SURE 静岡大学学術リポジトリ
Shizuoka University REpository

# Paradoxical effect in a three-candidate voter model


| | |
|---|---|
| メタデータ | 言語: eng<br>出版者:<br>公開日: 2022-02-18<br>キーワード (Ja):<br>キーワード (En):<br>作成者: Tainaka, Kei-ichi<br>メールアドレス:<br>所属: |
| URL | http://hdl.handle.net/10297/00028618 |

# Paradoxical effect in a three-candidate voter model

K. Tainaka
*Department of Physics, Ibaraki University, Mito, Ibaraki 310, Japan*

Received 18 November 1992; revised manuscript received 3 March 1993; accepted for publication 5 March 1993
Communicated by A.R. Bishop

By computer simulations, the spatial pattern in a voter system or an ecosystem is studied. This system consists of three candidates whose competitive power are as defined in the well-known paper–scissors–stone game. We find a kind of paradox: An initial damage and suppression of one candidate may later lead to an enhancement of the same candidate.

## 1. Introduction

Self-organized patterns in complex systems have received growing interest [1,2]. In the present work, we deal with the pattern dynamics in a voter system. Each voter supports a candidate (color), but often changes his color under various political conditions, such as, mass communication, demands of voters, and election campaign. Above all, mass communication crucially influences the result of the election. Our attention is, therefore, mainly directed to the role of mass communication.

We consider the biased voter model [3,4] in a two-dimensional lattice system. We first describe this model. Each lattice site is occupied by one voter. For an election, $N$ candidates (colors) are assumed to run. Each voter supports a candidate $i$ ($i=0, 1, ..., N$), where we use $i=0$ in the case that the voter supports no candidate. Voters change their color, when a consultation (collision) occurs between a pair of neighboring voters. By a collision of voters of colors $i$ and $j$, they change their colors into the color $i$ with probability $a_{ij}$, or into the color $j$ with probability $a_{ji}$, where $a_{ij}+a_{ji}\leqslant 1$. Moreover, we assume a single voter process taking place independently of the collision: the transition from color $i$ to $j$ spontaneously occurs with probability $b_{ij}$. The parameters $a_{ij}$ and $b_{ij}$ depend on the political conditions.

## 2. Model and basic equations

In the present article, we study the following special case ($N=3$),

$$V_1+V_2\to 2V_1\,, \tag{1a}$$

$$V_2+V_3\to 2V_2\,, \tag{1b}$$

$$V_3+V_1\to 2V_3\,, \tag{1c}$$

$$V_3\xrightarrow{b} V_1\,, \tag{1d}$$

where $V_i$ denotes the voter who supports color $i$ ($i=1$, 2, 3), and $b$ is the probability that reaction (1d) occurs. Note that the case $i=0$ is neglected. In this model, we put $a_{12}=a_{23}=a_{31}=1$, and $b_{31}=b$; otherwise $a_{ij}=b_{ij}=0$. System (1) has the following meaning: provided that reaction (1d) is neglected ($b=0$), then rule (1) is identical with the paper–scissors–stone game (PSSG) [5,6]. In this case, three candidates have a cyclic strength; there is no dominant color in the hierarchy. On the other hand, reaction (1d) is considered as an *external field*. An example of such a field is the effect of mass media: By the news of mass media, candidate 3 suffers damages, while candidate 1 receives advantages. Thus, the parameter $b$ measures the intensity of the effect of mass communication. Note that system (1) is an extension of a familiar model: if candidate 2 was absent, or reactions (1a) and (1b) were ignored, then (1) would be called the basic contact process [3,7,8]. Moreover, it should be noted that model (1) cor-

0375-9601/93/$ 06.00 © 1993 Elsevier Science Publishers B.V. All rights reserved.

responds to an ecosystem; for example, $V_1$ denotes plants, and $V_2$ and $V_3$ represent a herbivore and a carnivore, respectively. In this case, the parameter $b$ is the death rate of the carnivore.

Let $P_i$ be the density of color $i$ ($\Sigma_i P_i = 1$), then the basic equations for our voter system become

$$\dot{P}_1 = (P_{12} + P_{21}) - (P_{31} + P_{13}) + bP_3 , \tag{2a}$$

$$\dot{P}_2 = (P_{23} + P_{32}) - (P_{12} + P_{21}) , \tag{2b}$$

$$\dot{P}_3 = (P_{31} + P_{13}) - (P_{23} + P_{32}) - bP_3 , \tag{2c}$$

where the dots represent the derivative with respect to the time $t$, and $P_{ij}$ is the probability density to find a color $i$ at a site and a color $j$ at a nearest neighbor of that site ($i, j = 1, 2, 3$). Note the difference from the conditional probability. The relations $P_{ij} = P_{ji}$ and $\Sigma_j P_{ij} = P_i$ thus hold.

On the other hand, the evolution equations for the probability densities are

$$\tfrac{1}{4} z\dot{P}_{11} = P_{12} + (z-1)(P^2_{11} - P^1_{31}) + 2bP_{31} , \tag{3a}$$

$$\tfrac{1}{4} z\dot{P}_{22} = P_{23} + (z-1)(P^3_{22} - P^2_{12}) , \tag{3b}$$

$$\tfrac{1}{4} z\dot{P}_{33} = P_{31} + (z-1)(P^1_{33} - P^3_{23}) - 2bP_{33} , \tag{3c}$$

$$\tfrac{1}{2} z\dot{P}_{12} = -P_{12} + (z-1)(-P^2_{11} - P^1_{23} + P^3_{12} + P^2_{12}) + 2bP_{23} , \tag{3d}$$

$$\tfrac{1}{2} z\dot{P}_{23} = -P_{23} + (z-1)(-P^3_{22} - P^2_{31} + P^1_{23} + P^3_{23}) - 2bP_{23} , \tag{3e}$$

$$\tfrac{1}{2} z\dot{P}_{31} = -P_{31} + (z-1)(-P^1_{33} - P^3_{12} + P^2_{31} + P^1_{31}) + 2b(P_{33} - P_{31}) , \tag{3f}$$

where $P^i_{jk}$ denotes the probability density to find a color $i$ at a site and colors $j$ and $k$ at nearest neighbors of that site, and $z$ is the number of nearest neighbors ($z=4$ for a square lattice). It is worthwhile to note that (2) is derived from (3).

To solve the basic equations, one usually applies the mean-field approximation:

$$P_{ij} = P_i P_j . \tag{4}$$

Inserting (4) into (2), and setting all the time derivatives in (2) to be zero, we obtain the stationary solution as

$$P_1 = P_3 = \tfrac{1}{6}(2+b), \quad P_2 = \tfrac{1}{3}(1-b) . \tag{5}$$

## 3. Simulation results

Simulations are carried out for various values of $b$. The pattern dynamics of this system is self-organized into a stationary state, irrespective of the initial patterns. The density of each color in a stationary state does not depend on the size of the lattice. In fig. 1, the steady-state density of each color is plotted against $b$, where the straight lines are the prediction of the mean-field approximation (MFA). If $b=0$ (PSSG), three colors coexist with equal densities [5]. With increasing $b$, the population of color 3 increases, while colors 1 and 2 decrease. Thus, fig. 1 exhibits a kind of paradox that color 3 prevails the system. In spite of reaction (1d), color 1 is decreased. Moreover, it is found from this figure that color 2 disappears, when

$$b > b_1, \quad b_1 \sim 0.41 . \tag{6}$$

The mean-field theory (5) never explains the paradox nor predicts the value of $b_1$ ($b_1 = 1$ for MFA).

In the case $b > b_1$, color 2 disappears regardless of the initial patterns, so that the steady-state densities are represented as in fig. 2, which shows the steady-state densities under the condition that color 2 is absent (basic contact process). It is found from this figure that with the increase of $b$, color 3 conversely decreases. In particular, when

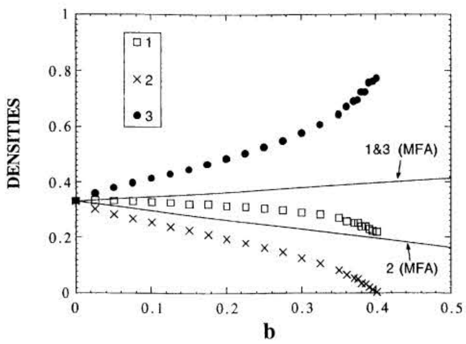


Fig. 1. The number (density) of each color in the stationary state is plotted against the intensity $b$ of the external field (effect of mass communication). The total number of sites of the square lattice is $160 \times 160$. Each plot is obtained by the time average in the period $200 < t \leqslant 500$, where $t$ is measured by the Monte Carlo step [5]. The straight lines represent the prediction of the mean-field approximation (5).

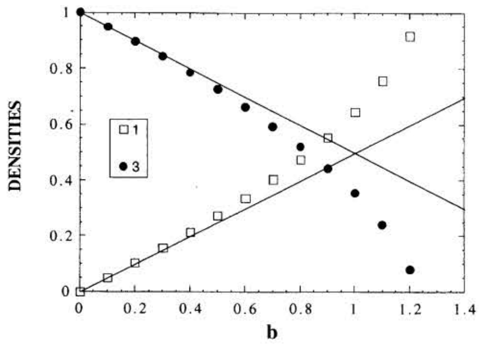


Fig. 2. The same as fig. 1 but for the case color 2 is absent (basic contact process) [3,7,8]. The straight lines are the prediction of the mean-field theory: $P_1=\frac{1}{2}b$, $P_3=1-\frac{1}{2}b$.

$$b>b_2, \quad b_2\sim 0.83\,, \tag{7}$$

the density of color 1 exceeds that of color 3. If $b$ is very large ($1.22\leqslant b$), color 1 occupies the total system [7]. These results of fig. 2 are reasonable, since color 1 becomes stronger with increasing $b$.

We can talk about the winner in our election. Suppose that only one person is elected, then it is found from fig. 1 that candidate 3 becomes the winner. Although he suffers losses by mass media, he wins the election. This paradox represents the indirect effect of mass communication. However, when the losses are too great ($b>b_2$), then candidate 1 wins the election.

## 4. Perturbation theory

It is well known that the mean-field theory is a first and crude approximation in systems with short-range interactions. Next, we describe the pair approximation (PA) given by Katori and Konno [8],

$$P^i_{jk}=P_{ij}P_{ik}/P_i\,. \tag{8}$$

Unfortunately, it is not easy to obtain the PA solution from (3) and (8). Now we apply a perturbation theory under the condition that $b$ is small. First, we get the PA solution in the case $b=0$ (PSSG). Setting all the time derivatives in (3) to be zero, and using (8), we find a stationary solution for $b=0$,

$$P_i=\tfrac{1}{3}, \quad P_{ii}=\tfrac{5}{27}, \quad P_{ij}=\tfrac{2}{27}\,, \tag{9}$$

where $i, j=1, 2, 3$ ($i\neq j$). Next, we expand $P_i$, $P_{ii}$, and $P_{ij}$ around solution (9),

$$P_i=\tfrac{1}{3}+q_ib, \quad P_{ii}=\tfrac{5}{27}+q_{ii}b, \quad P_{ij}=\tfrac{2}{27}+q_{ij}b\,. \tag{10}$$

By collecting terms to first order in $b$, we have

$$q_1=-\tfrac{1}{24}, \quad q_2=-\tfrac{7}{24}, \quad q_3=\tfrac{1}{3}\,, \tag{11}$$

and

$$q_{11}=-\tfrac{37}{216}, \quad q_{22}=-\tfrac{55}{216}, \quad q_{33}=\tfrac{11}{54}\,,$$

$$q_{12}=q_{23}=-\tfrac{1}{54}, \quad q_{31}=\tfrac{4}{27}\,. \tag{12}$$

It is found from (10) and (11) that the perturbation theory well explains the paradox that color 3 prevails the system.

We describe the mechanism of the paradox on the basis of perturbation theory. The spatial correlation plays an important role in this paradox. Here we define the ratio of the probability densities [9] as

$$R_{ij}\equiv P_{ij}/P_iP_j\,, \tag{13}$$

which represents how the distribution of voters is deviated from the random distribution (MFA). By using (10)–(12), we can calculate the $R_{ij}$ in explicit forms. It is found that the values of $R_{ij}$ are remarkably different from those for a random distribution (unity). In particular, $R_{22}$ is larger than unity ($R_{22}=\frac{5}{3}+\frac{5}{8}b$), whereas both $R_{21}$ and $R_{23}$ take smaller values than unity: $R_{21}=\frac{2}{3}+\frac{1}{2}b$, $R_{23}=\frac{2}{3}-\frac{1}{4}b$. These results reveal that voters of color 2 form contagious clusters with increasing $b$. Such clusters of color 2 explain the abnormal decrease (increase) of color 1 (color 3). The voters inside the cluster never interact, so that we can ignore color 2 as an approximation. If color 2 is absent, the system will be dominated by color 3 (see fig. 2 in the case $b<b_1$). Hence, the dynamically induced clusters of color 2 influence the populations of other colors.

## 5. Direct effect of mass media

Heretofore, we investigated the final stationary state, and exhibit the indirect effect of mass communication. In contrast, let us study the transient state, when the value of $b$ is suddenly increased at a certain time $t_0$ (we set $t_0=0$). An experiment of a jumping process from $b=0$ to 0.3 is performed: the simulation is carried out with $b=0.3$, where the ini-

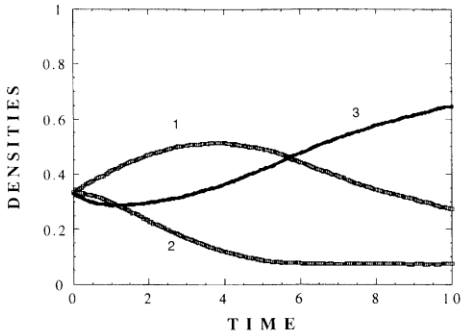


Fig. 3. The time-dependence of the population of each color. At time $t=0$, $b$ is suddenly increased from 0 to 0.3. Immediately after the mass media report the news, the news gives a straightforward effect: color 3 decreases, while color 1 increases.

tial condition is the stationary pattern for $b=0$. In fig. 3, the time dependence of the population of each color is depicted. This figure reveals that at first color 1 increases, but finally the population of color 3 exceeds that of color 1. Figure 3 therefore represents the direct effect of mass media: when mass media report the news just before the voting day, candidate 1 wins the election. The direct effect can be easily proved by eqs. (2): If $b=0$ ($t<0$), then $\dot{P}_i=0$ for any $i$. Immediately after the mass media report the news (after the jump of $b$), we have $\dot{P}_1 \sim bP_3$, so that $P_1$ should increase. On the other hand, the density $P_3$ must be decreased, since $\dot{P}_3 \sim -bP_3$. Thus, the direct effect comes from the fact that the single voter process (1d) takes place independently of the spatial structure.

## 6. Conclusions and discussions

The effect of damaging news by mass media on candidate 3 strongly depends upon when the news is reported. Candidate 3 loses the election if the news is reported just before the voting day (direct effect). On the other hand, he unexpectedly wins if the news is continuously reported sufficiently long before the voting day. This paradoxical effect of the mass media is qualitatively explained by perturbation theory: the dynamically induced clusters of color 2 bring about the decrease of color 1. However, the quantitative agreement between perturbation theory (11) and experiment (fig. 1) is still unsatisfactory, since fig. 1 reveals $q_1 \sim -0.04$, $q_2 \sim -0.77$, $q_3 \sim 0.81$. The investigation of the global pattern formation is thus essential to understand the winner. In our model, we assume a completely biased case: $a_{12}=a_{23}=a_{31}=1$. If we would modify the model, e.g. $a_{12}=a_{23}=a_{31}=0.8$ and $a_{ij}+a_{ji}=1$, a similar paradox is obtained. It should be added that in any case candidate 2 who is ignored by the mass media cannot be elected.

Previously, voter models were mainly considered as systems composed of two candidates (states). However, in this work, we deal with the three-state model. It is emphasized that there is an essential difference between two- and three-state models: In the case of the three-state model usually paradoxical results are observed [9], while they have been never reported in two-state systems. I think this difference closely relates to the principle of detailed balance: In the case of two states, detailed balance always holds under a stationary condition, whereas in the three-state system, the stationary condition does not require the validity of detailed balance [10]. Since real voter systems are inherently far from equilibrium (invalidity of detailed balance), the systems may exhibit such a paradox.

## References

[1] A.R. Bishop, G. Grüner and B. Nicolaenko, eds., Los Alamos Center for Nonlinear Studies Workshop on Spatio-temporal coherence and chaos in physical systems, Physica D 23 (1986).
[2] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. A 38 (1988) 364.
[3] R. Durrett, Lecture notes on particle systems and percolation (Wadsworth, Belmont, CA, 1988).
[4] I. Ferreira, Stoch. Process. Appl. 34 (1990) 25.
[5] K. Tainaka, J. Phys. Soc. Japan 57 (1988) 2588; Phys. Rev. Lett. 63 (1989) 2688;
K. Tainaka and Y. Itoh, Europhys. Lett. 15 (1991) 399.
[6] R. Fisch, Physica D 45 (1990) 19.
[7] R.C. Brower, M.A. Furman and M. Moshe, Phys. Lett. B 76 (1978) 213.
[8] M. Katori and N. Konno, J. Stat. Phys. 63 (1991) 115.
[9] K. Tainaka and S. Fukazawa, J. Phys. Soc. Japan 61 (1992) 1891.
[10] K. Tomita and H. Tomita, Prog. Theor. Phys. 74 (1974) 1731;
K. Tainaka, S. Kanno and T. Hara, Prog. Theor. Phys. 80 (1988) 199.