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FORCING THE MAPPING REFLECTION PRINCIPLE BY FINITE APPROXIMATIONS

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Abstract. Moore introduced the Mapping Reflection Principle and proved that the Bounded Proper Forcing Axiom implies that the size of the continuum is $\aleph_2$. The Mapping Reflection Principle follows from the Proper Forcing Axiom. To show this, Moore utilized forcing notions whose conditions are countable objects. Chodounský-Zapletal introduced the Y-Proper Forcing Axiom that is a weak fragments of the Proper Forcing Axiom but implies some important conclusions from the Proper Forcing Axiom, for example, the P-ideal Dichotomy. In this article, it is proved that the Y-Proper Forcing Axiom implies the Mapping Reflection Principle by introducing forcing notions whose conditions are finite objects.

1. Introduction

Cohen discovered the forcing method to show that it is consistent that the size of the continuum is larger than $\aleph_1$ [4, 5]. Solovay-Tennenbaum developed Cohen's method to the iterated forcing method, and showed that it is consistent that Suslin's Hypothesis holds [14]. Martin discovered a certain extension of Baire Category Theorem, called Martin's Axiom, and pointed out that Solovay-Tennenbaum's method indicated that, for any cardinal κ, it is consistent that Martin's Axiom holds and the size of the continuum is larger than κ [8]. Martin's Axiom is the assertion for some class of forcing notions, which is the class of forcing notions with the countable chain condition. Further developments of the iterated forcing method produced further extensions of Baire Category Theorem which are consistent relative to the consistency with some large cardinal axioms. Such extensions are called forcing axioms, which are similar assertions to Martin's Axiom but are ones applied for more wider class of forcing notions. The Proper Forcing Axiom and Martin's Maximum are useful and widely studied forcing axioms. By comparing classes of forcing notions, Martin's Maximum implies the Proper Forcing Axiom. Under the assumption that it is consistent that there exists a supercompact cardinal, Baumgartner proved that the Proper Forcing Axiom is consistent [7, Ch 31], and Foreman-Magidor-Shelah proved that Martin's Maximum is consistent [6]. Both axioms had been proved to be consistent (relative to the consistency of the existence of a supercompact cardinal) with the assertion that the size of the continuum is $\aleph_2$.

Foreman-Magidor-Shelah proved that Martin's Maximum decides the size of the continuum to be $\aleph_2$ [6]. Moreover, Todorčević and Veličković proved that the Proper Forcing Axiom implies that the size of the continuum is $\aleph_2$ ([2, 3.16 Theorem], [19]). After that, it was studied that weaker forcing axioms may decide the value of the continuum. Todorčević showed that the Bounded Martin's Maximum decides the

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size of the continuum to be $\aleph_2$ [16]. And at last, Moore proved that the Bounded Proper Forcing Axiom decides the size of the continuum to be $\aleph_2$ [10]. To show this, Moore introduced the assertion called the Mapping Reflection Principle. The Mapping Reflection Principle is implied from the Proper Forcing Axiom, and plays important roles to introduce many conclusions from the Proper Forcing Axiom. A notable application of the Mapping Reflection Principle is that there is a well-order of $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ of length ω_2, and $L(\mathcal{P}(\omega_1))$ satisfies the Axiom of Choice.

Zapletal showed that some forcing notions preserve the additivity of the measure zero ideal to be $\aleph_1$ [27]. The second author extended Zapletal's result to show that forcing notions in Zapletal's results preserve the covering number of the measure zero ideal to be $\aleph_1$, that is, forcing notions in Zapletal's results add no random reals [22, 24, 25, 26]. Chodounský-Zapletal introduced the properties of forcing notions, called Y-cc and Y-proper, and the new forcing axiom, called the Y-Proper Forcing Axiom, which is the forcing axiom for the class of forcing notions with the Y-proper condition [3]. They also proved that forcing notions with Y-cc or Y-proper add no random reals, and that it is consistent relative to the consistency of the existence of a supercompact cardinal that the Y-Proper Forcing Axiom holds, the covering number of the measure zero ideal is $\aleph_1$, and an entangled set of reals exists (hence Open Coloring Axioms due to Abraham-Rubin-Shelah and Todorčević respectively fail). They pointed out that, by following the argument for the Proper Forcing Axiom due to Todorčević (with a small change), the Y-Proper Forcing Axiom implies that the size of the continuum is $\aleph_2$.

In this paper, we show that the Y-Proper Forcing Axiom implies the Mapping Reflection Principle. It is better understood that the Y-Proper Forcing Axiom is a useful weak fragment of the Proper Forcing Axiom. In section 2, the Mapping Reflection Principle and the Y-proper forcing notions are reviewed. In section 3, it is proved that the Y-Proper Forcing Axiom implies the Mapping Reflection Principle.

2. Preliminaries

2.1. **The Mapping Reflection Principle.** In this article, for a cardinal θ, $H(\theta)$ denotes the set of all sets of hereditary cardinality less than θ, $[X]^\theta$ denotes the set of all subsets of a set X of cardinality θ, and $[X]^{<\theta}$ denotes the set of all subsets of a set X of cardinality less than θ. The Ellentuck topology on the set $[X]^{\aleph_0}$ is the topology generated by the sets of the form

$$[x, Z] := \left\{ Y \in [X]^{\aleph_0} : x \subseteq Y \subseteq Z \right\}$$

for some finite subset x of X and some infinite subset Z of X.

Definition 2.1 (Moore [10])**.** Σ is called an open stationary set mapping when there are an uncountable set X and a regular cardinal θ with $[X]^{\aleph_0} \in H(\theta)$ such that

- $\mathrm{dom}(\Sigma)$ is a club subset of the set of countable elementary submodels of $H(\theta)$,
- for every $M \in \mathrm{dom}(\Sigma)$,
  - $\Sigma(M)$ is an open subset of the space $[X]^{\aleph_0}$ equipped with the Ellentuck topology, and
  - $\Sigma(M)$ is M-stationary, i.e. for any club subset E of $[X]^{\aleph_0}$, if $E \in M$, then $E \cap \Sigma(M) \cap M \neq \emptyset$.

For an open stationary set mapping Σ, the parameters X and θ for Σ will be referred to as X_Σ and θ_Σ.

The Mapping Reflection Principle (MRP) is the statement that, for any open stationary set mapping Σ, there exists a reflecting sequence for Σ, which means a continuous $\in$-chain $\langle N_\nu : \nu \in \omega_1 \rangle$ in $\mathrm{dom}(\Sigma)$ such that, for all limit ordinals $\nu \in \omega_1$, there exists $\nu_0 < \nu$ such that, for any $\xi \in (\nu_0, \nu)$, $N_\xi \cap X_\Sigma \in \Sigma(N_\nu)$.

MRP has a large number of applications. For example, MRP implies that the assertion $\mho$ fails [12, §1], that there are no weak club guessing sequences [12, §1], that there is a well-order of $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ of length ω_2 [10, §4], that $\square(\kappa)$ fails for all regular cardinal $\kappa > \omega_1$ [10, §6], that the Singular Cardinal Hypothesis holds [20], that the Bounded Proper Forcing Axiom implies that every Aronszajn line contains a Countryman suborder [11], [12, §11], that Martin's Axiom implies that $\mathsf{ITP}(\lambda, \omega_2)$ holds for all cardinals $\lambda > \omega_1$ [15], and that $\mathfrak{p} > \aleph_1$ implies Measuring$_{<2^{\aleph_0}}$ [1].

It is proved that PFA implies MRP [10, §3]. In [10, §3], for a given open stationary set mapping Σ, the forcing notion $\mathbb{P}_\Sigma$ is defined in such a way that $\mathbb{P}_\Sigma$ is proper and adds a reflecting sequence for Σ by countable approximations (initial segments). Moore emphasized the fact that $\mathbb{P}_\Sigma$ adds no new reals, is not ω-proper, but is weakly ω-proper [12, §6]. By investigating Moore's $\mathbb{P}_\Sigma$, we proved that it is consistent relative to the consistency of the existence of a supercompact cardinal that MRP holds and there exists a Suslin tree [9].

2.2. **The Y-properness.**

Definition 2.2 (Chodounský-Zapletal [3, §1])**.** (1) A forcing notion $\mathbb{P}$ satisfies Y-cc if, for any large enough regular cardinal λ, any countable elementary submodel N of $H(\lambda)$ with $\mathbb{P} \in N$, and any $p \in \mathbb{P}$, there exists a filter $F \in N$ on the regular open algebra $\mathrm{RO}(\mathbb{P})$ of $\mathbb{P}$ such that the set $\left\{s \in \mathrm{RO}(\mathbb{P}) \cap N : p \leq_{\mathrm{RO}(\mathbb{P})} s\right\}$ is included in the set F as a subset.

(2) A forcing notion $\mathbb{P}$ satisfies Y-proper if, for any large enough regular cardinal λ, any countable elementary submodel N of $H(\lambda)$ with $\mathbb{P} \in N$, and any $p \in \mathbb{P} \cap N$, there exists an extension q of p in $\mathbb{P}$ such that q is $(M, \mathbb{P})$-generic, and, for any $r \leq_{\mathbb{P}} q$, there exists a filter $F \in N$ on $\mathrm{RO}(\mathbb{P})$ such that the set $\left\{s \in \mathrm{RO}(\mathbb{P}) \cap N : r \leq_{\mathrm{RO}(\mathbb{P})} s\right\}$ is included in the set F as a subset.

(3) The Y-Proper Forcing Axiom (YPFA) is the assertion that, for any Y-proper forcing notion $\mathbb{P}$ and any $\aleph_1$-many dense subsets $\{D_\alpha : \alpha \in \omega_1\}$, there exists a filter G on $\mathbb{P}$ which meets all D_α's.

A typical example of a Y-cc forcing notion is a σ-centered forcing notion. It is proved that a Y-cc forcing notion is ccc [3, §2]. The following lemma indicates that many partition forcing notions satisfy Y-cc.

Definition 2.3. (1) (Todorčević [17, Ch 7]) For a partition $[\omega_1]^2 = K_0 \cup K_1$, define the forcing notion $\mathbb{P}_{K_0}$ whose conditions are finite K_0-homogeneous subsets of ω_1, and $\leq_{\mathbb{P}_{K_0}} := \supseteq$. A partition $[\omega_1]^2 = K_0 \cup K_1$ is called ccc if $\mathbb{P}_{K_0}$ is ccc.

(2) For a partition $[\omega_1]^2 = K_0 \cup K_1$, define the forcing notion $\mathbb{Q}_{K_0}$ whose conditions are the finite subsets of ω_1, and, for each $p, q \in \mathbb{Q}_{K_0}$, the relation $q \leq_{\mathbb{Q}_{K_0}} p$ is defined by the assertion that, for any $\alpha \in q \setminus p$ and any $\beta \in p$, $\{\alpha, \beta\} \in K_0$.

If $\mathbb{P}_{K_0}$ is ccc, then some condition of $\mathbb{P}_{K_0}$ forces that there exists an uncountable K_0-homogeneous set. By the definition, for any pairwise disjoint linked (pairwise compatible) subset I of $\mathbb{Q}_{K_0}$ and any choice function $\langle \alpha_p : p \in I\rangle$ of I (that is, $\alpha_p \in p$ for each $p \in I$), the set $\{\alpha_p : p \in I\}$ is K_0-homogeneous. Therefore, if $\mathbb{Q}_{K_0}$ is ccc, then some condition of $\mathbb{Q}_{K_0}$ forces that there exists an uncountable K_0-homogeneous set.

Theorem 2.4. *For a partition $[\omega_1]^2 = K_0 \cup K_1$, if $\mathbb{Q}_{K_0}$ is ccc, then both $\mathbb{P}_{K_0}$ and $\mathbb{Q}_{K_0}$ are Y-cc.*

The following proof is similar to one in Chodounský-Zapletal [3, Theorem 3.1], which refers the proofs in [22, 25, 26].

Proof. We only show that $\mathbb{P}_{K_0}$ is Y-cc. The proof of Y-ccness of $\mathbb{Q}_{K_0}$ is same.

At first, we observe a key notion of the proof. For a finite subset a of ω_1, a subset $\mathcal{A}$ of $\mathbb{P}_{K_0}$ is called a-large if, for any countable subset b of ω_1, there exists $r \in \mathcal{A}$ such that $r \cap (a \cup b) = a$. We claim that, for any finite subset a of ω_1, the set

$$\left\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}_{K_0} \text{ is } a\text{-large}\right\}$$

is a centered subset of $\mathrm{RO}(\mathbb{P}_{K_0})$, that is, any finite subset has a non-zero lower bound.

To show this, let a be a finite subset of ω_1, $n \in \omega$, and $\mathcal{A}_i$ an a-large subset of $\mathbb{P}_{K_0}$ for each $i \in n$. Let us show that the Boolean calculation $\bigwedge_{i\in n} \left(\bigvee \mathcal{A}_i\right)$ in $\mathrm{RO}(\mathbb{P}_{K_0})$ is not zero. It suffices to prove that there exists a choice function $\langle p_i : i \in n\rangle$ of the set $\{\mathcal{A}_i : i \in n\}$ (that is, $p_i \in \mathcal{A}_i$ for each $i \in n$) such that $\langle p_i : i \in n\rangle$ has a common extension in $\mathbb{P}_{K_0}$. Since each $\mathcal{A}_i$ is a-large, we can build an uncountable subset $\left\{p_i^\xi : \xi \in \omega_1\right\}$ of $\mathcal{A}_i$, for each $i \in n$, such that the set $\left\{p_i^\xi : i \in n, \xi \in \omega_1\right\}$ forms a Δ-system with root a. For each $\xi \in \omega_1$, define $r^\xi := \left(\bigcup_{i\in n} p_i^\xi\right) \setminus a$. Then the set $\left\{r^\xi : \xi \in \omega_1\right\}$ is an uncountable pairwise disjoint subset of $\mathbb{Q}_{K_0}$. Since $\mathbb{Q}_{K_0}$ has no uncountable pairwise disjoint antichain in $\mathbb{Q}_{K_0}$, by applying Erdös-Dushnik-Miller's partition relation $\omega_1 \to (\omega_1, \omega)^2$ to the coloring defined by letting $\{\xi, \eta\}$ be colored 0 iff r^ξ and r^η are incompatible in $\mathbb{Q}_{K_0}$, there exists an infinite subset J of ω_1 such that the set $\left\{r^\xi : \xi \in J\right\}$ is pairwise compatible in $\mathbb{Q}_{K_0}$. Take n-many elements ξ_i, $i \in n$, of J. Then, for each $i \in n$, $p_i^{\xi_i} \in \mathcal{A}_i$, and $\bigcup_{i\in n} p_i^{\xi_i}$ is a condition of $\mathbb{P}_{K_0}$, and hence is a common extension of the set $\left\{p_i^{\xi_i} : i \in n\right\}$ in $\mathbb{P}_{K_0}$.

Let λ be a large enough regular cardinal, M^* a countable elementary submodel of $H(\lambda)$ which contains the set $\{K_0\}$, and $p \in \mathbb{P}_{K_0}$. By the previous observation, the set

$$\left\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}_{K_0} \text{ is } p \cap M^*\text{-large}\right\}$$

generates the filter F on $\mathrm{RO}(\mathbb{P}_{K_0})$. This F belongs to M^*. Let us show that, for every $s \in \mathrm{RO}(\mathbb{P}_{K_0}) \cap M^*$, if $p \leq_{\mathrm{RO}(\mathbb{P}_{K_0})} s$, then $s \in F$. To show this, let $s \in \mathrm{RO}(\mathbb{P}_{K_0}) \cap M^*$ be such that $p \leq_{\mathrm{RO}(\mathbb{P}_{K_0})} s$. Define $\mathcal{A} := \left\{q \in \mathbb{P}_{K_0} : q \leq_{\mathrm{RO}(\mathbb{P}_{K_0})} s\right\}$. $\mathcal{A}$ belongs to M^*, and, since $\mathbb{P}_{K_0}$ can be considered as a dense subset of $\mathrm{RO}(\mathbb{P}_{K_0})$, $\bigvee \mathcal{A} = s$. Since $p \leq_{\mathrm{RO}(\mathbb{P}_{K_0})} s$ and $s \in M^*$, it follows from elementarity of M^* that $\mathcal{A}$ is $p \cap M^*$-large. Therefore, $s = \bigvee \mathcal{A} \in F$. $\square$

A typical example of a non-ccc Y-proper forcing notion is the $\in$-collapse, which is the set of the finite $\in$-chains of countable elementary submodels of $H(\kappa)$ for some regular cardinal κ, ordered by $\supseteq$ [18, §7.1]. YPFA implies that every Aronszajn tree is special [21], that any ladder system coloring can be uniformized [23], that the P-ideal Dichotomy holds, that there are no S-spaces, and that Todorčević's five-element classifications of directed partial orders of size $\aleph_1$ is satisfied [3, §4], [25]. Chodounský-Zapletal proved that, if there exists a supercompact cardinal, YPFA can be forced. Moreover, it is consistent relative to the existence of a supercompact cardinal that YPFA holds, that the covering number of the measure zero ideal is equal to $\aleph_1$, and that there exists an entangled set of reals, which implies that there are two ccc forcing notions whose product is not ccc (hence Martin's Axiom fails), and that both of two Open Coloring Axioms fail [3, §6]. To prove this, they applied the forcing iteration by use of Neeman's side condition method [13].

3. YPFA implies MRP

Definition 3.1. Let Σ be an open stationary set mapping. Define the forcing notion $\mathbb{P}(\Sigma)$ which consists of finite subsets p of $\omega_1 \times \mathrm{dom}(\Sigma) \times \mathrm{dom}(\Sigma)$ such that

- for any $\langle \varepsilon, M_0, M_1 \rangle \in p$, $\varepsilon \in M_0 \in M_1$ and M_0 is a closure point of $\mathrm{dom}(\Sigma)$, that is, $M_0 = \bigcup(\mathrm{dom}(\Sigma) \cap M_0)$,
- for any $\{\langle \varepsilon, M_0, M_1 \rangle, \langle \varepsilon', M_0', M_1' \rangle\}$ in $[p]^2$, $\omega_1 \cap M_0 \neq \omega_1 \cap M_0'$ holds, and moreover,
  - if $\omega_1 \cap M_0 < \omega_1 \cap M_0'$, then $M_1 \in M_0'$, and
  - if $\varepsilon' < \omega_1 \cap M_0 < \omega_1 \cap M_0'$, then $M_0 \cap X_\Sigma \in \Sigma(M_0')$,

for any $p, q \in \mathbb{P}(\Sigma)$, $q \leq_{\mathbb{P}(\Sigma)} p$ iff $q \supseteq p$.

For $p, q \in \mathbb{P}(\Sigma)$, we say that q is an end-extension of p if $q \cap M_0 = p$ for some $\langle \varepsilon, M_0, M_1 \rangle \in q$.

In the rest of this section, we fix an open stationary set mapping Σ. Let λ be a regular cardinal which is greater than the cardinal $\left(2^{\left(2^{(\theta_\Sigma)}\right)}\right)^+$.

Lemma 3.2. *$\mathbb{P}(\Sigma)$ is proper.*

Proof. Let M^* be a countable elementary submodel of $H(\lambda)$ which contains the set $\{\Sigma, H(\theta_\Sigma)\}$, $p \in \mathbb{P}(\Sigma) \cap M^*$. Then the set $M_0 := M^* \cap H(\theta_\Sigma)$ is a closure point of $\mathrm{dom}(\Sigma)$. Take $M_1 \in \mathrm{dom}(\Sigma)$ such that $M_0 \in M_1$, and take $\varepsilon \in \omega_1 \cap M_0$ which is a large enough in such a way that the set $p^+ := p \cup \{\langle \varepsilon, M_0, M_1 \rangle\}$ is a condition of $\mathbb{P}(\Sigma)$, that is, for every $\langle \gamma, K_0, K_1 \rangle \in p^+ \cap M_0 = p$, $\omega_1 \cap K_1 < \varepsilon$. Then p^+ is an extension of p in $\mathbb{P}(\Sigma)$. We will show that p^+ is $(M^*, \mathbb{P}(\Sigma))$-generic.

Suppose that $\mathcal{D} \in M^*$ is a dense open subset of $\mathbb{P}(\Sigma)$, and q is an extension of p^+ in $\mathbb{P}(\Sigma)$. By extending q if necessary, we may assume that $q \in \mathcal{D}$. Take a finite subset x of $M_0 \cap X_\Sigma$ such that, for any $\langle \gamma, K_0, K_1 \rangle \in q$, if $\gamma < \omega_1 \cap M_0 < \omega_1 \cap K_0$, then $[x, M_0 \cap X_\Sigma] \subseteq \Sigma(K_0)$. Define

$$E := \Big\{ Y \in [X_\Sigma]^{\aleph_0} : \text{for any } y \in [Y]^{<\aleph_0}, \text{ there exists } r \in \mathcal{D} \text{ such that} \\ \bullet\ r \text{ is an end-extension of } q \cap M_0, \text{ and} \\ \bullet\ \text{for any } \langle \gamma, K_0, K_1 \rangle \in r \setminus (q \cap M_0), \\ x \cup y \subseteq K_0 \cap X_\Sigma \subseteq Y \Big\}.$$

Then $E \in M^* \cap H(\theta_\Sigma) = M_0$, and E is closed in $[X_\Sigma]^{\aleph_0}$ by the definition.

We claim that $M_0 \cap X_\Sigma$ belongs to E. To check this, let y be a finite subset of $M_0 \cap X_\Sigma$. Then q satisfies that

- $q \in \mathcal{D}$,
- q is an end-extension of $q \cap M_0$, and
- for any $\langle \gamma, K_0, K_1 \rangle \in q \setminus (q \cap M_0)$, $x \cup y \subseteq M_0 \cap X_\Sigma \subseteq K_0 \cap X_\Sigma$.

So by elementarity of M^*, there exists $r \in \mathcal{D} \cap M^*$ such that

- r is an end-extension of $q \cap M_0$, and
- for any $\langle \gamma, K_0, K_1 \rangle \in r \setminus (q \cap M_0)$, $x \cup y \subseteq K_0 \cap X_\Sigma$.

Since $r \in M^*$ and $M_0 = M^* \cap H(\theta_\Sigma)$,

- for any $\langle \gamma, K_0, K_1 \rangle \in r \setminus (q \cap M_0)$, $K_0 \cap X_\Sigma \subseteq M_0 \cap X_\Sigma$.

Since $M_0 \cap X_\Sigma \in E \in M^*$, E is unbounded. Thus E is club in $[X_\Sigma]^{\aleph_0}$. Since $\Sigma(M_0)$ is M_0-stationary, there exists $Y \in E \cap \Sigma(M_0) \cap M_0$. Since Y is countable, $Y \subseteq M_0 \cap X_\Sigma$. Take a finite subset y of Y such that $[y, Y] \subseteq \Sigma(M_0)$. By elementarity of M^*, there exists $r \in \mathcal{D} \cap M^*$ such that r is an end-extension of $q \cap M_0$, and, for any $\langle \gamma, K_0, K_1 \rangle \in r \setminus (q \cap M_0)$, $x \cup y \subseteq K_0 \cap X_\Sigma \subseteq Y$.

We claim that $q \cup r$ is a condition of $\mathbb{P}(\Sigma)$. Then $q \cup r$ is a common extension of q and r, and so the proof is completed. We will show here that, for any $\{\langle \gamma, K_0, K_1 \rangle, \langle \gamma', K_0', K_1' \rangle\}$ in $[q \cup r]^2$, if $\gamma' < \omega_1 \cap K_0 < \omega_1 \cap K_0'$, then $K_0 \cap X_\Sigma \in \Sigma(K_0')$. To show this, let $\{\langle \gamma, K_0, K_1 \rangle, \langle \gamma', K_0', K_1' \rangle\}$ in $[q \cup r]^2$ be such that $\gamma' < \omega_1 \cap K_0 < \omega_1 \cap K_0'$. Since both q and r are conditions of $\mathbb{P}(\Sigma)$, if $\{\langle \gamma, K_0, K_1 \rangle, \langle \gamma', K_0', K_1' \rangle\} \subseteq q$ or $\{\langle \gamma, K_0, K_1 \rangle, \langle \gamma', K_0', K_1' \rangle\} \subseteq r$, then $K_0 \cap X_\Sigma \in \Sigma(K_0')$. Since $r \in M^* \cap H(\theta_\Sigma) = M_0$ and $\langle \varepsilon, M_0, M_1 \rangle \in q$, the rest of cases is that $\langle \gamma, K_0, K_1 \rangle \in r \setminus q$ and $\langle \gamma', K_0', K_1' \rangle \in q \setminus r$. If $K_0' \neq M_0$, then by the role of x, $K_0 \cap X_\Sigma \in \Sigma(K_0')$. If $K_0' = M_0$, then by the role of y and the choice of r, $K_0 \cap X_\Sigma \in \Sigma(M_0) = \Sigma(K_0')$. $\square$

Definition 3.3. For an open stationary set mapping Σ, define a $\mathbb{P}(\Sigma)$-name $\dot{\mathcal{N}}(\Sigma)$ such that

$$\Vdash_{\mathbb{P}(\Sigma)} \text{“} \dot{\mathcal{N}}(\Sigma) := \left\{ M_0 : \exists \varepsilon\, \exists M_1 \left(\langle \varepsilon, M_0, M_1 \rangle \in \bigcup \dot{G}_{\mathbb{P}(\Sigma)} \right) \right\} \text{”},$$

here $\dot{G}_{\mathbb{P}(\Sigma)}$ is the canonical generic $\mathbb{P}(\Sigma)$-name.

By the definition of $\mathbb{P}(\Sigma)$, it is proved that

$$\Vdash_{\mathbb{P}(\Sigma)} \text{“} \dot{\mathcal{N}}(\Sigma) \text{ is linearly ordered by } \in \text{”}.$$

Proposition 3.4. $\Vdash_{\mathbb{P}(\Sigma)}$ “$\dot{\mathcal{N}}(\Sigma)$ *is an unbounded chain of countable elementary submodels of* $H(\theta_\Sigma)^V$”*, here* $H(\theta_\Sigma)^V$ *is* $H(\theta_\Sigma)$ *in the ground model.*

Proof. By the definition, $\Vdash_{\mathbb{P}(\Sigma)}$“ $\dot{\mathcal{N}}(\Sigma)$ forms a chain of countable elementary submodels of $H(\theta_\Sigma)^V$”. We will show the unboundedness.

Let $z \in H(\theta_\Sigma)$ and $p \in \mathbb{P}(\Sigma)$. Take a countable elementary submodel M^* of $H(\lambda)$ which contains the set $\{\Sigma, H(\theta_\Sigma), z, p\}$. Since $\mathrm{dom}(\Sigma)$ is a club subset of $H(\theta_\Sigma)$, $M_0 := M^* \cap H(\theta_\Sigma)$ is a closure point of $\mathrm{dom}(\Sigma)$. Take $M_1 \in \mathrm{dom}(\Sigma)$ such that $M_0 \in M_1$, and take a large enough ordinal ε in $\omega_1 \cap M^*$ in such a way that the set $q := p \cup \{\langle \varepsilon, M_0, M_1 \rangle\}$ is a condition in $\mathbb{P}(\Sigma)$, that is, for every $\langle \gamma, K_0, K_1 \rangle \in q \cap M_0 = p$, $\omega_1 \cap K_1 < \varepsilon$. Then q is an extension of p in $\mathbb{P}(\Sigma)$ and $q \Vdash_{\mathbb{P}(\Sigma)}$“ $z \in M_0 \in \dot{\mathcal{N}}(\Sigma)$”. $\square$

Lemma 3.5. $\Vdash_{\mathbb{P}(\Sigma)}$ "*$\dot{\mathcal{N}}(\Sigma)$ is closed, hence is a continuous $\in$-chain*".

Proof. Let $p \in \mathbb{P}(\Sigma)$, $\dot{N}_n$ a $\mathbb{P}(\Sigma)$-name for each $n \in \omega$ such that

$$p \Vdash_{\mathbb{P}(\Sigma)} \text{“} \left\{\dot{N}_n : n \in \omega\right\} \subseteq \dot{\mathcal{N}}(\Sigma) \ \&\ \forall n \in \omega \left(\dot{N}_n \in \dot{N}_{n+1}\right) \text{”}.$$

Let us show that

$$p \nVdash_{\mathbb{P}(\Sigma)} \text{“} \bigcup_{n\in\omega} \dot{N}_n \notin \dot{\mathcal{N}}(\Sigma) \text{”}.$$

At first, we claim that

$$p \nVdash_{\mathbb{P}(\Sigma)} \text{“} \sup_{n\in\omega} \left(\omega_1 \cap \dot{N}_n\right) \notin \left\{\omega_1 \cap M : M \in \dot{\mathcal{N}}(\Sigma)\right\} \text{”}.$$

Assume not. Since $\mathbb{P}(\Sigma)$ is proper, we can take an extension q of p and $\alpha \in \omega_1$ such that

$$q \Vdash_{\mathbb{P}(\Sigma)} \text{“} \sup_{n\in\omega} \left(\omega_1 \cap \dot{N}_n\right) = \alpha \text{”}.$$

By extending q if necessary, we may assume that there exists $\langle \varepsilon, M_0, M_1 \rangle \in q$ such that

$$q \Vdash_{\mathbb{P}(\Sigma)} \text{“ } M_0 \text{ is the least element of } \dot{\mathcal{N}}(\Sigma) \text{ with the property that } \alpha \leq \omega_1 \cap M_0 \text{”}.$$

It follows from our assumption that $\alpha < \omega_1 \cap M_0$. Take an extension q' of q in $\mathbb{P}(\Sigma)$, $m \in \omega$ and N such that

$$q' \Vdash_{\mathbb{P}(\Sigma)} \text{“ } q \cap M_0 \in \dot{N}_m = N \text{”}.$$

By extending q' if necessary, we may assume that, for some β and N', $\langle \beta, N, N' \rangle$ is in q'. Since M_0 is a closure point of $\mathbb{P}(\Sigma)$ and $\{N, \alpha\} \in M_0$, there exists $N'' \in \mathrm{dom}(\Sigma) \cap M_0$ such that $\{N, \alpha\} \in N''$. Define $r := q \cup \{\langle \beta, N, N'' \rangle\}$. Then r is a condition of $\mathbb{P}(\Sigma)$. r may not force that $\dot{N}_m = N$. However,

$r \Vdash_{\mathbb{P}(\Sigma)}$ "
- $\omega_1 \cap N < \alpha = \sup_{n\in\omega} \left(\omega_1 \cap \dot{N}_n\right)$, and
- for any $K \in \dot{\mathcal{N}}(\Sigma)$, if $\omega_1 \cap N < \omega_1 \cap K$, then $\alpha \in N'' \in K$,

therefore, it follows that $\left\{\dot{N}_n : n \in \omega\right\} \nsubseteq \dot{\mathcal{N}}(\Sigma)$",

which is a contradiction.

By extending p if necessary, we may assume that there exists $\langle \varepsilon, M_0, M_1 \rangle \in p$ such that

$$p \Vdash_{\mathbb{P}(\Sigma)} \text{“} \sup_{n\in\omega} \left(\omega_1 \cap \dot{N}_n\right) = \omega_1 \cap M_0 \text{”}.$$

At last, we claim that

$$p \Vdash_{\mathbb{P}(\Sigma)} \text{“} \bigcup_{n\in\omega} \dot{N}_n = M_0 \text{”}.$$

We notice that

$p \Vdash_{\mathbb{P}(\Sigma)}$ " for each $n \in \omega$, $\omega_1 \cap \dot{N}_n < \omega_1 \cap M_0$, hence $\dot{N}_n \in M_0$,
therefore, it follows that $\bigcup_{n\in\omega} \dot{N}_n \subseteq M_0$ ".

Let us show that

$$p \Vdash_{\mathbb{P}(\Sigma)} \text{“} \bigcup_{n\in\omega} \dot{N}_n \supseteq M_0 \text{”}.$$

Let q be an extension of p in $\mathbb{P}(\Sigma)$, and $z \in M_0$. Take an extension q' of q in $\mathbb{P}(\Sigma)$, $m \in \omega$ and N such that

$$q' \Vdash_{\mathbb{P}(\Sigma)} \text{“ } q \cap M_0 \in \dot{N}_m = N \text{ ”}.$$

Then $\omega_1 \cap N < \omega_1 \cap M_0$, hence $N \in M_0$. Since M_0 is a closure point of $\mathrm{dom}(\Sigma)$, we can take $N' \in \mathrm{dom}(\Sigma) \cap M_0$ such that $\{z, N\} \in N'$, and can take $\beta \in \omega_1 \cap N$ in such a way that $r := q \cup \{\langle \beta, N, N' \rangle\}$ is a condition in $\mathbb{P}(\Sigma)$. Then it is not sure that r forces that $\dot{N}_m = N$. However, it is true that

$$r \Vdash_{\mathbb{P}(\Sigma)} \text{“ } \omega_1 \cap N' < \omega_1 \cap M_0 = \sup_{n\in\omega} \left(\omega_1 \cap \dot{N}_n\right) \text{ and } \langle \beta, N, N' \rangle \in \bigcup \dot{G}_{\mathbb{P}(\Sigma)}, \\ \text{therefore, } z \in N' \subseteq \bigcup_{n\in\omega} \dot{N}_n \text{ ”}. \qquad \square$$

It follows from Lemmas 3.4 and 3.5 that

$$\Vdash_{\mathbb{P}(\Sigma)} \text{“ } \dot{\mathcal{N}}(\Sigma) \text{ is a cofinal continuous } \in\text{-chain of countable subsets of } H(\theta_\Sigma)^V \text{ ”}.$$

Since $\mathbb{P}(\Sigma)$ is proper, $\mathbb{P}(\Sigma)$ preserves ω_1. It follows that $\mathbb{P}(\Sigma)$ collapses θ_Σ to ω_1. The following lemma shows that YPFA implies MRP.

Lemma 3.6. *$\mathbb{P}(\Sigma)$ is Y-proper.*

Proof. As in Lemma 2.4, we observe a key notion. For a condition $p \in \mathbb{P}(\Sigma)$, a tuple $\langle \gamma, K_0, K_1 \rangle$ in p is called minimal if, for any $\langle \gamma', K_0', K_1' \rangle \in p \setminus \{\langle \gamma, K_0, K_1 \rangle\}$, $K_1 \in K_0'$. We note that each condition of $\mathbb{P}(\Sigma)$ has the unique minimal tuple. For a condition $p \in \mathbb{P}(\Sigma)$, a subset $\mathcal{A}$ of $\mathbb{P}(\Sigma)$ is called p-large if the set

$$\mathfrak{M}(\mathcal{A}) := \Big\{ M \in [H(\theta_\Sigma)]^{\aleph_0} : \text{there exists } r \in \mathcal{A} \text{ such that} \\ \bullet\ r \text{ is an end-extension of } p, \text{ and} \\ \bullet\ \text{for some } \varepsilon \text{ and } M_1,\ \langle \varepsilon, M, M_1 \rangle \text{ is} \\ \text{the minimal tuple of } r \setminus p \Big\}$$

is stationary in $[H(\theta_\Sigma)]^{\aleph_0}$.

Assume, for a while, that for any $p \in \mathbb{P}(\Sigma)$, the set

$$\Big\{ \bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}(\Sigma) \text{ is } p\text{-large} \Big\}$$

is a centered subset of $\mathrm{RO}(\mathbb{P}(\Sigma))$, and then let us show that $\mathbb{P}(\Sigma)$ is Y-proper. Let M^* be a countable elementary submodel of $H(\lambda)$ which contains the set $\{\Sigma, H(\theta_\Sigma)\}$, and $p \in \mathbb{P}(\Sigma) \cap M^*$. Define $M_0 := M^* \cap H(\theta_\Sigma)$. Take $M_1 \in \mathrm{dom}(\Sigma)$ such that $M_0 \in M_1$, and take a large enough ordinal ε in $\omega_1 \cap M^*$ in such a way that the set $q := p \cup \{\langle \varepsilon, M_0, M_1 \rangle\}$ is a condition of $\mathbb{P}(\Sigma)$, that is, for every $\langle \gamma, K_0, K_1 \rangle \in q \cap M_0 = p$, $\omega_1 \cap K_1 < \varepsilon$. By Lemma 3.2, q is $(M^*, \mathbb{P}(\Sigma))$-generic. We will show that this q works well.

Let r be an extension q in $\mathbb{P}(\Sigma)$. By our assumption, the set

$$\Big\{ \bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}(\Sigma) \text{ is } r \cap M^*\text{-large} \Big\}$$

generates the filter F on $\mathrm{RO}(\mathbb{P}(\Sigma))$. This F belongs to M^*. Let us show that, for every $s \in \mathrm{RO}(\mathbb{P}(\Sigma)) \cap M^*$, if $r \leq_{\mathrm{RO}(\mathbb{P}(\Sigma))} s$, then $s \in F$. To show this, let $s \in \mathrm{RO}(\mathbb{P}(\Sigma)) \cap M^*$ be such that $r \leq_{\mathrm{RO}(\mathbb{P}(\Sigma))} s$. Define $\mathcal{A} := \big\{ u \in \mathbb{P}(\Sigma) : u \leq_{\mathrm{RO}(\mathbb{P}(\Sigma))} s \big\}$. $\mathcal{A}$ belongs to M^*, hence $\mathfrak{M}(\mathcal{A}) \in M^*$. Moreover,

$\bigvee \mathcal{A} = s$. r witnesses that $M^* \cap H(\theta_\Sigma) = M_0 \in \mathfrak{M}(\mathcal{A})$. Thus $\mathfrak{M}(\mathcal{A})$ is stationary in $[H(\theta_\Sigma)]^{\aleph_0}$, and so $\mathcal{A}$ is $r \cap M^*$-large.

At the end, we show that, for any $p \in \mathbb{P}(\Sigma)$, the set

$$\left\{ \bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}(\Sigma) \text{ is } p\text{-large} \right\}$$

is a centered subset of $\mathrm{RO}(\mathbb{P}(\Sigma))$. This finishes the proof. Let $p \in \mathbb{P}(\Sigma)$. Define the assertion $\mathsf{Amalgable}(y, \langle \mathcal{A}_i : i \in n \rangle, \langle r_i : i \in n \rangle)$ which means that

- $y \in [X_\Sigma]^{<\aleph_0}$,
- each $\mathcal{A}_i$ is a p-large subset of $\mathbb{P}(\Sigma)$,
- for each $i \in n$,
  - $r_i \in \mathcal{A}_i$,
  - r_i is an end-extension of p, and
  - for any $\langle \varepsilon, M_0, M_1 \rangle \in r_i \setminus p$, $y \cup \{r_j : j < i\} \subseteq M_0$,

  and
- $\bigcup_{i \in n} r_i \in \mathbb{P}(\Sigma)$.

For each finite sequence $\langle \mathcal{A}_i : i \in n \rangle$ of p-large subsets of $\mathbb{P}(\Sigma)$, define

$$E(\langle \mathcal{A}_i : i \in n \rangle) := \Big\{ Y \in [X_\Sigma]^{\aleph_0} : \begin{array}[t]{l} \text{for any } y \in [Y]^{<\aleph_0}, \text{ there exists} \\ \langle r_i : i \in n \rangle \in \prod_{i \in n} \mathcal{A}_i \text{ such that} \\ \bullet\ \mathsf{Amalgable}(y, \langle \mathcal{A}_i : i \in n \rangle, \langle r_i : i \in n \rangle), \\ \text{and} \\ \bullet\ \text{for each } i \in n \text{ and } \langle \varepsilon, M_0, M_1 \rangle \in r_i \setminus p, \\ X_\Sigma \cap M_0 \subseteq Y \ \Big\}. \end{array}$$

It suffices to show that, for any finite sequence $\langle \mathcal{A}_i : i \in n \rangle$ of p-large subsets of $\mathbb{P}(\Sigma)$, $E(\langle \mathcal{A}_i : i \in n \rangle)$ is not empty.

By the definition, $E(\langle \mathcal{A}_i : i \in n \rangle)$ is closed in $[X_\Sigma]^{\aleph_0}$. By induction on the length n of the sequence $\langle \mathcal{A}_i : i \in n \rangle$, we show that $E(\langle \mathcal{A}_i : i \in n \rangle)$ is stationary in $[X_\Sigma]^{\aleph_0}$. Then it follows that $E(\langle \mathcal{A}_i : i \in n \rangle)$ is club in $[X_\Sigma]^{\aleph_0}$.

When $n = 1$, the set $E(\langle \mathcal{A}_0 \rangle)$ is of the form

$$\Big\{ Y \in [X_\Sigma]^{\aleph_0} : \text{for any } y \in [Y]^{<\aleph_0}, \text{ there exists } r \in \mathcal{A}_0 \text{ such that} \\ \bullet\ \mathsf{Amalgable}(y, \langle \mathcal{A}_0 \rangle, \langle r \rangle) \text{ holds, and} \\ \bullet\ \text{for each } \langle \varepsilon, M_0, M_1 \rangle \in r \setminus p,\ X_\Sigma \cap M_0 \subseteq Y \ \Big\}.$$

Let us show that $E(\langle \mathcal{A}_0 \rangle)$ is stationary in $[X_\Sigma]^{\aleph_0}$. To show this, let $\mathcal{C}$ be a club in $[X_\Sigma]^{\aleph_0}$. By recursion on $n \in \omega$ and by use of some book-keeping argument, since $\mathcal{A}_0$ is p-large and $\mathcal{C}$ is unbounded in $[X_\Sigma]^{\aleph_0}$, we can take a sequence $\langle r_n, M_0^n, C_n : n \in \omega \rangle$ such that

- for each $n \in \omega$,
  - r_n is in $\mathcal{A}_n$ and is an end-extension of p,
  - M_0^n is the second coordinate of the minimal tuple of $r_n \setminus p$, and
  - $C_n \in \mathcal{C}$ such that $C_n \subseteq C_{n+1}$ and, for all $\langle \varepsilon, M_0, M_1 \rangle \in r_n \setminus p$, $X_\Sigma \cap M_0 \subseteq C_n$,

  and
- for any $y \in \left[\bigcup_{n \in \omega} C_n\right]^{<\aleph_0}$, there exists $m \in \omega$ such that $y \subseteq M_0^m$.

Then $\bigcup_{n\in\omega} C_n$ is in both $\mathcal{C}$ and $E(\langle \mathcal{A}_0\rangle)$, which is what we want.

Assume that $\langle \mathcal{A}_i : i \in n+1\rangle$ is a sequence of p-large subsets of $\mathbb{P}(\Sigma)$ and $E(\langle \mathcal{A}_i : i \in n\rangle)$ is club in $[X_\Sigma]^{\aleph_0}$. Let us show that $E(\langle \mathcal{A}_i : i \in n+1\rangle)$ is stationary in $[X_\Sigma]^{\aleph_0}$. Since $\mathcal{A}_n$ is p-large, there are a countable elementary submodel M^* of $H(\lambda)$ and $q_n \in \mathcal{A}_n$ such that M^* has the set

$$\{\Sigma, H(\theta_\Sigma), p, \langle \mathcal{A}_i : i \in n+1\rangle\},$$

q_n is an end-extension of p, and $M^* \cap H(\theta_\Sigma)$ is the second coordinate of the minimal pair of $q_n \setminus p$. By elementarity of M^*, both $E(\langle \mathcal{A}_i : i \in n\rangle)$ and $E(\langle \mathcal{A}_i : i \in n+1\rangle)$ belong to $M^* \cap H(\theta_\Sigma)$. We shall show that $M^* \cap X_\Sigma$ belongs to $E(\langle \mathcal{A}_i : i \in n+1\rangle)$. Then it follows that $E(\langle \mathcal{A}_i : i \in n+1\rangle)$ is stationary in $[X_\Sigma]^{\aleph_0}$.

Let y be a finite subset of $M^* \cap X_\Sigma$. Take a finite subset x of $M^* \cap X_\Sigma$ such that, for any $\langle \varepsilon, M_0, M_1\rangle \in q_n \setminus p$, if $\varepsilon < \omega_1 \cap M^* < \omega_1 \cap M_0$, then $[x, M^* \cap X_\Sigma] \subseteq \Sigma(M_0)$. Since $E(\langle \mathcal{A}_i : i \in n\rangle)$ is a club subset of $[X_\Sigma]^{\aleph_0}$ in $M^* \cap H(\theta_\Sigma)$ and $\Sigma(M^* \cap H(\theta_\Sigma))$ is $M^* \cap H(\theta_\Sigma)$-stationary, there exists Y in the set

$$E(\langle \mathcal{A}_i : i \in n\rangle) \cap \Sigma(M^* \cap H(\theta_\Sigma)) \cap M^* \cap H(\theta_\Sigma)$$

such that $x \cup y \subseteq Y$. Since Y is in $\Sigma(M^* \cap H(\theta_\Sigma))$, we can take a finite subset z of Y such that $[z, Y] \subseteq \Sigma(M^* \cap H(\theta_\Sigma))$. Since Y is in $E(\langle \mathcal{A}_i : i \in n\rangle)$, by elementarity of M^*, there exists $\langle q_i : i \in n\rangle \in \left(\prod_{i\in n} \mathcal{A}_i\right) \cap M^*$ which satisfies the assertion $\mathsf{Amalgable}(x \cup y \cup z, \langle \mathcal{A}_i : i \in n\rangle, \langle q_i : i \in n\rangle)$. By a similar argument that q and r are compatible in $\mathbb{P}(\Sigma)$ in the proof of the properness, it is proved that the assertion $\mathsf{Amalgable}(y, \langle \mathcal{A}_i : i \in n+1\rangle, \langle q_i : i \in n+1\rangle)$ holds true in $H(\lambda)$ (in fact, $\mathsf{Amalgable}(x \cup y \cup z, \langle \mathcal{A}_i : i \in n+1\rangle, \langle q_i : i \in n+1\rangle)$ is affirmative). By elementarity of M^* again, there exists $\langle r_i : i \in n+1\rangle$ in the set $\left(\prod_{i\in n+1} \mathcal{A}_i\right) \cap M^*$ which satisfies the assertion $\mathsf{Amalgable}(y, \langle \mathcal{A}_i : i \in n+1\rangle, \langle r_i : i \in n+1\rangle)$. This finishes the proof. $\square$

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