THE INEQUALITY $\mathfrak{b} > \aleph_1$ CAN BE CONSIDERED AS AN ANALOGUE OF SUSLIN'S HYPOTHESIS

TERUYUKI YORIOKA

ABSTRACT. In [3], the author introduced a new chain condition, called the anti-rectangle refining property, of forcing notions and the statement $\neg\mathcal{C}(\text{arec})$ that every forcing notion with the anti-rectangle refining property has an uncountable antichain. We show that every forcing notion with the anti-rectangle refining property has an uncountable antichain. Since a typical example of a forcing notion with the anti-rectangle refining property is an Aronszajn tree, $\neg\mathcal{C}(\text{arec})$ is a generalization of Suslin's Hypothesis. We show that $\neg\mathcal{C}(\text{arec})$ implies that the bounding number is larger than $\aleph_1$, that is, this statement can be considered as an analogue of Suslin's Hypothesis.

1. INTRODUCTION

The author investigated several fragments of Martin's Axiom in [3]. Fragments of Martin's Axiom were studied mainly by Stevo Todorčević in 1980's, and many applications are discovered (see [2] and his many other articles). In this manuscript, we give a proof of one question in this area as follows.

We explain some notions in [3]. A forcing notion $\mathbb{P}$ has *the anti-rectangle refining property* if for any uncountable subset I and J of $\mathbb{P}$, there exists uncountable subsets I' and J' of I and J respectively such that for every $p \in I'$ and $q \in J'$, p and q are incompatible in $\mathbb{P}$. $\neg\mathcal{C}(\text{arec})$ is the statement that every forcing notion with the anti-rectangle refining property has an uncountable antichain. Since an Aronszajn tree has the anti-rectangle refining property, $\neg\mathcal{C}(\text{arec})$ can be considered a generalization of Suslin's Hypothesis. In fact, $\neg\mathcal{C}(\text{arec})$ implies Suslin's Hypothesis and that every (ω_1, ω_1)-gaps are indestructible. The author would like to find other examples of a generalization of Suslin's Hypothesis, that is, other statements about combinatorics on ω_1 which is deduced from $\neg\mathcal{C}(\text{arec})$. One candidate is the statement that the bounding number $\mathfrak{b}$ is larger than $\aleph_1$.

We had already known that $\mathcal{K}_2(\text{rec})$, which is a weak fragments of Martin's Axiom and implies $\neg\mathcal{C}(\text{arec})$, implies that $\mathfrak{b} > \aleph_1$. So it is naturally arisen a question that $\neg\mathcal{C}(\text{arec})$ implies $\mathfrak{b} > \aleph_1$. In this manuscript, we show a positive answer of this question, that is $\neg\mathcal{C}(\text{arec})$ implies that $\mathfrak{b} > \aleph_1$ in section 3.

A proof of the theorem is self contained in this manuscript, however I omit some proofs of well known results in section 2. All of them are written in [3] or [1].

Supported by Grant-in-Aid for JSPS Fellow, No. 18840022, Ministry of Education, Culture, Sports, Science and Technology.

2. A reason why we will prove as below

At first, we will see a proof that $\mathcal{K}_2(\mathrm{rec})$ implies $\mathfrak{b} > \aleph_1$. A partition $[\omega_1]^2 = K_0 \cup K_1$ has the rectangle refining property if for any uncountable subset I and J of ω_1, there exist uncountable subsets I' and J' of I and J respectively such that for every $\alpha \in I'$ and $\beta \in J'$, if $\alpha < \beta$, then $\{\alpha, \beta\} \in K_0$. We note that the rectangle refining property is a strong property than the countable chain condition. $\mathcal{K}_2(\mathrm{rec})$ is the statement that every partition $[\omega_1]^2 = K_0 \cup K_1$ with the rectangle refining property has an uncountable K_0-homogeneous set. We note that $\mathcal{K}_2(\mathrm{rec})$ is deduced from Martin's Axiom for $\aleph_1$-dense sets, and $\mathcal{K}_2(\mathrm{rec})$ implies $\neg\mathcal{C}(\mathrm{arec})$.

Let $F = \{f_\xi; \xi \in \omega_1\}$ be a set of strictly increasing functions from ω into ω such that for every ξ and η in ω_1, if $\xi < \eta$, then $f_\xi \leq^* f_\eta$, i.e. there exists $m \in \omega$ such that for all $n \geq m$, $f_\xi(n) \leq f_\eta(n)$. For this family, we define a partition $[\omega_1]^2 = K_0 \cup K_1$ by letting $\{\xi, \eta\} \in K_0$ iff there exists m and n in ω such that $f_\xi(m) < f_\eta(m)$ and $f_\eta(n) < f_\xi(n)$. We call that F is unbounded when for every function g in ω^ω, there exists $f \in F$ such that $f \not\leq^* g$. We note that if F is unbounded, then this partition has the rectangle refining property. (This follows from Lemma 3.2 below.) However, in [1, Lemma 16], if F is unbounded, since an uncountable subset of F is also unbounded, for every uncountable subset F' of F, there are two functions f and g in F such that g dominates f everywhere, i.e., for every $n \in \omega$, $f(n) \leq g(n)$. Therefore, $\mathcal{K}_2(\mathrm{rec})$ implies $\mathfrak{b} > \aleph_1$.

So to try to prove that $\neg\mathcal{C}(\mathrm{arec})$ implies $\mathfrak{b} > \aleph_1$, it seems to be natural to modify the argument above. Let $\mathbb{P}'$ be a forcing notion which consists of finite subsets σ of ω_1 such that the set $\{f_\xi; \xi \in \sigma\}$ is totally ordered by the dominance everywhere, i.e., for every $\xi \in \sigma$ and $n \in \omega$, $\max\{f_\zeta(n); \zeta \in \sigma \cap \xi\} \leq f_\xi(n)$, ordered by the reverse inclusion. As the above partition has the rectangle refining property, we note that $\mathbb{P}'$ has the anti-rectangle refining property if F is unbounded. So if we show that $\mathbb{P}'$ is ccc whenever F is unbounded, we conclude that F doesn't have to be unbounded. However, unfortunately, in general, $\mathbb{P}'$ does not have the ccc even if F is unbounded. For example, if the set $\{\{\xi_\zeta, \eta_\zeta\}; \zeta \in \omega_1\}$ is a subset of $\mathbb{P}'$ such that

- for any $\zeta < \zeta'$ in ω_1, $\xi_\zeta < \eta_\zeta < \xi_{\zeta'}$, and
- for any $\zeta \in \omega_1$, $f_{\xi_\zeta}(0) = 0$ and $f_{\eta_\zeta}(1) = 1$,

then it is an uncountable antichain in $\mathbb{P}'$.

In section 3, we define a forcing notion $\mathbb{P}$ which is a modification of $\mathbb{P}'$ and show that (Lemma 3.2) $\mathbb{P}$ has the anti-rectangle refining property whenever F is unbounded, and (Lemma 3.3) $\mathbb{P}$ has the countable chain condition whenever F is unbounded. This completes the proof of our theorem.

3. A proof

Throughout this section, let $F = \{f_\xi; \xi \in \omega_1\}$ be a set of strictly increasing functions from ω into ω such that for every ξ and η in ω_1, if $\xi < \eta$, then $f_\xi \leq^* f_\eta$. We define a forcing notion $\mathbb{P}$ which consists of finite subsets σ of ω_1 such that for every $\xi \in \sigma$ and $n \in \omega$, either $\max\{f_\zeta(n); \zeta \in \sigma \cap \xi\} \leq f_\xi(n)$ or $f_\xi(n) \in \{f_\zeta(n); \zeta \in \sigma \cap \xi\}$, ordered by the reverse inclusion.

Proposition 3.1. *Suppose that $F = \{f_\xi; \xi \in \omega_1\}$ is unbounded. Then there exists $e \in \omega$ such that for every $n \in \omega \setminus e$ and $k \in \omega$, the set $\{\xi \in \omega_1; f_\xi(n) \geq k\}$ is uncountable.*

Proof. Assume not, i.e. there exists an infinite set Z of natural numbers such that for every $n \in Z$, there exists $k_n \in \omega$ such that the set $\{\xi \in \omega_1; f_\xi(n) \geq k_n\}$ is countable. Let $\delta \in \omega_1$ be such that for all $n \in Z$, $\{\xi \in \omega_1; f_\xi(n) \geq k_n\}$ is a subset of δ. Let $\{n_i; i \in \omega\}$ be an increasing enumeration of Z, and we define a function g on ω by

$$g(m) := \max\left(\{f_\delta(m)\} \cup \{k_{n_i}; i \in m+1\} \cup \{g(i)+1; i \in m\}\right)$$

for each $m \in \omega$. We notice that for each $\xi \in \delta$, $f_\xi \leq^* g$. Moreover for each $\xi \in \omega_1 \setminus \delta$ and $m \in \omega$, since $m \leq n_m$,

$$f_\xi(m) \leq f_\xi(n_m) < k_{n_m} \leq g(m).$$

So F is bounded by g, which is a contradiction. □

Lemma 3.2. *If $F = \{f_\xi; \xi \in \omega_1\}$ is unbounded, then $\mathbb{P}$ has the anti-rectangle refining property.*

Proof. Let I and J be uncountable subsets of $\mathbb{P}$. By shrinking I and J if necessary, we may assume that

- I forms a Δ-system with a root μ, and J also forms a Δ-system with a root ν,
- all members of I has the same size, and all members of J also has the same size,
- for any $\sigma \in I$ and $\tau \in J$,

$$\max(\mu \cup \nu) < \min(\sigma \setminus \mu), \quad \max(\mu \cup \nu) < \min(\tau \setminus \nu), \quad (\sigma \setminus \mu) \cap (\tau \setminus \nu) = \emptyset,$$

- there exists $e \in \omega$, such that for every $\sigma \in I$ and $\tau \in J$ and $n \geq e$,

$$\max\left(\{f_\zeta(n); \zeta \in \mu \cup \nu\}\right) < \min\left(\{f_\xi(n); \xi \in \sigma \setminus \mu\}\right)$$

  and

$$\max\left(\{f_\zeta(n); \zeta \in \mu \cup \nu\}\right) < \min\left(\{f_\eta(n); \eta \in \tau \setminus \nu\}\right).$$

We notice that for every $A \in [\omega_1]^{\aleph_1}$, the set $\{f_\xi; \xi \in A\}$ is unbounded. So by the previous lemma, there exists $e_0 \geq e$ such that for every $k \in \omega$, the set

$$\{\sigma \in I; \min\left(\{f_\xi(e_0); \xi \in \sigma \setminus \mu\}\right) \geq k\}$$

is uncountable. Let J' be uncountable subset of J and $k_0 \in \omega$ such that for every $\tau \in J'$,

$$\max\left(\{f_\eta(e_0); \eta \in \tau\}\right) \leq k_0,$$

and then we take an uncountable subset I' of I such that for every $\sigma \in I'$,

$$\min\left(\{f_\xi(e_0); \xi \in \sigma \setminus \mu\}\right) > k_0.$$

Then we notice that for any $\sigma \in I'$ and $\tau \in J'$, since $e_0 \geq e$, if $\tau \not\subseteq \max(\sigma)+1$, then σ and τ are incompatible in $\mathbb{P}$.

Conversely, by the previous lemma, there exists $e_1 > e_0$ such that for every $k \in \omega$, the set

$$\{\tau \in J'; \min\left(\{f_\eta(e_1); \eta \in \tau \setminus \nu\}\right) \geq k\}$$

is uncountable. Let I'' be uncountable subset of I' and $k_1 \in \omega$ such that for every $\sigma \in I''$,

$$\max\left(\{f_\xi(e_1); \xi \in \sigma\}\right) \leq k_1,$$

and then we take an uncountable subset J'' of J' such that for every $\tau \in J''$,

$$\min\left(\{f_\eta(e_1); \eta \in \tau \setminus \nu\}\right) > k_1.$$

Then we notice that, since $e_1 \geq e$, for any $\sigma \in I''$ and $\tau \in J''$, if $\sigma \not\subseteq \max(\tau)+1$, then σ and τ are incompatible in $\mathbb{P}$.

By shrinking I'' and J'' if necessary, we may assume that for any $\sigma \in I''$ and $\tau \in J''$, either $\tau \not\subseteq \max(\sigma)+1$ or $\sigma \not\subseteq \max(\tau)+1$. Then for every $\sigma \in I''$ and $\tau \in J''$, σ and τ are incompatible in $\mathbb{P}$. $\square$

Lemma 3.3. *If $F = \{f_\xi; \xi \in \omega_1\}$ is unbounded, then $\mathbb{P}$ has the countable chain condition.*

Proof. Here, for each $\sigma \in \mathbb{P}$, letting $\langle \xi_i; i \in |\sigma| \rangle$ be an increasing enumeration of σ, we denote

$$\vec{\sigma} := \langle f_{\xi_i}; i \in |\sigma| \rangle,$$

which is a member of the set $(\omega^\omega)^{|\sigma|}$. Let I be an uncountable subset of $\mathbb{P}$. Without loss of generality, we may assume that

- I forms a Δ-system with a root μ,
- for every σ and τ in I, either $\max(\sigma) < \min(\tau \setminus \mu)$ or $\max(\tau) < \min(\sigma \setminus \mu)$,
- there exists $n_0 \in \omega$ such that for every $n \geq n_0$, $\sigma \in I$ and $\xi \in \sigma \setminus \mu$,
$$\max\{f_\zeta(n); \zeta \in \mu\} < f_\xi(n),$$
- there exists $k \in \omega$ such that for every $\sigma \in I$, $|\sigma| = k$,
- for every σ and τ in I, $\vec{\sigma}\restriction n_0 = \vec{\tau}\restriction n_0$, i.e. for each $j \in k$, the initial segment of the j-th element of $\vec{\sigma}$ of length n_0 is equal to the initial segment of the j-th element of $\vec{\tau}$ of length n_0.

Then there exists $\gamma \in \omega_1$ such that the set $\left\{\vec{\sigma}; \sigma \in I \cap [\gamma]^{<\aleph_0}\right\}$ is dense in the set $\{\vec{\sigma}; \sigma \in I\}$ as a subspace of the space $(\omega^\omega)^k$. We fix some (any) $\nu \in I \setminus [\gamma]^{<\aleph_0}$. For each $\sigma \in I$, we define two functions g_σ and h_σ on ω as follows: For each $n \in \omega$,

$$g_\sigma(n) := \max\{f_\xi(n); \xi \in \sigma\} \;\; (= \max\{f_\xi(n); \xi \in \sigma \setminus \mu\}),$$

and

$$h_\sigma(n) := \min\{f_\xi(n); \xi \in \sigma \setminus \mu\}.$$

We notice that for σ and τ in I, if $\max(\sigma) < \min(\tau \setminus \mu)$, then $g_\sigma \leq^* h_\tau$. So we can find $n_1 \geq n_0$ and $I' \in \left[I \setminus [\gamma]^{<\aleph_0}\right]^{\aleph_1}$ such that for every $\tau \in I'$ and $n \geq n_1$, $g_\nu(n) \leq h_\tau(n)$, and for every τ and τ' in I', $\vec{\tau}\restriction n_1 = \vec{\tau'}\restriction n_1$. Since F is unbounded and I' is uncountable, the set $\{h_\tau; \tau \in I'\}$ is unbounded. Hence there exists $n \geq n_1$ such that the set $\{h_\tau(n); \tau \in I'\}$ is infinite. Let

$$n_2 := \min\{n \in [n_1, \omega); \{h_\tau(n); \tau \in I'\} \text{ is infinite}\}.$$

By the minimality of n_2, we can take $\vec{t} \in (\omega^{n_2})^k$ and infinite $I'' \subseteq I'$ such that

- for all $\tau \in I''$, $\vec{t} \subseteq \vec{\tau}$, i.e. for every $j \in k$, the j-th member of $\vec{t}$ is an initial segment of the j-th member of $\vec{\tau}$,
- the set $\{h_\tau(n_2); \tau \in I''\}$ is infinite.

By our assumption, there exists $\sigma \in I \cap [\gamma]^{<\aleph_0}$ such that $\vec{t} \subseteq \vec{\sigma}$. Then there is $n_3 \geq n_2$ such that for every $n \geq n_3$, $g_\sigma(n) \leq g_\nu(n)$, and take $\tau \in I''$ such that $g_\nu(n_3) < h_\tau(n_2)$.

We will show that for every $n \geq n_2$, $g_\sigma(n) \leq h_\tau(n)$ holds. If $n_2 \leq n < n_3$, then

$$g_\sigma(n) < g_\sigma(n_3) \leq g_\nu(n_3) < h_\tau(n_2) \leq h_\tau(n),$$

so it is ok. If $n \geq n_3$, then since $n \geq n_3 \geq n_1$ and $\tau \in I'' \subseteq I'$,

$$g_\sigma(n) \leq g_\nu(n) \leq h_\tau(n).$$

We recall that $\vec{t} \in (\omega^{n_2})^k$ is an initial segment of both $\vec{\sigma}$ and $\vec{\tau}$, for every $n \geq n_2$, $g_\sigma(n) \leq h_\tau(n)$, and both σ and τ are members of $\mathbb{P}$. Therefore $\sigma \cup \tau$ is also a condition of $\mathbb{P}$, i.e. σ and τ are compatible in $\mathbb{P}$. $\square$

4. Acknowledgment

I would like to thank Stevo Todorčević for a discussion on the question in this manuscript during the international conference of Topology in 2007 December at Kyoto.

References

[1] S. Todorčević. *Remarks on cellularity in products*, Compositio Math. 57 (1986), no. 3, 357–372.

[2] S. Todorčević. *Partition Problems in Topology*, volume 84 of *Contemporary mathematics*, American Mathematical Society, Providence, Rhode Island, 1989.

[3] T. Yorioka. *Some weak fragments of Martin's Axiom related to the rectangle refining property*, to appear in Arch. Math. Logic.

Department of Mathematics, Shizuoka University, Ohya 836, Shizuoka, 422-8529, JAPAN

E-mail address: `styorio@ipc.shizuoka.ac.jp`