We can take an extension r of $p{\downarrow}\alpha$ in $\mathbb{P}_\alpha$ such that for each $M \in \operatorname{dom}(p(0)) \cap \mathcal{M}_{\alpha+1}$ with $\omega_1 \cap M = \omega_1 \cap N^*$ and $p(0, M) = \alpha + 1$, either there exists $k^M \in \omega$ such that

$$r \Vdash_{\mathbb{P}_\alpha} \text{“}\left\{n \in \omega; \dot{f}^\alpha_{\omega_1 \cap N^*}(\omega_1 \cap \dot{K}^M_n) = k^M\right\} \text{ is infinite”}$$

or

$$r \Vdash_{\mathbb{P}_\alpha} \text{“}\left\{n \in \omega; \dot{f}^\alpha_{\omega_1 \cap N^*}(\omega_1 \cap \dot{K}^M_n) = k\right\} \text{ is finite for every } k \in \omega\text{”}.$$

Let $l \in \omega$ be different from all such k^M's (note that $\operatorname{dom}(p(0)) \cap \mathcal{M}_{\alpha+1}$ is finite) and

$$r \Vdash_{\mathbb{P}_\alpha} \text{“} l \notin \dot{f}^{\alpha}_{\omega_1 \cap N^*}\left[\operatorname{ran}(p(\alpha,0))\right] \text{”}.$$

We define $r' \in \mathbb{P}_\alpha$ such that $\operatorname{dom}(r'(0)) := \operatorname{dom}(r(0))$, and for each $M \in \operatorname{dom}(r(0))$,

$$r'(0,M) := \begin{cases} \alpha+1 & \text{if } M \in \operatorname{dom}(p(0)) \text{ and } p(0,M) = \alpha+1, \\ r(0,M) & \text{otherwise.} \end{cases}$$

For each $M \in \operatorname{dom}(p(0)) \cap \mathcal{M}_{\alpha+1}$ (which is a subset of the set $\operatorname{dom}(r'(0))$) with $r'(0,M) = \alpha+1$, since p is a condition of $\mathbb{P}_{\alpha+1}$ and $p(0,M) = \alpha+1$, M satisfies the requirement (α-4) in Definition 3.1 Case 1 at stage $\alpha+1$. And for any other $M \in \operatorname{dom}(r'(0))$, M does not need to satisfy the requirement (α-4) in Definition 3.1 Case 1 at stage $\alpha+1$. Therefore the sequence

$$r'' := r' \frown \left\langle p(\alpha,0), p(\alpha,1) \cup \{\langle \omega_1 \cap N^*, l\rangle\} \right\rangle$$

is a condition of $\mathbb{P}_{\alpha+1}$, an extension of p with $\omega_1 \cap N^* \in \operatorname{dom}(r''(\alpha,1))$. □

4. $\mathbb{P}_\kappa$ may not force $\mathbf{MA}_{\aleph_1}$.

In this section, we prove that it is consistent that $\mathbb{P}_\kappa$ in Section 3 doesn't destroy any cardinal, and forces that $\mho$ fails, there are no weak club guessing ladder systems, $\mathfrak{p} = \operatorname{add}(\mathcal{N}) = 2^{\aleph_0} = \kappa$ holds, and $\mathrm{MA}_{\aleph_1}$ fails. To guarantee that $\mathbb{P}_\kappa$ may not force $\mathrm{MA}_{\aleph_1}$, we use a T-gap which is introduced by Stevo Todorčević, see in Section 2.4. The argument in this section can be applied to Suslin trees.

At first, we show that $\mathbb{P}_\kappa$ preserves any (ω_1,ω_1)-gaps. We note that the bounding number $\mathfrak{b}$, which is the smallest size of an unbounded family of ω^ω modulo finite, is not smaller than both $\mathfrak{p}$ and $\operatorname{add}(\mathcal{N})$. And $\mathfrak{b}$ is equal to the smallest cardinal λ such that there are (ω,λ)-gaps in $\mathcal{P}(\omega)/\text{fin}$. Therefore, $\mathbb{P}_\kappa$ destroys (ω,λ)-gaps for every $\lambda < \kappa$ [7].

Lemma 4.1. *Suppose that* CH *holds and* $(\mathcal{A},\mathcal{B}) = \langle a_\xi, b_\xi; \xi \in \omega_1\rangle$ *is an* (ω_1,ω_1)*-gap such that* $a_\xi \cap b_\xi = \emptyset$ *for each* $\xi \in \omega_1$*. Then* $\mathbb{P}_\kappa$ *preserves* $(\mathcal{A},\mathcal{B})$ *to form a gap.*

Proof. By CH, every $\mathbb{P}_\kappa$-name for a subset of ω_1 can be considered as a $\mathbb{P}_\alpha$-name for eventually all $\alpha \in \kappa$. Therefore it suffices to show that $\mathbb{P}_\alpha$ preserves $(\mathcal{A},\mathcal{B})$ to form a gap by induction on $\alpha \in (\kappa+1) \setminus \{0\}$. To show this, we show the following by induction on $\alpha \in (\kappa+1)\setminus\{0\}$.

($\star$) For every $\mathbb{P}_\alpha$-name $\dot{I}$ for an uncountable subset, $N^* \in \mathcal{M}^*_{\alpha+1}$ which contains $(\mathcal{A},\mathcal{B})$ and $\dot{I}$, $q \in \mathbb{P}_\alpha$ which satisfies that $N^* \cap H(\kappa) \in \operatorname{dom}(q(0))$ and $q(0, N^* \cap H(\kappa)) = \alpha$, and $\eta \in \omega_1 \setminus N^*$ such that $q \Vdash_{\mathbb{P}_\alpha}$ “$\eta \in \dot{I}$”, there are $r \in \mathbb{P}_\alpha \cap N^*$ and $\xi \in \omega_1 \cap N^*$ such that r is compatible with q in $\mathbb{P}_\alpha$, $r \Vdash_{\mathbb{P}_\alpha}$ “$\xi \in \dot{I}$” and

[7] There is another reason for this comment. Bell proved that for every $\lambda < \mathfrak{p}$, every σ-centered forcing has generic filters for any family of λ-many dense sets [**9**] (See also [**10**, 7.12 Theorem] and [**26**, Theorem 3.1]). Since a canonical forcing notion which generically adds an interpolation of an (ω,λ)-gap by finite approximations is σ-centered [**19**, Lemma 40, Corollary 62], $\mathbb{P}_\kappa$ destroys (ω,λ)-gaps for every $\lambda < \kappa$.

$$(a_\eta \cap b_\xi) \cup (a_\xi \cap b_\eta) \neq \emptyset.$$

By Kunen's and Todorčević's characterization of the gap-ness for (ω_1, ω_1)-pregaps, the preservation of (ω_1, ω_1)-gaps for each $\mathbb{P}_\alpha$ follows.

Basic stage: Let $\dot{I}$ be a $\mathbb{P}_1$-name for an uncountable subset of ω_1, $N^* \in \mathcal{M}_2^*$ which contains $(\mathcal{A}, \mathcal{B})$ and $\dot{I}$, $q \in \mathbb{P}_1$ which satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(q(0))$ and $q(0, N^* \cap H(\kappa)) = 1$, and $\eta \in \omega_1 \setminus N^*$ such that $q \Vdash_{\mathbb{P}_1}$"$\eta \in \dot{I}$". By extending q if necessary, we may assume that for each $M \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M = \omega_1 \cap N^*$ and each $M' \in \mathrm{dom}(q(0)) \cap M$, the set

$$\left({\Psi_{N^* \cap H(\kappa)}}^{-1} \circ \Psi_M\right)(M')$$

is a member of $\mathrm{dom}(q(0))$. We define the set

$J := \Big\{ \zeta \in \omega_1;$ there exists $r \in \mathbb{P}_1$ such that

– $\left\{\overline{M}; M \in \mathrm{dom}(r(0))\right\}$ end-extends the set

$$\left\{\overline{M}; M \in \mathrm{dom}(q(0)) \text{ and } \omega_1 \cap M < \omega_1 \cap N^*\right\},$$

– $\mathrm{dom}(r(0))$ includes $\mathrm{dom}(q(0)) \cap N^*$, and

– $r \Vdash_{\mathbb{P}_1}$"$\zeta \in \dot{I}$" $\Big\}$.

We note that J belongs to N^* and $\eta \in J$ (q witnesses this). Thus J is uncountable.

CLAIM 4.2.[8] *There exists $\xi \in J \cap N^*$ such that*

$$(a_\eta \cap b_\xi) \cup (a_\xi \cap b_\eta) \neq \emptyset.$$

PROOF OF CLAIM 4.2. If there are no such ξ, then for every $n \in a_\eta$ and $\xi \in J \cap N^*$, $n \notin b_\xi$ holds, and for every $n \in b_\eta$ and $\xi \in J \cap N^*$, $n \notin a_\xi$ holds. Let

$$x := \{n \in \omega; \forall \xi \in J,\ n \notin b_\xi\} = \omega \setminus \bigcup_{\xi \in J} b_\xi$$

and

$$y := \{n \in \omega; \forall \xi \in J,\ n \notin a_\xi\} = \omega \setminus \bigcup_{\xi \in J} a_\xi.$$

Then we note that both x and y are in N^*, and by the elementarity of N^*, $a_\eta \subseteq x$ [9] (and $b_\eta \subseteq y$). Since $\eta \in J$, $x \cap b_\eta = \emptyset$ (and $a_\eta \cap y = \emptyset$). Therefore, for every $\xi \in J \cap N^*$, since $a_\xi \subseteq^* a_\eta$ and $b_\xi \subseteq^* b_\eta$, it follows that $a_\xi \subseteq^* x$ and $x \perp b_\xi$. By the elementarity of

[8] A similar argument appears in [**30**, Proposition 3.2].

[9] Let $n \in a_\eta$. Then N^* thinks that $n \notin b_\xi$ holds for every $\xi \in J$. So by the elementarity, $n \notin b_\xi$ holds for every $\xi \in J$, from which it follows that $n \in x$.

N^* again, it follows that for every $\xi \in J$, $a_\xi \subseteq^* x$ and $x \perp b_\xi$, that is, x separates $(\mathcal{A}, \mathcal{B})$, which is a contradiction. $\square$

Then there exists $r \in \mathbb{P}_1 \cap N^*$ which witnesses $\xi \in J$ in N^*. Then as seen in the proof of Lemma 3.4, r is compatible with q in $\mathbb{P}_1$, which finishes the proof of $(\star)$ in this case.

Stages in C1: Suppose that $\alpha \in$ C1, $\dot{I}$ is a $\mathbb{P}_{\alpha+1}$-name for an uncountable subset of ω_1, $N^* \in \mathcal{M}^*_{\alpha+2}$ contains $(\mathcal{A}, \mathcal{B})$ and $\dot{I}$, $q \in \mathbb{P}_{\alpha+1}$ satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(q(0))$ and $q(0, N^* \cap H(\kappa)) = \alpha + 1$, and $\eta \in \omega_1 \setminus N^*$ such that $q \Vdash_{\mathbb{P}_{\alpha+1}}$"$\eta \in \dot{I}$". To argue for the general case, we may assume that $\omega_1 \cap N^*$ is in $\mathrm{dom}(q(\alpha, 1))$.

In this paragraph, we argue as in the proof of Lemma 3.4. By extending $q \downarrow \alpha$ if necessary, we can assume that for each ξ in the set $\mathrm{dom}(q(\alpha, 1)) \setminus (N^* \cup \{\omega_1 \cap N^*\})$, there exists a union e_ξ of finitely many intervals in $\xi \cap N^*$ such that

$$q{\downarrow}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} e_\xi = \left\{ i \in \xi \cap N^*; \dot{f}^\alpha_\xi(i) = q(\alpha, 1)(\xi) \right\} \text{”}.$$

By the requirement (α-4) in Definition 3.1 Case 1, we can take $q' \in \mathbb{P}_\alpha$, which is an extension of $q{\downarrow}\alpha$ in $\mathbb{P}_\alpha$, and $M \in \mathrm{dom}(q'(0)) \cap \mathcal{M}_{\alpha+1} \cap N^*$ such that

- $\{e_\xi; \xi \in \mathrm{dom}(q(\alpha, 1)) \setminus (N^* \cup \{\omega_1 \cap N^*\})\} \in M$,
- $q'(0, M) = \alpha$, and
- $q' \Vdash_{\mathbb{P}_\alpha}$"$\dot{f}^\alpha_{\omega_1 \cap N^*}(\omega_1 \cap M) \neq q(\alpha, 1)(\omega_1 \cap N^*)$".

Let $M^* \in \mathcal{M}^*_{\alpha+1} \cap N^*$ be such that $M = M^* \cap H(\kappa)$. We take an extension q'' of q' in $\mathbb{P}_\alpha$ and a union $e_{\omega_1 \cap N^*}$ of finitely many intervals in $\omega_1 \cap M^*$ such that

$$q'' \Vdash_{\mathbb{P}_\alpha} \text{“} e_{\omega_1 \cap N^*} = \left\{ i \in \omega_1 \cap M^*; f^\alpha_{\omega_1 \cap N^*}(i) = q(\alpha, 1)(\omega_1 \cap N^*) \right\} \text{”}.$$

We note again that $\sup(e_{\omega_1 \cap N^*}) < \omega_1 \cap M^*$.

We define the $\mathbb{P}_\alpha$-name $\dot{J}$ for a subset of ω_1 such that for all $r \in \mathbb{P}_\alpha$ and $\zeta \in \omega_1$,

$$r \Vdash_{\mathbb{P}_\alpha} \text{“} \zeta \in \dot{J} \text{”}$$

if and only if there exists $r' \in \mathbb{P}_{\alpha+1}$ such that

- $r'{\downarrow}\alpha = r$,
- $r'(\alpha, 0) \supseteq q(\alpha, 0) \cap M^*$ $(= q(\alpha, 0) \cap N^*)$,
- $r'(\alpha, 1) \supseteq q(\alpha, 1) \cap M^*$ $(= q(\alpha, 1) \cap N^*)$,
- for each $\xi \in \mathrm{dom}(q(\alpha, 1)) \setminus N^*$, $\mathrm{ran}(u_0) \cap e_\xi = \emptyset$, and
- $r' \Vdash_{\mathbb{P}_\alpha}$"$\zeta \in \dot{I}$".

We note that $\dot{J}$ belongs to M^* and, since $q'' \leq_{\mathbb{P}_\alpha} q{\downarrow}\alpha$,

$$q'' \Vdash_{\mathbb{P}_\alpha} \text{“} \eta \in \dot{J} \text{”}.$$

By the inductive hypothesis, there are $r \in \mathbb{P}_\alpha \cap M^*$ and $\xi \in \omega_1 \cap M^*$ such that r is compatible with q'' in $\mathbb{P}_\alpha$, $r \Vdash_{\mathbb{P}_\alpha}$"$\xi \in \dot{J}$" and

$$(a_\eta \cap b_\xi) \cup (a_\xi \cap b_\eta) \neq \emptyset.$$

By the definition of $\dot{J}$, there exists $r' \in \mathbb{P}_{\alpha+1} \cap M^*$ which witnesses that $r \Vdash_{\mathbb{P}_\alpha}$"$\xi \in \dot{J}$". By the choice of q'' and $r' \in M^*$, q and r' are compatible in $\mathbb{P}_{\alpha+1}$, which finishes the proof of $(\star)$ in this case.

Stages in C2: This proof is similar to the proof for a stage in C1.

Suppose that $\dot{I}$, N^*, q, η are as in the stage C1. By extending $q \downarrow \alpha$ and $q(\alpha, 1)$ if necessary, we may assume that for each ξ in the set $\mathrm{dom}(q(\alpha,1)) \setminus (N^* \cup \{\omega_1 \cap N^*\})$, there exists a finite subset e_ξ of $\omega_1 \cap N^*$ such that

$$q \downarrow \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} e_\xi = \dot{C}^\alpha_\xi \cap N^* \text{”}.$$

Next, we define q' as in the proof of Lemma 3.4. After that, we take an extension q'' of $q' \downarrow \alpha$ in $\mathbb{P}_\alpha$ and a finite subset $e_{\omega_1 \cap N^*}$ of $\omega_1 \cap N^*$ such that

$$q'' \Vdash_{\mathbb{P}_\alpha} \text{“} e_{\omega_1 \cap N^*} = \dot{C}^\alpha_{\omega_1 \cap N^*} \cap N^* \text{”}.$$

The rest of the proof in this case is the same as in the stage C1.

Stages in C3: Suppose that $\dot{I}$, N^*, q, η are as in the stage C1. We define the $\mathbb{P}_\alpha$-name $\dot{J}$ for a subset of ω_1 such that for all $r \in \mathbb{P}_\alpha$ and $\zeta \in \omega_1$,

$$r \Vdash_{\mathbb{P}_\alpha} \text{“} \zeta \in \dot{J} \text{”}$$

if and only if there exists $r' \in \mathbb{P}_{\alpha+1}$ such that $r' \downarrow \alpha = r$, $r'(\alpha, 0) = q(\alpha, 0)$ and $r' \Vdash_{\mathbb{P}_\alpha}$ "$\zeta \in \dot{I}$". We note that $\dot{J}$ belongs to N^* and by our assumption,

$$q \downarrow \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} \eta \in \dot{J} \text{”}.$$

By the inductive hypothesis, there are $r \in \mathbb{P}_\alpha \cap N^*$ and $\xi \in \omega_1 \cap N^*$ such that r is an extension of $q \downarrow \alpha$, $r \Vdash_{\mathbb{P}_\alpha}$"$\xi \in \dot{J}$" and

$$(a_\eta \cap b_\xi) \cup (a_\xi \cap b_\eta) \neq \emptyset.$$

By the definition of $\dot{J}$, there exists $r' \in \mathbb{P}_{\alpha+1} \cap N^*$ which witnesses that $r \Vdash_{\mathbb{P}_\alpha}$"$\xi \in \dot{J}$". Then $r' \in N^*$ is compatible with q in $\mathbb{P}_{\alpha+1}$, which finishes the proof of $(\star)$ in this case.

Stages in C4: This proof is similar to the stage C3. Suppose that $\dot{I}$, N^*, q, η are as in the stage C1. By extending q if necessary, we may assume that the length $q(\alpha, 0)$ is not shorter than $2 \cdot |q(\alpha, 1)|$. We define the $\mathbb{P}_\alpha$-name $\dot{J}$ for a subset of ω_1 such that for all $r \in \mathbb{P}_\alpha$ and $\zeta \in \omega_1$,

$$r \Vdash_{\mathbb{P}_\alpha} \text{“} \zeta \in \dot{J} \text{”}$$

if and only if there exists $r' \in \mathbb{P}_{\alpha+1}$ such that $r' \downarrow \alpha = r$, $r'(\alpha, 0) = q(\alpha, 0)$, the size of $r'(\alpha, 1)$ is equal to one of $q(\alpha, 1)$ and $r' \Vdash_{\mathbb{P}_\alpha}$"$\zeta \in \dot{I}$". The rest of the proof in this stage is the same as for stages C3.

Limit stages: Suppose that α is a limit ordinal, $\dot{I}$ (which is a $\mathbb{P}_\alpha$-name), N^* ($\in \mathcal{M}^*_\alpha$), q, η are as in the stage C1. As in the proof of Lemma 3.4, we have to consider two cases: α is of countable cofinality, or of uncountable cofinality. Here, we consider only the case that α is of uncountable cofinality. As seen in the proof of Lemma 3.4, the case that α is of countable cofinality is simpler than this case. We take $\beta \in \alpha \cap N^*$ such that $\max(\mathrm{supp}(q) \cap N^*) < \beta$ and for every $M' \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M' < \omega_1 \cap N^*$,

$$\sup(M' \cap N^* \cap \alpha) < \beta.$$

We define the $\mathbb{P}_\beta$-name $\dot{J}$ for a subset of ω_1 such that for all $r \in \mathbb{P}_\beta$ and $\zeta \in \omega_1$,

$$r \Vdash_{\mathbb{P}_\beta} \text{“}\zeta \in \dot{J}\text{”}$$

if and only if there exists $r' \in \mathbb{P}_\alpha$ such that

- $r \leq_{\mathbb{P}_\beta} r' {\downarrow} \beta$,
- $\left\{\overline{M}; M \in \mathrm{dom}(r'(0))\right\}$ end-extends the set

$$\left\{\overline{M}; M \in \mathrm{dom}(q(0)) \text{ and } \omega_1 \cap M < \omega_1 \cap N^*\right\},$$

- $\mathrm{dom}(r'(0))$ includes $\mathrm{dom}(q(0)) \cap N^*$,
- for each $M \in \mathrm{dom}(q(0))$ with $\omega_1 \cap M < \omega_1 \cap N^*$, there exists $M' \in \mathrm{dom}(q(0))$ such that $\omega_1 \cap M' = \omega_1 \cap M$ and $q(0, M') = \alpha$ if and only if there exists $M' \in \mathrm{dom}(r'(0))$ such that $\omega_1 \cap M' = \omega_1 \cap M$ and $r'(0, M') = \alpha$.

We note that $\dot{J} \in N^*$ and $q {\downarrow} \beta \Vdash_{\mathbb{P}_\beta} \text{“}\eta \in \dot{J}\text{”}$. By the inductive hypothesis, there are $r \in \mathbb{P}_\beta \cap N^*$ and $\xi \in \omega_1 \cap N^*$ such that r is compatible with $q {\downarrow} \beta$ in $\mathbb{P}_\beta$, $r \Vdash_{\mathbb{P}_\beta} \text{“}\xi \in \dot{J}\text{”}$ and

$$(a_\eta \cap b_\xi) \cup (a_\xi \cap b_\eta) \neq \emptyset.$$

By the definition of $\dot{J}$, there exists $r' \in \mathbb{P}_\alpha \cap N^*$ which witnesses that $r \Vdash_{\mathbb{P}_\beta} \text{“}\xi \in \dot{I}\text{”}$. Then, as seen in the proof of Lemma 3.4, q, r (as a condition of $\mathbb{P}_\alpha$) and r' are compatible in $\mathbb{P}_\alpha$, which finishes the proof of ($\star$) and the lemma. □

LEMMA 4.3. *Suppose that $(\mathcal{A}, \mathcal{B}) = \langle a_\xi, b_\xi; \xi \in \omega_1\rangle$ is an (ω_1, ω_1)-pregap which satisfies the property* (t). *Then for each $\alpha \in \kappa + 1 \setminus \{0\}$, $\mathbb{P}_\alpha$ preserves $(\mathcal{A}, \mathcal{B})$ to have the property* (t).

PROOF. This proof is very similar to the previous proof. By induction on $\alpha \in \kappa+1$, we show that

($\star$) for every $\mathbb{P}_\alpha$-name $\dot{I}$ for an uncountable subset, $N^* \in \mathcal{M}^*_{\alpha+1}$ which contains $(\mathcal{A}, \mathcal{B})$ and $\dot{I}$, $q \in \mathbb{P}_\alpha$ which satisfies that $N^* \cap H(\kappa) \in \mathrm{dom}(q(0))$ and $q(0, N^* \cap H(\kappa)) = \alpha$, and $\eta \in \omega_1 \setminus N^*$ such that $q \Vdash_{\mathbb{P}_\alpha} \text{“}\eta \in \dot{I}\text{”}$, there are $r \in \mathbb{P}_\alpha \cap N^*$ and $\xi \in \omega_1 \cap N^*$ such that r is compatible with q in $\mathbb{P}_\alpha$, $r \Vdash_{\mathbb{P}_\alpha} \text{“}\xi \in \dot{I}\text{”}$ and $\xi \lhd_{(\mathcal{A},\mathcal{B})} \eta$.

The only difference between this proof and the proof of Lemma 4.1 is Claim 4.2: If J is an uncountable subset of ω_1, N^* is a countable elementary submodel of $H(\theta)$ for some regular cardinal θ such that N^* contains $(\mathcal{A}, \mathcal{B})$, and $\eta \in J \setminus N^*$, then there exists $\xi \in J \cap N^*$ such that $\xi \triangleleft_{(\mathcal{A},\mathcal{B})} \eta$.

For such J, N^* and η, by the property (t) of $(\mathcal{A}, \mathcal{B})$, we can take a maximal $\triangleleft_{(\mathcal{A},\mathcal{B})}$-incomparable subset J' of J in N^*. Then J' is countable, hence J' is a subset of N^* and so $\eta \notin J'$. Therefore we can choose $\xi \in J'$ (which is also in $J \cap N^*$) such that $\xi \triangleleft_{(\mathcal{A},\mathcal{B})} \eta$, which finishes the proof. □

Therefore, we conclude the following.

Theorem 4.4. *Supposing that κ is an uncountable regular cardinal with $2^{<\kappa} = \kappa$,* CH *holds and there exists a T-gap, $\mathbb{P}_\kappa$ doesn't destroy any cardinal and forces that $\mho$ fails, $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = 2^{\aleph_0} = \kappa$ holds and there still exists a T-gap (hence $\mathrm{MA}_{\aleph_1}$ fails).*

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Teruyuki Yorioka
Department of Mathematics
Shizuoka University
Ohya 836
Shizuoka 422-8529, Japan
E-mail: styorio@ipc.shizuoka.ac.jp