Discontinuous homomorphisms on C(X) and the forcing axioms for EPC

DISSERTATION

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CHAPTER 1

Introduction

1. Kaplansky's Problem and Its Independency

In this thesis, we introduce properties of forcing notions, which are stronger than ccc. It is motivated by the classical problem, which is called Kaplansky problem. In 1949, I. Kaplansky [**26**] showed that any algebraic norm on $C(X, \mathbb{C})$ (the Banach algebra of complex-valued continuous functions on an infinite compact Hausdorff space X) is larger than or equivalent (i.e., provides the same topology) to the uniform norm. Then a natural question is raised: Is every algebraic norm on $C(X, \mathbb{C})$ equivalent to the uniform norm? In the present day, this problem is known to be independent from ZFC. W. Bade and P. Curtis [**2**] proved that this problem is equivalent to the assertion NDH (No Discontinuous Homomorphisms), that is, for each infinite compact Hausdorff space X, every homomorphism from $C(X, \mathbb{C})$ to any Banach algebra is continuous. H. G. Dales [**14**] and J. Esterle [**20**] independently gave the first contribution to this problem. Namely, they showed that the continuum hypothesis CH implies the negation of NDH. Hence $\neg$NDH is provable in ZFC + CH. On the other hand, H. Woodin [**34**] established the assertion *Woodin's condition* and showed that the assertion implies NDH and is consistent with ZFC + MA. Therefore, he showed that NDH can not be decided in ZFC.

Furthermore, H. Woodin [**33**] showed that $\neg$NDH + $\neg$CH is consistent with ZFC, so NDH can not be decided in ZFC + $\neg$CH. Concretely, H. Woodin showed that the finite support iteration of Cohen forcings of length ω_2 forces that NUB (No Ultrapower of $\mathbb{R}$ is Beta 1) fails, that is, there exists a non-trivial ultrapower of $\mathbb{R}$ with the property β_1. This is sufficient since $\neg$NUB implies $\neg$NDH (see [**13**, Theorem 5.7.13]). Thereafter, he raised the following problem: Is $\neg$NDH+MA+$\neg$CH consistent with ZFC? We shall present a partial solution to this question.

In this chapter, we introduce two classes of forcing notions called *eventual precaliber* $\aleph_1$ *(*$\mathrm{EPC}_{\aleph_1}$*)* and $\mathrm{EPC}^*_{\aleph_1}$. $\mathrm{EPC}^*_{\aleph_1}$ is a weakening of $\mathrm{EPC}_{\aleph_1}$ and is preserved under finite support iterations. Furthermore, the forcing axiom MA($\mathrm{EPC}_{\aleph_1}$ + "size $\leq \mathfrak{c}$") for $\mathrm{EPC}_{\aleph_1}$ forcings of size $\leq \mathfrak{c}$ implies MA($\mathrm{EPC}_{\aleph_1}$). Thus MA($\mathrm{EPC}^*_{\aleph_1}$ + size "$\leq \mathfrak{c}$") implies MA($\mathrm{EPC}_{\aleph_1}$).

We also introduce another class ProjCes(E) of forcing notions where $E \subset \omega_1$ is a stationary set. ProjCes(E) is preserved under finite support iterations, and

MA(ProjCes(E) + "size $\leq \mathfrak{c}$") implies MA(ProjCes(E)). Moreover, MA($\mathrm{EPC}^*_{\aleph_1}$ + ProjCes(E) + "size $\leq \mathfrak{c}$") implies MA($\mathrm{EPC}_{\aleph_1}$ + ProjCes(E)).

In Chapter 2, we shall show the following theorem based on Woodin's proof [**33**].

Main Theorem. *Let V be the ground model in which* CH *holds. In V, let $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \mid \xi \in \omega_2 \rangle$ be a finite support iteration of* $\mathrm{EPC}^*_{\aleph_1}$ + ProjCes(E) *forcing posets of size $\leq \aleph_1$. Let G be a $\mathbb{P}_{\omega_2}$-generic set over V. Then, in $V[G]$, there exists a non-principal ultrafilter U on $\mathcal{P}(\omega)$ such that $\mathbb{R}^\omega/U$ is β_1.*

Thus MA($\mathrm{EPC}^*_{\aleph_1}$ + ProjCes(E) + "size $\leq \aleph_1$") and MA($\mathrm{EPC}_{\aleph_1}$ + ProjCes(E)) are consistent with $\neg$NDH + $\neg$CH.

In Chapter 3, we raise a consequence of the Main Theorem which relates to Whitehead's conjecture in the group theory, which provides an example of an $\mathrm{EPC}_{\aleph_1}$ forcing. The main result seems to imply the properties $\mathrm{EPC}_{\aleph_1}$ and $\mathrm{EPC}^*_{\aleph_1}$ are quite stronger than ccc. We consider the following:

(1) The existence of a non-trivial example of $\mathrm{EPC}_{\aleph_1}$ forcing.
(2) The position of the property $\mathrm{EPC}_{\aleph_1}$ among well-known forcing properties

For (1), we shall show that the uniformization of a ladder system coloring is an example of $\mathrm{EPC}_{\aleph_1}$ forcing. Moreover, if the domain of a ladder system is a stationary-co-stationary $E \subset \omega_1$, then the uniformization is ProjCes($\omega_1 \setminus E$). This shows Whitehead's conjecture fails in the theory ZFC+$\mathrm{MA}_{\aleph_1}$($\mathrm{EPC}_{\aleph_1}$+ProjCes(E)). For(2), we shall show that any $\mathrm{EPC}^*_{\aleph_1}$ forcing preserves some set-theoretical objects that are preserved by Cohen forcing. Specifically, $\mathrm{EPC}_{\aleph_1}$ forcing preserves Luzin sets of size $\aleph_1$ and mad families of size $\aleph_1$ which were added by the forcing which adds a mad family. The following are the consequences of what we proved.

(1) MA($\mathrm{EPC}_{\aleph_1}$ + ProjCes(E)) is consistent with the same constellation of Cichoń-Blass diagram in Cohen model.
(2) MA(σ-centered) and MA($\mathrm{EPC}_{\aleph_1}$) do not imply each other.

For a more detailed description of the position of $\mathrm{EPC}_{\aleph_1}$, we use the Kunen forcing, which is defined by a pregap. We shall determine the relation between the property of Kunen forcing and the type of pregap.

1.1. Notation. For any set T of ordinals, let us introduce the abbreviations

$$\forall^\infty \alpha \in T,\ \cdots\alpha\cdots \quad \text{for} \quad \exists\beta \in T,\ \forall\alpha \in T \setminus \beta,\ \cdots\alpha\cdots$$

and

$$\exists^\infty \alpha \in T,\ \cdots\alpha\cdots \quad \text{for} \quad \forall\beta \in T,\ \exists\alpha \in T \setminus \beta,\ \cdots\alpha\cdots .$$

For any set X, $\mathcal{P}(X)$ denotes the power set of X. For any cardinal κ and any set X, $[X]^{<\kappa}$ and $[X]^\kappa$ denote the two sets of subsets of X of size $<\kappa$ and κ, respectively. Similarly, $X^{<\kappa}$ and X^κ denote two sets of sequence in X of length $<\kappa$ and κ, respectively.

For any totally ordered set $\langle T, < \rangle$ and a subset $X \subset T$, X is *cofinal* (*coinitial*) in T if for every $a \in T$, there exists $x \in X$ such that $a \leq x$ ($x \leq a$). $\mathrm{cof}(T)$ ($\mathrm{coi}(T)$) denotes the least cardinality of a cofinal (coinitial) subset in T.

2. Introduction of EPC

In this section, we introduce properties of forcing notions.

Definition 1. *A* forcing notion*, or simply,* forcing *is a partially ordered set. Its elements are often called* conditions*. If $q \leq p$, then q is called an* extension *of p. For a couple of conditions $p_0, \ldots, p_n$, their* common extension *is a condition q which extends all q_i's. Two elements p, q of a forcing notion $\mathbb{P}$ are* compatible *(*incompatible*) if there (does not) exists a common extension of them. A subset C of a forcing notion $\mathbb{P}$ is* centered *(*linked*) if every finite subset (pair of elements) of C have a common extension. In contrast, a subset A of a forcing notion is* antichain *if every pair of elements of A is incompatible.*

The following property of forcing notions is most commonly used.

Definition 2. *A forcing notion $\mathbb{P}$ has* countable chain condition (ccc) *if every antichain is countable, in other words, every uncountable subset has a compatible pair.*

The following three properties are well-known strengthening of ccc.

Definition 3. *A forcing notion $\mathbb{P}$ is* σ-centered *if $\mathbb{P}$ is the union of countably many centered sets.*

Definition 4. *A forcing notion $\mathbb{P}$ has* property (K) *if every uncountable sequence of $\mathbb{P}$ has an uncountable linked subsequence.*

Definition 5. *A forcing notion $\mathbb{P}$ is* precaliber $\aleph_1$ *($PC_{\aleph_1}$) if every uncountable sequence of $\mathbb{P}$ has a centered uncountable subsequence.*

The main theme of this paper is a strengthening of precaliber $\aleph_1$.

Definition 6. *A sequence $\langle p_\xi \mid \xi \in \kappa \rangle$ of conditions of a forcing notion $\mathbb{P}$ is* eventually centered *if*

$$\forall \alpha \in \kappa,\ \forall p \leq p_\alpha,\ \exists \delta(\alpha, p) \in \kappa,\ \{p\} \cup \{p_\xi \mid \xi \in \kappa \setminus \delta(\alpha, p)\} \textit{ is centered.}$$

Definition 7. *A forcing notion* $\mathbb{P}$ has eventual precaliber $\aleph_1$ ($\mathrm{EPC}_{\aleph_1}$) *if, for each sequence* $\langle p_\xi \in \mathbb{P} \mid \xi < \omega_1 \rangle$*, there exists* $T \in [\omega_1]^{\aleph_1}$ *such that* $\langle p_\xi \mid \xi \in T \rangle$ *is eventually centered.*

Definition 8. *A forcing notion* $\mathbb{P}$ has $\mathrm{EPC}^*_{\aleph_1}$ *if, for each sequence* $\langle p_\xi \in \mathbb{P} \mid \xi < \omega_1 \rangle$*, there exists* $T \in [\omega_1]^{\aleph_1}$ *and a eventually centered sequence* $\langle \tilde{p}_\xi \mid \xi \in T \rangle$ *such that* $\tilde{p}_\xi \leq p_\xi$ *for each* $\xi \in T$*.*

Notice that every $\mathrm{EPC}_{\aleph_1}$ forcing notion has $\mathrm{EPC}^*_{\aleph_1}$ and every $\mathrm{EPC}^*_{\aleph_1}$ forcing notion has precaliber $\aleph_1$, so they have the ccc. Cohen forcing $2^{<\omega}$ is an example of $\mathrm{EPC}_{\aleph_1}$, and we shall give other examples in chapter 3. We show that $\mathrm{EPC}^*_{\aleph_1}$ is preserved under finite support iterations.

Lemma 1. *Every finite support iteration of* $\mathrm{EPC}^*_{\aleph_1}$ *forcings has* $\mathrm{EPC}^*_{\aleph_1}$*.*

Proof. We prove the lemma by induction on the length of the iteration.

First, we check the successor step. Assume that $\mathbb{P}$ has $\mathrm{EPC}^*_{\aleph_1}$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for an $\mathrm{EPC}^*_{\aleph_1}$ forcing notion. Fix any sequence $\langle \langle p_\xi, \dot{q}_\xi \rangle \mid \xi \in \omega_1 \rangle \in (\mathbb{P} * \dot{\mathbb{Q}})^{\omega_1}$. Let $\dot{S}_0 := \{\langle \check{\xi}, p_\xi \rangle \mid \xi \in \omega_1\}$. Since $\mathbb{P}$ has the ccc, there exists $p^* \in \mathbb{P}$ such that $p^* \Vdash |\dot{S}_0| = \aleph_1$. By the assumption, we get $\mathbb{P}$-names $\dot{S}$ and $\dot{\tilde{q}}$ such that

$$p^* \Vdash \text{“}\dot{S} \in [\dot{S}_0]^{\aleph_1},\ \left\langle \dot{\tilde{q}}(\xi) \,\middle|\, \xi \in \dot{S} \right\rangle \text{ is eventually centered, and } \dot{\tilde{q}}(\xi) \leq \dot{q}_\xi \text{ for each } \xi \in \dot{S}\text{”}.$$

Define $T_0 = \{\xi \in \omega_1 \mid \exists p \leq p^*,\ p \leq p_\xi \wedge p \Vdash \check{\xi} \in \dot{S}\}$. Then, $p^* \Vdash \dot{S} \subset \check{T}_0$, so T_0 is uncountable. Select $\langle \hat{p}_\xi \mid \xi \in T_0 \rangle$ such that $\hat{p}_\xi \leq p^*, p_\xi$ and $\hat{p}_\xi \Vdash \check{\xi} \in \dot{S}$ for each $\xi \in T_0$. Fix a sequence $\left\langle \dot{\tilde{q}}_\xi \,\middle|\, \xi \in T_0 \right\rangle$ of $\mathbb{P}$-names such that $\hat{p}_\xi \Vdash \dot{\tilde{q}}(\check{\xi}) = \dot{\tilde{q}}_\xi$. Since $\mathbb{P}$ has $\mathrm{EPC}^*_{\aleph_1}$, there exist $T \in [T_0]^{\aleph_1}$ and a eventually centered sequence $\langle \tilde{p}_\xi \mid \xi \in T \rangle$ such that $\tilde{p}_\xi \leq \hat{p}_\xi$ for each $\xi \in T$.

We proceed to show that $\left\langle \left\langle \tilde{p}_\xi, \dot{\tilde{q}}_\xi \right\rangle \,\middle|\, \xi \in T \right\rangle$ is eventually centered. Fix any $\alpha \in T$ and $\langle p, \dot{q} \rangle \leq \left\langle \tilde{p}_\alpha, \dot{\tilde{q}}_\alpha \right\rangle$. Then, we get $p' \leq p$ and $\delta \in \omega_1$ such that $p' \Vdash \check{\delta} = \dot{\delta}(\alpha, \dot{q})$. Fix any $\Gamma \in [T \setminus (\delta \cup \delta(\alpha, p'))]^{<\omega}$. Then, there exists $p^\dagger \leq p'$ such that, for each $\gamma \in \Gamma$, $p^\dagger \leq \tilde{p}_\gamma$. Thus, $p^\dagger \Vdash \check{\Gamma} \subset \dot{S} \setminus \dot{\delta}(\alpha, \dot{q})$ and hence there exists a $\mathbb{P}$-name $\dot{q}^\dagger$ such that $p^\dagger \Vdash \dot{q}^\dagger \leq \dot{q} \wedge \forall \gamma \in \check{\Gamma},\ \dot{q}^\dagger \leq \dot{\tilde{q}}_\gamma$. Therefore $\left\langle p^\dagger, \dot{q}^\dagger \right\rangle$ is a common extension of $\{\langle p, \dot{q} \rangle, \left\langle \tilde{p}_\gamma, \dot{\tilde{q}}_\gamma \right\rangle \mid \gamma \in \Gamma\}$.

Second, we check the limit steps. Fix any $\langle p_\alpha \in \mathbb{P}_\gamma \mid \alpha \in \omega_1 \rangle$ with γ limit. Pick $T \in [\omega_1]^{\aleph_1}$ and $S \in [\gamma]^{<\omega}$ such that $\mathrm{supp}(p_\alpha) \cap \mathrm{supp}(p_\beta) = S$ for each $\alpha \neq \beta$ in T. Pick $T_1 \in [T]^{\aleph_1}$ and $\langle \tilde{p}_\alpha \mid \alpha \in T_1 \rangle$ such that $\langle \tilde{p}_\alpha \restriction (\max S + 1) \mid \alpha \in T_1 \rangle$ is eventually centered in $\mathbb{P}_{\max S+1}$, $\tilde{p}_\alpha \restriction (\max S + 1) \leq p_\alpha \restriction (\max S + 1)$, and $\tilde{p}_\alpha \restriction (\gamma \setminus \max S + 1) = p_\alpha \restriction (\gamma \setminus \max S + 1)$ for each $\alpha \in T_1$. We show that $\langle \tilde{p}_\alpha \mid \alpha \in T_1 \rangle$ is eventually centered. Fix any $\alpha \in T_1$ and $q \leq \tilde{p}_\alpha$. Take $\delta \in \omega_1$ s.t. $\{q \restriction (\max S + 1), \tilde{p}_\xi \restriction (\max S + 1) \mid \xi \in$

$T_1 \setminus \delta\}$ is centered. Now, fix $\delta' \in \omega_1$ such that $\forall \xi \in T_1 \setminus \delta'$, $\mathrm{supp}(q) \cap \mathrm{supp}(\tilde{p}_\xi) \setminus (\max S + 1) = \emptyset$. Then, $\{q, \tilde{p}_\xi \mid \xi \in T_1 \setminus (\delta \cup \delta')\}$ is centered. □

$\mathrm{EPC}_{\aleph_1}$ also has a good property.

LEMMA 2. $\mathrm{MA}(\mathrm{EPC}_{\aleph_1} +$ *"size* $< \mathfrak{c}$*")* *implies* $\mathrm{MA}(\mathrm{EPC}_{\aleph_1})$. *Here,* $\mathfrak{c} = 2^{\aleph_0}$.

PROOF. Assume that $\mathrm{MA}(\mathrm{EPC}_{\aleph_1})$ fails. Let $\mathbb{P}$ be an $\mathrm{EPC}_{\aleph_1}$ forcing and $\mathcal{D} = \langle D_\alpha \mid \alpha < \lambda \rangle$ be a sequence of length $\lambda < \mathfrak{c}$ of dense sets without generic filters. Fix an elementary submodel M of large enough H_κ of size λ such that $\mathbb{P} \in M$ and that $\{D_\alpha \mid \alpha < \lambda\}$ and ω_1 are subsets of M. We shall show that $\mathbb{P} \cap M$ is a counter-example for $\mathrm{MA}(\mathrm{EPC}_{\aleph_1} +$ "size $< \mathfrak{c}$").

At first, we shall see that $\mathbb{P} \cap M$ is $\mathrm{EPC}_{\aleph_1}$. Fix any sequence $\bar{p} = \langle p_\alpha \mid \alpha < \omega_1 \rangle$ in $\mathbb{P} \cap M$. $\bar{p}$ is also a sequence in $\mathbb{P}$, so there is $T \in [\omega_1]^{\aleph_1}$ such that $\bar{p} \restriction T$ is eventually centered. Fix any $\alpha \in T$ and $q \leq p_\alpha$. Fix any $\xi_0, \ldots, \xi_{n-1} \in T \setminus \delta(\alpha, q)$. Then $\{q, p_{\xi_0}, \ldots, p_{\xi_{n-1}}\}$ is a finite subset of $\mathbb{P} \cap M$ and has a common extension in $\mathbb{P}$. By elementarity, it has a common extension in $\mathbb{P} \cap M$. Thus $\bar{p}$ is eventually centered.

We proceed to show that $\mathcal{D} \restriction M = \langle D_\alpha \cap M \mid \alpha < \lambda \rangle$ is a sequence of dense sets in $\mathbb{P} \cap M$ which has no generic filters. Let $\alpha \in \omega_1$ and $p \in \mathbb{P} \cap M$. Then there exists $q \in D_\alpha$ such that $q \leq p$. By elementarity and $D_\alpha \in M$, we can assume that $q \in M$. So each $D_\alpha \cap M$ is dense in $\mathbb{P} \cap M$. Suppose that there exists a $\mathcal{D} \restriction M$-generic filter G on $\mathbb{P} \cap M$. Then, $\{p \in \mathbb{P} \mid \exists q \in G,\ q \leq p\}$ is a $\mathcal{D}$-generic filter on $\mathbb{P}$, a contradiction.

Therefore $\mathbb{P} \cap M$ is a counter-example for $\mathrm{MA}(\mathrm{EPC}_{\aleph_1} +$ "size $< \mathfrak{c}$"). □

COROLLARY 1. $\mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} +$ *"size* $< \mathfrak{c}$*")* *implies* $\mathrm{MA}(\mathrm{EPC}_{\aleph_1})$.

The following lemmata are necessary for the proof of the main theorem.

LEMMA 3. *Let* $\langle p_\alpha \mid \alpha \in \omega_1 \rangle$ *be a eventually centered sequence and* $\alpha \in \omega_1$. *Let* $T \in [\omega]^{\aleph_1}$. *Then,* $D_\alpha(T) = \{q \leq p_\alpha \mid \exists \xi \in T, q \leq p_\xi\}$ *is dense below* p_α. *Furthermore,*

$$\Vdash_{\mathbb{P}} \forall \beta \in \omega_1,\ \forall T \in [\omega_1]^{\aleph_1},\ D_\beta(T) \text{ is dense in } \check{\mathbb{P}} \text{ below } p_\beta.$$

PROOF. We only prove the latter statement. Fix any $q_0 \in \mathbb{P}$, $\beta \in \omega_1$, a $\mathbb{P}$-name $\dot{T}$ for an uncountable set of ω_1, and $r \leq p_\beta$. Pick $q_1 \leq q_0$ and $\xi \in \omega_1 \setminus \delta(\beta, r)$ such that $q_1 \Vdash \check{\xi} \in \dot{T}$. Then, we get a common extension of r and p_ξ. □

LEMMA 4. *Let* $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ *be an eventually centered sequence. Fix* $\alpha_0 < \omega_1$. *Then,* p_{α_0} *forces that* $\{\xi \in \omega_1 \mid p_\xi \in \dot{G}\}$ *is uncountable.*

PROOF. Fix any $\beta \in \omega_1$ and $p \leq p_{\alpha_0}$. Choose $\xi \in \omega_1 \setminus (\delta(\alpha_0, p) \cup \beta)$ and a common extension q of p_ξ and p. Then, q forces that $\check{p}_\xi \in \dot{G}$. Therefore, p_{α_0} forces that, for each $\beta \in \omega_1$, there exists $\xi \in \omega_1 \setminus \beta$ such that $\check{p}_\xi \in \dot{G}$. □

3. Projections on countable elementary submodels

We shall introduce another property of forcing notions.

Definition 9. *For a forcing* $\mathbb{P}$ *and for a set* X*,* $p^{(X)} \in X \cap \mathbb{P}$ *is* a projection of $p \in \mathbb{P}$ on X *if, for any extension* $q \in X \cap \mathbb{P}$ *of* $p^{(X)}$*,* $q \parallel p$*. We say that* $\mathbb{P}$ projects into X *if every* $p \in \mathbb{P}$ *has a projection* $p^{(X)}$ *on* X*.*

Note that, if densely many $p \in \mathbb{P}$ has a projection on X, then all $q \in \mathbb{P}$ has a projection.

Lemma 5. *Let* $\mathbb{P}$ *be a forcing notion and* κ *be a large enough regular cardinal such that* $\mathbb{P} \in H_\kappa$ *and that* $H_\kappa \models$ *"*$\mathbb{P}$ *is ccc" iff* $\mathbb{P}$ *is ccc. If* $\mathbb{P}$ *projects into some countable* $N \prec H_\kappa$ *with* $\mathbb{P} \in N$*, then* $\mathbb{P}$ *has ccc.*

Proof. Suppose not, and let N be a countable elementary submodel of H_κ such that $\mathbb{P} \in N$ and that $\mathbb{P}$ projects into N. Then $N \models$ "$\mathbb{P}$ is not ccc" by elementarity. Thus there exists an uncountable antichain $A \subset \mathbb{P}$ in N. Pick $p \in A \setminus N$. Let $p^{(N)} \in \mathbb{P} \cap N$ be a projection of p on N. Pick $q \in A \cap N$ such that $q \parallel p^{(N)}$. Then $q \parallel p$, which is a contradiction. □

Definition 10. *Let* $\mathbb{P} \in H_\kappa$ *be a partial order. Let* N *be a countable elementary submodel of* H_κ *which has* $\mathbb{P}$ *as an element. Then a filter* $d \subset \mathbb{P} \cap N$ is a generic filter over N *if and only if* d *meets all dense subsets of* $\mathbb{P}$ *in* N*.*

Similar to the standard forcing theory for transitive models, the above definition is equivalent even if "all dense subsets" were replaced by "all maximal antichains". If $\mathbb{P}$ is ccc, $\mathbb{P} \in N_0 \prec N_1 \prec H_\kappa$, and $d \subset \mathbb{P} \cap N_1$ is a generic filter over N_1, then $d \cap N_0$ meets all maximal antichains of $\mathbb{P}$ in N_0 since they are also members of N_1 and subsets of N_0. Moreover, $d \cap N_0$ is a filter: If $p, q \in d \cap N_0$, then, since

$$\{r \in \mathbb{P} \mid r \leq p, q \text{ or } \forall r_0 \leq p, q,\ r_0 \perp r\} \in N_0.$$

is a dense set, there exists $r \in d \cap N_0$ which is a common extension of p, q.

The purpose of introducing projections of conditions on a countable elementary submodel is to obtain the productivity of generic filters over (non-transitive) countable elementary submodels. This is used only in the last part of the main lemma of this thesis.

Lemma 6. *Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be forcing notions,* κ *a large enough regular cardinal, and* $N_0 \prec N_1 \prec H_\kappa$ *countable elementary submodels with* $\mathbb{P}, \mathbb{Q} \in N_0$*. Let* $d_0 \subset \mathbb{P} \cap N_0$ *and* $d_1 \subset \mathbb{Q} \cap N_1$ *be generic filters over* N_0 *and* N_1*, respectively. Suppose that* $d_0 \in N_1$ *and that* $\mathbb{Q}$ *projects into* N_0*. Then,* $(d_0 \times d_1) \cap N_0 \subset (\mathbb{P} \times \mathbb{Q}) \cap N_0$ *is a generic filter over* N_0*.*

Proof. As we have already seen, $d_1 \cap N_0$ is a filter and so is $(d_0 \times d_1) \cap N_0 = d_0 \times (d_1 \cap N_0)$. Thus we shall prove only the genericity. Let $A \subset \mathbb{P} \times \mathbb{Q}$ be a maximal antichain in N_0 with respect to the product order. Let

$$D_1 = \{p_1 \in \mathbb{Q} \mid \exists p_0 \in d_0,\, \exists \bar{r} \in A,\, (p_0, p_1) \leq \bar{r}\}.$$

Then $D_1 \in N_1$ is dense: Fix any $p_1 \in \mathbb{Q}$. Then a projection $p_1^{(N_0)} \in N_0$ of p_1 on N_0 exists and

$$D_0 = \{p_0 \in \mathbb{P} \mid \exists p_1' \leq p_1^{(N_0)},\, \exists \bar{r} \in A,\, (p_0, p_1') \leq \bar{r}\} \in N_0$$

is a dense subset of $\mathbb{P}$. So we pick $p_0 \in d_0 \cap D_0$. Then, by elementarity, there exists $p_1' \leq p_1^{(N_0)}$ in N_0 and $\bar{r} \in A \cap N_0$ such that $(p_0, p_1') \leq \bar{r}$. Since $p_1' \parallel p_1$, pick a common extension q_1 of p_1' and p_1. Then, $q_1 \in D_1$.

Thus we get $p_1 \in d_1 \cap D_1$. There exists $p_0 \in d_0$ and the unique $(r_0, r_1) \in A$ such that $(p_0, p_1) \leq (r_0, r_1)$. Since $\mathbb{Q}$ has the ccc by Lemma 5, the antichain $A_1 = \{s_1 \in \mathbb{Q} \mid \exists s_0 \in \mathbb{P},\, p_0 \leq s_0 \wedge (s_0, s_1) \in A\} \in N_0$ is countable and $r_1 \in A_1 \subset N_0$. Since

$$N_0 \models \text{“there exists a unique } r \in \mathbb{P} \text{ such that } p_0 \leq r \text{ and } (r, r_1) \in A\text{”},$$

we have $r_0 \in N_0$. In conclusion, $\bar{r} \in A \cap (d_0 \times d_1) \cap N_0$. □

Corollary 2. *Let $\mathbb{P}$ be a forcing notion and κ a large enough regular cardinal. Let $N_0 \prec \cdots \prec N_m \prec H_\kappa$ be countable elementary submodels that has $\mathbb{P}$ and, for each $i \leq m$, $d_i \subset \mathbb{P} \cap N_i$ be a generic filter over N_i, moreover that is a member of N_{i+1} whenever $i+1 \leq m$. Suppose that $\mathbb{P}$ projects into N_0. Then, $\Pi_{i \leq m} d_i \cap N_0 \subset \mathbb{P}^{m+1} \cap N_0$ is a generic filter over N_0.*

Proof. Use Lemma 6 m times repeatedly. □

Definition 11. *Let $\mathbb{P}$ be a forcing notion and $E \subset \omega_1$. $\mathbb{P}$* has a ces(E) projection *if, for any large enough regular cardinal κ and any countable $N \prec H_\kappa$ with $\mathbb{P}, E \in N$ and with $\omega_1 \cap N \in E$, $\mathbb{P}$ projects into N. Let* ProjCes(E) *be the class of all forcing notions that has a* ces(E) *projection.*

Any countable forcing is in ProjCes(ω_1). We shall show that, for stationary-co-stationary $E \subset \omega_1$, the uniformization of a coloring of a ladder system on $\omega_1 \setminus E$ has a ces(E) projection in section 1 of chapter 3. Note the following:

(1) For uncountable sets $A \subset B$ and for any club sets $\mathcal{C}_A \subset [A]^{\leq\aleph_0}$ and $\mathcal{C}_B \subset [B]^{\leq\aleph_0}$, the lifting $\{X \subset B \mid X \cap A \in \mathcal{C}_A\}$ is club in $[B]^{\leq\aleph_0}$ and the restriction $\{X \cap A \mid X \in \mathcal{C}_B\}$ contains a club set in $[A]^{\leq\aleph_0}$ (e.g., see [**25**, Theorem 8.27]).

(2) For a club set $\mathcal{C} \subset [\omega_1]^{\leq\aleph_0}$, since $\omega_1 = \{\alpha \mid \alpha < \omega_1\}$ and $\mathcal{C}$ are mutually $\subset$-cofinal, $\mathcal{C} \cap \omega_1$ is a club set in ω_1.

(3) For any infinite regular cardinal κ and $x \in [H_\kappa]^{\leq\aleph_0}$, by the elementary chain theorem and the Löwenheim-Skolem theorem,

$$\{N \mid N \prec H_\kappa,\ |N| = \aleph_0,\ x \subset N\}$$

is a club set in $[H_\kappa]^{\leq\aleph_0}$ (e.g., see [**32**, 2.3 Theorem], [**12**, Theorem 3.1.13], [**25**, Theorem 12.1]).

(4) In particular, for any stationary set $E \subset \omega_1$, there exists a countable $N \prec H_\kappa$ such that $\mathbb{P}, E \in N$ and $\omega_1 \cap N \in E$.

(5) By Lemma 5, for any stationary set $E \subset \omega_1$, every ProjCes(E) forcing is ccc.

To facilitate our argument about the property ProjCes(E), we introduce the following property.

Definition 12. *Let $\mathbb{P}$ be a forcing notion and $E \subset \omega_1$. $\mathbb{P}$ has* a club(E) projection *if, for every ordering $\bar{p} = \langle p_\xi \mid \xi < \lambda\rangle$ of $\mathbb{P}$ (we mean that only* $\mathrm{ran}(\bar{p}) = \mathbb{P}$, *no need of injectivity) with $\lambda \geq \omega_1$, there exists a club set $\mathcal{C} \subset [\lambda]^{\leq\aleph_0}$ such that, for every $C \in \mathcal{C}$ with $\omega_1 \cap C \in E$, $\mathbb{P}$ projects into $\{p_\xi \mid \xi \in C\}$.*

For a countable forcing $\mathbb{P}$ and an ordering $\langle p_\xi \mid \xi < \lambda\rangle$ ($\lambda \geq \omega_1$), select $\langle \xi_p \mid p \in \mathbb{P}\rangle$ such that $p_{\xi_p} = p$. Then $\mathcal{C} = \{C \subset \lambda \mid \{\xi_p \mid p \in \mathbb{P}\} \subset C\}$ is a club subset (even is a tail subset) that witnesses that $\mathbb{P}$ has a club(E) projection.

Lemma 7. *Let $\mathbb{P}$ be a forcing notion and $E \subset \omega_1$ a stationary set. Then, the following are equivalent.*

(1) $\mathbb{P}$ has a club(E) *projection.*

(2) $\mathbb{P}$ has a ces(E) *projection.*

Proof. 1 $\Longrightarrow$ 2**:** Suppose that $\mathbb{P}$ has a club(E) projection. Pick a large enough regular cardinal κ such that $H_\kappa \models$ "$\mathbb{P}$ has a club(E) projection". Fix any countable $N \prec H_\kappa$ such that $\mathbb{P}, E \in N$ and that $\omega_1 \cap N \in E$. Fix an ordering $\bar{p} = \langle p_\xi \mid \xi < \lambda\rangle$ ($\lambda \geq \omega_1$) of $\mathbb{P}$ in N. There exists a club set $\mathcal{C} \subset [\lambda]^{\leq\aleph_0}$ in N such that, for every $C \in \mathcal{C}$ with $\omega_1 \cap C \in E$, $\mathbb{P}$ projects into $\{p_\xi \mid \xi \in C\}$. Then $C_N = \bigcup(\mathcal{C} \cap N) = \lambda \cap N$ is a member of $\mathcal{C}$ and satisfies $\omega_1 \cap C_N = \omega_1 \cap N \in E$. Thus $\mathbb{P}$ projects into $\{p_\xi \mid \xi \in C_N\} = \mathbb{P} \cap N$.

2 $\Longrightarrow$ 1**:** Suppose that $\mathbb{P}$ has a ces(E) projection. Let $\bar{p} = \langle p_\xi \mid \xi < \lambda\rangle$ ($\lambda \geq \omega_1$) be an ordering of $\mathbb{P}$. Fix a club set

$$\mathcal{C} \subset \{\lambda \cap N \mid N \prec H_\kappa,\ |N| = \aleph_0,\ \bar{p}, \mathbb{P}, E \in N\}$$

in $[\lambda]^{\leq\aleph_0}$. Fix any countable $N \prec H_\kappa$ such that $\bar{p}, \mathbb{P}, E \in N$ and $\lambda \cap N \in \mathcal{C}$. Suppose that $\omega_1 \cap \lambda \cap N = \omega_1 \cap N \in E$. Since $\mathbb{P}$ has a ces(E) projection, $\mathbb{P}$ projects into $\mathbb{P} \cap N = \{p_\xi \mid \xi \in \lambda \cap N\}$.

□

By the above proof,

(1) in the definition of club(E) projection, "for every ordering" can be replaced by "for some ordering" and,

(2) in the definition of ces(E) projection, "any countable $N \prec H_\kappa$ with $\mathbb{P}, E \in N$ and with ..." can be replaced by "there exists $x \in [H_\kappa]^{\leq \aleph_0}$ such that, any countable $N \prec H_\kappa$ with $\{\mathbb{P}, E\} \cup x \subset N$ and with ...".

Lemma 8. *For any stationary set $E \subset \omega_1$, every finite support iteration of* ProjCes(E) *forcings is* ProjCes(E).

Proof. We prove the lemma by induction on the length of the iteration.

First, we check the successor step. Assume that $\mathbb{P}$ is ProjCes(E) and forces that $\dot{\mathbb{Q}}$ is ProjCes(E), or equivalently, has club(E) projection. Let $\langle \dot{q}_\xi \mid \xi < \lambda \rangle$ be a sequence of $\mathbb{P}$ names that is an ordering of $\dot{\mathbb{Q}}$. Pick a $\mathbb{P}$ name $\dot{\mathcal{C}}$ for a club set in $[\lambda]^{\leq \aleph_0}$ that witnesses that $\dot{\mathbb{Q}}$ has the club(E) projection. Since $\mathbb{P}$ is ccc, there exists a club set $\mathcal{C}_0$ in $[\lambda]^{\leq \aleph_0}$ that is forced to be a subset of $\dot{\mathcal{C}}$ (e.g., see [**32**, 1.8 Theorem]). Let κ be a large enough regular cardinal. Fix any countable $N \prec H_\kappa$ with $\mathbb{P} * \dot{\mathbb{Q}}, E \in N$ such that $\omega_1 \cap N \in E$. Since κ is large enough, we can assume that $\langle \dot{q}_\xi \mid \xi < \lambda \rangle$, $\dot{\mathcal{C}}$, and $\mathcal{C}_0$ are members of N. Fix any $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$. Our goal is to find a projection of $(p, \dot{q})$ on N. Let $C = \bigcup(\mathcal{C}_0 \cap N) = \lambda \cap N \in \mathcal{C}_0$. Since $\omega_1 \cap C \in E$, we have

$$p \Vdash \text{“}\dot{q} \text{ has a projection on } \{\dot{q}_\xi \mid \xi \in C\}\text{”}.$$

So there exists $p' \leq p$ and $\xi^{(C)} \in C$ such that

$$p' \Vdash \text{“}\dot{q}_{\xi^{(C)}} \text{ is a projection of } \dot{q} \text{ on } \{\dot{q}_\xi \mid \xi \in C\}\text{”}.$$

Note that $\dot{q}_{\xi^{(C)}} \in N$. Let $p'^{(N)}$ be a projection of p' on N. Then $(p'^{(N)}, \dot{q}_{\xi^{(C)}})$ is a projection of $(p, \dot{q})$ on N: Fix any $(r, \dot{s}) \in (\mathbb{P} * \dot{\mathbb{Q}}) \cap N$ and assume that $(r, \dot{s}) \leq (p'^{(N)}, \dot{q}_{\xi^{(C)}})$. Since

$$N \models \text{“}\exists r' \leq r,\ \exists \xi < \lambda,\ r' \Vdash \text{“}\dot{s} = \dot{q}_\xi\text{””},$$

pick $r' \leq r$ in N and $\xi \in \lambda \cap N$ such that $(r', \dot{q}_\xi) \leq (r, \dot{s})$. Since $p'^{(N)}$ is a projection of p', there exists a common extension $p'' \in \mathbb{P}$ of r' and p' (and, of course, p). Then,

$$p'' \Vdash \text{“}\dot{q}_\xi \parallel \dot{q}\text{”}.$$

since $\xi \in C$ and $\dot{q}_{\xi^{(C)}}$ is a projection of $\dot{q}$. Pick $p^* \leq p''$ and $\dot{q}^*$ such that

$$p^* \Vdash \text{“}\dot{q}^* \text{ is a common extension of } \dot{q}_\xi \text{ and } \dot{q}\text{”}.$$

Thus, $(p^*, \dot{q}^*)$ is a common extension of $(r, \dot{s})$ and $(p, \dot{q})$.

Consider the case $\mathrm{cof}(\gamma) \geq \omega$. Pick $N \prec H_\kappa$ with $\mathbb{P}_\gamma, E \in N$ and with $\omega_1 \cap N \in E$. Fix any $p \in \mathbb{P}_\gamma$. Pick $\delta \in \gamma \cap N$ such that $\max(\mathrm{supp}(p) \cap \delta) = \max(\mathrm{supp}(p) \cap N)$. Let $(p|\delta)^N \in \mathbb{P}_\delta \cap N$ be a projection of $p|\delta$ on N and let $p^N = i_{\delta,\gamma}((p|\delta)^N)$ where

$i_{\delta,\gamma}\colon \mathbb{P}_\delta \to \mathbb{P}_\gamma$ is the cannonical complete embedding. Fix any $q \leq p^N$ in N. Then, $q|\delta \leq (p|\delta)^N$ is in N and hence a common extension $r \in \mathbb{P}_\delta$ of $q|\delta$ and $p|\delta$ exists. Since $\mathrm{supp}(p) \cap \mathrm{supp}(q) \setminus \delta \subset \mathrm{supp}(p) \cap N \setminus \delta = \emptyset$, the condition $\tilde{r} \in \mathbb{P}_\gamma$ defined by

$$\tilde{r}(\xi) = \begin{cases} r(\xi) & (\xi < \delta) \\ q(\xi) & (\xi \in \mathrm{supp}(q) \setminus \delta) \\ p(\xi) & (\text{otherwise}) \end{cases}$$

is a common extension of q and p. □

LEMMA 9. *For any stationary set $E \subset \omega_1$,* MA(ProjCes(E)+ *"size $< \mathfrak{c}$") implies* MA(ProjCes(E))

PROOF. Assume that MA(ProjCes(E) + "size $< \mathfrak{c}$"). Fix a forcing notion $\mathbb{P}$ that has a ces(E) projection. Let $\bar{p} = \langle p_\xi \mid \xi < \mu \rangle$ be an ordering of $\mathbb{P}$. Let $\mathcal{D} = \langle D_\alpha \mid \alpha < \lambda \rangle$ be a sequence of dense sets in $\mathbb{P}$ of length $\lambda < \mathfrak{c}$. Our goal is to find a $\mathcal{D}$-generic filter on $\mathbb{P}$. Let κ be a large enough regular cardinal and select $M \prec H_\kappa$ of size λ such that $\mathbb{P}, \mathcal{D}, \lambda, \bar{p}, E \in M$ and that $\lambda \subset M$. Let μ_M be the order type of $\mu \cap M$ and $c\colon \mu \cap M \to \mu_M$ be the transitive collapse. Note that $c \restriction \lambda$ is the identity map.

We shall only show that $\mathbb{P} \cap M$ has a club(E) projection since the rest of the proof is completely the same as the one of Lemma 2. Note that $\bar{q} = \left\langle p_{c^{-1}(\xi)} \mid \xi < \mu_M \right\rangle$ is an ordering of $\mathbb{P} \cap M$. Select a club set

$$\mathcal{C} \subset \{c^{\rightarrow}(\mu \cap N) \mid N \prec M,\, |N| = \aleph_0,\, \bar{p}, \mathbb{P}, E \in N\}$$

where $c^{\rightarrow}(\mu \cap N)$ is the image of $\mu \cap N$ by c. Then $\mathcal{C}$ is a club set in $[\mu_M]^{\leq \aleph_0}$. Fix any $c^{\rightarrow}(\mu \cap N) \in \mathcal{C}$ such that $\omega_1 \cap c^{\rightarrow}(\mu \cap N) = \omega_1 \cap N \in E$. Since $\mathbb{P}$ projects into $\mathbb{P} \cap N = \{p_\xi \mid \xi \in \mu \cap N\} = \{p_{c^{-1}(\xi)} \mid \xi \in c^{\rightarrow}(\mu \cap N)\}$, so does $\mathbb{P} \cap M$ since $p \parallel_{\mathbb{P}} q \implies p \parallel_{\mathbb{P} \cap M} q$ by elementarity $M \prec H_\kappa$. □

Combining proofs of Lemmata 2 and 9 naively, we obtain the following.

LEMMA 10. *For any stationary set $E \subset \omega_1$,* MA($\mathrm{EPC}_{\aleph_1}$+ProjCes(E)+*size* $\leq \aleph_1$) *implies* MA($\mathrm{EPC}_{\aleph_1}$ + ProjCes(E))

H. Woodin [**33**] established the next theorem.

THEOREM 1 ([**33**, Theorem 4.]). *Let $\mathbb{C}_{\omega_2}$ be the finite support iteration of the Cohen forcing $\mathbb{C}$ of length ω_2. Let G be a $\mathbb{C}_{\omega_2}$-generic set over the ground model V in which* CH *holds. Then, in $V[G]$, there exists a non-principal ultrafilter U on $\mathcal{P}(\omega)$ such that $\mathbb{R}^\omega/U$ is β_1.*

Our main theorem is a generalization of Woodin's theorem.

Main Theorem (Repetition). *Let* $E \subset \omega_1$ *be a stationary set. Let* $\mathbb{P}_{\omega_2} = \left\langle \left\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \right\rangle \,\middle|\, \xi \in \omega_2 \right\rangle$ *be a finite support iteration of* $\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)$ *forcing posets of size* $\leq \aleph_1$. *Let* G *be a* $\mathbb{P}_{\omega_2}$*-generic over the ground model* V *in which* CH *holds. Then, in* $V[G]$*, there exists a non-principal ultrafilter* U *on* $\mathcal{P}(\omega)$ *such that* $\mathbb{R}^\omega/U$ *is* β_1.

CHAPTER 2

Discontinuous homomorhpisms on $C(X)$ and $\mathrm{EPC}_{\aleph_1}$

1. Real Closed Fields

We aim to force that $\mathbb{R}^\omega/U$ is a β_1-field for some ultrafilter U on $\mathcal{P}(\omega)$. Thus it is very helpful to investigate properties of real closed fields.

Most of the contents in this section are derived from Woodin [**33**] and we shall provide proofs of the necessary lemmata that are omitted in his paper.

1.1. β_1-Fields and A Discontinuous Homomorphism from $C(X)$. We consider the notion of gaps in ordered fields which plays an important role throughout this article. Let us begin the argument by defining gaps and several order properties. The following two definitions are based on [**16**].

Definition 13. *Let $\langle T, <\rangle$ be a totally ordered set and let A, B, and S be subsets of T. We write $A \ll B$ if $a < b$ for all $a \in A$ and $b \in B$, in which case we say that the pair $\langle A, B\rangle$ is a* pregap *on T. We say that $x \in T$* interpolates *$\langle A, B\rangle$ if $a < x < b$ for all $a \in A$ and $b \in B$. If no $x \in T$ interpolates $\langle A, B\rangle$, then $\langle A, B\rangle$ is called a* gap*. Furthermore, if $\mathrm{cof}(A) = \kappa$ and $\mathrm{coi}(B) = \lambda$, then $\langle A, B\rangle$ is called a $\langle\kappa, \lambda\rangle$*-gap*. When $\kappa, \lambda < \aleph_1$, we say that $\langle A, B\rangle$ is a* countable gap*. Otherwise, we say that $\langle A, B\rangle$ is an* uncountable gap*.* The pregap defined by *$x \in T$* on *S is the pregap $\langle\{y \in S \mid y < x\}, \{y \in S \mid x < y\}\rangle$.*

Definition 14. *Let $\langle K, <\rangle$ be an ordered field over $\mathbb{R}$. We say that K is*

- *an α_1*-field *if, for each $X \subset K$, both the cofinality* $\mathrm{cof}(X)$ *of X and the coitintiality* $\mathrm{coi}(X)$ *of X are less than $\aleph_1$,*
- *a β_1*-field *if there exists a chain of α_1-fields $\langle K_\nu \mid \nu < \lambda\rangle$ such that $K = \bigcup_{\nu<\lambda} K_\nu$, and*
- *an η_1*-field *if there is no countable gap in K.*

It is easily seen that every subfield of an α_1-field and every subfield of a β_1-field is also an α_1-field and a β_1-field, respectively. Since the set of real numbers $\mathbb{R}$ has the countable chain condition, $\mathbb{R}$ is an α_1-field. Since the ultraproduct $\mathbb{R}^\omega/U$ of $\mathbb{R}$ over a non-principal ultrafilter U on $\mathcal{P}(\omega)$ is $\aleph_1$-saturated, it is an η_1-field. Recall that a model is $\aleph_1$-saturated if any countable type is realized. η_1 and a model-theoretic terminology $\aleph_1$-saturation are equivalent conditions for real closed fields, by cell decomposition (see Note 1 and the end of this section). The following represents that the property β_1 implies "small".

LEMMA 11 ([**16**, Theorem 2.30]). *Let K be a β_1-field, and let L be a real-closed η_1-field. Then there is an embedding of K in L.*

The following theorem is due to J. Esterle and B. E. Johnson; for example, see [**13**],[**15**], or [**33**].

THEOREM 2 ([**13**, Theorem 5.7.13]). *If there is a non-principal ultrafilter U on $\mathcal{P}(\omega)$ such that $\mathbb{R}^\omega/U$ is β_1, then there is a discontinuous homomorphism from $C(X)$ into a Banach algebra for any infinite compact Hausdorff space X.*

Our definition of property β_1 is different from Woodin's one [**33**]. The relation between the two definitions will be considered at the end of the subsection 1.3.

1.2. The Perspective of Model Theory. We shall research real closed fields with model theoretical arguments.

Let $\mathcal{L}_{\mathsf{OR}} := \{0, 1, +, \cdot, =, <\}$ be the language of ordered rings. For every language $\mathcal{L}$, $\mathsf{form}_{\mathcal{L}}$ denotes the set of all formulas of $\mathcal{L}$. In this paper, every structure is denoted by a blackboard bold letter, and its domain is denoted by a Roman letter, e.g., the domain of a structure $\mathbb{F}$ is F.

DEFINITION 15. *Let $\mathbb{F}$ be a real closed field and $S \subset F$. Define $\mathcal{L}_{\mathsf{OR}}(S) := \mathcal{L}_{\mathsf{OR}} \cup S$ where we regard each element of S as a constant symbol that is interpreted as itself in $\mathbb{F}$. For each $1 \leq k < \omega$, define*

$$\mathrm{Def}^k(\mathbb{F}, S) := \left\{ \{\bar{a} \in \mathbb{F}^k \mid \mathbb{F} \models \varphi(\bar{a})\} \mid \varphi(\bar{v}) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}(S)} \right\} \text{ and}$$
$$\mathrm{Func}^k(\mathbb{F}, S) := \left\{ G \in \mathrm{Def}^{k+1}(\mathbb{F}, S) \,\middle|\, G\colon F^k \to F \right\}.$$

We show later that $\mathrm{Func}^0(\mathbb{F}, S)$ can be defined as the real closure of S in $\mathbb{F}$ (see Lemma 12).

A *real closed field* is defined to be a field that is *real* (i.e., every sum of squares is not equal to -1) and has no proper real algebraic extension. Field theory allows a reformulation of this notion in the first-order language, see [**29**, p. 94, Corollary 3.3.5]. We let RCF be the $\mathcal{L}_{\mathsf{OR}}$-theory of real closed fields. The following is a basic fact of RCF, see [**29**].

THEOREM 3 (Quantifier elimination for RCF). RCF *eliminates quantifiers, which means that for every formula $\varphi(\bar{v}) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}}$, there exists a quantifier free formula $\psi(\bar{v}) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}}$ such that $\mathsf{RCF} \models \varphi(\bar{v}) \leftrightarrow \psi(\bar{v})$.*

THEOREM 4. RCF *is model-complete, which means that if $\mathbb{F}$ and $\mathbb{H}$ are models of* RCF *and $\mathbb{H}$ is submodel of $\mathbb{F}$, then $\mathbb{H}$ is an elementary submodel of $\mathbb{F}$.*

THEOREM 5. RCF *is complete, which means that* $\mathsf{RCF} \vdash \varphi$ *or* $\mathsf{RCF} \vdash \neg\varphi$ *for each sentence* $\varphi \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}}$.

Furthermore, real closures are easily obtained by model-completeness.

DEFINITION 16. *Let* $\mathbb{M}$ *be an* $\mathcal{L}$*-structure and* $S \subset M$. *We say that* $y \in M$ *is* uniquely defined over S *if there is an* $\mathcal{L}$*-formula* $\varphi(\bar{v}, w)$ *and* $\bar{a} \in S^{<\omega}$ *such that*

$$\mathbb{M} \models \varphi(\bar{a}, y) \wedge \forall v(\varphi(\bar{a}, v) \rightarrow v = y).$$

LEMMA 12. *Let* $\mathbb{F}$ *be a real closed field and* $S \subset \mathbb{F}$. *Then,*

$$R^{\mathbb{F}}(S) = \{y \in \mathbb{F} \mid \ y \text{ is uniquely defined with respect to the language } \mathcal{L}_{\mathsf{OR}}(S)\}$$

where $R^{\mathbb{F}}(S)$ *denotes the real closure of* S *under* $\mathbb{F}$.

PROOF. It is clear that $\mathbb{Q}(S) \subset R^{\mathbb{F}}(S)$ where $\mathbb{Q}(S)$ is the minimal field containing $\mathbb{Q} \cup S$.

First, fix any $y \in R^{\mathbb{F}}(S)$. Then, since y is algebraic over S, there is a non-zero polynomial $p(X) \in \mathbb{Q}(S)[X]$ and $m \in \omega$ such that

$$R^{\mathbb{F}}(S) \models \text{“}y \text{ is the } m\text{-th root of } p(X)\text{”}.$$

By elementarity,

$$\mathbb{F} \models \text{“}y \text{ is the } m\text{-th root of } p(X)\text{”}.$$

The m-th root of $p(X)$ is unique so that y is uniquely defined over S.

Second, fix any $y \in \mathbb{F}$ such that

$$\mathbb{F} \models \varphi(\bar{a}, y) \wedge \forall v(\varphi(\bar{a}, v) \rightarrow v = y)$$

for some $\varphi(\bar{u}, v) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}}$ and $\bar{a} \in S^{<\omega}$. Then,

$$\mathbb{F} \models \exists u \left(\varphi(\bar{a}, u) \wedge \forall v(\varphi(\bar{a}, v) \rightarrow v = u)\right)$$

and hence, by the elementarity,

$$R^{\mathbb{F}}(S) \models \exists u \left(\varphi(\bar{a}, u) \wedge \forall v(\varphi(\bar{a}, v) \rightarrow v = u)\right).$$

Thus, there is $y' \in R^{\mathbb{F}}(S)$ such that

$$R^{\mathbb{F}}(S) \models \varphi(\bar{a}, y').$$

Again, by elementarity,

$$\mathbb{F} \models \varphi(\bar{a}, y') \wedge \forall v(\varphi(\bar{a}, v) \rightarrow v = y)$$

so that $y = y' \in R^{\mathbb{F}}(S)$. □

LEMMA 13. *Let* $\mathbb{F} \subset \mathbb{H}$ *be real closed fields,* $S \subset \mathbb{H}$ *and assume that* $\mathbb{H} = R^{\mathbb{H}}(\mathbb{F}(S))$. *Then* $\mathbb{H} = \{G(\bar{x}) \mid m < \omega,\ G \in \mathrm{Func}^m(\mathbb{H}, F),\ \bar{x} \in S^m\}$.

Proof. The direction $\supset$ is trivial. Fix any $a \in \mathbb{H}$ and assume that a is uniquely defined by $\varphi(u, \bar{x}, \bar{y})$ where $\bar{x} \in \mathbb{F}^{<\omega}$ and $\bar{y} \in S^m$. Then $A = \{\bar{z} \in \mathbb{H} \mid \mathbb{H} \models \exists! u \varphi(u, \bar{x}, \bar{z})\} \in \mathrm{Def}^m(\mathbb{H}, F)$ is non-empty because $\mathbb{H} \models \exists \bar{v} \exists! u \varphi(u, \bar{x}, \bar{v})$. Let $G(\bar{z})$ be the unique element u which satisfies $\varphi(u, \bar{x}, \bar{z})$ if $\bar{z} \in A$ and let $G(\bar{z}) = 0$ otherwise. Then $G \in \mathrm{Func}^m(\mathbb{H}, F)$ and $G(\bar{y}) = a$. □

We fix a real closed field $\mathbb{F}$ and a subset $S \subset F$ for the rest of this subsection.

Definition 17 ([**29**, p. 102, Definition 3.3.29]). *Let $\mathcal{C} \subset \bigcup_{1 \leq k < \omega} \mathrm{Def}^k(\mathbb{F}, S)$ be the minimal collection such that*

(C1) $\{a\} \in \mathcal{C}$ for all $a \in R^{\mathbb{F}}(S)$,
(C2) $(a, +\infty) \in \mathcal{C}$ and $(-\infty, a) \in \mathcal{C}$ for all $a \in R^{\mathbb{F}}(S)$,
(C3) $(a, b) \in \mathcal{C}$ for all $a, b \in R^{\mathbb{F}}(S)$,
(C4) $G \in \mathcal{C}$ for all cells $A \in \mathcal{C} \cap \mathrm{Def}^k(\mathbb{F}, S)$ and continuous functions $G\colon A \to F$ in $\mathrm{Def}^{k+1}(\mathbb{F}, S)$,
(C5) $\{\langle \bar{a}, b\rangle \in A \times F \mid b < G(\bar{a})\} \in \mathcal{C}$ and $\{\langle \bar{a}, b\rangle \in A \times F \mid G(\bar{a}) < b\} \in \mathcal{C}$ for all cells $A \in \mathcal{C} \cap \mathrm{Def}^k(\mathbb{F}, S)$ and all continuous functions $G\colon A \to F$ in $\mathrm{Def}^{k+1}(\mathbb{F}, S)$,
(C6) $\{\langle \bar{a}, b\rangle \in A \times F \mid G_1(\bar{a}) < b < G_2(\bar{a})\} \in \mathcal{C}$ for all cells $A \in \mathcal{C} \cap \mathrm{Def}^k(\mathbb{F}, S)$ and continuous functions $G_1, G_2\colon A \to F$ in $\mathrm{Def}^{k+1}(\mathbb{F}, S)$ that satisfy $G_1 < G_2$ on A.

Any element of $\mathcal{C}$ is called a cell.

Note 1. *The following are fundamental facts about* RCF.

(1) *For each $X \in \mathrm{Def}^k(\mathbb{F}, S)$, there is a quantifier free formula $\varphi \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}(S)}$ such that*

$$X = \{a \in F^k \mid \mathbb{F} \models \varphi(a)\}.$$

(2) *(Uniformization) For each $A \in \mathrm{Def}^{k+1}(\mathbb{F}, S)$, there exists a function $G \in \mathrm{Func}^k(F, S)$ on F^k such that $G \restriction \mathrm{dom}\, A \subset A$.*([**29**, p. 101, Corollary 3.3.26])
(3) *(Cell decomposition) For each $A \in \mathrm{Def}^k(\mathbb{F}, S)$, there are disjoint cells $C_1, \ldots, C_m \in \mathrm{Def}^k(\mathbb{F}, S)$ such that $A = C_1 \cup \ldots \cup C_m$.*([**29**, p. 103, Theorem 3.3.31])
(4) *For each $G \in \mathrm{Func}^k(F, S)$, there are cells $C_1, \ldots, C_m \in \mathcal{C}$ partitioning F^k such that each $G \restriction C_i$ is continuous.*([**18**, Ch3. (2.11)])
(5) *For every definable continuous function on a definable bounded closed set, its image is closed and bounded.*([**29**, Corollary 3.3.20])

Fix a natural number $k \geq 1$. Note that $\mathrm{Def}^k(\mathbb{F}, S)$ is a Boolean algebra with respect to the subset relation and the set operations. Fix an ultrafilter U on $\mathrm{Def}^k(\mathbb{F}, S)$ from now on. Then $f =_U g :\iff \{\bar{x} \in F^k \mid f(\bar{x}) = g(\bar{x})\} \in U$ is an equivalence relation on $\mathrm{Func}^k(\mathbb{F}, S)$. For each function $f \in \mathrm{Func}^k(F, S)$, define

$[f]_U := \{g \in \mathrm{Func}^k(F,S) \mid g =_U f\}$. Let $\mathrm{Ult}^{\mathbb{F}}(S,U) := \{[f]_U \mid f \in \mathrm{Func}^k(\mathbb{F},S)\}$. We define

(1) 0 is the equivalence class of the function $0\colon F^k \to \{0\}$,
(2) 1 is the equivalence class of the function $1\colon F^k \to \{1\}$,
(3) $[f]_U < [g]_U :\iff \{\bar{x} \in F^k \mid f(\bar{x}) < g(\bar{x})\} \in U$,
(4) $[f]_U + [g]_U := [f+g]_U$ for each $f, g \in \mathrm{Func}^k(F,S)$, and
(5) $[f]_U \cdot [g]_U := [f \cdot g]_U$ for each $f, g \in \mathrm{Func}^k(F,S)$.

It is easily seen that $<$, $+$, and $\cdot$ are well-defined, and that $\left\langle \mathrm{Ult}^{\mathbb{F}}(S,U), 0, 1, +, \cdot, < \right\rangle$ is a structure for $\mathcal{L}_{\mathsf{OR}}$. The analogue of Łos's theorem for $\mathrm{Ult}^{\mathbb{F}}(S,U)$ holds due to the uniformization of real closed fields.

Theorem 6. *Let* $f_1, \ldots, f_m \in \mathrm{Func}^k(\mathbb{F},S)$ *and* $\varphi(v_1, \ldots, v_m) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}}$. *Then*

$$\mathrm{Ult}^{\mathbb{F}}(S,U) \models \varphi([f_1]_U, \ldots, [f_m]_U) \iff \{\bar{x} \in F^k \mid \mathbb{F} \models \varphi(f_1(\bar{x}), \ldots, f_m(\bar{x}))\} \in U$$

In particular, $\mathrm{Ult}^{\mathbb{F}}(S,U)$ *is a real closed field.*

Proof. As the standard proof of Łos's theorem, we shall proceed by induction on the complexity of φ. The negation step and the conjunction step are very simple and are similar to the proof of Łos's theorem, so we consider the existential quantifier step. Let us assume that

$$\mathrm{Ult}^{\mathbb{F}}(S,U) \models \varphi_0([f]_U, [f_1]_U, \ldots, [f_m]_U) \iff \{\bar{x} \in F^k \mid \mathbb{F} \models \varphi_0(f(\bar{x}), f_1(\bar{x}), \ldots, f_m(\bar{x}))\} \in U$$

for all $f \in \mathrm{Func}^k(\mathbb{F},S)$. We have

$$\begin{aligned} &\mathrm{Ult}^{\mathbb{F}}(S,U) \models \exists v \varphi_0(v, [f_1]_U, \ldots, [f_m]_U) \\ \iff\ & \mathrm{Ult}^{\mathbb{F}}(S,U) \models \varphi_0([f]_U, [f_1]_U, \ldots, [f_m]_U) \text{ for some } f \in \mathrm{Func}^k(\mathbb{F},S) \\ \iff\ & \{\bar{x} \in F^k \mid \mathbb{F} \models \varphi_0(f(\bar{x}), f_1(\bar{x}), \ldots, f_m(\bar{x}))\} \in U \text{ for some } f \in \mathrm{Func}^k(\mathbb{F},S) \\ \iff\ & \{\bar{x} \in F^k \mid \mathbb{F} \models \exists v \varphi_0(v, f_1(\bar{x}), \ldots, f_m(\bar{x}))\} \in U \end{aligned}$$

The last equivalence follows from uniformization in Note 1. □

For each $l < k$, $\mathrm{id}_l \in \mathrm{Func}^k(\mathbb{F},S)$ denotes the projection $\mathrm{id}_l(x_0, \ldots, x_{k-1}) = x_l$. We shall assume that $S = \mathbb{F}_0$ is a real closed subfield of $\mathbb{F}$.

Lemma 14.

$$\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U) = R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)}(\mathbb{F}_0([\mathrm{id}_0], \ldots, [\mathrm{id}_{k-1}])).$$

In particular, the transcendental degree $\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)/\mathbb{F}_0)$ *of* $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)$ *over* $\mathbb{F}_0$ *is less than or equal to* k.

PROOF. Fix any $f \in \mathrm{Func}^k(\mathbb{F}, F_0)$. Pick $\varphi \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}(F_0)}$ which defines f, so that

$$f(\bar{x}) = y \iff \mathbb{F} \models \varphi(\bar{x}, y)$$

and

$$\mathbb{F} \models \forall \bar{u} \exists v (\varphi(\bar{u}, v) \wedge \forall w (\varphi(\bar{u}, w) \to v = w)).$$

Then, $\{\bar{x} \in F^k \mid \mathbb{F} \models \varphi(\bar{x}, f(\bar{x}))\} = \{\bar{x} \in F^k \mid \mathbb{F} \models \varphi(\mathrm{id}_0(\bar{x}), \ldots, \mathrm{id}_{k-1}(\bar{x}), f(\bar{x}))\} = F^k \in U$. Therefore,

$$\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U) \models \varphi([\mathrm{id}_0]_U, \ldots, [\mathrm{id}_{k-1}]_U, [f]_U).$$

Since $\mathbb{F}_0$ is a common elementary submodel of $\mathbb{F}$ and $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$,

$$\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U) \models \forall \bar{u} \exists v (\varphi(\bar{u}, v) \wedge \forall w (\varphi(\bar{u}, w) \to v = w))$$

and hence $[f]_U$ is uniquely defined over $\mathbb{F}_0([\mathrm{id}_0]_U, \ldots, [\mathrm{id}_{k-1}]_U)$. □

Note that, by cell decomposition, U is generated by cells. We shall consider how properties of U characterize properties of the field extension $\mathbb{F}_0 \leq \mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$. First, we shall see that the generators of U characterize an algebraic property of the extension.

LEMMA 15. *The following are equivalent.*

(1) $\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)/\mathbb{F}_0) = k$
(2) *U is generated by cells that are constructed by (C2), (C3), (C5), and (C6).*
(3) *U is generated by open sets with respect to the product topology of the order topology.*

PROOF. Since the implication $2 \implies 3$ is trivial, we shall show $3 \implies 1 \implies 2$. For $3 \implies 1$, we assume that U is generated by open sets. Suppose that

$$\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U) \models \sum_{n\colon k \to l} a_n \prod_{i<k} [\mathrm{id}_i]_U{}^{n(i)} = 0$$

where each a_n is in F_0 and $l \in \omega$. Here, any natural number l is identified with the set of non-negative integers less than l, and each $a \in F_0$ is identified with the equivalent class of the constant function with value a. Then,

$$\left\{ \langle x_0, \ldots, x_{k-1} \rangle \in F^k \;\middle|\; \mathbb{F} \models \sum_{n\colon k \to l} a_n \prod_{i<k} x_i^{n(i)} = 0 \right\} \in U.$$

Since U is generated by open sets, there is an open set $A \in U$ such that if $\langle x_0, \ldots, x_{k-1} \rangle \in A$ then $\sum_{n\colon k \to l} a_n \prod_{i<k} x_i^{n(i)} = 0$. Thus, each a_n is 0. Therefore $[\mathrm{id}_0]_U, \ldots, [\mathrm{id}_{k-1}]_U$ are independent.

For $1 \implies 2$, we assume that $C \in U$ where C is generated using (C1) or (C4). If (C1) is used, then $\{a\} \times F \times \cdots \times F \in U$, so that $[\mathrm{id}_0]_U = [a]_U$. Thus

$\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)/\mathbb{F}_0) < k$. If (C4) is used, then $G \times F \times \cdots \times F \in U$ for some $A \in \mathcal{C} \cap \mathrm{Def}^l(\mathbb{F}, F_0)$ and $G\colon A \to F$ in $\mathrm{Def}^{l+1}(\mathbb{F}, F_0)$, so that $[\mathrm{id}_l]_U$ is uniquely defined over $R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)}(\mathbb{F}_0([\mathrm{id}_0]_U, \ldots, [\mathrm{id}_{l-1}]_U))$. Thus, $\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)/\mathbb{F}_0) < k$. □

Next, we shall see that U characterizes an order property of the extension $\mathbb{F}_0 \leq \mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$. Note that an ultrafilter $U \subset \mathrm{Def}^k(\mathbb{F}, F_0)$ is principal iff $\{\bar{x}\} \in U$ for some $\bar{x} \in F_0^k$: If U is principal, then some single non-empty set $A \in \mathrm{Def}^k(\mathbb{F}, F_0)$ generates U. Then, by model-completeness, there exists $\bar{x} \in A \cap F_0^k$. Since U is an ultrafitler, $A \setminus \{\bar{x}\} \in U$ or $\{\bar{x}\} \in U$ but, since A generates U, $A \setminus \{\bar{x}\} \in U$ is not the case. Thus $A = \{\bar{x}\} \in U$.

Lemma 16. *If $\langle A, B\rangle$ is a gap in $\mathbb{F}_0$, then $U_{\langle A,B\rangle} := \{C \in \mathrm{Def}^1(\mathbb{F}, F_0) \mid \exists x \in A \cup \{-\infty\},\ \exists y \in B \cup \{+\infty\},\ (x, y) \subset C\}$ is a non-principal ultrafilter in the Boolean algebra $\mathrm{Def}^1(\mathbb{F}, F_0)$. Conversely, if U is a non-principal ultrafilter in $\mathrm{Def}^1(\mathbb{F}, F_0)$, then $U = U_{\langle A_U, B_U\rangle}$ where $A_U = \{a \in F_0 \mid \{x \in F \mid a < x\} \in U\}$ and $B_U = \{b \in F_0 \mid \{x \in F \mid x < b\} \in U\}$.*

Proof. Suppose that $\langle A, B\rangle$ is a gap in $\mathbb{F}_0$. Since the case when $A = \emptyset$ or $B = \emptyset$ is simpler, we assume that both A and B are non-empty. First, we shall show that $U_{\langle A,B\rangle}$ is non-principal. Note that $\{(x, y) \mid x \in A,\ y \in B\}$ generates $U_{\langle A,B\rangle}$. Fix any $x \in A$ and $y \in B$. Then $x < (x+y)2^{-1} < y$ and hence either there exists $x' \in A$ such that $(x + y)2^{-1} \leq x'$ or there exists $y' \in B$ such that $y' \leq (x + y)2^{-1}$. We assume that the former case occurs. Then $(x', y) \subsetneq (x, y)$ and $(x', y) \in U_{\langle A,B\rangle}$. Thus $U_{\langle A,B\rangle}$ is non-principal. Next, let us proceed to the maximality of $U_{\langle A,B\rangle}$. Fix any $C \in U_{\langle A,B\rangle}$. Using cell decomposition, there exist finitely many elements $a_i^-, a_i^+, a_j, b^-, b^+ \in F_0$ such that

$$C = \bigcup_i (a_i^-, a_i^+) \cup \bigcup_j \{a_j\} \cup (b^-, +\infty) \cup (-\infty, b^+).$$

Since $\langle A, B\rangle$ is a gap, there exist $x \in A$ and $y \in B$ such that $a_i^-, a_i^+, a_j, b^-, b^+ \notin (x, y)$ for each i and j. Thus either (x, y) is included in C or it is disjoint from C. In the former case, $C \in U$, and otherwise $F_0 \setminus C \in U$.

Suppose that U is a non-principal ultrafilter in $\mathrm{Def}^1(\mathbb{F}, F_0)$. Since U is closed under intersections and $\emptyset \notin U$, $\langle A_U, B_U\rangle$ is a pregap. Towards a contradiction, let us assume that $c \in F_0$ interpolates $\langle A_U, B_U\rangle$. Then, one of $\{x \in F_0 \mid x < c\}$, $\{x \in F_0 \mid c < x\}$, and $\{c\}$ is in U. If $\{c\} \in U$, then U is principal, a contradiction. If $\{x \in F_0 \mid x < c\} \in U$, then $c \in B_U$, a contradiction. The case $\{x \in F_0 \mid c < x\} \in U$ is similar. □

Lemma 17. *Let U be an ultrafilter on $\mathrm{Def}^k(\mathbb{F}, F_0)$. If $\langle A, B\rangle$ is a gap in $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$, then the set of all $\{\langle \bar{x}, y\rangle \mid \bar{x} \in X,\ H_A(\bar{x}) < y < H_B(\bar{x})\}$ where $X \in U$,*

$[H_A]_U \in A \cup \{[-\infty]_U\}$, *and* $[H_B]_U \in B \cup \{[+\infty]_U\}$ *generates a non-principal ultrafilter* $W_{\langle A,B\rangle}$ *on* $\mathrm{Def}^{k+1}(\mathbb{F}, F_0)$. *Here,* $+\infty$ *and* $-\infty$ *denote constant functions to* $+\infty$ *and* $-\infty$*, respectively.*

Proof. Let us assume that $\langle A, B\rangle$ is a gap in $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$. We consider the case that both A and B are non-empty. Fix any $X \in \mathrm{Def}^{k+1}(\mathbb{F}, F_0)$. By cell decomposition, we get F_0-definable cells C_j $(j < k_C)$, D_j $(j < k_D)$, E^+_j $(j < k_{E^+})$, and E^-_j $(j < k_{E^-})$ and F_0-definable functions H^C_j $(j < k_C)$, $H^{D,-}_j$, $H^{D,+}_j$ $(j < k_D)$, $H^{E^-}_j$ $(j < k_{E^-})$, $H^{E^+}_j$ $(j < k_{E^+})$ such that

$$X = \bigcup_{j<k_C} \{\langle \bar{x}, y\rangle \in C_j \times F \mid H^C_j(\bar{x}) = y\} \cup \bigcup_{j<k_D} \{\langle \bar{x}, y\rangle \in D_j \times F \mid H^{D,-}_j(\bar{x}) < y < H^{D,+}_j(\bar{x})\}$$
$$\cup \bigcup_{j<k_{E^-}} \{\langle \bar{x}, y\rangle \in E^-_j \times F \mid y < H^{E^-}_j(\bar{x})\} \cup \bigcup_{j<k_{E^+}} \{\langle \bar{x}, y\rangle \in E^+_j \times F \mid H^{E^+}_j(\bar{x}) < y\}.$$

By reordering $\mathcal{C} = \{C_j, D_j, \dots \mid j < k_C, j < k_D, \dots\}$, we shall assume that C_j $(j < l_C)$, D_j $(j < l_D)$, E^-_j $(j < l_{E^-})$, and E^+_j $(j < l_{E^+})$ are only members of U in $\mathcal{C}$. Let $\mathcal{H}$ be the set of definable functions we have chosen. Since $\langle A, B\rangle$ is a gap in $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$ and since $\mathcal{H}$ is finite, there are $[H^-]_U \in A$ and $[H^+]_U \in B$ such that the open interval $([H^-]_U, [H^+]_U)$ does not contain $[H]_U$ for any $H \in \mathcal{H}$. If $[H^{D,-}_j]_U \le [H^-]_U < [H^+]_U \le [H^{D,+}_j]_U$ for some $j < l_D$, if $[H^{E^+}_j]_U \le [H^-]_U$ for some $j < l_{E^+}$, or if $[H^+]_U \le [H^{E^-}_j]_U$ for some $j < l_{E^-}$, then $X \in W_{\langle A,B\rangle}$. Otherwise, define

$$C = \bigcap_{j<l_C} C_j \cap \bigcap_{l_C \le j<k_C} \left(F^k \setminus C_j\right) \cap \bigcap_{j<l_D} D_j \cap \bigcap_{l_D \le j<k_D} \left(F^k \setminus D_j\right)$$
$$\cap \bigcap_{j<l_{E^-}} E^-_j \cap \bigcap_{l_{E^-} \le j<k_{E^-}} \left(F^k \setminus E^-_j\right) \cap \bigcap_{j<l_{E^+}} E^+_j \cap \bigcap_{l_{E^+} \le j<k_{E^+}} \left(F^k \setminus E^+_j\right).$$

Then $C \in U$. Let

$$Y = \bigcap_{j<l_C} \{\langle \bar{x}, y\rangle \in C \times F \mid H^C_j(\bar{x}) \ne y\} \cap \bigcap_{j<l_D} \{\langle \bar{x}, y\rangle \in C \times F \mid y \le H^{D,-}_j(\bar{x}) \text{ or } H^{D,+}_j(\bar{x}) \le y\}$$
$$\cap \bigcap_{j<l_{E^-}} \{\langle \bar{x}, y\rangle \in C \times F \mid H^{E^-}_j(\bar{x}) \le y\} \cap \bigcap_{j<l_{E^+}} \{\langle \bar{x}, y\rangle \in C \times F \mid y \le H^{E^+}_j(\bar{x})\}.$$

Then $Y \subset F^{k+1} \setminus X$ and $\{\bar{x} \in F^k \mid \forall y \in F,\ H^-(\bar{x}) < y < H^+(\bar{x}) \implies \langle \bar{x}, y\rangle \in Y\} \in U$. Thus $Y \in W_{\langle A,B\rangle}$ and hence $F^{k+1} \setminus X \in W_{\langle A,B\rangle}$. Therefore $W_{\langle A,B\rangle}$ is an ultrafilter.

If $\{\langle \bar{x}_0, y_0\rangle\} \in W_{\langle A,B\rangle}$, then the constant function $H(\bar{x}) := y_0$ interpolates $\langle A, B\rangle$ and is definable over F_0, a contradiction. Thus $W_{\langle A,B\rangle}$ is non-principal. □

Let $\langle A_l, B_l\rangle$ be the pregap in $R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)}(\mathbb{F}_0([\mathrm{id}_0]_U,\dots,[\mathrm{id}_{l-1}]_U))$ defined by $[\mathrm{id}_l]_U$ for each $l < k$. That is, $A_l = \{[G]_U \in R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)}(\mathbb{F}_0([\mathrm{id}_0]_U,\dots,[\mathrm{id}_{l-1}]_U)) \mid \{\bar{x} \in F^k \mid \mathbb{F} \models G(\bar{x}) < x_l\} \in U\}$ and B_l is defined similarly. We define $U_l := \{\{\langle x_0,\dots,x_{l-1}\rangle \mid \exists x_l,\dots x_{k-1}, \langle x_0,\dots,x_{k-1}\rangle \in A\} \mid A \in U\}$. Note that U_l is an ultrafilter on $\mathrm{Def}^l(\mathbb{F}, F_0)$. We regard $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U_l)$ as a subfield of $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U_{l'})$ for each $l \le l' \le k$ by the embedding $[G]_{U_l} \mapsto [\tilde{G}]_{U_{l'}}$ where $\tilde{G}(x_0,\dots,x_{l'-1}) = G(x_0,\dots,x_{l-1})$. Then $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U_l) = R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U_{l'})}(\mathbb{F}_0([\mathrm{id}_0]_U,\dots,[\mathrm{id}_{l-1}]_U))$.

Lemma 18. *The following are equivalent.*

- *(i)* $\mathbb{F}_0$ *is cofinal in* $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$.
- *(ii) Both of* $\mathrm{cof}(A_l)$ *and* $\mathrm{coi}(B_l)$ *are infinite for all* $l < k$.
- *(iii) For each* $A \in U$*, there is a bounded cell* $C \in U$ *such that* $\mathrm{Cl}(C) \subset A$ *where* $\mathrm{Cl}(C)$ *is the closure of* C.
- *(iv) For each* $G \in \mathrm{Func}^k(F, F_0)$*, there exists* $A \in U$ *such that* $G\restriction_A$ *is bounded.*

Before proceeding to the proof, we remark that $\mathrm{Cl}(A) \in \mathrm{Def}^k(\mathbb{F}, F_0)$ for any $A \in \mathrm{Def}^k(\mathbb{F}, F_0)$. To see this, let $\varphi(\bar{v})$ be the definition of A. Then,

$$\mathrm{Cl}(A) = \{\bar{w} \in F \mid F \models \forall r(r > 0 \to \exists \bar{v}(\varphi(\bar{v}) \wedge (w_1 - v_1)^2 + \cdots + (w_k - v_k)^2 < r))\}.$$

So, both the interior $\mathrm{Int}(A)$ and the boundary $\partial(A)$ of A are also F_0-definable.

Proof. First, we shall show that (i) implies (ii). We assume that some $\mathrm{coi}(B_l)$ is finite. If $\mathrm{coi}(B_l) = 0$, then $[\mathrm{id}_l]_U$ bounds $R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)}(\mathbb{F}_0([\mathrm{id}_0]_U,\dots,[\mathrm{id}_{l-1}]_U))$, so it bounds $\mathbb{F}_0$. If $\mathrm{coi}(B_l) = 1$, then $(\min(B_l) - [\mathrm{id}_l]_U)^{-1}$ bounds $R^{\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0,U)}(\mathbb{F}_0([\mathrm{id}_0]_U,\dots,[\mathrm{id}_{l-1}]_U))$, so it bounds $\mathbb{F}_0$.

Second, we shall assume that (ii) holds, and prove (iii) by induction on k. Suppose that $k = 1$ and fix any $A \in U$. By cell decomposition, A can be decomposed into open intervals (some of them may have no endpoints) and singletons. Remark that endpoints of F_0-definable open intervals and elements of F_0-definable singletons are uniquely defined over F_0, hence they are members of F_0 by Lemma 12. If some singleton is a member of U, since the closure of a singleton is a singleton, the case $k = 1$ is done. Otherwise, there exists $x \in F_0 \cup \{-\infty\}$ and $y \in F_0 \cup \{+\infty\}$ such that $(x, y) \subset A$ and $(x, y) \in U$. Note that $x \in A_0 \cup \{-\infty\}$ and that $y \in B_0 \cup \{+\infty\}$. Since $\mathrm{cof}(A_0)$ and $\mathrm{coi}(B_0)$ are infinite, there exist $a \in A_0$ and $b \in B_0$ such that $x < a < b < y$. Thus, $\mathrm{Cl}((a, b)) = [a, b] \subset (x, y) \subset A$ and $(a, b) \in U$. We proceed to the inductive step. So we shall assume that the claim holds for l and we shall consider the case $k = l + 1$. Fix any $A \in U$. We choose a cell $C \in U$ such that $C \subset A$ by cell decomposition. Since $\{\langle x_0,\dots x_l\rangle \mid G(x_0,\dots,x_{l-1}) < x_l < H(x_0,\dots,x_{l-1})\} \in U$ for some $[G]_{U_l} \in A_l$ and $[H]_{U_l} \in B_l$, we shall assume that $\{y \in F \mid \langle \bar{x}, y\rangle \in C\}$ is bounded for each $\bar{x} \in F^l$. So C is not generated by (C5). Consider the case when there are $C_0 \in U_l$ and functions $G, H \in \mathrm{Func}^l(\mathbb{F}, F_0)$ that is continuous on C_0 such that

$$C = \{\langle x_0, \dots, x_l\rangle \in C_0 \times F \mid G(x_0, \dots, x_{l-1}) < x_l < H(x_0, \dots, x_{l-1})\}.$$

The other case, which means that C is generated by (C4), is simpler. Now, we have $[G]_{U_l} < [\mathrm{id}_l]_U < [H]_{U_l}$. Since A_l and B_l are infinite, there are functions $G', H' \in \mathrm{Func}^l(\mathbb{F}, F_0)$ such that

$$[G]_{U_l} < [G']_{U_l} < [\mathrm{id}_l]_U < [H']_{U_l} < [H]_{U_l}.$$

Choose a cell $C_0' \in U_l$ such that G' and H' are continuous on C_0' (see Note 1 (iv)),

$$G(x_0, \dots, x_{l-1}) < G'(x_0, \dots, x_{l-1}) < H'(x_0, \dots, x_{l-1}) < H(x_0, \dots, x_{l-1})$$

for all $\langle x_0, \dots, x_{l-1}\rangle \in C_0'$, and $C_0' \subset C_0$. By the induction hypothesis, we pick a bounded cell $C_1 \in U_l$ such that $\mathrm{Cl}(C_1) \subset C_0'$. Note that $G' \restriction \mathrm{Cl}(C_1)$ and $H' \restriction \mathrm{Cl}(C_1)$ are bounded (see [**29**, Corollary 3.3.20]). Let

$$\tilde{C} := \{\langle x_0, \dots, x_l\rangle \in C_1 \times F \mid G'(x_0, \dots, x_{l-1}) < x_l < H'(x_0, \dots, x_{l-1})\}.$$

Then, $\tilde{C} \in U$, $\tilde{C}$ is bounded, and

$$\mathrm{Cl}(\tilde{C}) = \{\langle x_0, \dots, x_l\rangle \in \mathrm{Cl}(C_1) \times F \mid G'(x_0, \dots, x_{l-1}) \le x_l \le H'(x_0, \dots, x_{l-1})\}$$

is a subset of C (see [**18**, Ch.6 (1.7)]). This finishes the implication (ii) $\implies$ (iii).

Third, we shall show that (iii) implies (iv). Fix any $G \in \mathrm{Func}^k(\mathbb{F}, F_0)$. Choose a cell $C \in U$ such that $G \restriction C$ is continuous, see 4 of Note 1. By the assumption, we pick a bounded cell $C' \in U$ such that $\mathrm{Cl}(C') \subset C$. Then, $G \restriction \mathrm{Cl}(C')$ is bounded.

Fourth, we shall show that (iv) implies (i). Fix any $G \in \mathrm{Func}^k(\mathbb{F}, F_0)$. By the assumption, pick $A \in U$ such that $G \restriction A$ is bounded (see [**29**, Corollary 3.3.20]). This statement can be written in $\mathbb{F}_0$. Thus there is $x \in F_0$ such that $G(\bar{z}) < x$ for all $\bar{z} \in A$. Therefore $[G]_U < x$. □

NOTE 2. *If $C_0, \dots, C_n$ are cells that is constructed using (C1) or (C4) at least once, then $\mathrm{Int}(C_0 \cup \dots \cup C_n) = \emptyset$. To see this, for $i < m$, fix any $a_i, b_i \in F$ such that $a_i < b_i$. Suppose that, for the construction of C_k, (C1) or (C4) is used at the i_k-th step. For each $i < m$, select $x_i \in \bigcap_{i=i_k} \{x \in (a_i, b_i) \mid (x_0, \dots, x_{i-1}, x) \notin \pi_i C_k\}$ where $\pi_i \colon F^m \to F^{i+1}$ is the projection map on the first $i+1$ coordinates. Then the sequence $\bar{x} = \langle x_i \mid i \le m\rangle$ is in $(\Pi_{i<m}(a_i, b_i)) \setminus \bigcup_{k \le n} C_k$.*

LEMMA 19. *For any $A \in \mathrm{Def}^m(\mathbb{F}, F_0)$, $\mathrm{Int}(\partial(A)) = \emptyset$.*

PROOF. $\partial(A)$ does not contain any open cell (see [**18**, Ch.3 (2.3), Ch.4 (1.1) and (1.10)]). So any cell decomposition of $\partial(A)$ consists of cells $C_0, \dots, C_n$ that are constructed using (C1) or (C4) at least once. Thus $\mathrm{Int}(\partial(A)) = \mathrm{Int}(C_0 \cup \dots \cup C_n) = \emptyset$. □

By Lemmata 15 and 18, we have the next lemma. H. Woodin [**33**] mentions it without proof.

LEMMA 20. *Let $\mathbb{F}$ be a real closed field and fix a real closed subfield $\mathbb{F}_0$. Suppose that U is an ultrafilter on $\mathrm{Def}^k(\mathbb{F}, F_0)$. Then, the following are equivalent.*

(1) $\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)/\mathbb{F}_0) = k$ *and $\mathbb{F}_0$ is cofinal in* $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_0, U)$.

(2) U is generated by closures of bounded open cells in U.

PROOF. To see that 2 implies 1, suppose that U is generated by closures of bounded open cells. For each $A \in U$, there exists an open cell C such that $\mathrm{Cl}(C) \subset A$ and that $\mathrm{Cl}(C) \in U$. Since $\mathrm{Cl}(C) = C \cup \partial(C)$ and $\partial(C)$ has empty interior (see Lemma 19), $\partial(C) \notin U$ and hence $C \in U$. Thus U satisfies (iii) of Lemma 18 and 3 of Lemma 15.

Conversely, we assume that 1 holds. Fix any $A \in U$. By Lemma 18, pick a bounded cell $C \in U$ such that $\mathrm{Cl}(C) \subset A$. Since U is generated by open sets, C has non-empty interior. By Note 2, since C is bounded, C is generated using only (C3) and (C6). Thus C is an open cell. □

For the proof of the main theorem, we also require the following fact.

LEMMA 21 ([**18**, Ch 8. (3.10)]). *Let $f\colon A \to C$ be a definable continuous map from a definable closed subset A of a definable set B into a definable set C that is definably contractible to a point $c \in C$. Then f can be extended to a continuous definable function $\tilde{f}\colon B \to C$.*

Here, "C is definably contractible to a point $c \in C$" means that there exists a definable homotopy $H\colon C \times [0, 1] \to F$ that contracts to c, that is, a definable continuous function such that $x \mapsto H(x, 0)$ defines the identity map on C and that $x \mapsto H(x, 1)$ defines the constant map to $\{c\}$. For a definable interval $I \subset F$ and a point $c \in I$, the function $H(x, t) = x(1 - t) + ct$ on $I \times [0, 1]$ is a definable homotopy that witnesses that I is contractible to a point $c \in I$.

Let us consider the case when f is a continuous definable function on the closure of a cell C into a bounded set in F. Then the image of f is a closed interval (see [**18**, Ch.1 (3.6), Ch.3 (2.9), Ch.6 (1.11)]). So we have the following corollary.

COROLLARY 3. *Every bounded continuous definable function on the closure of a cell in F^m can be extended to a bounded continuous definable function on F^m.*

1.3. γ_1-extensions and β_1-real Fields. We shall define a key notion of an extension of real closed fields.

DEFINITION 18. *Suppose that $\mathbb{H} \subset \mathbb{F}$ are real closed fields. For each $\bar{a} \in F^n$,* $\mathrm{tp}^{\mathbb{F}}(\bar{a}/\mathbb{H})$ *denotes $\{A \cap H^n \mid \bar{a} \in A \in \mathrm{Def}^n(\mathbb{F}, H)\}$. Notice that* $\mathrm{tp}^{\mathbb{F}}(\bar{a}/\mathbb{H})$ *is an ultrafilter on* $\mathrm{Def}^n(\mathbb{H}, H)$. *We say that $Y \subset F$ is γ_1* over *$\mathbb{H}$ if* $\mathrm{tp}^{\mathbb{F}}(\bar{a}/\mathbb{H})$ *is countably generated for each $\bar{a} \in (Y \setminus H)^{<\omega}$. $\mathbb{F}$ is a γ_1-*extension *of $\mathbb{H}$ if F is γ_1 over $\mathbb{H}$.*

The notation "tp" comes from the model-theoretical term "type". We identify each element of $\mathrm{tp}^{\mathbb{F}}(\bar{a}/\mathbb{H})$ with the formula that defines it. If two extensions $\mathbb{G}$ and $\mathbb{F}$ of a real closed field $\mathbb{H}$ have a tuple of common elements $\bar{a} \in (\mathbb{G} \cap \mathbb{F})^{<\omega}$, then $\mathrm{tp}^{\mathbb{F}}(\bar{a}/\mathbb{H}) = \mathrm{tp}^{\mathbb{G}}(\bar{a}/\mathbb{H})$. So we often omit $\mathbb{F}$ in $\mathrm{tp}^{\mathbb{F}}(\bar{a}/\mathbb{H})$.

If $\langle A, B\rangle$ is the gap on $\mathbb{H}$ defined by $a \in F \setminus H$, then, by cell decomposition, $\{(x, y) \subset \mathbb{H} \mid x \in A,\ y \in B\}$ generates $\mathrm{tp}^{\mathbb{F}}(a/\mathbb{H})$. This implies the following lemma.

LEMMA 22. *Let $\mathbb{H} \subset \mathbb{F}$ be real closed fields. $a \in \mathbb{F}$ is γ_1 over $\mathbb{H}$ iff a does not define an uncountable gap on $\mathbb{H}$.*

LEMMA 23. *Let $\mathcal{H}$ be a chain of γ_1-extensions of $\mathbb{F}$. Then, $\bigcup \mathcal{H}$ is a γ_1-extension of $\mathbb{F}$.*

PROOF. Fix any $\{a_1, ..., a_n\} \subset \bigcup \mathcal{H}$. Then, $\{a_1, ..., a_n\} \subset H$ for some $H \in \mathcal{H}$. Thus $\mathrm{tp}(\langle a_1, \ldots, a_n\rangle /\mathbb{F})$ is coutably generated. □

The next lemma is a straightforward generalization of a well-known fact about atomic extensions in model theory. This was proved by Alex Kruckman (private communication).

LEMMA 24. *Let $\langle \mathbb{H}_\alpha \mid \alpha < \lambda\rangle$ be a $\subset$-increasing sequence such that $\mathbb{H}_{\alpha+1}$ is γ_1 over $\mathbb{H}_\alpha$ for each $\alpha \in \lambda$ and $\mathbb{H}_\gamma = \bigcup_{\xi<\gamma} \mathbb{H}_\xi$ for each limit $\gamma < \lambda$. Then, $\bigcup_{\alpha<\lambda} \mathbb{H}_\alpha$ is γ_1 over $\mathbb{H}_0$.*

PROOF. By Lemma 23, it suffices to show that if $\mathbb{H}_0 \subset \mathbb{H}_1 \subset \mathbb{H}_2$ are γ_1-extensions, then $\mathbb{H}_2$ is γ_1 over $\mathbb{H}_0$. Fix any $\bar{a} \in \mathbb{H}_2^m$. Let $\langle A_n \in \mathrm{Def}^m(\mathbb{H}_1, H_1) \mid n < \omega\rangle$ be a $\subset$-decreasing sequence that generates $\mathrm{tp}(\bar{a}/\mathbb{H}_1)$. Let $\varphi_n(\bar{x}, \bar{b}_n)$ be a formula that defines A_n with parameters $\bar{b}_n \in H_1^{m_n}$. For each $n \in \omega$, let $\langle B_{n,i} \in \mathrm{Def}^{m_n}(\mathbb{H}_0, H_0) \mid i < \omega\rangle$ be a $\subset$-decreasing sequence that generates $\mathrm{tp}(\bar{b}_n/\mathbb{H}_0)$. Let $\psi_{n,i}(\bar{x}, \bar{d}_{n,i})$ be a formula that defines $B_{n,i}$ where $\bar{d}_{n,i} \in H_0^{<\omega}$. We define

$$C_{n,i} := \{\bar{x} \in H_0 \mid \mathbb{H}_0 \models \exists \bar{y}(\varphi_n(\bar{x}, \bar{y}) \wedge \psi_{n,i}(\bar{y}, \bar{d}_{n,i}))\}$$

for each $n, i \in \omega$. We will show that $\langle C_{n,i} \mid n, i \in \omega\rangle$ generates $\mathrm{tp}(\bar{a}/\mathbb{H}_0)$. So let $A \in \mathrm{tp}(\bar{a}/\mathbb{H}_0)$ and assume that $\varphi(\bar{x}, \bar{d}_0)$ defines A. Since $A \in \mathrm{tp}(\bar{a}/\mathbb{H}_1)$, we take $n \in \omega$ such that $A_n \subset A$. Thus,

$$\mathbb{H}_1 \models \varphi_n(\bar{x}, \bar{b}_n) \to \varphi(\bar{x}, \bar{d}_0)$$

and hence

$$\mathrm{EDiag}(\mathbb{H}_1) \models \varphi_n(\bar{x}, \bar{b}_n) \to \varphi(\bar{x}, \bar{d}_0)$$

where $\mathrm{EDiag}(\mathbb{H}_1)$ denotes the elementary diagram of $\mathbb{H}_1$. By compactness, there is a formula $\theta(\bar{b}, \bar{b}_n, \bar{d}_1) \in \mathrm{EDiag}(\mathbb{H}_1)$ such that $\bar{b} \in \mathbb{H}_1 \setminus \mathbb{H}_0$, $\bar{d}_1 \in \mathbb{H}_0$, and

$$\theta(\bar{b}, \bar{b}_n, \bar{d}_1) \models \varphi_n(\bar{x}, \bar{b}_n) \to \varphi(\bar{x}, \bar{d}_0).$$

Now, $\bar{b}$ is not mentioned in $\varphi_n(\bar{x}, \bar{b}_n) \to \varphi(\bar{x}, \bar{d}_0)$, so that

$$\exists\bar{w}(\theta(\bar{w}, \bar{b}_n, \bar{d}_1)) \models \varphi_n(\bar{x}, \bar{b}_n) \to \varphi(\bar{x}, \bar{d}_0).$$

Moreover, $\bar{b}_n$ is not mentioned in $\varphi(\bar{x}, \bar{d}_0)$, so that

$$\exists\bar{y}(\varphi_n(\bar{x}, \bar{y}) \land \exists\bar{w}(\theta(\bar{w}, \bar{y}, \bar{d}_1))) \models \varphi(\bar{x}, \bar{d}_0).$$

Since $\exists\bar{w}(\theta(\bar{w}, \bar{y}, \bar{d}_1)) \in \mathrm{tp}(\bar{b}_n/\mathbb{H}_0)$, there is $i \in \omega$ such that

$$\mathbb{H}_0 \models \psi_{n,i}(\bar{y}, \bar{d}_{n,i}) \to \exists\bar{w}(\theta(\bar{w}, \bar{y}, \bar{d}_1)).$$

Thus,

$$\mathbb{H}_0 \models \exists\bar{y}(\varphi_n(\bar{x}, \bar{y}) \land \psi_{n,i}(\bar{y}, \bar{d}_{n,i})) \to \varphi(\bar{x}, \bar{d}_0).$$

Therefore, $C_{n,i} \subset A$. □

Lemma 25. *Let $\mathbb{H}$ be a real closed subfield of a real closed field $\mathbb{F}$. Suppose that $\mathbb{F} = R^{\mathbb{F}}(\mathbb{H}(x))$. If $\{x\}$ is γ_1 over $\mathbb{H}$, then so is $\mathbb{F}$.*

Proof. Consider the case the type of the gap on $\mathbb{H}$ defined by x is $\langle\omega, \omega\rangle$. Other cases are similar. Let $\langle\langle a_j, b_j\rangle \mid j < \omega\rangle$ be the gap. Fix any $\bar{y} \in \mathbb{F}^m$. By Lemma 12, each y_i is uniquely defined by some $\varphi_i(u, x) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}(\mathbb{H}\cup\{x\})}$, that is, $\mathbb{F} \models \varphi_i(y_i, x) \land \forall u(\varphi_i(u, x) \to u = y_i)$. Let $C_j = \left\{\bar{z} \in \mathbb{H}^m \mid \mathbb{H} \models \exists w\left(\bigwedge_{i<m} \varphi_i(z_i, w) \land a_j < w < b_j\right)\right\}$ for each $j < \omega$. We shall show that $\{C_j \mid j < \omega\}$ is a generator of $\mathrm{tp}^{\mathbb{F}}(\bar{y}/\mathbb{H})$.

Fix any $C \in \mathrm{tp}^{\mathbb{F}}(\bar{y}/\mathbb{H})$ which is defined by $\varphi_C(\bar{u})$. Let

$$D = \left\{ w \in \mathbb{H} \;\middle|\; \mathbb{H} \models \forall\bar{u}\left(\left(\bigwedge_{i<m} \varphi_i(u_i, w)\right) \to \varphi_C(\bar{u})\right)\right\}.$$

Since $D \in \mathrm{tp}^{\mathbb{F}}(x/\mathbb{H})$, there exists a $j < \omega$ such that $(a_j, b_j) \subset D$. Then, $C_j \subset C$. □

Woodin [**33**] used the property γ_1 to define the property β_1 in his sense.

Definition 19 ([**33**]). *Let $\mathbb{H}$ be a real closed subfield of a real closed field $\mathbb{F}$. $\mathbb{F}$ is* W-β_1 over $\mathbb{H}$ *if $\mathbb{F} = \bigcup_\alpha \mathbb{H}_\alpha$ where $\langle\mathbb{H}_\alpha \mid \alpha < \lambda\rangle$ is a continuously $\subset$-increasing sequence of real closed fields such that, for each $\xi < \lambda$,*

(1) $\mathbb{H}_0 = \mathbb{H}$
(2) $\mathbb{H}_{\xi+1} = R^{\mathbb{H}_{\xi+1}}(\mathbb{H}_\xi(Y))$ for some countable subset $Y \subset \mathbb{H}_{\xi+1}$, and
(3) $\mathbb{H}_{\xi+1}$ is γ_1 over $\mathbb{H}_\xi$.

$\mathbb{F}$ is W-β_1-real field *if it contains $\mathbb{R}$ and is W-β_1 over $\mathbb{R}$.*

In more detail about Woodin's result [**33**], he showed that, under CH, the finite support iteration of Cohen forcing $\mathbb{C}$ of length ω_2 forces that some ultrapower $\mathbb{R}^\omega/U$ is a W-β_1-real field. Woodin also defined a real closed η_1-field as an $\aleph_1$-saturated real closed field. It is known that, for real closed fields, $\aleph_1$-saturation is equivalent to our definition of η_1 (e.g., see [**12**, 5.4]). By the following lemma, the two properties W-β_1-real and β_1 are equivalent for ultrapowers of the reals over ω since they are η_1. For that reason, we do not distinguish β_1 and W-β_1 when we discuss ultrapowers of the reals.

Lemma 26. *(1) The unique real closed β_1-η_1-field $\mathcal{E}$, which is called* the Esterle algebra*, exists. More specifically, if K is a real closed β_1-η_1-field over $\mathbb{R}$, then there exists an $\mathbb{R}$-algebra isomorphism from K onto $\mathcal{E}$ (see* [**16**, Corollary 2.33 and introduction of Ch.2]*).*
(2) $\mathcal{E}$ is W-β_1 (see [**16**, Theorem 2.36] *and Lemma 25).*
(3) Any W-β_1-real field is β_1.

proof of (3). We assume that $\mathbb{F}$ is W-β_1. Since $\mathbb{R}$ is β_1, $\mathbb{R}$ is embeddable in any real closed η_1-field ([**16**, Theorem 2.30]). By Lemma 4 in [**33**], $\mathbb{F}$ can be embedded in any η_1 real closed field. In particular, it can be embedded in $\mathcal{E}$. Since any real closed subfield of any β_1 real closed field is β_1, $\mathbb{F}$ is also β_1. □

2. The Main Lemma

In this section, V always stands for the ground model.
For $M \subset \mathbb{R}^\omega$, define

$$M^\wedge := \{F(g_1, \ldots, g_m) \in \mathbb{R}^\omega \mid m \in \omega,\ g_1, \ldots, g_m \in M,\ F \in \mathrm{Func}^m(\mathbb{R}, \emptyset)\}.$$

where $F(g_1, \ldots, g_m)(n) = F(g_1(n), \ldots, g_m(n))$ for each $n < \omega$. We say that a filter base $\mathcal{F} \subset \mathcal{P}(\omega)$ is an *M-ultrafilter* if it totally orders M, that is, for each $f, g \in M$, one of the three sets $\{i \in \omega \mid f(i) < g(i)\}$, $\{i \in \omega \mid f(i) = g(i)\}$, and $\{i \in \omega \mid f(i) > g(i)\}$ includes an element of $\mathcal{F}$. To avoid redundancy, we abuse the notation by using $\mathcal{F}$ itself for the filter obtained by $\subset$-upward closure of $\mathcal{F}$. Define the $\mathcal{L}_{\mathsf{OR}}$-structure $\langle M^\wedge/\mathcal{F}, 0, 1, +, \cdot, < \rangle$ similarly to $\left\langle \mathrm{Ult}^{\mathbb{F}}(S, U), 0, 1, +, \cdot, < \right\rangle$. This structure was studied in Woodin [**33**]. If $\mathcal{F}$ is an $M^\wedge$-ultrafilter, then $M^\wedge/\mathcal{F}$ is a real closed ordered field by the following lemma, which is the analogue of Łos's theorem for $M^\wedge/\mathcal{F}$.

Lemma 27. *Let $M \subset \mathbb{R}^\omega$ and let $\mathcal{F} \subset \mathcal{P}(\omega)$ be an $M^\wedge$-ultrafilter. Let $f_1, \ldots, f_m \in M^\wedge$ and $\varphi(v_1, \ldots, v_m)$ be an $\mathcal{L}_{\mathsf{OR}}$-formula. Then*

$$M^\wedge/\mathcal{F} \models \varphi([f_1]_\mathcal{F}, \ldots, [f_m]_\mathcal{F}) \iff \{i < \omega \mid \mathbb{R} \models \varphi(f_1(i), \ldots, f_m(i))\} \in \mathcal{F}$$

Proof. As the standard proof of Łos's theorem, we shall proceed by induction on the complexity of φ. The conjunction step is straightforward. The negation step works by the following claim.

CLAIM 1. *Fix $k<\omega$. For $\psi(\bar{v})\in\mathcal{L}_{\mathsf{OR}}$ and $\bar{f}\in(M^\wedge)^k$, either $\{i<\omega\mid\mathbb{R}\models\psi(\bar{f}(i))\}\in\mathcal{F}$ or $\{i<\omega\mid\mathbb{R}\models\neg\psi(\bar{f}(i))\}\in\mathcal{F}$.*

PROOF. Induct on k. The case $k=0$ is clear. Assume that the claim holds for k and fix $(\bar{f},g)\in(M^\wedge)^{k+1}$ and $\varphi(\bar{v},w)\in\mathcal{L}_{\mathsf{OR}}$. By cell decomposition, there exist $F_l^-,F_l^+,G_j^-,G_{j'}^+,H_n\in\mathrm{Func}^k(\mathbb{R},\emptyset)$ and $\psi_l^F(\bar{v}),\psi_j^{G^-}(\bar{v}),\psi_j^{G^+}(\bar{v}),\psi_n^H(\bar{v})\in\mathcal{L}_{\mathsf{OR}}$ such that

$$\begin{aligned}&\mathbb{R}\models\varphi(\bar{r},s)\\ \iff&\mathbb{R}\models\bigvee_{l<L}\left(\psi_l^F(\bar{r})\wedge F_l^-(\bar{r})<s<F_l^+(\bar{r})\right)\vee\bigvee_{j<J^-}\left(\psi_j^{G^-}(\bar{r})\wedge G_j^-(\bar{r})<s\right)\\ &\qquad\vee\bigvee_{j<J^+}\left(\psi_j^{G^+}(\bar{r})\wedge s<G_j^+(\bar{r})\right)\vee\bigvee_{n<N}\left(\psi_n^H(\bar{r})\wedge H_n(\bar{r})=s\right).\end{aligned}$$

By the assumption, either $\{i<\omega\mid\psi(\bar{f}(i))\}\in\mathcal{F}$ or $\{i<\omega\mid\neg\psi(\bar{f}(i))\}\in\mathcal{F}$ for each $\psi(\bar{v})\in\mathcal{L}_{\mathsf{OR}}$. If $\{i<\omega\mid\neg\psi(\bar{f}(i))\}\in\mathcal{F}$ for every $\psi(\bar{v})\in\Psi:=\{\psi_l^F(\bar{v}),\psi_{j^-}^{G^-}(\bar{v}),\psi_{j^+}^{G^+}(\bar{v}),\psi_n^H(\bar{v})\mid l<L,\ j^-<J^-,\ j^+<J^+,\ n<N\}$, then $\{i<\omega\mid\mathbb{R}\models\neg\varphi(\bar{f}(i),g(i))\}\in\mathcal{F}$. Otherwise, removing $\psi\in\Psi$ such that $\{i<\omega\mid\neg\psi(\bar{f}(i))\}\in\mathcal{F}$, we shall assume $\{i<\omega\mid\psi(\bar{f}(i))\}\in\mathcal{F}$ for all $\psi\in\Psi$. Since $F^-(\bar{f}),\dots,H_n(\bar{f}),g\in M^\wedge$ and $\mathcal{F}$ is an $M^\wedge$-ultrafilter, $\mathcal{F}$ determines whether g is a member of $\bigcup_l\left(F_l^-(\bar{f}),F_l^+(\bar{f})\right)\cup\bigcup_j\left(G_j^-(\bar{f}),+\infty\right)\cup\bigcup_j\left(-\infty,G_j^+(\bar{f})\right)\cup\bigcup_n\{H_n(\bar{f})\}$, that is, whether $\{i<\omega\mid\mathbb{R}\models\varphi(\bar{f}(i),g(i))\}$ is a member of $\mathcal{F}$. □

The existential quantifier step works by the following claim.

CLAIM 2. *For $\varphi(\bar{u},v)\in\mathcal{L}_{\mathsf{OR}}$ and $\bar{g}\in(M^\wedge)^m$,*

$$(\exists f\in M^\wedge,\{i<\omega\mid\mathbb{R}\models\varphi(\bar{g}(i),f(i))\}\in\mathcal{F})\iff\{i<\omega\mid\mathbb{R}\models\exists x\varphi(\bar{g}(i),x)\}\in\mathcal{F}.$$

PROOF. ($\Longrightarrow$) is clear. Applying uniformization in Note 1 to $A:=\{\langle\bar{r},s\rangle\in\mathbb{R}^{m+1}\mid\mathbb{R}\models\varphi(\bar{r},s)\}$, we have ($\Longleftarrow$). □

□

NOTE 3. *Let $M\subset M^*\subset\mathbb{R}^\omega$. Let $\mathcal{F}$ and $\mathcal{F}^*$ be an $M^\wedge$-ultrafilter and an $(M^*)^\wedge$-ultrafilter, respectively, such that $\mathcal{F}\subset\mathcal{F}^*$. Define $\mathbb{G}=M^\wedge/\mathcal{F}$ and $\mathbb{G}^*=(M^*)^\wedge/\mathcal{F}^*$. Then $\mathbb{G}$ can be considered as a subfield of $\mathbb{G}^*$ via the embedding $[x]_{\mathcal{F}}\mapsto[x]_{\mathcal{F}^*}$.*

NOTE 4 (See [**33**]). *We shall consider the case $M^*=M\cup\{g_0,\dots,g_{m-1}\}$ for some $\bar{g}=\langle g_0,\dots,g_{m-1}\rangle\in(\mathbb{R}^\omega)^m$ in the situation of Note 3. Let $W=\mathrm{tp}^{\mathbb{G}^*}([g_0]_{\mathcal{F}^*},\dots,[g_{m-1}]_{\mathcal{F}^*}/\mathbb{G})$. Then,*

$$\iota([F(f_0,\dots,f_{k-1},\bar{g})]_{\mathcal{F}^*})=F^{\mathrm{Ult}^{\mathbb{G}^*}(\mathbb{G},W)}([f_0]_{\mathcal{F}},\dots,[f_{k-1}]_{\mathcal{F}},[\mathrm{id}_0]_W,\dots,[\mathrm{id}_{m-1}]_W)$$

for $F \in \mathrm{Func}^{k+m}(\mathbb{R}, \emptyset)$ *and* $f_0, \dots, f_{k-1} \in M$ *defines an isomorphism* $\iota\colon \mathbb{G}^* \to \mathrm{Ult}^{\mathbb{G}^*}(\mathbb{G}, W)$*. Here,* $F^{\mathrm{Ult}^{\mathbb{G}^*}(\mathbb{G},W)}$ *is the function in* $\mathrm{Func}^{k+m}(\mathrm{Ult}^{\mathbb{G}^*}(\mathbb{G}, W), \emptyset)$ *with the same definition of* F*. Furthermore,* ι *fixes* $\mathbb{G}$ *and* $\iota([g_k]_{\mathcal{F}^*}) = [\mathrm{id}_k]_W$*.*

Definition 20. *A filter* $\mathcal{F} \subset \mathcal{P}(\omega)$ *is* countably generated over a filter base $\mathcal{B} \subset \mathcal{P}(\omega)$ *if there exists a countable set* $\{\tau_n \subset \omega \mid n < \omega\}$ *such that* $\mathcal{F} = \{A \subset \omega \mid \exists n < \omega,\ \exists X \in \mathcal{B},\ \tau_n \cap X \subset A\}$*. A forcing notion* $\mathbb{P}$ *has* the γ_1 property *if, for an ultrafilter* U *on* $\mathcal{P}(\omega)$*, for a* $\mathbb{P}$*-name* $\dot{M}$ *of a subset of* $\mathbb{R}^\omega$ *that contains* $\mathbb{R}^\omega \cap V$*, and for a* $\mathbb{P}$*-name* $\dot{\mathcal{F}}$ *for an* $\dot{M}^\wedge$*-ultrafilter that is countably generated over* U*,* $\dot{M}^\wedge/\dot{\mathcal{F}}$ *is* γ_1 *over* $(\mathbb{R}^\omega \cap V)/U$ *in any generic extension by* $\mathbb{P}$*.*

Note that some ccc forcings do not have the γ_1 property. Hechler forcing is an example since a dominating real defines a $((\mathrm{cof}(\mathbb{R}^\omega/U))^V, 0)$ gap in $(\mathbb{R}^\omega \cap V)/U$. H. Woodin [**33**] proved the next theorem, essentially.

Theorem 7 ([**33**, Theorem 4.]). *(*CH*) Let* $\mathcal{K}$ *be a class of forcing notions that is closed under finite support iterations. Suppose that* ZFC *proves that every* $\mathbb{P} \in \mathcal{K}$ *has the ccc and the* γ_1 *property. Let* $\mathbb{P}_{\omega_2} = \left\langle \left\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \right\rangle \,\middle|\, \xi \in \omega_2 \right\rangle$ *be a finite support iteration of forcing notions in* $\mathcal{K}$ *such that each* $\mathbb{P}_\xi$ *forces that* $|\dot{\mathbb{Q}}_\xi| \leq \aleph_1$*. Let* G *be a* $\mathbb{P}_{\omega_2}$*-generic set over* V*. Then, in* $V[G]$*, there exists an ultrafilter* U *on* $\mathcal{P}(\omega)$ *such that* $\mathbb{R}^\omega/U$ *is* β_1*.*

Thanks to this theorem, to prove our main result, it is enough to show that every $\mathrm{EPC}^*_{\aleph_1}$ forcing has the γ_1 property. First, we shall show that any eventual precaliber $\aleph_1$ forcing has the 1-dimensional version of the γ_1 property, which is useful to prove the general version.

Lemma 28. *(*CH*) Let* $E \subset \omega_1$ *be a stationary set. Let* $\mathbb{P}$ *be an* $\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)$ *forcing notion. Let* U *be an ultrafilter on* ω*. In the forcing extension* $V^{\mathbb{P}}$ *with* $\mathbb{P}$*, let* $\mathbb{R}^\omega \cap V \subset \dot{M} \subset \mathbb{R}^\omega$ *and let* $U \subset \dot{\mathcal{F}} \subset \mathcal{P}(\omega)$ *be a* $\dot{M}^\wedge$*-ultrafilter that is countably generated over* U*. Then, every singleton* $\{[\dot{g}]_{\dot{\mathcal{F}}}\} \subset \dot{M}^\wedge/\dot{\mathcal{F}}$ *is* γ_1 *over* $\mathbb{F} := (\mathbb{R}^\omega \cap V)/U$*, that is,* $[\dot{g}]_{\dot{\mathcal{F}}}$ *does not define an uncountable gap.*

Proof. Since CH is assumed in the ground model, it is enough to show that the cofinality of $\{h \in (\mathbb{R}^\omega)\check{} \mid h <_{\dot{\mathcal{F}}} \dot{g}\}$ is not ω_1 in the extension model. Fix any $p \in \mathbb{P}$. Assume that p forces that $\langle \dot{g_\alpha} \in (\mathbb{R}^\omega)\check{} \mid \alpha < \omega_1 \rangle$ is a $<_{\check{U}}$-increasing sequence $<_{\dot{\mathcal{F}}}$-below $\dot{g}$ and $\dot{\mathcal{F}}$ is generated by $\check{U} \cup \{\dot{\tau}_n \mid n \in \omega\}$. It can be assumed that $\{\dot{\tau}_n \mid n \in \omega\}$ is $\subset$-decreasing. We shall show that there exist $h \in \mathbb{R}^\omega$ and $q \leq p$ such that $q \Vdash \forall \alpha \in \omega_1,\ \dot{g}_\alpha <_{\check{U}} \check{h} \leq_{\dot{\mathcal{F}}} \dot{g}$.

Select a sequence $\langle n_\alpha \in \omega, g_\alpha \in \mathbb{R}^\omega, p_\alpha \leq p, \sigma_\alpha \in U \mid \alpha < \omega_1 \rangle$ such that each p_α forces that $\dot{g_\alpha} = \check{g_\alpha}$ and $\dot{g_\alpha}(i) < \dot{g}(i)$ for all $i \in \check{\sigma}_\alpha \cap \dot{\tau}_{n_\alpha}$. Pick $T \in [\omega_1]^{\aleph_1}$,

$\langle \tilde{p}_\alpha \le p_\alpha \mid \alpha \in T\rangle$, and $\langle \delta(\alpha, q) \in \omega_1 \mid \alpha \in T, q \le \tilde{p}_\alpha\rangle$ such that each $\{q, \tilde{p}_\xi \mid \xi \in T \setminus \delta(\alpha, q)\}$ is centered. Shrinking T, we assume that $n_\alpha = n$ for all $\alpha \in T$. Define

$$h(i) = \inf\{\sup\{g_\xi(i) \mid i \in \sigma_\xi, \xi \in T \setminus \alpha\} \mid \alpha \in \omega_1\}.$$

Fix $\alpha \in T$.

Claim 3. $\tilde{p}_\alpha \Vdash \forall i \in \dot{\tau}_n, \check{h}(i) \le \dot{g}(i)$

Proof. Suppose not, and fix $q \le \tilde{p}_\alpha$, $i \in \omega$, and $x \in \mathbb{Q}$ such that $q \Vdash \check{i} \in \dot{\tau}_n \wedge \dot{g}(\check{i}) < \check{x} < \check{h}(\check{i})$. Since $x < h(i)$, there exists $S \in [T]^{\aleph_1}$ such that $i \in \sigma_\xi$ and $x < g_\xi(i)$ for each $\xi \in S$. Pick $\beta \in S \setminus \delta(\alpha, q)$ and $r \le q, \tilde{p}_\beta$. Then, r forces that $\dot{g}_\beta = \check{g}_\beta$, $\dot{g}_\beta(j) < \dot{g}(j)$ for all $j \in \check{\sigma}_\beta \cap \dot{\tau}_n$, $\dot{g}(i) < \check{g}_\beta(i)$, and $i \in \check{\sigma}_\beta \cap \dot{\tau}_n$, which is a contradiction. ☐

Claim 4. $\tilde{p}_\alpha \Vdash \forall \xi < \omega_1 (\dot{g}_\xi <_{\check{U}} \check{h})$

Proof. Otherwise, fix $q \le \tilde{p}_\alpha$ and $\xi \in \omega_1$ such that $q \Vdash \check{h} <_{\check{U}} \dot{g}_\xi$. Pick $\beta \in T \setminus (\xi \cup \delta(\xi, q))$ and a common extension $r \in \mathbb{P}$ of q and $\tilde{p}_\beta$. Then $r \Vdash \check{h} <_U \dot{g}_\xi \le_U \dot{g}_\beta = \check{g}_\beta$. Thus we have $h <_U g_\beta$, and let $\sigma' := \{i \in \omega \mid h(i) < g_\beta(i)\}$. For each $i \in \sigma'$, select $\eta_i \in \omega_1$ such that $\sup\{g_\gamma(i) \mid i \in \sigma_\gamma, \gamma \in T \setminus \eta_i\} < g_\beta(i)$. Pick $\gamma \in T \setminus (\beta \cup \sup_{i \in \sigma'} \eta_i)$. Then for all $i \in \sigma' \cap \sigma_\gamma$, $g_\gamma(i) < g_\beta(i)$. However, since p_β and p_γ are compatible, $g_\beta < g_\gamma$. This is a contradiction. ☐

Thus, $[h]_{\dot{\mathcal{F}}}$ interpolates $\langle [\dot{g_\alpha}]_{\dot{\mathcal{F}}}, [\dot{g}]_{\dot{\mathcal{F}}} \mid \alpha < \omega_1\rangle$. By the same argument, any pair of $[\dot{g}]_{\dot{\mathcal{F}}}$ and a decreasing ω_1-sequence in $\mathbb{F}$ above $[\dot{g}]_{\dot{\mathcal{F}}}$ is interpolated by some $x \in \mathbb{F}$. Thus, $[\dot{g}]_{\dot{\mathcal{F}}}$ is γ_1 over $\mathbb{F}$. ☐

Let us prove the general version. To facilitate our proof, we introduce more notation.

Definition 21. *Let* $\mathbb{F} := \mathbb{R}^\omega/U$. *For each* $A \in \mathrm{Def}^k(\mathbb{F}, F)$ *which is defined by* $\phi(\bar{v}, [\bar{f}]_U)$, *and for each* $i \in \omega$, *let* $A[i] := \{\bar{r} \in \mathbb{R} \mid \mathbb{R} \models \phi(\bar{r}, \bar{f}(i))\}$.

Note 5. *The above notation* $A[i]$ *depends on the choice of a formula* φ *and parameters* $\bar{f}$. *Hence we need to take care that* $A[i]$ *is* not *well-defined, which means that, even if* $A = B$, *it may happen that* $\langle A[i] \mid i < \omega\rangle \ne \langle B[i] \mid i < \omega\rangle$.

Note 6. *If* $A = \{\bar{x} \in (\mathbb{R}^\omega/U)^m \mid \mathbb{R}^\omega/U \models \varphi(\bar{x}, [\bar{f}]_U)\}$ *has a definable property* P *by a single formula, e.g., being cell, bounded, open, function, continuous function, etc., then* $\rho = \{i \in \omega \mid \mathbb{R} \models A[i]$ *has the property* $P\} \in U$. *Let*

$$\bar{\bar{f}}(i) = \begin{cases} \bar{f}(i) & (i \in \rho) \\ \bar{f}(\min \rho) & (i \notin \rho) \end{cases}$$

Then $\tilde{A} = \{\bar{x} \in (\mathbb{R}^\omega/U)^m \mid \mathbb{R}^\omega/U \models \varphi(\bar{x}, [\bar{\bar{f}}]_U)\}$ *is equal to* A *and* $\tilde{A}[i]$ *has the property* P *for every* $i < \omega$.

DEFINITION 22. *Let S be a set and T be an ordered set. For functions $f, g\colon S \to T$, define $[f,g] := \{\langle s,t\rangle \in S \times T \mid f(s) \le t \le g(s)\}$. For any set $S_0 \subset S$ and any binary relation R on T, define $S_0[fRg] := \{s \in S_0 \mid f(s)Rg(s)\}$.*

MAIN LEMMA. *Assume* CH. *Let $\mathbb{P}$ be an* EPC$^*_{\aleph_1}$ *forcing.*

(1) Suppose that $\mathbb{P}$ does not have the γ_1 property. Let U be an ultrafilter on ω and $\mathbb{F} = \mathbb{R}^\omega/U$. In $V^{\mathbb{P}}$, there exist $\{g_1,\dots,g_m\} \subset \mathbb{R}^\omega$ and a $((\mathbb{R}^\omega \cap V) \cup \{g_1,\dots,g_m\})^\wedge$-ultrafilter $\mathcal{F} \supset U$ that is countably generated over U such that, letting $\mathbb{G} = ((\mathbb{R}^\omega \cap V) \cup \{g_1,\dots,g_m\})^\wedge/\mathcal{F}$,

(a) $\mathrm{tp}^{\mathbb{G}}([g_1]_{\mathcal{F}},\dots,[g_m]_{\mathcal{F}}/\mathbb{F})$ is not countably generated in $\mathrm{Def}^m(\mathbb{F},F)$ but

(b) it is generated by a $\subset_U$-decreasing sequence $\langle Z_l \mid l < \omega\rangle$ of sets in $V^\omega_{\omega+\omega}/U$ that is not equal to $[\emptyset]_U$.

(2) Let $E \subset \omega_1$ be a stationary set and assume that $\mathbb{P}$ also has $\mathrm{ProjCes}(E)$. Then $\mathbb{P}$ has the γ_1 property.

Proofs of (1) and (2) of the main lemma are based on ones of [**33**, Theorem 2.] and [**33**, Theorem 1.], respectively. The original proof frequently uses the countability of Cohen forcing and the pigeonhole principle. However, in general, we cannot apply such an argument to EPC$^*_{\aleph_1}$ + ProjCes(E) forcings. Many difficulties come from this difference.

Note that the conjunction (a) and (b) of (1) of the main lemma does not directly imply a contradiction since each Z_l may not be a member of $\mathrm{Def}^m(\mathbb{F},F)$. Woodin [**33**] proposed a question about it.

PROBLEM 1 ([**33**]). *Fix $m \in \omega$. Suppose that M be a model of* ZFC−*Replacement and let $\mathbb{F} = (\mathbb{R})^M = \{r \in M \mid M \models$ "r is a real"$\}$. Suppose $\langle A_k \mid k < \omega\rangle$ is a sequence of elements of M such that:*

(1) $M \models A_{k+1} \subset A_k \subset \mathbb{F}^m$ for each k and

(2) $U = \{A \in \mathrm{Def}^m(\mathbb{F},F) \mid \exists k,\ M \models A_k \subset A\}$ is an ultrafilter.

Must U be countably generated by elements of $\mathrm{Def}^m(\mathbb{F},F)$*?*

If this question is affirmative, (1) of the main lemma implies that $\mathbb{P}$ have the γ_1 property.

PROOF OF THE MAIN LEMMA. **(1):** Fix an EPC$^*_{\aleph_1}$ forcing $\mathbb{P}$ and suppose that $\mathbb{P}$ does not have the γ_1 property. Let U be an ultrafilter on ω and $\mathbb{F} = \mathbb{R}^\omega/U$. We work in $V^{\mathbb{P}}$. Since $\mathbb{P}$ does not have the γ_1 property, there exist M, $\mathcal{F}$, and a $\subset$-decreasing sequence $\langle \tau_n \subset \omega \mid n < \omega\rangle$ such that

(1) $\mathbb{R}^\omega \cap V \subset M \subset \mathbb{R}^\omega$,

(2) $\mathcal{F} \supset U$ is a filter on ω that is generated by $\langle \tau_n \subset \omega \mid n < \omega\rangle$ over U, and

(3) $\mathbb{G} := M^\wedge/\mathcal{F}$ is not γ_1 over $\mathbb{F}$.

Pick a minimal $\{g_1,\dots,g_m\}\subset M$ such that $((\mathbb{R}^\omega\cap V)\cup\{g_1,\dots g_m\})^\wedge/\mathcal{F}$ is not γ_1 over $\mathbb{F}$. We assume that $M=(\mathbb{R}^\omega\cap V)\cup\{g_1,\dots,g_m\}$. By the minimality, for every $X\subsetneq\{g_1,\dots,g_m\}$, $((\mathbb{R}^\omega\cap V)\cup X)^\wedge/\mathcal{F}\neq\mathbb{G}$. Thus $\mathrm{trdeg}(\mathbb{G}/\mathbb{F})=m$ (see Note 4).

Now, define

$$W:=\{A\in\mathrm{Def}^m(\mathbb{F},F)\mid\langle g_1,\dots,g_m\rangle\in_{\mathcal{F}}A\}=\mathrm{tp}^{\mathbb{G}}(\langle[g_1]_{\mathcal{F}},\dots,[g_m]_{\mathcal{F}}\rangle/\mathbb{F})$$

where $\langle g_1,\dots,g_m\rangle\in_{\mathcal{F}}A$ denotes $\{i\in\omega\mid\langle g_1(i),\dots,g_m(i)\rangle\in A[i]\}\in\mathcal{F}$. Then, W is uncountably generated. We shall construct a $\subset_U$-decreasing sequence $\vec{Z}=\langle Z_l\in((V_{\omega+\omega})^\omega\cap V)/U\mid l<\omega\rangle$ that generates the uncountably generated ultrafilter W on $\mathrm{Def}^m(\mathbb{F},F)$. This is the goal of the proof of (i). By Lemma 28, we have $m>1$ and $\mathbb{F}$ is cofinal in $\mathbb{G}$. In particular, by Lemma 20, W is generated by closures of bounded open cells. Let $n=m-1$. We define

$$\begin{aligned}&M_0:=(\mathbb{R}^\omega\cap V)\cup\{g_1,\dots g_n\},\\&\mathbb{G}_0:=M_0^\wedge/\mathcal{F},\ \text{and}\\&W_0:=\{A\in\mathrm{Def}^n(\mathbb{F},F)\mid A\times F\in W\}=\mathrm{tp}^{\mathbb{G}_0}(\langle[g_1]_{\mathcal{F}},\dots,[g_n]_{\mathcal{F}}\rangle/\mathbb{F}),\end{aligned}$$

Then, by the minimality of m and Lemma 24, $\mathbb{G}_0$ is γ_1 over $\mathbb{F}$ and $\mathbb{G}$ is not γ_1 over $\mathbb{G}_0$. Note that W_0 is also generated by closures of bounded open cells. Fix a $\subset$-decreasing sequence $\langle\Theta_k\mid k<\omega\rangle$ of closures of bounded open cells in $\mathrm{Def}^n(\mathbb{F},F)$ that generates W_0. Assume that each Θ_k is defined by $\theta_k(\bar{v},\bar{f}_{\theta_k})$. By Lemmata 18, 22, and 25 and Note 4, g_m defines an uncountable gap in $\mathbb{G}_0$ whose sides have infinite cofinality and coinitiality. By Lemma 13, such a gap can be taken to be the form $\langle\Phi_\alpha([g_1]_{\mathcal{F}},\dots,[g_n]_{\mathcal{F}}),\Psi_\beta([g_1]_{\mathcal{F}},\dots,[g_n]_{\mathcal{F}})\mid\alpha<\kappa,\ \beta<\lambda\rangle$ where each Φ_α and Ψ_β are in $\mathrm{Func}^n(\mathbb{G}_0,F)$ and κ and λ are regular cardinals with at least one of them uncountable. By 4 and 5 in Note 1, we can assume that each Φ_α and each Ψ_β are continuous and bounded on some Θ_k. By Corollary 3, each Φ_α and each Ψ_β can be assumed to be bounded and continuous on G_0^n. By $(\mathrm{CH})^V$, both of κ and λ are less than or equal to $\aleph_1$. We shall assume that $\kappa=\omega$ and $\lambda=\omega_1$. Even if the type of the gap is $\langle\omega_1,\omega\rangle$ or $\langle\omega_1,\omega_1\rangle$, the following argument works. For each $k<\omega$ and $\bar{x}$, define $\Phi'_k(\bar{x})=\max_{l\le k}\Phi_l(\bar{x})$ and then $\Phi_k([g_1]_{\mathcal{F}},\dots,[g_n]_{\mathcal{F}})\le\Phi'_k([g_1]_{\mathcal{F}},\dots,[g_n]_{\mathcal{F}})<[g_m]_{\mathcal{F}}$ for each $k<\omega$ and Φ'_k is continuous, bounded, and definable over F. Thus we can assume that $\langle\Phi_k(\bar{x})\mid k<\omega\rangle$ is a non-decreasing sequence for each $\bar{x}$ and hence $\langle\{\langle\bar{x},y\rangle\mid\bar{x}\in\Theta_k\wedge\Phi_k(\bar{x})\le y\}\mid k\in\omega\rangle$ is a $\subset$-decreasing sequence. This guarantees that $\langle Z_l\mid l<\omega\rangle$ will be $\subset$-decreasing. Note that, for each $C\in W_0$, $k<\omega$ and $\alpha<\omega_1$, there exists $k'<\omega$ and $\alpha'<\omega_1$ such that $(\Theta_{k'}\times F)\cap[\Phi_{k'},\Psi_{\alpha'}]\subset(C\times F)\cap(\Phi_k,\Psi_\alpha)$ and hence, by Lemma 17, $\langle(\Theta_k\times F)\cap[\Phi_k,\Psi_\alpha]\mid k\in\omega,\ \alpha\in\omega_1\rangle$ generates W. Notice that $\langle\Theta_k[i],\Phi_k[i]\restriction V,\Psi_\alpha[i]\restriction V\mid i<\omega\rangle$ is in V for each k and each α. By Note 6, we can assume that, for every $i\in\omega$, $\Theta_k[i]$ is the closure of an open cell, $\Phi_k[i]$ and $\Psi_\alpha[i]$ are bounded and countinuous functions while keeping them definable over F. For each $\alpha \in \omega_1$, pick $N_\alpha \in \omega$ and $\sigma_\alpha \in U$ such that $\tau_{N_\alpha} \cap \sigma_\alpha \subset \{i \in \omega \mid g_m(i) < \Psi_\alpha[i](g_1(i), \ldots, g_n(i))\}$ and pick the definition $\psi_\alpha(\bar{v}, w, \bar{f}_{\psi_\alpha})$ of $\dot{\Psi}_\alpha$.

From now on, we work in the ground model until the end of the proof of the main lemma. Let us assume that the results above are forced by $p \in \mathbb{P}$ and we get $\mathbb{P}$-names $\dot{M}$, $\dot{\mathcal{F}}$, $\dot{g}_1 \ldots \dot{g}_m$, $\dot{W}$, etc. Fix $p_\alpha \leq p$, $N_\alpha \in \omega$, $\sigma_\alpha \in U$, $\Psi^V_\alpha \in \mathrm{Func}^n(F, F)$, and $\psi_\alpha(\bar{v}, w, \bar{f}_{\psi_\alpha}) \in \mathsf{form}_{\mathcal{L}_{\mathsf{OR}}(\mathbb{F})}$ for each $\alpha < \omega_1$ such that

$$p_\alpha \Vdash \dot{\sigma}_\alpha = \check{\sigma}_\alpha\ ,\ \dot{N}_\alpha = \check{N}_\alpha,\ \check{\Psi}^V_\alpha = \dot{\Psi}_\alpha \restriction V,\ \text{and}\ \check{\psi}_\alpha(\bar{v}, w, \check{\bar{f}}_{\psi_\alpha}) = \dot{\psi}_\alpha(\bar{v}, w, \dot{\bar{f}}_{\psi_\alpha}).$$

Fix $T \in [\omega_1]^{\aleph_1}$ and an eventually centered $\langle \tilde{p}_\alpha \mid \alpha \in T \rangle$ such that $\tilde{p}_\alpha \leq p_\alpha$ for each α. Shrinking T, we assume that $N = N_\alpha$ for each $\alpha \in T$. Fix $\alpha_0 \in T$. For each $\alpha \in T$ and $l \in \omega$, let $\dot{X}^\alpha_l$ be a $\mathbb{P}$-name for a function on ω such that

$$\tilde{p}_{\alpha_0} \Vdash \forall i \in \check{\sigma}_\alpha,\ \dot{X}^\alpha_l(i) = (\dot{\Theta}_l[i] \times \check{\mathbb{R}}) \cap [\dot{\Phi}_l[i], \check{\Psi}^V_\alpha[i]].$$

Note that

$$(\star) \qquad \tilde{p}_{\alpha_0} \Vdash \forall l \in \omega,\ \left\langle \dot{X}^\alpha_l(i) \;\middle|\; \alpha \in \check{T}, i \in \check{\sigma}_\alpha \right\rangle \in V.$$

Claim 5.

$$\tilde{p}_{\alpha_0} \Vdash \forall l < \omega, \left\{ i \in \omega \;\middle|\; \bigcup \left\{ \bigcap \left\{ \dot{X}^\alpha_l(i) \;\middle|\; \alpha \in \check{T} \setminus \beta, i \in \check{\sigma}_\alpha \right\} \;\middle|\; \beta \in \check{T} \right\} \neq \emptyset \right\} \in \check{U}.$$

Proof. Suppose not. By $(\star)$, we take a condition $q_0 \leq \tilde{p}_{\alpha_0}$ and a natural number $l < \omega$ such that

$$q_0 \Vdash \left\{ i \in \omega \;\middle|\; \bigcup \left\{ \bigcap \left\{ \dot{X}^\alpha_l(i) \;\middle|\; \alpha \in \check{T} \setminus \beta, i \in \check{\sigma}_\alpha \right\} \;\middle|\; \beta \in \check{T} \right\} = \emptyset \right\} \in \check{U}.$$

In particular, the above set meets $\{i \in \dot{\tau}_N \mid \dot{\Phi}_l[i](\dot{g}_1(i), \ldots, \dot{g}_n(i)) \leq \dot{g}_m(i), \langle \dot{g}_1(i), \ldots \dot{g}_n(i) \rangle \in \dot{\Theta}_l[i]\}$ in the extension by q_0.

Thus we get a condition $q_1 \leq q_0$, a natural number $i \in \omega$, tuples of functions $\bar{f}_\phi, \bar{f}_\theta \in \mathbb{R}^\omega$, and formulas ϕ, θ such that q_1 forces that:

(1) $\check{\phi}(\bar{v}, w, \check{\bar{f}}_\phi)$ defines $\dot{\Phi}_l$,
(2) $\check{\theta}(\bar{v}, \check{\bar{f}}_\theta)$ defines $\dot{\Theta}_l$,
(3) $i \in \dot{\tau}_N$, so we have $\dot{g}_m(i) \leq \dot{\Psi}_\alpha[i](\dot{g}_1(i), \ldots, \dot{g}_n(i))$ whenever $\alpha \in \check{T}$ and $i \in \dot{\sigma}_\alpha$,
(4) $\dot{\Phi}_l[i](\dot{g}_1(i), \ldots, \dot{g}_n(i)) \leq \dot{g}_m(i)$,
(5) $\langle \dot{g}_1(i), \ldots \dot{g}_n(i) \rangle \in \dot{\Theta}_l[i]$, and
(6) $\bigcap \left\{ \dot{X}^\alpha_l(i) \;\middle|\; \alpha \in \check{T} \setminus \beta, i \in \check{\sigma}_\alpha \right\} = \emptyset$ for all $\beta \in \check{T}$.

By 1 to 5, for every $r \leq q_1$, we have

$$r \Vdash \mathbb{R} \models \exists \bar{v} \exists w \exists u \left(\check{\phi}(\bar{v}, w, \check{\bar{f}}_\phi(i)) \wedge \check{\theta}(\bar{v}, \check{\bar{f}}_\theta(i)) \wedge \bigwedge_{\alpha \in \check{A}} \exists w_\alpha \left(\dot{\psi}_\alpha(\bar{v}, w_\alpha, \dot{\bar{f}}_{\psi_\alpha}(i)) \wedge w \leq u \leq w_\alpha \right) \right)$$

for all $A \in [T]^{<\omega}$ with $r \Vdash \forall\alpha \in \check{A}(i \in \dot{\sigma}_\alpha)$. Note that this $\mathbb{R}$ is the set of reals defined in the extension. Let us find a finite subset B of T and an extension q_2 of q_1 which are counter-examples to the above.

Let $\Phi \in \mathrm{Func}^n(\mathbb{R},\mathbb{R})$ be a function defined by $\phi(\bar{v}, w, \bar{f}_\phi(i))$ and let $\Theta \in \mathrm{Def}^n(\mathbb{R},\mathbb{R})$ be a set defined by $\theta(\bar{v}, \bar{f}_\theta(i))$. Define $X^\alpha := \left\{\langle \bar{r}, s\rangle \in \mathbb{R}^m \;\middle|\; \bar{r} \in \Theta \wedge \Phi(\bar{r}) \leq s \leq \Psi^V_\alpha[i](\bar{r})\right\}$ for each $\alpha \in T$ with $i \in \sigma_\alpha$. Note that $q_1 \Vdash \check{\Phi} = \dot{\Phi}_l[i]\restriction V \wedge \check{\Theta} = \dot{\Theta}_l[i] \wedge \forall\alpha \in \check{T}(i \in \check{\sigma}_\alpha \to \dot{X}^\alpha_l(i) = \check{X}^\alpha)$.

Fix $\beta \in T \setminus \delta(\alpha_0, q_1)$. Then, $\bigcap\{X^\alpha \mid \alpha \in T \setminus \beta, i \in \sigma_\alpha\} = \emptyset$. For each $\alpha \in T \setminus \beta$, q_1 and $\tilde{p}_\alpha$ are compatible, and hence X^α is a bounded closed cell. Fix $\gamma_0 \in T \setminus \beta$ such that $i \in \sigma_{\gamma_0}$. Since $\bigcap\{X^\alpha \cap X^{\gamma_0} \mid \alpha \in T \setminus \beta, i \in \sigma_\alpha\} = \emptyset$ and X^{γ_0} is a compact space, we can select $A \in [T \setminus \beta]^{<\omega}$ such that $\bigcap_{\gamma\in A} X^\gamma \cap X^{\gamma_0} = \emptyset$ and $i \in \sigma_\gamma$ for all $\gamma \in A$. Set $B = A \cup \{\gamma_0\}$. Thus $\bigcap_{\gamma\in B}(\Theta \times F) \cap [\Phi, \Psi^V_\gamma[i]] = \emptyset$. Pick a common extension q_2 of $\{q_1, \tilde{p}_\gamma \mid \gamma \in B\}$. Now,

$$\mathbb{R} \models \neg\exists\bar{v}\exists w\exists u\left(\phi(\bar{v}, w, \bar{f}_\phi(i)) \wedge \theta(\bar{v}, \bar{f}_\theta(i)) \wedge \bigwedge_{\gamma\in B} \exists w_\gamma\left(\psi_\gamma(\bar{v}, w_\gamma, \bar{f}_{\psi_\gamma}(i)) \wedge w \leq u \leq w_\gamma\right)\right).$$

By the absoluteness of "$\models$" and by the completeness of RCF,

$$q_2 \Vdash \mathbb{R} \models \neg\exists\bar{v}\exists w\exists u\left(\check{\phi}(\bar{v}, w, \check{\bar{f}}_\phi(i)) \wedge \check{\theta}(\bar{v}, \check{\bar{f}}_\theta(i)) \wedge \bigwedge_{\gamma\in B} \exists w_\gamma\left(\check{\psi}_\gamma(\bar{v}, w_\gamma, \check{\bar{f}}_{\psi_\gamma}(i)) \wedge w \leq u \leq w_\gamma\right)\right).$$

However, q_2 forces that $\check{\sigma}_\gamma = \dot{\sigma}_\gamma$ and $\check{\psi}_\gamma(\bar{v}, w, \check{\bar{f}}_{\psi_\gamma}) = \dot{\psi}_\gamma(\bar{v}, w, \dot{\bar{f}}_{\psi_\gamma})$ whenever $\gamma \in B$. Thus we have

$$q_2 \Vdash \mathbb{R} \models \neg\exists\bar{v}\exists w\exists u\left(\check{\phi}(\bar{v}, w, \check{\bar{f}}_\phi(i)) \wedge \check{\theta}(\bar{v}, \check{\bar{f}}_\theta(i)) \wedge \bigwedge_{\gamma\in B} \exists w_\gamma\left(\dot{\psi}_\gamma(\bar{v}, w_\gamma, \dot{\bar{f}}_{\psi_\gamma}(i)) \wedge w \leq u \leq w_\gamma\right)\right)$$

and $q_2 \Vdash \forall\gamma \in \check{B}(i \in \dot{\sigma}_\gamma)$, which is a contradiction. □

For each $l, i \in \omega$, fix names $\dot{Z}_{l,i}$ and $\dot{Z}_l$ such that

$$\tilde{p}_{\alpha_0} \Vdash \dot{Z}_{l,i} := \bigcup\left\{\bigcap\left\{\dot{X}^\alpha_l(i) \;\middle|\; \alpha \in \check{T} \setminus \beta, i \in \check{\sigma}_\alpha\right\} \;\middle|\; \beta \in \check{T}\right\},\ \textit{and}$$
$$\tilde{p}_{\alpha_0} \Vdash \dot{Z}_l = \left[\left\langle \dot{Z}_{l,i} \;\middle|\; i \in \omega\right\rangle\right]_{\check{U}} \in (V^\omega_{\omega+\omega}/U)\check{}.$$

Then, $\tilde{p}_{\alpha_0} \Vdash \emptyset \neq \dot{Z}_l \subset \mathbb{F}^m$, which means $\tilde{p}_{\alpha_0} \Vdash \exists f \in (V^\omega_{\omega+\omega}),\ [f]_{\check{U}} \in \dot{Z}_l$ and $\tilde{p}_{\alpha_0} \Vdash \forall[f]_{\check{U}} \in \dot{Z}_l,\ \{i \in \omega \mid f(i) \in \check{\mathbb{R}}\} \in \check{U}$. Furthermore, $\tilde{p}_{\alpha_0} \Vdash$ "$\left\langle \dot{Z}_l \;\middle|\; l < \omega\right\rangle$ is decreasing" by the selection of $\left\langle \dot{\Theta}_l, \dot{\Phi}_l \;\middle|\; l < \omega\right\rangle$. Let $\mathcal{N} = V^\omega_{\omega+\omega}/U$.

NOTE 7. *Since* $\tilde{p}_{\alpha_0} \Vdash$ " $\exists x \in \check{\mathcal{N}} \left(\check{\mathcal{N}} \models \bigwedge_{l \leq L} x \in \dot{Z}_l\right)$ *for each* $L < \omega$", *a set of formulae* $\{$"$x \in \dot{Z}_l$" $\mid l < \omega\}$ *is a countable type of* $\mathcal{N}$ *in the extension by* $\tilde{p}_{\alpha_0}$.

NOTE 8. *In the extension by* $\tilde{p}_{\alpha_0}$, *the following are equivalent.*

(P) $\left\{i \in \omega \mid \langle \bar{f}(i), g(i) \rangle \in \dot{Z}_{l,i}\right\}$ *is a member of* $\check{U}$

(Q) $\left\{i \in \omega \mid \forall^\infty \alpha \in \check{T} \left(i \in \check{\sigma}_\alpha \to \bar{f}(i) \in \dot{\Theta}_l[i] \wedge \dot{\Phi}_l[i](\bar{f}(i)) \leq g(i) \leq \check{\Psi}^V_\alpha[i](\bar{f}(i))\right)\right\}$ *is a member of* $\check{U}$.

(R) The intersection of

- $\left\{i \in \omega \mid \bar{f}(i) \in \dot{\Theta}_l[i],\ \dot{\Phi}_l[i](\bar{f}(i)) \leq g(i)\right\}$ *and*
- $\left\{i \in \omega \mid \forall^\infty \alpha \in \check{T} \left(i \in \check{\sigma}_\alpha \implies g(i) \leq \check{\Psi}^V_\alpha[i](\bar{f}(i))\right)\right\}$

is a member of $\check{U}$.

The equivalence (P) $\iff$ *(Q) is by the definition of* $\dot{Z}_{l,i}$ *and the implication (R)* $\implies$ *(Q) is clear. For the implication (Q)* $\implies$ *(R), we assume that the second statement holds. For* i *in the set stated in (Q), select* $\beta_i \in \check{T}$ *which witnesses* $\forall^\infty \alpha \in \check{T}(\cdots)$. *Let* $\beta \in \check{T} \setminus \sup_i \beta_i$. *Then,* $\check{\sigma}_\beta \subset \left\{i \in \omega \mid \bar{f}(i) \in \dot{\Theta}_l[i],\ \dot{\Phi}_l[i](\bar{f}(i)) \leq g(i)\right\}$. *Thus the third statement holds.*

Let us define

$$\Psi(i)(\bar{r}) = \sup\{\inf\{\Psi^V_\alpha[i](\bar{r}) \mid \alpha \in T \setminus \beta, i \in \sigma_\alpha\} \mid \beta \in T\} \in \mathbb{R} \cup \{+\infty, -\infty\}$$

and, for each $\bar{f} \in (\mathbb{R}^\omega)^n$, let $\Psi(\bar{f})$ denote the function on ω such that $\Psi(\bar{f})(i) = \Psi(i)(\bar{f}(i))$ for each $i \in \omega$.

NOTE 9. *The sequence* $\left\langle s^{\bar{r}}_\beta \;\middle|\; \beta \in T\right\rangle := \left\langle \inf\{\Psi^V_\alpha[i](\bar{r}) \mid \alpha \in T \setminus \beta, i \in \sigma_\alpha\} \;\middle|\; \beta \in T\right\rangle$ *is non-decreasing for each* $\bar{r} \in \mathbb{R}^n$. *Since* $\mathbb{R} \cup \{+\infty, -\infty\}$ *has the countable chain condition, the sequence stops at some point in* $\mathbb{R} \cup \{-\infty, +\infty\}$. *In particular,* $x \leq \sup_\beta s^{\bar{r}}_\beta$ *if and only if* $\exists \beta \in T (x \leq s^{\bar{r}}_\beta)$.

Notes 8 and 9 imply that

$$\tilde{p}_{\alpha_0} \Vdash \forall l \in \omega,\ \dot{Z}_l = \{\langle [\bar{f}]_{\check{U}}, [g]_{\check{U}} \rangle \in \check{\mathbb{F}}^m \mid\ [\bar{f}]_{\check{U}} \in \dot{\Theta}_l \wedge \dot{\Phi}_l([\bar{f}]_{\check{U}}) \leq [g]_{\check{U}} \leq [\check{\Psi}(\bar{f})]_{\check{U}}\}.$$

Remark that each $\dot{Z}_l$ may not be definable with parameters in F.

CLAIM 6.

$$\tilde{p}_{\alpha_0} \Vdash \exists^\infty \alpha \in \check{T}, \exists l \in \omega,\ \forall [\bar{f}]_{\check{U}} \in \dot{\Theta}_l,\ [\check{\Psi}(\bar{f})]_{\check{U}} \leq \dot{\Psi}_\alpha \restriction V([\bar{f}]_{\check{U}}).$$

PROOF. Towards a contradiction, let us assume that there exist $q_0 \leq \tilde{p}_{\alpha_0}$ and $\beta \in T$ such that

$$q_0 \Vdash \forall \alpha \in \check{T} \setminus \check{\beta}, \forall l \in \omega,\ \exists [\bar{f}]_{\check{U}} \in \dot{\Theta}_l,\ \dot{\Psi}_\alpha \restriction V([\bar{f}]_{\check{U}}) < [\check{\Psi}(\bar{f})]_{\check{U}}.$$

Consider the statement $\dot\Psi_\alpha\restriction V([\bar f]_{\check U}) < [\check\Psi(\bar f)]_{\check U}$ in the extension by q_0. By the definition of Ψ, this implies that

$$\left\{ i\in\omega \;\middle|\; \exists\gamma\in\check T,\ \forall\xi\in\check T\setminus\gamma,\ \left(i\in\check\sigma_\xi \implies \dot\Psi_\alpha[i]\restriction V(\bar f(i)) < \check\Psi^V_\xi[i](\bar f(i))\right)\right\}\in\check U.$$

Recall that $\dot\Psi_\alpha\restriction V$ and $\left\langle \check\Psi^V_\xi, \check\sigma_\xi \;\middle|\; \xi\in\check T\right\rangle$ are in V. Taking the complement,

$$\bigcap_{\gamma\in\check T}\bigcup_{\xi\in\check T\setminus\gamma}\{i\in\check\sigma_\xi \mid \check\Psi^V_\xi(i)(\bar f(i)) \le \dot\Psi_\alpha[i]\restriction V(\bar f(i))\}\notin\check U.$$

If $\{i\in\omega \mid \check\Psi^V_\xi[i](\bar f(i)) \le \dot\Psi_\alpha[i]\restriction V(\bar f(i))\}\in\check U$ for uncountably many $\xi\in\check T$, then the set above is in $\check U$. Thus we have $\dot\Psi_\alpha\restriction V([\bar f]_{\check U}) < [\check\Psi(\bar f)]_{\check U} \implies \forall^\infty\xi\in\check T,\ \dot\Psi_\alpha\restriction V([\bar f]_{\check U}) < \check\Psi^V_\xi([\bar f]_{\check U})$ and hence

$$q_0\Vdash\forall\alpha\in\check T\setminus\check\beta, \forall l\in\omega,\ \exists[\bar f]_{\check U}\in\dot\Theta_l,\ \exists\gamma\in\check T,\ \forall\xi\in\check T\setminus\gamma, \dot\Psi_\alpha\restriction V([\bar f]_{\check U}) < \check\Psi^V_\xi([\bar f]_{\check U}).$$

Fix $\alpha\in T\setminus\beta$. Then there exist $\gamma\in T$ and $q_1\le q_0$ such that

$$q_1\Vdash\forall l\in\omega,\ \exists[\bar f]_{\check U}\in\dot\Theta_l,\ \forall\xi\in\check T\setminus\check\gamma,\ \dot\Psi_\alpha\restriction V([\bar f]_{\check U}) < \check\Psi^V_\xi([\bar f]_{\check U}).$$

Fix $\xi\in T\setminus(\delta(\alpha_0,q_1)\cup\alpha\cup\gamma)$ and a common extension $r\in\mathbb P$ of q_1 and $\tilde p_\xi$. Then,

$$r\Vdash\forall l\in\omega,\ \exists[\bar f]_{\check U}\in\dot\Theta_l,\ \dot\Psi_\alpha\restriction V([\bar f]_{\check U}) < \check\Psi^V_\xi([\bar f]_{\check U}) \quad \textit{and}$$
$$r\Vdash\exists l\in\omega,\ \forall[\bar f]\in\dot\Theta_l,\ \check\Psi^V_\xi([\bar f]_{\check U}) = \dot\Psi_\xi\restriction V([\bar f]_{\check U}) < \dot\Psi_\alpha\restriction V([\bar f]_{\check U}),$$

which is a contradiction. □

Claim 7. $\tilde p_{\alpha_0}\Vdash$ "$\left\langle \dot Z_l \;\middle|\; l<\omega\right\rangle$ *generates* $\dot W$".

Proof. In the forcing extension by $\tilde p_{\alpha_0}$, fix any $\alpha\in\omega_1$ and $k\in\omega$. Then, by Claim 6, there exist $\beta\in T\setminus\alpha$ and $k_0\in\omega\setminus k$ such that $\forall[\bar f]_{\check U}\in\dot\Theta_{k_0},\ [\check\Psi(\bar f)]_{\check U}\le\dot\Psi_\beta\restriction V([\bar f]_{\check U})\le\dot\Psi_\alpha\restriction V([\bar f]_{\check U})$. Then, $\dot Z_{k_0}\subset(\dot\Theta_k\times F)\cap[\dot\Phi_k,\dot\Psi_\alpha]$. □

$\dot{\vec Z} = \left\langle \dot Z_l \;\middle|\; l<\omega\right\rangle$ has been constructed in the forcing extension by $\tilde p_{\alpha_0}$. (i) has been shown.

(2): Assuming that $\mathbb P$ also has $\mathrm{ProjCes}(E)$, we continue the argument from the proof of (1). Our goal is to show a contradiction. Define $D_l := \{p\le\tilde p_{\alpha_0} \mid p$ decides $\dot Z_l$, $\dot\Theta_l$, and $\dot\Phi_l\}$ for each $l\in\omega$. Since $\tilde p_{\alpha_0}\Vdash$ "$\dot g_m\notin V$", there are no atoms below $\tilde p_{\alpha_0}$. Pick a maximal anti-chain $A_l\subset D_l$ with $A_{l+1}\subset A_l\!\downarrow\, := \{q\in\mathbb P \mid \exists r\in A_l,\ q\le r\wedge q\ne r\}$ for each $l\in\omega$.

Recall that $\mathcal N = V^\omega_{\omega+\omega}/U$. Choose a sequence $\langle Z,\Theta,\Phi\rangle : \bigcup_{l<\omega}A_l \to \mathcal N\times \mathrm{Def}^n(\mathbb F,F)\times\mathrm{Func}^n(\mathbb F,F)$ such that

$$r\Vdash\check Z(r)=\dot Z_l\wedge\check\Theta(r)=\dot\Theta_l\wedge\check\Phi(r)=\dot\Phi_l\restriction V$$

for each $l \in \omega$ and each $r \in A_l$. Fix a large enough regular cardinal κ and a sequence of countable elementary submodels $X \prec Y_0 \prec Y_1 \prec \cdots \prec Y_{j_0+1} \prec H_\kappa$ such that

- X has U, $\mathbb{P}$, $\tilde{p}_{\alpha_0}$, $\mathcal{N}$, Z, Φ, Θ, Ψ, $\langle \tilde{p}_\xi \mid \xi \in T \rangle$, $\langle A_l \mid l < \omega \rangle$, $\left\langle \Psi^V_\alpha, \sigma_\alpha \,\middle|\, \alpha \in T \right\rangle$, and the sequences of names $\left\langle \dot{\Theta}_l, \dot{\Phi}_l, \dot{Z}_l \,\middle|\, l < \omega \right\rangle$ and $\left\langle \dot{\Psi}_\alpha, \dot{\sigma}_\alpha \,\middle|\, \alpha \in \omega_1 \right\rangle$ as elements,
- $\bigcup_{l<\omega} A_l \subset X$,
- $E \in Y_0$ and $\omega_1 \cap Y_0 \in E$,
- $X \in Y_0$, so $\alpha_X := \min(T \setminus X) \in Y_0$ and $S \in Y_0$ for some finite set $S \subset \mathbb{F}$ such that $\Psi^V_{\alpha_X} \in \mathrm{Func}^n(\mathbb{F}, S)$,
- $j_0 = |S|$,
- each Y_{j+1} has Y_j and a Y_j-generic filter d_j for $\mathbb{P}$ as elements, and
- each d_j has $\tilde{p}_{\alpha_0}$ and $\tilde{p}_{\alpha_X}$ as elements.

Since $\mathbb{P}$ has a $\mathrm{ces}(E)$ projection, $\mathbb{P}$ projects into Y_0. Thus, by Lemma 6, $\Pi_{i<j} d_i \cap Y_0 \subset \mathbb{P}^j \cap Y_0$ is a generic filter over Y_0. [1] Set

- $\mathbb{F}_X = \mathbb{F} \cap X$,
- $\mathbb{F}_j = \mathbb{F} \cap Y_j$ for each $j \leq j_0 + 1$,
- $r_{j,l} \in d_j \cap A_l$ for each $j \leq j_0$ and each $l \in \omega$
- $T_j = T \cap Y_j$ for each $j \leq j_0 + 1$,
- $S^d = \{\xi \in S \mid \tilde{p}_\xi \in d\}$ for each $S \subset \omega_1$ and each $d \subset \mathbb{P}$, and
- $Z^V(r;\xi) = (\Theta(r) \times F) \cap [\Phi(r), \Psi^V_\xi]$ (note that $Z^V \in X$).

Note that, for each for each $j \leq j_0$, each $N \in \{X, Y_0, \ldots, Y_j\}$, and each $S \in [T]^{\aleph_1} \cap N$, $S^{d_j} \cap N$ is cofinal in $\omega_1 \cap N$. Indeed, for each $\alpha \in \omega_1 \cap N$, it can be shown that $D = \{q \leq \tilde{p}_{\alpha_0} \mid \exists \xi \in S \setminus \alpha,\, q \Vdash \text{“}\tilde{p}_\xi \in \dot{G}_{\mathbb{P}}\text{”}\} \in N$ is dense as in Lemma 4. We pick $r \in d_j \cap N$ (for $r \in N$, use ccc) and $\xi \in S \setminus \alpha$ such that $r \Vdash \text{“}\tilde{p}_\xi \in \dot{G}_{\mathbb{P}}\text{”}$. Thus $\{q \in \mathbb{P} \mid q \leq \tilde{p}_\xi\} \in N$ is dense below r, $\tilde{p}_\xi \in d_j$ and hence $\xi \in S^{d_j} \setminus \alpha$.

Recall that $m = \mathrm{trdeg}(\mathbb{G}, \mathbb{F})$ and $n = m - 1$. Note that $\mathrm{Def}^m(\mathbb{F}, F) \cap Y_j = \mathrm{Def}^m(\mathbb{F}, F_j)$. Define

$$W_0(j) = \{A \in \mathrm{Def}^n(\mathbb{F}, F_j) \mid \exists l \in \omega,\ \Theta(r_{j,l}) \subset A\} \qquad \textit{and}$$
$$W^{\Pi}(j) = \left\{ C \in \mathrm{Def}^{m(j+1)}(\mathbb{F}, F_0) \,\middle|\, \exists \bar{k}, \exists \xi_j \in T_0^{d_j}, \cdots, \exists \xi_0 \in T_0^{d_0}, \Pi_{i \leq j} Z^V(r_{i,k_i}; \xi_i) \subset C \right\}.$$

Then each $W_0(j)$ is an ultrafilter. Indeed, if $A \in \mathrm{Def}^n(\mathbb{F}, F_j)$, then

$$Y_j \models \text{“ } \tilde{p}_{\alpha_0} \Vdash \text{“ } \exists l < \omega,\, (\dot{\Theta}_l \subset \check{A}) \vee (\dot{\Theta}_l \cap A = \emptyset)\text{””}$$

and hence

$$\{r \leq \tilde{p}_{\alpha_0} \mid \exists l < \omega,\, r \Vdash \text{“}\dot{\Theta}_l \subset \check{A}\text{”} \vee r \Vdash \text{“}\dot{\Theta}_l \cap \check{A} = \emptyset\text{”}\} \in Y_j$$

[1] This is the only use of $\mathrm{ProjCes}(E)$, and it is used only in the last part of this proof. So it is expected that $\mathrm{ProjCes}(E)$ can be omitted from this proof.

is a dense set and hence there exist $l<\omega$ and $r\in d_j$ such that $r\Vdash$ "$\dot\Theta_l\subset\check A$" or $r\Vdash$ "$\dot\Theta_l\cap\check A=\emptyset$". We assume that the former case occurs. Since d_j is a filter and $r,r_{j,l}\in d_j$, there exists a common extension $s\in d_j$ such that $s\Vdash$ "$\dot\Theta_l\subset\check A$ and $\dot\Theta_l=\check\Theta(r_{j,l})$". So $A\in W_0(j)$. By a similar argument, $W^\Pi(0)$ is an ultrafilter and each $W^\Pi(j)$ is a filter. We shall implicitly use argument like the above "dense argument with elementarity".

Since, for each $j\le j_0$, by dense argument with elementarity, $\langle Z(r_{j,l})\mid l<\omega\rangle$ forms a countable type in $\mathcal N$ (see Note 7), hence we select $\bar e:=\langle \bar e_j\mid j\le j_0\rangle=\left\langle \bar b_j^\frown b'_j\;\middle|\; j\le j_0\right\rangle\in\bigcap_{l<\omega}(Z(r_{0,l})\cap Y_1)\times\cdots\times(Z(r_{j_0,l})\cap Y_{j_0+1})$. Note that $\bar e_j\in\mathbb F^m_{j+1}$. Set $\bar c:=\langle\bar c_j\mid j\le j_0\rangle=\left\langle \bar b_j^\frown\Psi^V_{\alpha_X}(\bar b_j)\;\middle|\; j\le j_0\right\rangle$. To show that $W^\Pi(j_0)$ is not an ultrafilter, we first prove the following claim.

Claim 8. *Every $C\in W^\Pi(j)\cap X$ contains $\bar c=\left\langle\left\langle\bar b_i,\Psi^V_{\alpha_X}(\bar b_i)\right\rangle\;\middle|\; i\le j\right\rangle$.*

Proof. Pick $\bar k$ and $\xi_i\in T_0^{d_i}$ such that $\Pi_{i\le j}Z^V(r_{i,k_i};\xi_i)\subset C$. For each $i\le j$, by pigeonhole principle, any common extension $r\in d_i$ of r_{i,k_i} and $\tilde p_{\xi_i}$ forces that

$$\exists k\ge k_i,\ \exists^\infty\xi\in T^{\dot G},\ (\dot\Theta_k\times F)\cap[\dot\Phi_k,\dot\Psi_\xi]\subset Z^V(r_{i,k_i};\xi_i).$$

By dense argument with elementarity, there exist $k'_i\ge k_i$ and $S_i\in[T]^{\aleph_1}\cap Y_0$ such that

$$Y_0\models\text{" }\forall\eta\in S_i,\ Z^V(r_{i,k'_i};\eta)\subset Z^V(r_{i,k_i};\xi_i)\text{ "}.$$

So we have

$$Y_0\models\text{" }\forall\eta_0\in S_0,\cdots,\forall\eta_j\in S_j,\ \Pi_{i\le j}Z^V(r_{i,k'_i};\eta_i)\subset C\text{ "}.$$

By elementarity $X\prec Y_0$, we assume that $\langle S_i\mid i\le j\rangle\in X$. Fix any $i\le j$. Pick $\eta_i\in S_i^{d_i}\cap X$. Then, there exists $s\in d_i$ (not a member of X) that extends r_{i,k'_i}, $\tilde p_{\eta_i}$, and $\tilde p_{\alpha_X}$ such that

$$Y_i\models\text{"}s\Vdash\text{" }\exists k<\omega,\ \dot\Theta_k\subset F[\dot\Phi_{k'_i}<\dot\Psi_{\alpha_X}<\dot\Psi_{\eta_i}]=F[\check\Phi(r_{i,k'_i})<\check\Psi^V_{\alpha_X}<\check\Psi^V_{\eta_i}]\text{ ""}.$$

By elementary dense argument, pick $k<\omega$ such that $\Theta(r_{i,k})\subset F[\Phi(r_{i,k'_i})<\Psi^V_{\alpha_X}<\Psi^V_{\eta_i}]$. Since $\bar b_i\in\Theta(r_{i,k})\cap\Theta(r_{i,k'_i})$, $\bar c_i=\left\langle\bar b_i,\Psi^V_{\alpha_X}(\bar b_i)\right\rangle\in Z^V(r_{i,k'_i};\eta_i)$. Therefore, $\bar c\in C$. □

Claim 9. *$W^\Pi(j_0)$ is not an ultrafilter on* $\mathrm{Def}^{m(j_0+1)}(\mathbb F,F_0)$.

Proof. Otherwise, $W^\Pi(j_0)\cap X$ is also an ultrafilter on $\mathrm{Def}^{m(j_0+1)}(\mathbb F,F_X)$. By the previous claim, $C\in W^\Pi(j_0)\cap X$ if and only if $\bar c\in C$. It is easily seen that $[H]_{W^\Pi(j_0)\cap X}\mapsto H(\bar c)$ defines an isomorphism $\mathrm{Ult}^{\mathbb F}(\mathbb F_X,W^\Pi(j_0)\cap X)\to R^F(\mathbb F_X(\bar c))$

(for surjectivity, see Lemma 13). Thus $\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_X, W^\Pi(j_0) \cap X) \simeq R^F(\mathbb{F}_X(\bar{c})) \subset R^F(\mathbb{F}_X(\bar{b}, S))$ and hence

$$\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_X, W^\Pi(j_0) \cap X)/\mathbb{F}_X) \leq \mathrm{trdeg}(R^F(\mathbb{F}_X(\bar{b}, S))/\mathbb{F}_X) \leq n(j_0+1) + j_0.$$

However, $W^\Pi(j_0) \cap X$ is generated by open cells, so $\mathrm{trdeg}(\mathrm{Ult}^{\mathbb{F}}(\mathbb{F}_X, W^\Pi(j_0) \cap X)/\mathbb{F}_X) = m(j_0+1)$, which is a contradiction. □

Define $\tilde{\jmath} := \max\{j < j_0 \mid W^\Pi(j) \text{ is an ultrafilter on } \mathrm{Def}^{m(j+1)}(\mathbb{F}, F_X)\}$ and $\bar{a} := \langle \bar{e}_j \mid 0 \leq j \leq \tilde{\jmath}\rangle$. Note that $\bar{a} \in \bigcap W^\Pi(\tilde{\jmath}) \cap Y_{\tilde{\jmath}+1}$ To facilitate our argument, for $A \subset F^{m_A}$, $B \subset F^{m_B}$, and $E \subset F^{m_A+m_E+m_B}$, we define

$$(A)E := \{\bar{y} \in F^{m_E+m_B} \mid A \times \{\bar{y}\} \subset E\} \quad \text{and}$$
$$E(B) := \{\bar{y} \in F^{m_A+m_E} \mid \{\bar{y}\} \times B \subset E\}$$

through the proof. For $\bar{a} \in F^{<\omega}$, we abbreviate $(\bar{a})E = (\{\bar{a}\})E$ and $E(\bar{a}) = E(\{\bar{a}\})$.

Claim 10.

$$\left\langle \left\langle [\Phi(r_{\tilde{\jmath}+1,k})]_{W_0(\tilde{\jmath}+1)}, [\Psi^V_\alpha]_{W_0(\tilde{\jmath}+1)} \right\rangle \,\middle|\, k < \omega,\, \alpha \in T_0^{d_{\tilde{\jmath}+1}} \right\rangle$$

is not a gap in $\mathrm{Ult}(R^F(\mathbb{F}_0(\bar{a})), W_0(\tilde{\jmath}+1))$.

Proof. Toward a contradiction, we assume that it is a gap. We will show that $W^\Pi(\tilde{\jmath}+1)$ is an ultrafilter. Fix any $C \in \mathrm{Def}^{m(\tilde{\jmath}+2)}(\mathbb{F}, F_0)$. Then, $(\bar{a})C \in \mathrm{Def}^m(R^F(\mathbb{F}_0(\bar{a})), F_0)$. By the assumption, the set

$$\{(A \times F) \cap [\Phi(r_{\tilde{\jmath}+1,k}), \Psi^V_\alpha] \mid A \in W_0(\tilde{\jmath}+1),\, k < \omega,\, \alpha \in T_0^{d_{\tilde{\jmath}+1}}\}$$

generates an ultrafilter on $\mathrm{Def}^m(\mathbb{F}, R^F(\mathbb{F}_0(\bar{a})))$ (see Lemma 17). Thus there exist $l \in \omega$ and $\xi_{\tilde{\jmath}+1} \in T_0^{d_{\tilde{\jmath}}}$ such that

$$Z^V(r_{\tilde{\jmath}+1,l}; \xi_{\tilde{\jmath}+1}) \subset (\bar{a})C \quad \textit{or}$$
$$Z^V(r_{\tilde{\jmath}+1,l}; \xi_{\tilde{\jmath}+1}) \subset F^m \setminus (\bar{a})C = (\bar{a})(F^m \setminus C).$$

We assume that the former case occurs and we shall show that $C \in W^\Pi(\tilde{\jmath}+1)$. Since $\bar{a} \in C(Z(r_{\tilde{\jmath}+1,l}; \xi_{\tilde{\jmath}+1})) \in \mathrm{Def}^{m(\tilde{\jmath}+1)}(\mathbb{F}, F_0)$, hence $C(Z(r_{\tilde{\jmath}+1,l}; \xi_{\tilde{\jmath}+1})) \in W^\Pi(\tilde{\jmath})$. Select $k_0, \ldots, k_{\tilde{\jmath}} \in \omega$ and $\bar{\xi} \in \Pi_{i \leq \tilde{\jmath}} T_0^{d_i}$, such that

$$\Pi_{i \leq \tilde{\jmath}} Z^V(r_{i,k_i}; \xi_i) \subset C(Z^V(r_{\tilde{\jmath}+1,l}; \xi_{\tilde{\jmath}+1})).$$

Thus

$$\Pi_{i \leq \tilde{\jmath}+1} Z^V(r_{i,k_i}; \xi_i) \subset C$$

and hence $C \in W^\Pi(\tilde{\jmath}+1)$. □

Fix $\Lambda_0 \in \mathrm{Func}^n(\mathbb{F}, R^{F_0}(\bar{a}))$ such that

$$[\Phi(r_{\tilde{\jmath}+1,k})]_{W_0(\tilde{\jmath}+1)} < [\Lambda_0]_{W_0(\tilde{\jmath}+1)} < [\Psi^V_\alpha]_{W_0(\tilde{\jmath}+1)} \tag{1}$$

for each $k \in \omega$ and each $\alpha \in T_0^{d_{\tilde{\jmath}+1}}$. Let $\lambda_0(\bar{x}, y, \bar{f}_{\lambda_0}(\bar{a}))$ be the definition of Λ_0 where $\bar{f}_{\lambda_0} \in (\mathrm{Func}^{m(\tilde{\jmath}+1)}(\mathbb{F}, F_0))^{<\omega}$. Define $A := \{\bar{z} \in F^{m(\tilde{\jmath}+1)} \mid \mathbb{F} \models \forall \bar{x} \exists ! y \lambda_0(\bar{x}, y, \bar{f}_{\lambda_0}(\bar{z}))\}$ and $f := \{\langle \bar{u}, \bar{x}, y\rangle \mid \mathbb{F} \models (\bar{u} \in A \to (\lambda_0(\bar{x}, y, \bar{f}_{\lambda_0}(\bar{u}))) \wedge (\bar{u} \notin A \to y = 0)\} \in \mathrm{Func}^{m(\tilde{\jmath}+1)+n}(\mathbb{F}, F_0)$. Let $\Lambda(\bar{u})(\bar{x}) = f(\bar{u}, \bar{x})$ and then $\Lambda(\bar{a}) = \Lambda_0$. Note that $\Pi_{i \leq \tilde{\jmath}+1} d_i \cap Y_0$ is a Y_0-generic. Let $\mathbb{P}^{\tilde{\jmath}+2} = \Pi_{j \leq \tilde{\jmath}+1} \mathbb{P}_j$ where each $\mathbb{P}_j = \mathbb{P}$. Let $\iota_j : V^{\mathbb{P}_j} \to V^{\mathbb{P}^{\tilde{\jmath}+2}}$ be the cannonical embedding of names and, for a $\mathbb{P}_j$ name $\dot{x}$, let $\dot{x}^j = \iota_j(\dot{x})$. To show that

$$\bar{r} \Vdash_{\mathbb{P}^{\tilde{\jmath}+2}} \text{“}\forall \alpha \in T^{\dot{G}^{\tilde{\jmath}+1}}, \exists l < \omega, \forall \bar{u} \in \Pi_{i \leq \tilde{\jmath}} \dot{Z}^i_l, \forall \bar{x} \in \dot{\Theta}^{\tilde{\jmath}+1}_l, \Lambda(\bar{u})(\bar{x}) < \dot{\Psi}^{\tilde{\jmath}+1}_\alpha(\bar{x})\text{”}$$

for some $\bar{r} \in \Pi_{i \leq \tilde{\jmath}+2} d_{\tilde{\jmath}+1} \cap Y_0$, we assume that, toward a contradiction,

$$\bar{r} \Vdash_{\mathbb{P}^{\tilde{\jmath}+2}} \text{“}\exists \alpha \in T^{\dot{G}^{\tilde{\jmath}+1}}, \forall l < \omega, \exists \bar{u} \in \Pi_{i \leq \tilde{\jmath}} \dot{Z}^i_l, \exists \bar{x} \in \dot{\Theta}^{\tilde{\jmath}+1}_l, \dot{\Psi}^{\tilde{\jmath}+1}_\alpha(\bar{x}) \leq \Lambda(\bar{u})(\bar{x})\text{”}$$

for some $\bar{r} \in \Pi_{i \leq \tilde{\jmath}+1} d_{\tilde{\jmath}+1} \cap Y_0$. By dense argument with elementarity, pick $\alpha \in T_0^{d_{\tilde{\jmath}+1}}$ and $\bar{r}' \leq \bar{r}$ in $\Pi_{i \leq \tilde{\jmath}+1} d_i \cap Y_0$ such that

$$\bar{r}' \Vdash_{\mathbb{P}^{\tilde{\jmath}+2}} \text{“}\forall l < \omega, \exists \bar{u} \in \Pi_{i \leq \tilde{\jmath}} \dot{Z}^i_l, \exists \bar{x} \in \dot{\Theta}^{\tilde{\jmath}+1}_l, \dot{\Psi}^{\tilde{\jmath}+1}_\alpha(\bar{x}) \leq \Lambda(\bar{u})(\bar{x})\text{”}.$$

On the other hand, since $[\Lambda(\bar{a})]_{W_0(\tilde{\jmath}+1)} < [\Psi^V_\alpha]_{W_0(\tilde{\jmath}+1)}$, there exists $l_0 < \omega$ such that

$$\mathbb{F} \models \forall \bar{x} \in \Theta(r_{\tilde{\jmath}+1,l_0}), \Lambda(\bar{a})(\bar{x}) < \Psi^V_\alpha(\bar{x}).$$

So $\{\bar{u} \in F^{m\tilde{\jmath}} \mid \mathbb{F} \models \forall \bar{x} \in \Theta(r_{\tilde{\jmath}+1,l_0}), \Lambda(\bar{u})(\bar{x}) < \Psi_\alpha(\bar{x})\} \in W^\Pi(\tilde{\jmath})^+ = W^\Pi(\tilde{\jmath})$. Pick $l \geq l_0$ such that

$$\forall \bar{u} \in \Pi_{i \leq \tilde{\jmath}} Z(r_{i,l}),\ \forall \bar{x} \in \Theta(r_{\tilde{\jmath}+1,l}), \Lambda(\bar{u})(\bar{x}) < \Psi^V_\alpha(\bar{x}). \tag{2}$$

Let $s_i \leq r'_i, r_{i,l}$ for $i \leq \tilde{\jmath}+1$. Furthermore, let $s_{\tilde{\jmath}+1}$ extends $\tilde{p}_\alpha$. Then, $\bar{s}$ forces that;

(1) $\exists \bar{u} \in \Pi_{i \leq \tilde{\jmath}} \dot{Z}^i_l, \exists \bar{x} \in \dot{\Theta}^{\tilde{\jmath}+1}_l, \dot{\Psi}_\alpha(\bar{x}) \leq \Lambda(\bar{u})(\bar{x})$,
(2) $\dot{Z}^i_l = Z(r_{i,l})$ for $i \leq \tilde{\jmath}$,
(3) $\dot{\Theta}^{\tilde{\jmath}+1}_l = \Theta(r_{\tilde{\jmath}+1,l})$, and
(4) $\dot{\Psi}_\alpha = \Psi^V_\alpha$,

which contradicts to (2).

Therefore, by pigeonhole principle, there exists $\bar{r} \in \Pi_{i \leq \tilde{\jmath}+1} d_i \cap Y_0$ such that

$$\bar{r} \Vdash_{\mathbb{P}^{\tilde{\jmath}+2}} \text{“}\exists l < \omega, \exists^\infty \alpha \in T^{\dot{G}^{\tilde{\jmath}+1}}, \forall \bar{u} \in \Pi_{i \leq \tilde{\jmath}} \dot{Z}^i_l, \forall \bar{x} \in \dot{\Theta}^{\tilde{\jmath}+1}_l, \Lambda(\bar{u})(\bar{x}) < \dot{\Psi}^{\tilde{\jmath}+1}_\alpha(\bar{x})\text{”}.$$

Since $r_{i,l} \in d_i \cap Y_0$ for $i \leq \tilde{\jmath}+1$,

$$\bar{s} \Vdash_{\mathbb{P}^{\tilde{\jmath}+2}} \text{“}\exists^\infty \alpha \in T^{\dot{G}^{\tilde{\jmath}+1}}, \forall \bar{u} \in \Pi_{i \leq \tilde{\jmath}} Z(r_{i,l}), \forall \bar{x} \in \dot{\Theta}^{\tilde{\jmath}+1}_l, \Lambda(\bar{u})(\bar{x}) < \dot{\Psi}^{\tilde{\jmath}+1}_\alpha(\bar{x})\text{”} \tag{3}$$

for some $\bar{s} \in \Pi_{i\leq\tilde{\jmath}+1} d_{\tilde{\jmath}+1} \cap Y_0$ and some $l < \omega$. By elementarity $Y_0 \prec Y_{\tilde{\jmath}+1}$, the above (3) holds in $Y_{\tilde{\jmath}+1}$. Since $\bar{a} \in \Pi_{i\leq\tilde{\jmath}} Z(r_{i,l})$, we have

$$Y_{\tilde{\jmath}+1} \models \text{“}\bar{s} \Vdash_{\mathbb{P}^{\tilde{\jmath}+2}} \text{“}\exists^\infty \alpha \in T^{\dot{G}^{\tilde{\jmath}+1}},\ \forall \bar{x} \in \dot{\Theta}_l^{\tilde{\jmath}+1},\ \Lambda(\bar{a})(\bar{x}) < \dot{\Psi}_\alpha^{\tilde{\jmath}+1}(\bar{x})\text{””}$$

and hence

$$Y_{\tilde{\jmath}+1} \models \text{“}s_{\tilde{\jmath}+1} \Vdash \text{“}\exists^\infty \alpha \in T^{\dot{G}},\ \forall \bar{x} \in \dot{\Theta}_l,\ \Lambda(\bar{a})(\bar{x}) < \dot{\Psi}_\alpha(\bar{x})\text{””}.$$

Thus, since $\left\langle [\dot{\Psi}_\alpha]_{\dot{W}_0} \;\middle|\; \alpha < \omega_1 \right\rangle$ is a decreasing sequence in the forcing extension,

$$Y_{\tilde{\jmath}+1} \models \text{“}s_{\tilde{\jmath}+1} \Vdash \text{“}\forall \alpha < \omega_1,\ [\Lambda(\bar{a})]_{\dot{W}_0} < [\dot{\Psi}_\alpha]_{\dot{W}_0}\text{””}.$$

On the other hand, some $p \in d_{\tilde{\jmath}+1}$ forces that

$$\forall l < \omega,\ [\dot{\Psi}_l]_{\dot{W}_0} < [\Lambda(\bar{a})]_{\dot{W}_0}.$$

Indeed, if not, for some $l, k < \omega$, some $p \in d_{\tilde{\jmath}+1}$ forces that

(1) $\forall \bar{x} \in \dot{\Theta}_k$, $\Lambda(\bar{a})(\bar{x}) \leq \dot{\Psi}_l(\bar{x})$, that
(2) $\dot{\Theta}_k = \Theta(r_{\tilde{\jmath}+1}, k)$, and that
(3) $\dot{\Phi}_l = \Phi(r_{\tilde{\jmath}+1}, l)$,

which contradicts to (1).

Therefore, in the extension by some $q \leq \tilde{p}_{\alpha_0}$, $\Lambda(\bar{a})([\dot{g}_1]_{\dot{\mathcal{F}}}, \ldots, [\dot{g}_n]_{\dot{\mathcal{F}}})$ interpolates $\left\langle \dot{\Phi}_k([\dot{g}_1]_{\dot{\mathcal{F}}}, \ldots, [\dot{g}_n]_{\dot{\mathcal{F}}}), \dot{\Psi}_\alpha([\dot{g}_1]_{\dot{\mathcal{F}}}, \ldots, [\dot{g}_n]_{\dot{\mathcal{F}}}) \;\middle|\; k < \omega,\ \alpha < \omega_1 \right\rangle$, which is a contradiction. □

By Theorem 7, we have the main theorem as a corollary of the main lemma. By a standard bookkeeping argument, we have the next corollary.

COROLLARY 4. *For any stationary set $E \subset \omega_1$, the following are consistent relative to* ZFC*:*

(1) $\neg\mathrm{NUB} + \mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E) + \text{“}size \leq \aleph_1\text{”}) + \neg\mathrm{CH}$
(2) $\neg\mathrm{NUB} + \mathrm{MA}(\mathrm{EPC}_{\aleph_1} + \mathrm{ProjCes}(E)) + \neg\mathrm{CH}$
(3) $\neg\mathrm{NDH} + \mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E) + \text{“}size \leq \aleph_1\text{”}) + \neg\mathrm{CH}$
(4) $\neg\mathrm{NDH} + \mathrm{MA}(\mathrm{EPC}_{\aleph_1} + \mathrm{ProjCes}(E)) + \neg\mathrm{CH}$

Furthermore, if Woodin's question (see Problem 1) is affirmative, then the condition $\mathrm{ProjCes}(E)$ *can be omitted from the above.*

CHAPTER 3

Examples and preservation properties of EPC

1. An example: Ladder system coloring uniformization

In this section, we introduce an example of a forcing that has both properties $\mathrm{EPC}_{\aleph_1}$ and $\mathrm{ProjCes}(E)$, which is obtained by the uniformization of a ladder system coloring.

DEFINITION 23. *For* $E \subset \omega_1 \cap \mathrm{Lim}$, *a sequence* $\vec{C} := \langle C_\alpha \mid \xi \in E\rangle$ *is* a ladder system on E *if each* C_α *has order type* ω *and its supremum is* α. $\vec{\zeta_\alpha} := \langle \zeta_{\alpha,n} \mid n < \omega\rangle$ *denotes the increasing enumeration of* C_α. *We define* $C_{\alpha,N} := \{\zeta_{\alpha,n} \mid n \geq N\}$ *for each* $\alpha \in E$ *and each* $N \in \omega$. *For* $\nu \leq \omega$, a ν-coloring of a ladder system $\vec{C}$ on E *is a sequence* $\vec{l} = \langle l_\xi \colon C_\xi \to \nu \mid \xi \in E\rangle$. *A* ν*-coloring* $\vec{l}$ *can be uniformized if there exists* $\varphi \colon \omega_1 \to \nu$ *such that* $\{\xi \in C_\alpha \mid \varphi(\xi) \neq l_\alpha(\xi)\}$ *is finite for each* $\alpha \in E$. *A coloring denotes a* ν*-coloring for some* $\nu \leq \omega$.

DEFINITION 24. *Let* E, $\vec{l}$, *and* $\vec{C}$ *be as in Definition 23.* The uniformization of a ladder system coloring $\vec{l}$ on E *is the forcing*

$$\mathbb{P}(\vec{l}) := \left\{ p \colon E \xrightarrow{\text{finite partial}} \omega \;\middle|\; \bigcup_{\xi \in \mathrm{dom}(p)} l_\xi \restriction C_{\xi,p(\xi)} \text{ is a function} \right\}$$

with the relation $q \leq p :\Longleftrightarrow q \supset p$. *For each* $p \in \mathbb{P}(\vec{l})$, *we define* $l^p = \bigcup_{\xi \in \mathrm{dom}(p)} l_\xi \restriction C_{\xi,p(\xi)}$. *Let*

$$\mathrm{ULC}(E) = \{\mathbb{P}(\vec{r}) \mid \vec{r} \text{ is a ladder system } \nu\text{-coloring on } E \text{ for some } \nu \leq \omega\}.$$

ULC was introduced by K. J. Devlin and S. Shelah [**17**], and it is partially motivated by Whitehead's conjecture, which asserts every Whitehead group, which is an abelian group A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$, is a free abelian group.

THEOREM 8 (([**19**, Ch. XIII], [**17**, 5.2 Theorem])).
- *There exists a non-free Whitehead group of size* $\aleph_1$ *if and only if there exists a ladder system* $\vec{C}$ *on some stationary set* $E \subset \omega_1$ *such that every 2-coloring of* $\vec{C}$ *can be uniformized.*

- *For any stationary set $E \subset \omega_1$,* MA(ULC(E)) + ¬CH *implies that every coloring of any ladder system on E can be uniformized.*

Note that $p, q \in \mathbb{P}(\vec{l})$ are compatible if and only if both $p \cup q$ and $l^p \cup l^q$ are functions.

THEOREM 9. *Let $\vec{l}$ be a ladder system coloring on $\omega_1 \cap \mathrm{Lim}$. Then, the uniformization $\mathbb{P}(\vec{l})$ of $\vec{l}$ is $\mathrm{EPC}_{\aleph_1}$, that is, for each sequence $\vec{p} = \left\langle p_\alpha \in \mathbb{P}(\vec{l}) \;\middle|\; \alpha < \omega_1 \right\rangle$, there exists $T \in [\omega_1]^{\aleph_1}$ such that*

$$\forall r \in \mathbb{P}(\vec{l})(\exists \beta \in T(r \parallel p_\beta) \implies \exists \gamma_0 < \omega_1(\{p_\xi \mid \xi \in T \setminus \gamma_0\} \cup \{r\} \text{ is centered.}))$$

To prove this theorem, we use the next lemma.

LEMMA 29. *For any sequence $\vec{K} = \left\langle K_\epsilon \in [\omega_1]^{<\aleph_0} \;\middle|\; \epsilon < \omega_1 \right\rangle$ and any countable model $N \prec H_\kappa$ with $\vec{K} \in N$, there exists $\alpha \in \omega_1 \setminus N$ such that $\omega_1 \cap N \notin K_\alpha$.*

PROOF. Fix $\epsilon \in \omega_1 \setminus N$. Pick $\delta \in \omega_1 \cap N$ with $K_\epsilon \cap N = K_\epsilon \cap \delta$. Note that $K_\epsilon \cap N \in N$. Then, for each $\gamma \in (\omega_1 \setminus \delta) \cap N$, by elementarity, $N \models \exists \alpha \in \omega_1 \setminus \gamma(K_\alpha \cap \gamma = K_\epsilon \cap N)$. So

$$N \models \forall \gamma \in \omega_1 \setminus \delta(\exists \alpha \in \omega_1 \setminus \gamma(K_\alpha \cap \gamma = K_\epsilon \cap N)).$$

Thus we get a strictly increasing sequence $\bar{\gamma} = \langle \gamma_\xi \mid \xi < \omega_1 \rangle \in \omega_1^{\omega_1} \cap N$ such that $\delta < \gamma_0$ and that, for each $\eta < \omega_1$, $K_{\gamma_\eta} \cap \sup_{\xi < \eta} \gamma_\xi = K_\epsilon \cap N$. Then $\left\langle K_{\gamma_\xi} \setminus \delta \;\middle|\; \xi < \omega_1 \right\rangle$ is a $\ll$-increasing sequence (see Definition 13). Therefore, $\omega_1 \cap N \notin K_{\gamma_\xi}$ for some $\xi < \omega_1$. □

Note that $\sup(\mathrm{dom}(l^p)) = \max(\mathrm{dom}(p))$ for each non-empty $p \in \mathbb{P}$. We proceed to the proof of Theorem 9.

PROOF OF THEOREM 9. Let $\vec{p} = \left\langle p_\alpha \in \mathbb{P}(\vec{l}) \;\middle|\; \alpha < \omega_1 \right\rangle$ be a sequence of conditions. Let κ be a large enough regular cardinal and fix a countable elementary submodel $N \prec H_\kappa$ that contains $\vec{p}$, $\vec{C}$, and $\vec{l}$. Fix $\alpha \in \omega_1 \setminus N$ with $\omega_1 \cap N \notin \mathrm{dom}(p_\alpha)$ byLemma 29. Then $p_{\text{fixed}} := p_\alpha \restriction N$ and $l_{\text{fixed}} := l^{p_\alpha \restriction (\omega_1 \setminus N)} \restriction N$ are in N because they are finite sequences of elements of N. Pick $\delta \in \omega_1 \cap N$ greater than $\max(\mathrm{dom}(p_{\text{fixed}}) \cup \mathrm{dom}(l_{\text{fixed}}))$. By a similar argument to the proof of Lemma 29, we have

$$N \models \forall \beta \in \omega_1 \setminus \delta \left(\exists \gamma \in \omega_1 \setminus \beta \left(p_{\text{fixed}} = p_\gamma \restriction \beta \wedge l_{\text{fixed}} = l^{p_\gamma \restriction (\omega_1 \setminus \delta)} \restriction \beta\right)\right).$$

Thus there exists an increasing sequence $\bar{\gamma} = \langle \gamma_\eta \mid \eta < \omega_1 \rangle \in \omega_1^{\omega_1} \cap N$ such that $p_{\text{fixed}} = p_{\gamma_0} \restriction \delta$, $l_{\text{fixed}} = l^{p_{\gamma_0}} \restriction \delta$,

$$p_{\text{fixed}} = p_{\gamma_{\eta'}} \restriction \sup\left(\delta \cup \text{dom}(p_{\gamma_\eta})\right) + 1, \quad \textit{and}$$
$$l_{\text{fixed}} = l^{p_{\gamma_{\eta'}} \restriction (\omega_1 \setminus \delta)} \restriction \sup\left(\delta \cup \text{dom}(p_{\gamma_\eta})\right) + 1$$

for all $\eta < \eta' < \omega_1$. So we get $T \in [\omega_1]^{\aleph_1} \cap N$ such that

(1) $p_{\text{fixed}} = p_\gamma \restriction \delta$ for all $\gamma \in T$,
(2) $l_{\text{fixed}} = l^{p_\gamma \restriction (\omega_1 \setminus \delta)} \restriction \delta$ for all $\gamma \in T$,
(3) $\langle \text{dom}(p_\gamma) \setminus \delta \mid \gamma \in T \rangle$ is $\ll$-increasing, and
(4) $\left\langle \text{dom}(l^{p_\gamma \restriction (\omega_1 \setminus \delta)}) \setminus \delta \;\middle|\; \gamma \in T \right\rangle$ is $\ll$-increasing.

Suppose that $\beta \in T$, $r \in \mathbb{P}(\vec{l})$, and $r \parallel p_\beta$. Then both $r \cup p_\beta$ and $l^r \cup l^{p_\beta}$ are functions. Pick $\gamma(\beta, r) \in T$ such that

$$\text{dom}\,(r) \cup \text{dom}\,(p_\beta) \ll \left(\text{dom}\left(p_{\gamma(\beta,r)}\right) \setminus \delta\right) \cup \left(\text{dom}\left(l^{p_{\gamma(\beta,r)} \restriction (\omega_1 \setminus \delta)}\right) \setminus \delta\right).$$

We shall show that $\{r\} \cup \{p_\beta \mid \beta \in T \setminus \gamma(\beta, r)\}$ is centered. Fix any $\Gamma \in [T \setminus \gamma(\beta, r)]^{<\aleph_0}$. Note that, for each $\gamma \in \Gamma$, the domains of $p_\gamma \restriction (\omega_1 \setminus \delta)$ and $l^{p_\gamma \restriction (\omega_1 \setminus \delta)} \restriction (\omega_1 \setminus \delta)$ are disjoint from the domains of $r \cup p_\beta$ and $l^r \cup l^{p_\beta}$, respectively. Then, both

$$\begin{aligned} r \cup \bigcup_{\gamma \in \Gamma} p_\gamma &= r \cup \bigcup_{\gamma \in \Gamma} p_\gamma \restriction \delta \cup \bigcup_{\gamma \in \Gamma} p_\gamma \restriction (\omega_1 \setminus \delta) \\ &= r \cup p_\beta \restriction \delta \cup \bigcup_{\gamma \in \Gamma} p_\gamma \restriction (\omega_1 \setminus \delta) \end{aligned}$$

and

$$\begin{aligned} l^r \cup \bigcup_{\gamma \in \Gamma} l^{p_\gamma} &= l^r \cup \bigcup_{\gamma \in \Gamma} l^{p_\gamma \restriction \delta} \cup \bigcup_{\gamma \in \Gamma} l^{p_\gamma \restriction (\omega_1 \setminus \delta)} \restriction \delta \cup \bigcup_{\gamma \in \Gamma} l^{p_\gamma \restriction (\omega_1 \setminus \delta)} \restriction (\omega_1 \setminus \delta) \\ &= l^r \cup l^{p_\beta \restriction \delta} \cup l^{p_\beta \restriction (\omega_1 \setminus \delta)} \restriction \delta \cup \bigcup_{\gamma \in \Gamma} l^{p_\gamma \restriction (\omega_1 \setminus \delta)} \restriction (\omega_1 \setminus \delta) \end{aligned}$$

are functions. Therefore $\{p_\xi \mid \xi \in \Gamma\} \cup \{r\}$ has a common extension. □

Corollary 5. *Any uniformization of a ladder system coloring on any* $E \subset \omega_1 \cap \text{Lim}$ *is* $\text{EPC}_{\aleph_1}$.

Proof. Let $\vec{l}$ be a ladder system coloring on E. Let $\vec{r}$ be a ladder system coloring on $\omega_1 \cap \text{Lim}$ that extends $\vec{l}$. Fix any sequence $\vec{p} = \langle p_\alpha \mid \alpha < \omega_1 \rangle$ in $\mathbb{P}(\vec{l})$. Then $\vec{p}$ is a sequence in $\mathbb{P}(\vec{r})$. So there exists $T \in [\omega_1]^{\aleph_1}$ such that $\vec{p} \restriction T$ is eventually centered in $\mathbb{P}(\vec{r})$. For $\Gamma \in [\mathbb{P}(\vec{l})]^{<\aleph_0}$, Γ has common extension iff both $\bigcup \Gamma$ and $\bigcup_{p \in \Gamma} l^p$ are functions. Thus $\vec{p} \restriction T$ is also evenrually centered in $\mathbb{P}(\vec{l})$. □

THEOREM 10. *Let $E \subset \omega_1$ be a stationary set whose complement $E^c = \omega_1 \setminus E$ is also stationary. For any ladder system coloring $\vec{l}$ on E, $\mathbb{P}(\vec{l})$ has a* $\mathrm{ces}(E^c)$ *projection.*

PROOF. Let $\kappa \in \mathrm{Reg}$ be large enough. Suppose that $N \prec H_\kappa$ is countable and $\mathbb{P}, E^c \in N$ and $\omega_1 \cap N \in E^c$. Fix any $p \in \mathbb{P}(\vec{l})$. Then $p \restriction N \in N$ and, since $\omega_1 \cap N \notin E$, $l^p \restriction N \in N$. By elementarity, there exists $p^{(N)} \leq p \restriction N$ in N such that $l^p \restriction N \subset l^{p^{(N)}}$. We shall show that $p^{(N)}$ is a projection of p on N. Fix any $q \leq p^{(N)}$ in N. Then $q, l^q \subset N$, $p \restriction N \subset q$, and $l^p \restriction N \subset l^q$. Thus both $p \cup q$ and $l^p \cup l^q$ are functions. □

So, in conjunction with our Main Theorem, we have the consequence:

COROLLARY 6. $\mathrm{ZFC} + \neg\mathrm{NUB} +$ *Whitehead's conjecture fails* $+ \neg\mathrm{CH}$ *is consistent relative to* ZFC.

2. Preservation of set theoretical objects and cardinal invariants

2.1. Cichoń-Blass diagram. Cichoń-Blass diagram in Figure 1 is the diagram of relationships between cardinal invariants of the continuum. Not all cardinal invariants in the diagram are defined in this paper. We shall define some invariants only when they are necessary.

Cohen forcing raise *the covering number of meager sets* $\mathrm{cov}(\mathcal{M})$, that is the least cardinality of a family of meager sets in the set $\mathbb{R}$ of reals that covers $\mathbb{R}$. Since Cohen forcing is $\mathrm{EPC}_{\aleph_1}$ and $\mathrm{ProjCes}(\omega_1)$, for each stationary set $E \subset \omega_1$, $\mathrm{MA}(\mathrm{EPC}_{\aleph_1} + \mathrm{ProjCes}(E)) + \neg\mathrm{CH}$ implies $\mathrm{cov}(\mathcal{M}) > \aleph_1$ [**6**]. On the other hand, in the Cohen model (an extension by Cohen forcing of a ground model in which CH holds), the least cardinality of mad families $\mathfrak{a}$, the groupwise density number $\mathfrak{g}$, and the uniformity of meager sets $\mathrm{non}(\mathcal{M})$ are $\aleph_1$, see [**9**, Theorem 3.3.22], [**23**, Proposition 6.], and [**11**, Theorem 1.18.]. In this section, we shall define $\mathrm{non}(\mathcal{M})$ and $\mathfrak{a}$ and show that it is consistent that these invariants are $\aleph_1$ with $\mathrm{MA}(\mathrm{EPC}_{\aleph_1}) + \neg\mathrm{CH}$.

2.2. Preservation of the uniformity of meager sets.

DEFINITION 25 (E.g., see Bartoszyński, Judah [**9**]). The uniformity of meager sets $\mathrm{non}(\mathcal{M})$ *is the minimum size of non-meager sets in the set of reals.*

DEFINITION 26 (E.g., see Bartoszyński, Judah [**9**]). *An uncountable set X of reals is* a Luzin set *if it has a countable intersection with every nowhere dense set.*

Note the following:

- The word "every nowhere dense set" in the definition of Luzin sets can be replaced with "every meager set".
- Every Luzin set is non-meager.
- Every uncountable subset of a Luzin set is also a Luzin set.
- Cohen forcing $\mathbb{C}_{\omega_1}$ adds a Luzin set of size $\aleph_1$, See LEMMA 8.2.6 in [**9**].

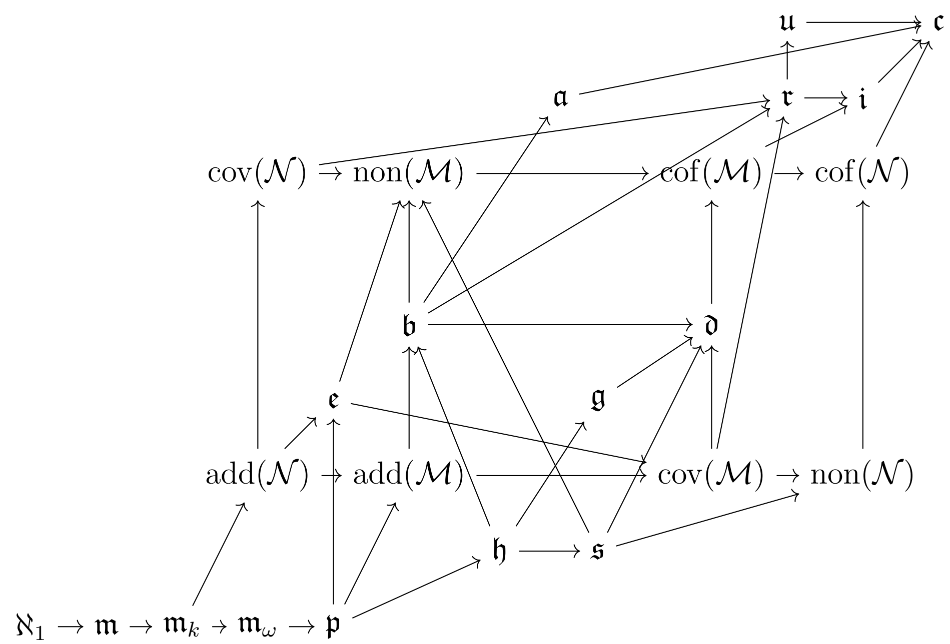


FIGURE 1. Cichoń's diagram and the Blass diagram combined. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$ [**21**].

THEOREM 11. *Every* $\mathrm{EPC}^*_{\aleph_1}$ *forcing preserves Luzin sets of size* $\aleph_1$.

PROOF. Let $\mathbb{P}$ be an $\mathrm{EPC}^*_{\aleph_1}$ forcing notion and $X = \{x_\alpha \mid \alpha < \omega_1\}$ be a Luzin set. Suppose that, toward a contradiction,

$$p \Vdash \text{“}\check{X} \text{ is not a Luzin set”}$$

for some $p \in \mathbb{P}$. Then we can select a name $\dot{F} \in V^{\mathbb{P}}$ for a nowhere dense set that has an uncountable intersection with X in the extension by p. Let $\dot{T} \in V^{\mathbb{P}}$ be a name such that $p \Vdash$ “$\dot{T} = \left\{\alpha \;\middle|\; x_\alpha \in \dot{F}\right\}$”. Define $S_0 = \left\{\xi < \omega_1 \;\middle|\; q \Vdash \text{“}\check{\xi} \in \dot{T}\text{” for some } q \leq p\right\}$ and, for every $\xi \in S_0$, select $p_\xi \leq p$ which forces that “$\check{\xi} \in \dot{T}$”. Pick an uncountable subset S of S_0 and an eventually centered sequence $\langle \tilde{p}_\xi \mid \xi \in S\rangle$ where $\tilde{p}_\xi \leq p_\xi$. Let $\{U_n \mid n < \omega\}$ be a set of an open basis of the set of reals. For each $\xi < \omega_1$, select $n_\xi < \omega$ such that $X{\restriction}(S \setminus \xi)$ is dense in U_{n_ξ}. By the pigeonhole principle, select $n < \omega$ such that $n_\xi = n$ for cofinal ξ's. Then $X{\restriction}(S \setminus \xi)$ is dense in U_n for each $\xi < \omega_1$. Let $\alpha_0 = \min S$. Since $\dot{F}$ is nowhere dense in the extension, we pick a non-empty open set $W \subset U_n$ and $q \leq \tilde{p}_{\alpha_0}$ such that $q \Vdash$ “$\dot{F} \cap \check{W} = \emptyset$”. $X{\restriction}(S \setminus \delta(q, \alpha_0))$ is dense in U_n, so it is also dense in W. Pick $\beta \in S \setminus \delta(q, \alpha_0)$ such that $x_\beta \in W$ and a common extension r of q and $\tilde{p}_\beta$. Then r forces that "$\check{x}_\beta \in \dot{F}$" and that "$\dot{F} \cap \check{W} = \emptyset$". It contradicts $x_\beta \in W$. □

THEOREM 12. *Every* $\mathrm{EPC}^*_{\aleph_1}$ *forcing of size* $\aleph_1$ *preserves non-meager sets.*

PROOF. Let $\mathbb{P} = \{p_\alpha \mid \alpha \in \omega_1\}$ be an $\mathrm{EPC}^*_{\aleph_1}$ forcing of size $\aleph_1$. Let X be a Polish space and $A \subset X$ be a non-meager set. Let $\dot{F}_n$ for $n \in \omega$ be $\mathbb{P}$-names for closed nowhere dense subsets of X and suppose towards a contradiction that $\mathbb{P} \Vdash$ "$\check{A} \subset \bigcup_n \dot{F}_n$".

For each $p \in \mathbb{P}$ and each $n \in \omega$, let $D(p,n) = \left\{x \in X \;\middle|\; p \Vdash \text{“}\check{x} \in \dot{F}_n\text{”}\right\}$; this is a closed nowhere dense set since $p \Vdash$ "$\check{D}(p,n) \subset \dot{F}_n$". For each $\beta \in \omega_1$, select $x_\beta \in A \setminus \bigcup_{\alpha \le \beta} \bigcup_{n<\omega} D(p_\alpha, n)$. For each $\beta \in \omega_1$, pick $q_\beta \in \mathbb{P}$ and $n_\beta \in \omega$ such that $q_\beta \Vdash$ "$\check{x}_\beta \in \dot{F}_{n_\beta}$". Let $T \in [\omega_1]^{\aleph_1}$ and $n \in \omega$ such that $n = n_\beta$ for each $\beta \in T$. Let $S \in [T]^{\aleph_1}$ and $r_\beta \le q_\beta$ for $\beta \in S$ be as in the definition of $\mathrm{EPC}^*_{\aleph_1}$. Let $\alpha_0 = \min(S)$ and $r_{\alpha_0} = p_{\beta_0}$. Let $\mathcal{B}$ be a countable open basis for X and let $\mathcal{B}_0 = \left\{O \in \mathcal{B} \;\middle|\; r \Vdash \text{“}\dot{F}_n \cap \check{O} = \emptyset\text{” for some } r \le r_{\alpha_0}\right\}$. For each $O \in \mathcal{B}_0$, select r_O as in the definition of $\mathcal{B}_0$. Fix $\beta \in T \setminus \left(\sup_{O \in \mathcal{B}_0} \delta(\alpha_0, r_O) \cup \beta_0\right)$. Since $x_\beta \notin D(p_{\beta_0}, n) = D(r_{\alpha_0}, n)$, "$\check{x}_\beta \notin \dot{F}_n$" is forcable below r_{α_0} and hence $r \Vdash$ "$\check{O} \cap \dot{F}_n = \emptyset$" for some open neighborhood O of x_{β_0} and some $r \le r_{\alpha_0}$. Thus $O \in \mathcal{B}_0$. A common extension of r_β and r_O forces $\check{x}_\beta \in \dot{F}_n$ and $\check{x}_\beta \notin \dot{F}_n$ simultaneously, a contradiction. □

2.3. Preservation of the least cardinality of mad families.

DEFINITION 27. *A family* $\mathcal{F} \subset [\omega]^{\aleph_0}$ *of infinite subsets of* ω *is an* almost disjoint family *if the intersection of any distinct pair* $A, B \in \mathcal{F}$ *is finite. A family* $\mathcal{A} \subset [\omega]^{\aleph_0}$ *is a* maximal almost disjoint (mad) family *if it is maximal with respect to the order* $\subset$ *among almost disjoint families.* $\mathfrak{a}$ *is the least cardinality of infinite mad families.*

A mad family preserved by the Cohen forcing can be constructed in a model of CH [**28**]. This is a reason that $\mathfrak{a} = \aleph_1$ in the Cohen model. To construct a mad family which is preserved by any $\mathrm{EPC}_{\aleph_1}$ forcing, we use the following forcing poset that adds a mad family.

DEFINITION 28 (Hechler [**22**]). *For ordinal* γ*, we define*

$$\mathbb{A}_\gamma = \left\{p\colon F_p \times n_p \to 2 \;\middle|\; F_p \in [\gamma]^{<\aleph_0},\ n_p < \omega\right\}.$$

For conditions $q, p \in \mathbb{A}_\gamma$*,* $q \le p$ *iff* $q \supset p$ *and* $|q^{-1}[\{1\}] \cap F_p \times \{i\}| \le 1$ *for all* $i \in n_q \setminus n_p$*.*

THEOREM 13 (Hechler [**22**]). $\mathbb{A}_\gamma$ *adds a maximal almost disjoint family*

$$\dot{\mathcal{A}} = \left\{\dot{A}_\alpha = \left\{n \in \omega \;\middle|\; p(\alpha, n) = 1 \text{ for some } p \in \dot{G}_{\mathbb{A}_\gamma}\right\} \;\middle|\; \alpha < \gamma\right\}$$

whenever $\gamma \geq \omega_1$.

The framework of the proof of Theorem 13 does not change to one of Theorem 14 which is about the preservation of the maximality of $\dot{A}$ in the extension with $\mathbb{A}_\gamma$ followed by $\mathrm{EPC}_{\aleph_1}$ forcing.

PROOF. First, we shall show that each $\dot{A}_\alpha$ is forced to be infinite. Fix any $p \in \mathbb{A}_\gamma$, $\alpha < \gamma$, and $n < \omega$ and we shall show that

$$p \Vdash \text{“}\dot{A}_\alpha \setminus n \text{ is non-empty”}.$$

Fix any $q \leq p$. Fix $i \in \omega$ greater than both of n and n_q, and choose $r \leq q$ with $r(\alpha, i) = 1$. Then $r \Vdash$ “$i \in \dot{A}_\alpha \setminus n$”.

Second, we shall show that distinct $\dot{A}_\alpha$ and $\dot{A}_\beta$ are forced to have a finite intersection. Toward a contradiction, suppose $p \Vdash$ “$\dot{A}_\alpha \cap \dot{A}_\beta$ is infinite”. Extending p, we may assume that $\alpha, \beta \in F_p$. By the assumption,

$$p \Vdash \text{“there is } q \in \dot{G}_{\mathbb{A}_\gamma} \text{ and } i > n_p \text{ such that } q(\alpha, i) = q(\beta, i) = 1\text{”}.$$

So we get $r \leq p$ and $i < n_p$ such that $r(\alpha, i) = r(\beta, i) = 1$, this contradicts the definition of the order relation of $\mathbb{A}_\gamma$.

Finally, we shall show the maximality of $\dot{\mathcal{A}}$. Toward a contradiction, fix a name $\dot{B} \in V^{\mathbb{A}_\gamma}$ for an infinite set of natural numbers and we assume that

$$p \Vdash \text{“}\dot{A}_\alpha \cap \dot{B} \text{ is finite for each } \alpha < \gamma\text{”}.$$

Select a sequence $\left\langle p_\alpha \leq p,\ \tau_\alpha \in [\omega]^{<\aleph_0} \;\middle|\; \alpha < \gamma \right\rangle$ such that

$$p_\alpha \Vdash \text{“}\dot{A}_\alpha \cap \dot{B} = \tau_\alpha\text{”}$$

for each $\alpha < \gamma$. Without loss of generality, we may assume that $\alpha \in F_{p_\alpha}$ for each $\alpha < \gamma$. By the delta system lemma and the pigeonhole principle, we pick $T \in [\gamma]^{\aleph_1}$, $R \in [\gamma]^{<\aleph_0}$, $n < \omega$, $q \colon R \times n \to 2$ and $\tau \in [\omega]^{<\aleph_0}$ such that

(1) $\tau_\alpha = \tau$ and $n_{p_\alpha} = n$ for each $\alpha \in T$,
(2) $\langle F_{p_\alpha} \mid \alpha < \gamma \rangle$ forms a delta system with root R, and
(3) $p_\alpha \restriction (R \times n) = q$ for each $\alpha \in T$.

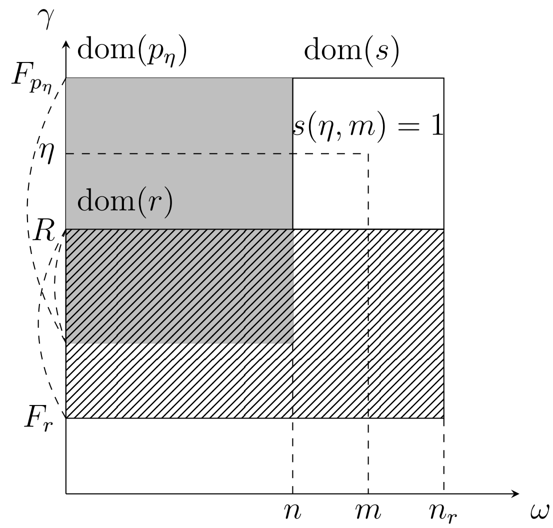
Let $\alpha_0 = \min T$. Refer to Figure 2 for discussion from this point forward. Choose $m > \max(\tau \cup n)$ and $r \leq p_{\alpha_0}$ such that $m < n_r$ and that

$$r \Vdash \text{“}\check{m} \in \dot{B} \text{ and } \bigcup_{\alpha \in R} \dot{A}_\alpha \cap \dot{B} \subset \check{m}\text{”}.$$

Remark that $r(\alpha, m) = 0$ for each $\alpha \in R$ since $r(\alpha, m) = 1$ implies $r \Vdash$ “$\check{m} \in \dot{A}_\alpha$”. Pick $\eta \in T \setminus R$ such that $F_{p_\eta} \cap F_r = R$. Notice that p_η and r are compatible since $p_\eta \restriction (\mathrm{dom}(p_\eta) \cap \mathrm{dom}(r)) = p_\eta \restriction (R \times n) = q \geq r$. Let $s \colon (F_{p_\eta} \setminus R) \times (n_r \setminus n) \to 2$ be a function such that $s(\xi, i) = 1$ if and only if $(\xi, i) = (\eta, m)$. Then $t = p_\eta \cup s \cup r \colon (F_{p_\eta} \cup$

$F_r) \times n_r \to 2$ is a common extension of r and p_η. We have $t \Vdash$ "$\check{m} \in \dot{A}_\eta \cap \dot{B}$", this contradicts $p_\eta \Vdash$ "$\dot{A}_\eta \cap \dot{B} = \check{\tau}$" and $m > \max \tau$. $\square$

FIGURE 2. Domains of p_η, r, and s.



We shall prove that $\mathbb{A}_\gamma$ is an example of $\mathrm{EPC}_{\aleph_1}$ forcing.

LEMMA 30. *$\mathbb{A}_\gamma$ is $\mathrm{EPC}^*_{\aleph_1}$.*

PROOF. Fix any ω_1-sequence $\langle p_\alpha \in \mathbb{A}_\gamma \mid \alpha \in \omega_1 \rangle$. By the delta system lemma and the pigeonhole principle, we take $T \in [\omega_1]^{\aleph_1}$, $R \in [\gamma]^{<\aleph_0}$, and $n < \omega$ such that

(1) $\langle F_{p_\alpha} \mid \alpha \in T \rangle$ forms a delta system with root R,
(2) $p_\alpha \restriction (R \times n) = p_\beta \restriction (R \times n)$ for each $\alpha, \beta \in T$, and
(3) $n_{p_\alpha} = n$.

We shall show that $\langle p_\alpha \mid \alpha \in T \rangle$ is an eventually centered sequence.

Fix any $\alpha_0 \in T$ and $q \le p_{\alpha_0}$. Pick $\delta < \omega_1$ such that $F_q \cap F_{p_\xi} = R$ whenever $\xi \in T \setminus \delta$. Fix any finite subset $\Gamma \subset T \setminus \delta$. Define $s = q \cup \bigcup_{\xi \in \Gamma} p_\xi \cup s_0$ where $s_0 \colon (\bigcup_{\xi \in \Gamma} F_{p_\xi} \setminus R) \times (n_q \setminus n) \to \{0\}$. Then, s is a common extension of $\{q\} \cup \{p_\xi \mid \xi \in \Gamma\}$. $\square$

THEOREM 14. *Let $\mathbb{A} = \mathbb{A}_{\omega_1}$ be the forcing adding a mad family of size $\aleph_1$ and $\dot{\mathbb{P}}$ be an $\mathbb{A}$-name for an $\mathrm{EPC}^*_{\aleph_1}$ forcing notion. Then, the two step iteration $\mathbb{Q} = \mathbb{A} * \dot{\mathbb{P}}$ forces that $\mathfrak{a} = \aleph_1$. In practice, $\dot{\mathbb{P}}$ preserves the mad family $\dot{\mathcal{A}} = \left\{ \dot{A}_\alpha = \left\{ n \in \omega \;\middle|\; p(\alpha, n) = 1 \text{ for some } p \in \dot{G}_{\mathbb{A}} \right\} \;\middle|\; \alpha < \omega_1 \right\}$ added by $\mathbb{A}$ in the extension with $\mathbb{A}$.*

PROOF. Suppose not, we take $\langle p, \dot{q}\rangle \in \mathbb{Q}$ and a name $\dot{B}$ such that

$$\langle p, \dot{q}\rangle \Vdash \text{“}\dot{B} \subset [\omega]^{\aleph_0} \text{ and each } \dot{B} \cap \dot{A}_\alpha \text{ is finite”}.$$

Select $\left\langle \langle p_\alpha, \dot{q}_\alpha\rangle \in \mathbb{Q},\ \tau_\alpha \in [\omega]^{<\aleph_0} \,\middle|\, \alpha < \omega_1 \right\rangle$ below $\langle p, \dot{q}\rangle$ such that

$$\langle p_\alpha, \dot{q}_\alpha\rangle \Vdash \text{“}\dot{B} \cap \dot{A}_\alpha = \check{\tau}_\alpha\text{”}$$

for each $\alpha < \omega_1$.

Let $\dot{S}_0 = \{\langle \check{\xi}, p_\xi\rangle \mid \xi \in \omega_1\}$. There exists $p^* \in \mathbb{A}$ with $p^* \leq p$ such that $p^* \Vdash$ “$\dot{S}_0 = \{\xi < \omega_1 \mid p_\xi \in \dot{G}\}$ is uncountable” since $\mathbb{A}$ has the ccc. Pick $\mathbb{A}$-names $\dot{S}$ and $\dot{\dot{q}}_\xi$ for $\xi < \omega_1$ such that

$$p^* \Vdash \text{“ } \dot{S} \in [\dot{S}_0]^{\aleph_1} \text{ and } \left\langle \dot{\dot{q}}_\xi \leq \dot{q}_\xi \,\middle|\, \xi \in \dot{S} \right\rangle \text{ is eventually centered”}.$$

Define $T_0 = \left\{ \xi \in \omega_1 \,\middle|\, \text{there exists a common extension } p \text{ of } p^* \text{ and } p_\xi \text{ that forces } \check{\xi} \in \dot{S} \right\}$. Then p^* forces that $\dot{S} \subset \check{T}_0$ and hence T_0 is uncountable. Select $\langle \tilde{p}_\xi \leq p^* \mid \xi \in T_0\rangle$ such that $\tilde{p}_\xi \leq p_\xi$, that $\tilde{p}_\xi \Vdash$ “$\check{\xi} \in \dot{S}$”, and that $\xi \in F_{\tilde{p}_\xi}$.

By the delta system lemma and the pigeonhole principle, pick $T \in [T_0]^{\aleph_1}$, $\tau \in [\omega]^{<\aleph_0}$, and $n \in \omega$ such that

(1) $\left\langle \tilde{p}_\alpha, \dot{\dot{q}}_\alpha \right\rangle \Vdash$ “$\dot{B} \cap \dot{A}_\alpha = \check{\tau}$” for each $\alpha \in T$,
(2) $\langle F_{\tilde{p}_\alpha} \mid \alpha \in T\rangle$ forms a delta system with root R,
(3) $n_{\tilde{p}_\alpha} = n$ for each $\alpha \in T$, and
(4) $\tilde{p}_\alpha \restriction (R \times n) = \tilde{p}_\beta \restriction (R \times n)$ for each $\alpha, \beta \in T$.

Let $\alpha_0 = \min T$. Pick $\langle p^0, \dot{q}^0\rangle \leq \left\langle \tilde{p}_{\alpha_0}, \dot{\dot{q}}_{\alpha_0} \right\rangle$ and $m > \max(\tau \cup n)$ such that $n_{p^0} > m$ and that

$$\left\langle p^0, \dot{q}^0 \right\rangle \Vdash \text{“}\check{m} \in \dot{B} \text{ and } \bigcup_{\alpha \in \check{R}} (\dot{B} \cap \dot{A}_\alpha) \subset \check{m}\text{”}.$$

Note that $p^0(\alpha, m) = 0$ for each $\alpha \in R$. Pick $p^1 \leq p^0$ and $\delta < \omega_1$ such that $p^1 \Vdash$ “$\check{\delta} = \dot{\delta}(\check{\alpha}_0, \dot{q}_0)$ ” and that $m < n_{p^1}$. Pick $\eta \in T \setminus (\delta \cup R)$ such that $F_{p^1} \cap F_{\tilde{p}_\eta} = R$. Recall that $\eta \in F_{\tilde{p}_\eta}$. Define $s = p^1 \cup \tilde{p}_\eta \cup s^0$ where $s^0 \colon (F_{\tilde{p}_\eta} \setminus R) \times (n_{p^1} \setminus n) \to 2$ such that $s^0(\xi, i) = 1$ if and only if $\langle \xi, i\rangle = \langle \eta, m\rangle$. Note that $s \in \mathbb{A}$ is a common extension of p^1 and $\tilde{p}_\eta$ and that $s \Vdash$ “$m \in \dot{A}_\eta$ and $\eta \in \dot{S} \setminus \dot{\delta}(\check{\alpha}_0, \dot{q}_0)$”. There exists $\dot{q}^+$ such that $s \Vdash$ “$\dot{q}^+$ is a common extension of $\dot{q}_0$ and $\dot{\dot{q}}_\eta$”. Thus, $\langle s, \dot{q}^+\rangle$ is a common extension of $\langle p^1, \dot{q}_0\rangle$ and $\left\langle \tilde{p}_\eta, \dot{\dot{q}}_\eta \right\rangle$ and hence $\langle s, \dot{q}^+\rangle \Vdash$ “$m \in \dot{B} \cap \dot{A}_\eta = \tau \subset m$”, which is a contradiction. □

Other than the mad family added by $\mathbb{A}$, there is a kind of mad families which is preserved by $\mathrm{EPC}^*_{\aleph_1}$ of size $\leq \aleph_1$.

Definition 29. *For a family $A \subset \mathcal{P}(\omega)$, define*

$$\mathcal{I}^+(A) = \{x \subset \omega \mid x \text{ is not covered by finitely many elements of } A\}.$$

A family $A \subset [\omega]^{\aleph_0}$ is tight *if for every countable collection $\{b_n \in \mathcal{I}^+(A) \mid n \in \omega\}$, there is $a \in A$ such that, for all $n \in \omega$, $|a \cap b_n| = \aleph_0$.*

Note that any tight almost disjoint family is mad. When $\mathfrak{c} = \mathfrak{b}$, tight mad families exist, see Corollary 3.4 in [**24**].

Theorem 15. *Any $\mathrm{EPC}^*_{\aleph_1}$ forcing $\mathbb{P}$ of size $\aleph_1$ preserves tight families.*

Proof. Let $\mathbb{P} = \{p_\alpha \mid \alpha \in \omega_1\}$. Suppose that A is a tight family and $\dot{b}_n$ for $n \in \omega$ are $\mathbb{P}$-names for elements of $\mathcal{I}^+(A)$. Then, for $p \in \mathbb{P}$ and $n \in \omega$, $c(p, n) = \left\{ i \in \omega \;\middle|\; q \Vdash \text{“}\check{i} \in \dot{b}_n\text{” for some } q \leq p \right\}$ is a member of $\mathcal{I}^+(A)$. Since $C_\alpha = \{c(p_\xi, n) \mid \xi \leq \alpha,\, n < \omega\}$ is a countable family of members of $\mathcal{I}^+(A)$, there exists $a_\alpha \in A$ such that, for each $c \in C_\alpha$, $|a_\alpha \cap c| = \aleph_0$. Suppose that, towards a contradiction,

$$\mathbb{P} \Vdash \text{“for each } a \in \check{A}, \text{ there exists } n < \omega \text{ such that } |a \cap \dot{b}_n| < \aleph_0\text{”}.$$

Thus

$$p_\alpha \Vdash \text{“}|\check{a}_\alpha \cap \dot{b}_n| < \aleph_0 \text{ for some } n\text{”}.$$

Choose a sequence $\left\langle q_\alpha \leq p_\alpha,\ n_\alpha \in \omega,\ \tau_\alpha \in [\omega]^{<\aleph_0} \;\middle|\; \alpha < \omega_1 \right\rangle$ such that $q_\alpha \Vdash$ “$\check{a}_\alpha \cap \dot{b}_{n_\alpha} = \check{\tau}_\alpha$” for each $\alpha \in \omega_1$. Select $T \in [\omega_1]^{\aleph_1}$, n, τ such that $n_\beta = n$ and that $\tau_\beta = \tau$ for $\beta \in T$. Let $S \in [T]^{\aleph_1}$ and $r_\alpha \leq q_\alpha$ for $\alpha \in S$ be as in the definition of $\mathrm{EPC}^*_{\aleph_1}$. Let $\alpha_0 = \min(S)$ and $r_{\alpha_0} = p_\beta$. Then, for each $i \in c(p_\beta, n)$, select an extension s_i of $r_{\alpha_0} = p_\beta$ which forces $\check{i} \in \dot{b}_n$. Pick $\gamma \in T \setminus \left(\sup_{i \in c(r_{\alpha_0}, n)} \delta(\alpha_0, s_i) \cup \beta \right)$. Then, $|a_\gamma \cap c(p_\beta, n)| = \aleph_0$. So we can select $i \in a_\gamma \cap c(p_\beta, n) \setminus \tau$. Then a common extension s of r_γ and s_i forces $\check{i} \in \dot{b}_n \cap \check{a}_\gamma = \check{\tau}$, a contradiction. □

2.4. Preservation of the groupwise density number. The groupwise density number $\mathfrak{g}$ was introduced in Blass-Laflamme [**7**] to provide a formulation of combinatorial properties of filters. An example of combinatorial properties is that, whenever every two non-principal ultrafilters U and V are given, there exists a finite-to-one function $f\colon \omega \to \omega$ such that $f(U) = f(V)$. The consistency of these properties was initially given by the forcing method by Blass-Shelah [**10**]. Blass-Laflamme [**7**] defined the groupwise density number $\mathfrak{g}$ and showed that these properties are consequences of $\mathfrak{u} < \mathfrak{g}$ where $\mathfrak{u}$ is called the ultrafilter number, which is the least cardinality of a family of infinite sets of natural numbers that generates a non-principal ultrafilter on ω.

DEFINITION 30. *A family* $\mathscr{G} \subset [\omega]^{\aleph_0}$ *of infinite sets of natural numbers is* groupwise dense *if*

(1) $\mathscr{G}$ *is* $\subset^*$*-downward closed and*
(2) For every increasing sequence $\vec{n} = \langle n_i \mid i < \omega \rangle$*, there exists* $X \in [\omega]^{\aleph_0}$ *such that* $\bigcup_{i \in X} [n_i, n_{i+1}) \in \mathscr{G}$*.*

The groupwise density number $\mathfrak{g}$ *is defined as follows:*

$$\mathfrak{g} = \min \left\{ |G| \colon G \text{ is a family of groupwise dense families such that } \bigcap G = \emptyset \right\}.$$

The cofinality of a non-trivial ultrapower $\mathbb{R}^\omega/U$ of $\mathbb{R}$ is an upper bound for the groupwise density number.

THEOREM 16 (Blass, Mildenberger THEOREM 3.1 in [**8**]). *If* U *is a non-principal ultrafilter on* ω*, then* $\mathfrak{g} \leq \mathrm{cof}(\omega^\omega/U)$*.*

Notice that our main result, which is the consistency of $\mathrm{MA}(\mathrm{EPC}_{\aleph_1} + \mathrm{ProjCes}(E))$ + "some $\mathbb{R}^\omega/U$ is β_1" + $\mathfrak{c} = \aleph_2$, is given by the ω_2-length finite support iteration of $\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)$ forcing notions of size $\leq \aleph_1$.

COROLLARY 7. *Assume that CH holds and* $E \subset \omega_1$ *is stationary. Any finite support iteration* $\mathbb{P}$ *of* ω_2*-length* $\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)$ *forcing notions of size* $\leq \aleph_1$ *forces* $\mathfrak{g} = \aleph_1$*.*

PROOF. If $\mathrm{cof}(\mathbb{R}^\omega/U) > \aleph_1$ and $\mathbb{R}^\omega/U$ is the union $\bigcup_{\alpha<\omega_1} R_\alpha$ of a continuously $\subset$-increasing sequence of ordered fields, then some R_α is not α_1. So $\mathrm{cof}(\mathbb{R}^\omega/U) > \aleph_1$ implies that $\mathbb{R}^\omega/U$ is not β_1. Thus, in the extension model, $\mathrm{cof}(\mathbb{R}^\omega/U) = \aleph_1$ for some non-principal ultrafilter $U \subset \mathcal{P}(\omega)$. Since $\mathbb{R}^\omega/U$ and ω^ω/U are mutually cofinal, hence $\mathrm{cof}(\omega^\omega/U) = \mathrm{cof}(\mathbb{R}^\omega/U) = \aleph_1$. By Theorem 16, $\mathfrak{g} = \aleph_1$. □

COROLLARY 8. $\mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E) +$ *"size* $< \aleph_2$*"*$)$ + "$\mathrm{non}(\mathcal{M}) = \mathfrak{a} = \mathfrak{g} = \aleph_1$" + "$\mathfrak{c} = \mathrm{cov}(\mathcal{M}) = \aleph_2$" *and* $\mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} +$ *"size* $< \aleph_2$*"*$)$ + "$\mathrm{non}(\mathcal{M}) = \mathfrak{a} = \aleph_1$" + "$\mathfrak{c} = \mathrm{cov}(\mathcal{M}) = \aleph_2$" *is consistent relative to* ZFC*.*

PROOF. Our ground model is the forcing extension by $\mathbb{C}_{\omega_1}\dot{\mathbb{A}}_{\omega_1}$ in which CH holds. By the standard bookkeeping, we force that $\mathrm{MA}(\mathrm{EPC}^*_{\aleph_1}) + \mathfrak{c} = \aleph_2$ or $\mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)) + \mathfrak{c} = \aleph_2$ by a finite support iteration of $\mathrm{EPC}^*_{\aleph_1}$ forcings or $\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)$ forcings. Since a Luzin set and a mad family which are of size $\aleph_1$ exist in our ground model and since any $\mathrm{EPC}^*_{\aleph_1}$ forcing notion preserves them, in the extension model, $\mathrm{non}(\mathcal{M}) = \mathfrak{a} = \aleph_1$ holds. In addition, in the model for $\mathrm{MA}(\mathrm{EPC}^*_{\aleph_1} + \mathrm{ProjCes}(E)) + \mathfrak{c} = \aleph_2$ that we constructed, $\mathfrak{g} = \aleph_1$ by the previous colollary. □

2.5. Mutual Independency between $\mathrm{MA}(\sigma$**-centered**$)$ **and** $\mathrm{MA}(\mathrm{EPC}_{\aleph_1})$**.** We proceed to the independency of $\mathrm{MA}(\sigma\text{-centered})$ from $\mathrm{MA}(\mathrm{EPC}_{\aleph_1})$.

DEFINITION 31. *A family* $\mathcal{S} \subset [\omega]^{\aleph_0}$ *has* the strong finite intersection property *iff for every finite subset* $F \subset \mathcal{S}$, $\bigcap F$ *is infinite. An infinite set* $K \in [\omega]^{\aleph_0}$ *is* a pseudo-intersection of $\mathcal{S}$ *iff for every* $S \in \mathcal{S}$, $K \setminus S$ *is finite.* The pseudo-intersection number $\mathfrak{p}$ *is the least cardinality of* $\mathcal{S} \subset [\omega]^{\aleph_0}$ *which has the strong finite intersection property but has no pseudo-intersection.*

THEOREM 17 (Bell [**5**] e.g., see Theorem III.3.61 in [**28**]). $\mathfrak{p} = \min\{\kappa \mid \mathrm{MA}_\kappa(\sigma\text{-centered}) \text{ fails}\}$.

THEOREM 18 (Barnet [**3**]). *There exists a model of* ZFC + MA(σ*-centered*) + *"a non-uniformizable ladder system coloring exists".*

By Theorem 9, Figure 1, Theorem 17, Theorem 18 and the Main Theorem, we have the following.

COROLLARY 9. *Both of* MA(EPC$_{\aleph_1}$ + *"size* $< \aleph_2$*"*) *and* MA(EPC$^*_{\aleph_1}$) *does not imply* MA(σ*-centered*). *Furthermore,* MA(σ*-centered*) *does not imply both of* MA(EPC$_{\aleph_1}$ + *"size* $< \aleph_2$*"*) *and* MA(EPC$^*_{\aleph_1}$)*.*

3. Kunen forcing

Using Kunen forcings, we give a more detailed description of the position of EPC$_{\aleph_1}$ and EPC$^*_{\aleph_1}$ within other well-known forcing properties (see Section 2 in Chapter 1). In particular, we give examples of EPC$^*_{\aleph_1}$ forcing notions which are not EPC$_{\aleph_1}$ and precaliber $\aleph_1$ forcing notions that are not EPC$^*_{\aleph_1}$.

Each Kunen forcing is defined according to a pregap, hence we shall start with defining what is pregap.

DEFINITION 32. *For* ω*-sequences* f *and* g *of reals and for a natural number* $k < \omega$, *define*

$$f <^k g :\iff \forall n \geq k,\ f(n) < g(n) \qquad \textit{and}$$
$$f <^* g :\iff \exists k < \omega,\ f <^k g.$$

DEFINITION 33. *Let* κ *and* λ *be regular cardinals. Let* $(\mathcal{F}, \mathcal{G}) = (f_\alpha, g_\beta \mid \alpha < \kappa,\ \beta < \lambda)$ *be a pair of a* κ*-sequence and a* λ*-sequence of* ω*-sequences of rationals.*

- $(\mathcal{F}, \mathcal{G})$ *is a* (κ, λ)*-pregap if* $f_\alpha <^* g_\beta$ *for all* $\alpha < \kappa$ *and* $\beta < \lambda$.
- $(\mathcal{F}, \mathcal{G})$ *is filled by* $h \in \mathbb{R}^\omega$ *if* $f_\alpha <^* h <^* g_\beta$ *for all* $\alpha < \kappa$ *and* $\beta < \lambda$.
- $(\mathcal{F}, \mathcal{G})$ *is a* (κ, λ)*-gap in* $\mathbb{R}^\omega$ *(in* $\mathbb{Q}^\omega$*) if it is a* (κ, λ)*-pregap which is not filled by any* $h \in \mathbb{R}^\omega$ *(*$h \in \mathbb{Q}^\omega$*).*
- *A* κ*-pregap (*κ*-gap) is a* (κ, κ)*-pregap ((*κ, κ*)-gap).*
- *For relations* $R, S \in \{\neq, <, \leq, \ldots\}$, $(R\kappa, S\lambda)$*-pregap ((*$R\kappa, S\lambda$*)-gap) is a* (κ', λ')*-pregap ((*κ', λ'*)-gap) for some regular cardinals* $\kappa' R \kappa$ *and* $\lambda' S \lambda$.

In this section, we fix a (κ, λ)-pregap $(\mathcal{F}, \mathcal{G}) = (f_\alpha, g_\beta \mid \alpha < \kappa,\ \beta < \lambda)$ in $({}^\omega\mathbb{Q}, <^*)$. The Kunen forcing generates an interpolation of $(\mathcal{F}, \mathcal{G})$:

Definition 34. *Let $\mathbb{K}(\mathcal{F}, \mathcal{G})$ be the set*

$$\{(L_p, R_p, s_p) \in [\kappa]^{<\aleph_0} \times [\lambda]^{<\aleph_0} \times \mathbb{Q}^{<\omega} \mid \forall(\alpha, \beta) \in L_p \times R_p,\ f_\alpha(k) <^{|s_p|} g_\beta(k)\}.$$

For $p, q \in \mathbb{K}(\mathcal{F}, \mathcal{G})$, $q \leq p$ iff

(1) $L_q \supset L_p$, $R_q \supset R_p$, $s_q \supset s_p$,
(2) *for each* $(\alpha, \beta) \in L_p \times R_p$, $\forall k \in |s_q| \setminus |s_p|$, $f_\alpha(k) < s_q(k) < g_\beta(k)$.

The following is essentially the only case in which the Kunen poset is not ccc.

Definition 35. *An ω_1-gap $(\mathcal{F}, \mathcal{G})$ is* special *if there exists $k < \omega$ such that*

- $\forall \alpha < \omega_1,\ f_\alpha <^k g_\alpha$ *($(\mathcal{F}, \mathcal{G})$ is well-formed over k) and*
- $\forall^{\neq} \alpha, \beta < \omega_1,\ \exists n \geq k,\ f_\alpha(n) \geq g_\beta(n)$ *or* $f_\beta(n) \geq g_\alpha(n)$

Definition 36. *A pregap $(\mathcal{F}', \mathcal{G}')$ is* equivalent *to $(\mathcal{F}, \mathcal{G})$ if $\mathcal{F}$ and $\mathcal{F}'$ are mutually cofinal and $\mathcal{G}$ and $\mathcal{G}'$ are mutually coinitial.*

Combining already known facts (e.g., see [**27**]), we summarize the relation between the form of a pregap and its Kunen poset and prove the unknown parts.

Theorem 19. **(O):** *The following are equivalent:*

(1) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is ccc.*
(2) $(\mathcal{F}, \mathcal{G})$ *is not equivalent to a special ω_1-gap.*

(A): *The following are equivalent:*

(1) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is (K).*
(2) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is* $PC_{\aleph_1}$.
(3) $(\kappa, \lambda) \neq (\omega_1, \omega_1)$ *or some* $f \in {}^\omega\mathbb{R}$ *fills* $(\mathcal{F}, \mathcal{G})$.

(B): *The following are equivalent:*

(1) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is σ-centered.*
(2) $\min(\kappa, \lambda) \leq \omega$ *or some* $f \in {}^\omega\mathbb{R}$ *fills* $(\mathcal{F}, \mathcal{G})$.

(C): *The following are equivalent:*

(1) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is* $\mathrm{EPC}^*_{\aleph_1}$.
(2) $\kappa \neq \omega_1$ *and* $\lambda \neq \omega_1$.

(D): *The following are equivalent:*

(1) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is* $\mathrm{EPC}_{\aleph_1}$.
(2) $\mathbb{K}(\mathcal{F}, \mathcal{G})$ *is countable.*
(3) $\max(\kappa, \lambda) \leq \aleph_0$.

The equivalence (O) is a classical result by Kunen, see [**27**].

Lemma 31 (For (A) (1) $\to$ (3)). *If $(\mathcal{F}, \mathcal{G})$ is an ω_1-gap in $({}^\omega\mathbb{R}, <^*)$, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ does not have property (K).*

	Forms of pregaps	Property of Kunen p.o.
(O)	*Not equivalent to a special ω_1-gap*	*ccc*
(A)	*Not an ω_1-gap in $({}^{\omega}\mathbb{R}, <^*)$*	*(K)* $PC_{\aleph_1}$
(B)	*Not a $(\geq \omega_1, \geq \omega_1)$-gap in $({}^{\omega}\mathbb{R}, <^*)$*	*σ-centered*
(C)	*$(\neq \omega_1, \neq \omega_1)$-pregap*	$\mathrm{EPC}^*_{\aleph_1}$
(D)	*countable pregap*	$\mathrm{EPC}_{\aleph_1}$ *Countable*

Proof. We assume that, toward a contradiction, $(\mathcal{F}, \mathcal{G})$ is an ω_1-gap in $({}^{\omega}\mathbb{R}, <^*)$ such that $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is (K). By pigeonhole principle, pick $T \in [\omega_1]^{\aleph_1}$ and $k < \omega$ such that $(\mathcal{F}{\restriction}T, \mathcal{G}{\restriction}T)$ is well-formed over k. Fix $s \in {}^{k}\mathbb{Q}$ and let $p_\alpha = (\{\alpha\}, \{\alpha\}, s)$ for each $\alpha < \omega_1$. Then $\bar{p} = (p_\alpha \mid \alpha \in T)$ is an ω_1-sequence in $\mathbb{K}(\mathcal{F}, \mathcal{G})$. Since $\mathbb{K}(\mathcal{F}, \mathcal{G})$ has property (K), there exists $S \in [T]^{\aleph_1}$ such that $\bar{p}{\restriction}S$ is linked. Then, for each $\alpha, \beta \in S$, $p_\alpha \wedge p_\beta = (\{\alpha, \beta\}, \{\alpha, \beta\}, s)$ is a condition, and hence $\forall^{\neq}\alpha, \beta \in S,\ f_\alpha <^k g_\beta$. Let $f(n) = \sup_{\alpha \in S} f_\alpha(n)$ if $n \geq k$, else $f(n) = 0$. Then $f_\alpha \leq^k f \leq^k g_\beta$ for each $\alpha, \beta \in S$. Since $(f_\alpha, g_\alpha \mid \alpha \in S)$ is a gap in $\mathbb{R}^\omega$, there exists $\alpha < \omega_1$ such that $f <^* f_\alpha$ or $g_\alpha <^* f$, a contradiction. □

The following three lemmata are proved in [**27**].

Lemma 32 (For (A) (3) → (2)). *If a side of $(\mathcal{F}, \mathcal{G})$ is greater than ω_1, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is $PC_{\aleph_1}$.*

Lemma 33 (For (A) (3) → (2) and (B) (2) → (1)). *If a side of $(\mathcal{F}, \mathcal{G})$ is countable, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is σ-centered.*

Lemma 34 (For (A) (3) → (2) and (B) (2) → (1)). *If $(\mathcal{F}, \mathcal{G})$ is filled by $h \in {}^{\omega}\mathbb{R}$, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is σ-centered.*

We have now successfully proven (A). We shall proceed to finish to prove (B).

Lemma 35 (For (B) (1) → (2)). *If $(\mathcal{F}, \mathcal{G})$ is a $(\geq \omega_1, \geq \omega_1)$-gap in $({}^{\omega}\mathbb{R}, <^*)$, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is not σ-centered.*

Proof. Let $(\mathcal{F}, \mathcal{G}) = (f_\alpha, g_\beta \mid \alpha < \kappa,\ \beta < \lambda)$ be an (κ, λ)-gap in $({}^{\omega}\mathbb{R}, <^*)$ where $\lambda \geq \kappa \geq \omega_1$ are regular. Toward a contradiction, we assume that $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is σ-centered. Cover $\mathbb{K}(\mathcal{F}, \mathcal{G}) = \bigcup_{n<\omega} K_n$ by a countable family of centered sets. Let $\bar{0}_k \colon k \to \{0\}$ for each $k < \omega$. For each $(\alpha, \beta) \in \kappa \times \lambda$, select $k_{\alpha,\beta} \in \omega$ such that $f_\alpha <^{k_{\alpha,\beta}} g_\beta$ and let $p_{\alpha,\beta} = (\{\alpha\}, \{\beta\}, \bar{0}_{k_{\alpha,\beta}}) \in \mathbb{K}(\mathcal{F}, \mathcal{G})$. We pick $T \in [\kappa]^\kappa$, $(S_\alpha \in [\lambda]^\lambda \mid \alpha \in T)$, and $k, n \in \omega$ such that, for each $\alpha \in T$ and each $\beta \in S_\alpha$,

(1) $p_{\alpha,\beta} \in K_n$ and
(2) $k_{\alpha,\beta} = k$.

Fix $\alpha_0 \in T$ and let $g(n) = \inf_{\beta \in S_{\alpha_0}} g_\beta(n)$ for each $n < \omega$. Then, for each $(\xi, \eta) \in \kappa \times \lambda$, there exists $(\alpha, \beta) \in (\kappa \setminus \xi) \times (\lambda \setminus \eta)$ such that $\alpha \in T$ and $\beta \in S_\alpha$. Note that $(\{\alpha, \alpha_0\}, \{\beta, \beta'\}, \bar{0}_k)$ is a condition for each $\beta' \in S_{\alpha_0}$, and hence

$$f_\xi <^* f_\alpha \leq^k g <^* g_\beta <^* g_\eta$$

Thus g fills the gap, a contradiction. □

LEMMA 36 (For (C) (2) → (1)). *If $(\mathcal{F}, \mathcal{G})$ is a $(\neq \omega_1, \neq \omega_1)$-pregap, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is* $\mathrm{EPC}^*_{\aleph_1}$.

PROOF. Let $(\mathcal{F}, \mathcal{G}) = (f_\alpha, g_\beta \mid \alpha < \kappa,\ \beta < \lambda)$. The case $\min(\kappa, \lambda) \leq \omega$ is simple by pigeonhole principle, so we may assume that $\kappa, \lambda \geq \omega_2$. Fix any $\bar{p} = (p_\alpha \in \mathbb{K}(\mathcal{F}, \mathcal{G}) \mid \alpha < \omega_1)$. Then there exists $(\gamma, \delta) \in \kappa \times \lambda$ such that $\sup_{\xi < \omega_1} L_{p_\xi} < \gamma$ and that $\sup_{\xi < \omega_1} R_{p_\xi} < \delta$. Pick $T \in [\omega_1]^{\aleph_1}$, $k < \omega$ and $s \in {}^k\mathbb{Q}$ such that, for each $\alpha \in T$, $|s_{p_\alpha}| \leq k$ and $\forall (\xi, \eta) \in L_{p_\alpha} \times R_{p_\alpha}$, $f_\xi <^k f_\gamma <^k< g_\delta <^k g_\eta$. For each $\alpha \in T$, let $\tilde{p}_\alpha = (L_{p_\alpha} \cup \{\gamma\}, R_{p_\beta} \cup \{\delta\}, s_\alpha)$ where s_α is an extension of s_{p_α} such that $s_\alpha(i) = 2^{-1}(\max\{f_\xi(i) \mid \xi \in L_{p_\alpha}\} + \min\{g_\xi(i) \mid \xi \in R_{p_\alpha}\})$ for $|s| \leq i < k$. Then $\tilde{p}_\alpha \leq p_\alpha$ is a condition. We proceed to prove $(\tilde{p}_\alpha \mid \alpha \in T)$ is eventually centered. Fix any $\alpha \in T$, $q \leq \tilde{p}_\alpha$, and finite $\Gamma \subset T$. Then $r = \left(\bigcup_{\xi \in \Gamma} L_{\tilde{p}_\xi} \cup L_q, \bigcup_{\xi \in \Gamma} R_{\tilde{p}_\xi} \cup R_q, s_q \right)$ an condition which extends q and each $\tilde{p}_\xi$ $(\xi \in \Gamma)$. □

LEMMA 37 (For (C) (1) → (2)). *If $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is $\mathrm{EPC}^*_{\aleph_1}$, then $(\mathcal{F}, \mathcal{G})$ is a $(\neq \omega_1, \neq \omega_1)$-pregap.*

PROOF. Toward a contradiction, let us assume that $(\mathcal{F}, \mathcal{G})$ is (ω_1, λ)-pregap and that $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is $\mathrm{EPC}^*_{\aleph_1}$. Let $p_\alpha = (\{\alpha\}, \emptyset, ())$ for each $\alpha \in \omega_1$. Pick $T \in [\omega_1]^{\aleph_1}$ and eventually centered sequence $(\tilde{p}_\alpha \leq p_\alpha \mid \alpha \in T)$. Shrinking T, we assume that $\forall \alpha \in T$, $m = |s_{\tilde{p}_\alpha}|$. Pick $\alpha \in T$ and let $\gamma = \max L_{\tilde{p}_\alpha}$. Select $S \in [T \setminus \gamma]^{\aleph_1}$, $k \in \omega$, $l \geq \max\{m, k\}$ and a rational number $\varepsilon > 0$ such that, for each $\eta \in L_{\tilde{p}_\alpha}$, each $\delta \in R_{\tilde{p}_\alpha}$, and each $\xi \in S$, $f_\eta <^k f_\gamma <^k f_\xi, g_\delta$ and $f_\gamma(l) + \varepsilon < f_\xi(l), g_\delta(l)$. Let $q \leq \tilde{p}_\alpha$ such that $s_q(l) = f_\alpha(l) + \varepsilon$. Then q is incompatible to $\tilde{p}_\xi$ for each $\xi \in S$, a contradiction. □

LEMMA 38 (For (D) (3) → (2)). *If a pregap $(\mathcal{F}, \mathcal{G})$ is countable, then $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is countable.*

PROOF. Immediate. □

LEMMA 39 (For (D) (1) → (3)). *If $\mathbb{K}(\mathcal{F}, \mathcal{G})$ is $\mathrm{EPC}_{\aleph_1}$, then $(\mathcal{F}, \mathcal{G})$ is countable.*

PROOF. The proof is similar to and simpler than Lemma 37. □

4. Questions

H. Woodin [**33**] asked whether the existence of discontinuous homomorphisms on C(X) is relatively consistent with ZFC + MA + ¬CH. The property eventual precaliber $\aleph_1$ is quite stronger than ccc (actually, stronger than precaliber $\aleph_1$), hence $\mathrm{MA}(\mathrm{EPC}_{\aleph_1})$ is quite weaker than MA, a fortiori, so is $\mathrm{MA}(\mathrm{EPC}_{\aleph_1} + \mathrm{ProjCes}(E))$. Thus the consistency of the existence of discontinuous homomorphisms on $C(X)$ with $\mathrm{MA} + \neg\mathrm{CH}$ remains open.

Problem 2. *The following remain open.*

(1) *The consistency of* $\mathrm{MA} + \neg\mathrm{NDH} + \neg\mathrm{CH}$.
(2) *Woodin's question (see Problem 1).*
(3) *The consistency of* $\mathrm{MA}(\mathrm{EPC}_{\aleph_1}) + \neg\mathrm{NUB} + \neg\mathrm{CH}$.
(4) *The consistency of full ladder system coloring uniformization* $+ \neg\mathrm{NUB} + \neg\mathrm{CH}$.

Note that $2 \implies 3 \implies 4$.

In Table 1, we summarize known and unknown consistency results of each combination of truth values of the continuum hypothesis (CH), automatic continuity of homomorphisms of $C(X)$ (NDH), and Whitehead's conjecture (WhC).

Table 1. Consistency of each combination of truth values of CH, NDH, and WhC

CH	NDH	WhC	consistency
T	T	Any	inconsistent (Dales [**14**], Esterle [**20**])
T	F	T	consistent (consequences of $V = L$ (Shelah [**30**]))
T	F	F	consistent (Shelah [**31**])
F	T	T	?
F	T	F	consistent (MA + NDH is consistent by Woodin [**34**])
F	F	T	?
F	F	F	consistent (our result, Devlin and Shelah [**17**])

CHAPTER 4

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Bibliography

[1] Aoki, Yushiro. Discontinuous homomorphisms on C(X) with the negation of CH and a weak forcing axiom, submitted in 2022.

[2] Bade, W. G.; Curtis, P. C. , Jr. Homomorphisms of commutative Banach algebras. Amer. J. Math. 82 (1960), 589–608.

[3] Barnett, Janet Heine. Weak variants of Martin's axiom. Fund. Math. 141 (1992), no. 1, 61–73.

[4] Bartoszyński, Tomek; Judah, Haim. Set theory. On the structure of the real line. A K Peters, Ltd., Wellesley, MA, 1995.

[5] Bell, Murray G. On the combinatorial principle $P(\mathfrak{c})$. Fund. Math. 114 (1981), no. 2, 149–157.

[6] Blass, Andreas. Combinatorial cardinal characteristics of the continuum. Handbook of set theory. Vols. 1, 2, 3, 395–489, Springer, Dordrecht, 2010.

[7] Blass, Andreas; Laflamme, Claude. Consistency results about filters and the number of inequivalent growth types. J. Symbolic Logic 54 (1989), no. 1, 50–56.

[8] Blass, Andreas; Mildenberger, Heike On the cofinality of ultrapowers. J. Symbolic Logic 64 (1999), no. 2, 727–736.

[9] Bartoszyński, Tomek; Judah, Haim. Set theory. On the structure of the real line. A K Peters, Ltd., Wellesley, MA, 1995.

[10] Blass, Andreas; Shelah, Saharon. There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed. Ann. Pure Appl. Logic 33 (1987), no. 3, 213–243.

[11] Brendle, Jörg. Aspects of iterated forcing: the Hejnice lectures, Lecture notes, 2010.

[12] Chang, C. C.; Keisler, H. J. Model theory. Second edition Stud. Logic Found. Math., 73 North-Holland Publishing Co., Amsterdam–New York–Oxford, 1977. xii+554 pp.

[13] Dales, H. Garth. Banach algebras and automatic continuity. London Mathematical Society Monographs. New Series, 24. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

[14] Dales, H. Garth. A discontinuous homomorphism from $C(X)$. Amer. J. Math. 101 (1979), no. 3, 647–734.

[15] Dales, H. Garth. Norming infinitesimals of large fields. Foundations of mathematics, 1–29, Contemp. Math., 690, Amer. Math. Soc., Providence, RI, 2017.

[16] Dales, H. Garth.; Woodin, W. Hugh. Super-real fields. Totally ordered fields with additional structure. London Mathematical Society Monographs. New Series, 14. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[17] Devlin, Keith J.; Shelah, Saharon. A weak version of $\diamond$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. Israel J. Math. 29 (1978), no. 2–3, 239–247.

[18] van den Dries, Lou. Tame topology and o-minimal structures. London Mathematical Society Lecture Note Series, 248. Cambridge University Press, Cambridge, 1998.

[19] Eklof, Paul C.; Mekler, Alan H. Almost free modules. Set-theoretic methods North-Holland Math. Library, 46 North-Holland Publishing Co., Amsterdam, 1990.

[20] Esterle, Jean. Injection de semi-groupes divisibles dans des algèbres de convolution et construction d'homomorphismes discontinus de $C(K)$. (French) Proc. London Math. Soc. (3) 36 (1978), no. 1, 59–85.

[21] Goldstern, Martin; Kellner, Jakob; Mejía, Diego A.; Shelah, Saharon. Preservation of splitting families and cardinal characteristics of the continuum. Israel J. Math. 246 (2021), no. 1, 73–129.

[22] Hechler, Stephen H. Short complete nested sequences in $\beta N \setminus N$ and small maximal almost-disjoint families. General Topology and Appl. 2 (1972), 139–149.

[23] Hrušák, Michael. MAD families and the rationals. Comment. Math. Univ. Carolin. 42 (2001), no. 2, 345–352.

[24] Guzmán-González, Osvaldo; Hrušák, Michael; Martínez-Ranero, Carlos Azarel; Ramos-García, Ulises Ariet. Generic existence of MAD families.J. Symb. Log.82(2017), no.1, 303–316.

[25] Jech, Thomas. Set theory. The third millennium edition, revised and expanded Springer Monogr. Math. Springer–Verlag, Berlin, 2003. xiv+769 pp.

[26] Kaplansky, Irving. Normed algebras. Duke Math. J. 16 (1949), 399–418.

[27] Kibedi, Francisco Guillermo Justo. Maximal Saturated Linear Orders. Doctral Thesis, 2016.

[28] Kunen, Kenneth. Set theory. Studies in Logic (London), 34. College Publications, London, 2011.

[29] Marker, David. Model theory. An introduction. Graduate Texts in Mathematics, 217. Springer-Verlag, New York, 2002.

[30] Shelah, Saharon A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals. Israel J. Math. 21 (1975), no. 4, 319–349.

[31] Shelah, Saharon. Whitehead groups may be not free, even assuming CH. I. Israel J. Math. 28 (1977), no. 3, 193–204.

[32] Shelah, Saharon. Proper and improper forcing. Second edition. Perspect. Math. Logic. Springer–Verlag, Berlin, 1998. xlviii+1020 pp.

[33] Woodin, W. Hugh. A discontinuous homomorphism from $C(X)$ without CH. J. London Math. Soc. (2) 48 (1993), no. 2, 299–315.

[34] Woodin, W. Hugh. Discontinuous homomorphisms of $C(\Omega)$ and set theory. Thesis (Ph.D.)-University of California, Berkeley. 1984. 63 pp.