Doctoral Dissertation

Dowker spaces with normal products from combinatorial principles

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Shizuoka University
Graduate School of Science and Technology
Educational Division
Department of Information Science and Technology

Yosuke Kasai

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Chapter 1

Introduction

A normal space is a topological space which is a quite important and a natural space as a metric space. Recall that a product of normal spaces is not generally normal as for example ordinal spaces ω_1 and $\omega_1 + 1$ are normal but their product $\omega_1 \times (\omega_1 + 1)$ is not normal (Dieudonné, 1939 [9]). Historically, many researchers have attempted to find conditions for the normality in products of normal spaces. In a stream of such investigations, the *Dowker's problem* has arisen. This asks the equivalence of normality and *binormality*. A binormal space is a space for which the product with the unit interval $[0, 1]$ is normal. By the importance of binormality in homotopy theory, many researchers hoped that Dowker's problem is affirmative. A normal space which provides a counterexample for the problem is called a *Dowker space* (i.e. a normal but not a binormal space). In 1951, C. H. Dowker has discovered the following characterization of countably paracompact spaces:

Theorem 1.0.1 ([10])**.** *For every normal space X, the followings are equivalent:*

1. *X is countably paracompact;*
2. *$X \times M$ is normal for all compact metric space M;*
3. *$X \times [0, 1]$ is normal (i.e. X is binormal).*

Therefore the Dowker's problem turns to ask whether normal spaces are always countably paracompact. As we mention in chapter 2, Dowker also gave a useful equivalent condition for coutable paracomactness in the same paper.

Now it is known that a Dowker space exists. Like many counterexamples in topology, some extraordinary techniques have been used to construct Dowker spaces. In 1955, M. E. Rudin has constructed the first example of a Dowker space using an ω_1-Suslin tree [23]. So now we can say that the consistency of the existence of a Dowker space was proved relatively early. But notice that the consistency of the existence of a Suslin tree was not known then, and was proved by S. Tennenbaum in 1968 [31]. Hence the next subject was to find a Dowker space which is constructed only using ZFC. Such space is called a *ZFC Dowker space*. The existence of a ZFC Dowker space had become a big problem in topology. About 20 years later since Dowker's problem arises, Rudin has constructed the first ZFC Dowker space. Thus finally Dowker's problem was negatively solved. However, it remains a quite important question: Is there a "small" ZFC Dowker space?. Rudin's ZFC Dowker space and its basis are of size $\aleph_\omega^{\aleph_0}$ which is quite large. By small we mean not only the cardinalities of the spaces but also for topological properties, e.g. the cardinality of local basis, local cardinality , density and etc. The existence of a ZFC Dowker space itself has already been solved. But to find ZFC Dowker spaces with such small properties still now stand as extreme problems. These are called *small Dowker space problems*. By its definition, the possible minimum cardinality of Dowker spaces is $\aleph_1$ and the following are the best current solutions of small ZFC Dowker spaces: (1) Of size $2^{\aleph_0}$ given by Z. T. Balogh [2]; (2) Of size $\aleph_{\omega+1}$ given by M. Kojman and S. Shelah [18]. It is also unknown what the minimum cardinality of a ZFC Dowker space is. Note that the space of (2) is obtained as a subspace of Rudin's first ZFC Dowker space. While these results on small cardinalities are provided, it is still completely open that the existence of a ZFC Dowker space which has any of the followings: (1) Separable; (2) First countable; (3) Locally compact (countable), etc. Second countability cannot be equipped to Dowker spaces since metric spaces are paracompact (or more easily from (2) of theorem 1.0.1).

While the small Dowker space problem is difficult, by assuming certain set theoretical hypothesis, many and various small Dowker spaces have been constructed. The following table exhibits a small part of such examples which are constructed from combinatorial principles:

Given by	Underlying set	Description
Rudin, 1955 [23]	$R \times \omega$	ω_1-Suslin tree R.
de Caux, 1976 [8]	$\omega_1 \times \omega$	$\clubsuit$. Collectionwise normal, hereditary separable.
Bešlagić, 1992 [6]	$\omega_1 \times \omega_1$	$\Diamond$. This is the product of perfectly normal (hence countably paracompact) spaces.
Good, 1996 [7]	ω_1	$\clubsuit^*$. First countable, locally compact, etc.
Szeptycki, 2010 [29]	ω_2	$\Diamond^*(E^{\omega_2}_{\omega_1})$, the square is normal.

Even then, only few examples of Dowker spaces with normal products are known. As listed, Szeptycki's space has a normal square and in fact so is Rudin's first ZFC Dowker space (proved in [14]). Our motivation is to give more examples of Dowker spaces with normal products. Finally we have proved the following results:

1. Rudin's (truly) first Dowker spaces from Suslin trees may have Dowker products under some combinatorial assumptions;

2. There is a collectionwise normal Dowker space with Dowker square.

Hence we newly presented two more examples of Dowker spaces with normal products. Both normality in products are proved using two properties:

- λ-*additivity*, that is, every union of less than λ many closed subsets is closed. In particular, every λ-additive regular space of size $\leq \lambda$ is normal;

- There are no pairs of disjoint dominating (which is defined in the chapter 2) closed subsets.

However each space in our results must have size at least $\aleph_2$. Not only that, Szeptycki's space and Rudin's ZFC Dowker space have size $\aleph_2$ and $\aleph_\omega^{\aleph_0}$ respectively. We conclude the paper by explaining a necessary condition on Dowker spaces to have normal products. Related to this, we state a problem that asks the existence of a Dowker space of size $\aleph_1$ which satisfies the necessary condition.

Lastly, related to our motivation, we would like to summarize several facts around Dowker spaces with the sight of normality in products. For every statement ϕ that describes a topological property, let (ϕ) denote the class of all topological spaces with the property ϕ and $\mathcal{P}(\phi)$ the class of all topological spaces X such that $X \times Y$ is normal for all $Y \in (\phi)$. Using this notation, we can arrange important theorems on the normality in products as follows:

1. $\mathcal{P}$(compact & metric) = (countably paracompact & normal) (C. H. Dowker, [10]);

2. $\mathcal{P}$(compact) = (paracompact & Hausdorff) (H. Tamano, [30]);

3. $\mathcal{P}$(metrizable) = (Morita's P-space) (K. Morita, [19]);

4. $\mathcal{P}(\mathcal{P}$(normal)) = (discrete) (Morita's first conjecture [20]);

5. $\mathcal{P}(\mathcal{P}$(metrizable)) = (metrizable) (Morita's second conjecture [20]);

6. $\mathcal{P}$(countably paracompact & normal)
   = (σ-locally compact & metric) (Morita's third conjecture [20]).

Note that: (1) Perfectly normal spaces are countably paracompact [10]. (2) A normal submetrizable space is perfect if and only if it is a Morita's P-space [1]. Again, notice that Dowker spaces are not metrizable. But it is not known that there is a *submetrizable* (i.e. the topology contains a metrizable sub-topology) Dowker space. Back to looking the list. Previously described, Balogh has constructed a ZFC Dowker space of size $2^{\aleph_0}$ in 1996. Two years later, he developed another construction of a ZFC Dowker space of size $2^{\aleph_0}$ [4]. Using the latter method, he solved Morita's second and third conjectures [3, 5]. Furthermore, before Balogh's results, Rudin solved the first conjecture with a technique on Dowker spaces [26]. Remember that researches on Dowker spaces originated from Dowker's theorem which provides a translation of the countable paracompactness in terms of normality in products. By a chain of relevancy, we may say that investigations on normality in products make a substantial contribution to Dowker spaces and vice versa.

Chapter 2

Preliminaries

2.1 Basics on Dowker spaces

In this thesis, we use $\overline{\overline{X}}$ to indicate the cardinality of a set X. First let us recall basic definitions on topological spaces.

Definition 2.1.1. *Suppose that $\langle X, \tau \rangle$ is a topological space.*

- *A sequence $\langle F_i \rangle_{i\in\Lambda} \subseteq \mathcal{P}(X)$ is separated if and only if there is a pairwise disjoint sequence $\langle U_i \rangle_{i\in\Lambda} \subseteq \tau$ s.t. $(\forall i \in \Lambda)[F_i \subseteq U_i \in \tau]$.*
- *X is normal if and only if every pair of two disjoint closed subsets is separated.*
- *X is countably paracompact if and only if every countable open cover of X has a locally finite refinement.*
- *X is countably metacompact if and only if every countable open cover of X has a point finite refinement.*
- *A sequence $\langle U_i \rangle_{i\in\Lambda}$ of subsets of X is an open expansion of $\langle F_i \rangle_{i\in\Lambda}$ if and only if $(\forall i \in \Lambda)[F_i \subseteq U_i]$ and $\bigcap_{i\in\Lambda} F_i = \bigcap_{i\in\Lambda} U_i$.*

Countable paracompactness and countable metacompactness are also understood via countable sequences of closed subsets and those open expansions. The following definitions have been introduced by M. E. Rudin and M. Starbird [27].

Definition 2.1.2.

- *We say that a space X has CPN if and only if every decreasing sequence $\langle F_n \rangle_{n\in\omega}$ of τ-closed subsets with $\bigcap_{n\in\omega} F_n = \varnothing$ has an open expansion.*

- *We say that a space X has CP if and only if for every decreasing sequence $\langle F_n\rangle_{n\in\omega}$ of τ-closed subsets with $\bigcap_{n\in\omega} F_n = \varnothing$, there is a sequence $\langle H_n\rangle_{n\in\omega}$ of τ-closed subsets such that $\langle \mathrm{int} H_n\rangle_{n\in\omega}$ expands $\langle F_n\rangle_{n\in\omega}$.*

Clearly CP implies CPN. The next is a part of Dowker's characterizations of countably paracompact spaces.

Theorem 2.1.3 ([10])**.** *If X is a normal space, then X is countably paracompact if and only if X has CPN.*

Theorem 2.1.3 is interpreted as "countable paracompactness and countable metacompactness coincide in normal spaces" by the next theorem. Generally, countable paracompactness is a notion stronger than countable metacompactness.

Theorem 2.1.4 ([15])**.** *For every topological space X:*

1. *X is countably paracompact if and only if X has CP;*
2. *X is countably metacompact if and only if X has CPN.*

In this paper, all arguments on Dowker-ness will be conducted using the property CPN. If $\vec{C}$ is a decreasing sequence of countably many closed subsets of X with $\bigcap(\vec{C}) = \varnothing$, then we say that $\vec{C}$ is a sequence for ¬CPN of X.

The next two lemmas provide some useful sufficient conditions for Dowker-ness which will work on spaces in chapter 4.

Lemma 2.1.5 (e.g. lemma 3.1 of [21])**.** *Let X be a normal space which has no pairs of disjoint closed subsets of size $\overline{\overline{X}}$. Then X is Dowker if $\omega < \mathrm{cf}(\overline{\overline{X}})$ and X has a sequence for ¬CPN consisting of size $\overline{\overline{X}}$ subsets.*

Proof. Fix a normal space X as above and let $\vec{C} = \langle C\rangle_{n\in\omega}$ be a sequence of closed subsets such that $(\forall n \in \omega)[C_{n+1} \subseteq C_n \in [X]^{\overline{\overline{X}}}]$ and $\bigcap(\vec{C}) = \varnothing$. By theorem 2.1.3, to conclude that X is Dowker, we prove $\vec{C}$ has no open expansion. Towards a contradiction, suppose that $\vec{U} = \langle U_n\rangle_{n\in\omega}$ is an open expansion of $\vec{C}$ i.e. $(\forall n \in \omega)[C_n \subseteq U_n : \text{open}]$ and $\bigcap(\vec{U}) = \varnothing$. By $X = \bigcup\{X \setminus U_n \mid n \in \omega\}$ and the uncountable cofinality of $\overline{\overline{X}}$, $\overline{\overline{X \setminus U_n}} = \overline{\overline{X}}$ for some $n \in \omega$. But then $\langle C_n, X \setminus U_n\rangle$ forms a pair of disjoint closed subsets of size $\overline{\overline{X}}$. This contradicts the assumption of X. ■

Analogously, we can extend lemma 2.1.5 to product spaces. Before stating this, we need a definition.

Definition 2.1.6. *We say that a subset Z of $X \times Y$ is dominating if and only if there is no pair $\langle W, W'\rangle \in [X]^{<\overline{\overline{X}}} \times [Y]^{<\overline{\overline{Y}}}$ such that $Z \subseteq (W \times Y) \cup (X \times W')$.*

Lemma 2.1.7. *Suppose that $\omega < \min(\mathrm{cf}(\overline{\overline{X}}), \mathrm{cf}(\overline{\overline{Y}}))$ and the product $X \times Y$ is normal having no pairs of disjoint closed and dominating subsets. Then $X \times Y$ is Dowker if $X \times Y$ has a sequence for $\neg$CPN consisting of dominating subsets.*

Proof. We can prove this similarly to of lemma 2.1.5. Let $\vec{C}$ be a sequence for $\neg$CPN of $X \times Y$ and $\vec{V}$ be an open expansion of $\vec{C}$. By $X \times Y = \bigcup\{(X \times Y) \setminus V_n \mid n \in \omega\}$ and $\omega < \min(\mathrm{cf}(\overline{\overline{X}}), \mathrm{cf}(\overline{\overline{Y}}))$, $(X \times Y) \setminus V_n$ is dominating for some $n \in \omega$. This leads to a contradiction. ■

2.2 λ-additive spaces

The λ-*additivity*, which has been introduced in [28] by R. Sikorski, plays a substantial role to realize our results.

Definition 2.2.1. *Let λ be an infinite cardinal. A topological space X is said to be λ-additive if and only if $\mathrm{Cl}_X(\bigcup(\mathfrak{A})) = \bigcup\{\mathrm{Cl}_X(A) \mid A \in \mathfrak{A}\}$ for all family $\mathfrak{A}$ of less than λ-many subsets of X.*

Note. (1) Every topological space is $\aleph_0$-additive. (2) A space is a P-*space* (different from Morita's sense) if and only if it is $\aleph_1$-additive. (3) A completely regular space is a P-space if and only if every G_δ subset is open. Couples of algebraic characterizations of P-spaces are established in [13] pp. 62-63 (cf. also [12] p. 350).

Lemma 2.2.2.

1. *Every subspace of a λ-additive space is λ-additive.*
2. *X is λ-additive if and only if every union of less than λ-many closed subsets of X is closed.*
3. *X is λ-additive if and only if every intersection of less than λ-many open subsets of X is open.*

To prove our main theorems, we shall use λ-additivity in products of λ-additive spaces. The product is of course the Tikhonov product, but we would like to explain the λ-additivity of products in a more general situation.

Definition 2.2.3. *Let μ, ρ be cardinals and $\{X_\alpha\}_{\alpha<\rho}$ be a sequence of topological spaces. Then $< \mu$-box product of $\{X_\alpha\}_{\alpha<\rho}$ denoted by ${}^{<\mu}\square_{\alpha<\rho} X_\alpha$ is the topological space on $\prod_{\alpha<\rho} X_\alpha$ whose the topology is generated by $\{\prod_{\alpha<\rho} U_\alpha \mid (\forall \alpha < \rho)[U_\alpha$ is open in $X_\alpha] \wedge \overline{\overline{\{\alpha < \rho \mid U_\alpha \neq X_\alpha\}}} < \mu\}$ as a basis.*

Note. ${}^{<\omega}\square_{\alpha<\rho} X_\alpha$ (resp. ${}^{<\rho^+}\square_{\alpha<\rho} X_\alpha$) is the same as the usual Tikhonov (resp. box) product of $\{X_\alpha\}_{\alpha<\rho}$. The name of *middle box product* is also used to mention spaces of the type ${}^{<\mu}\square_{\alpha<\rho} X_\alpha$ when μ is not specified.

Lemma 2.2.4. *Suppose that λ, ρ and μ are cardinals which satisfy either $\lambda \leq \mathrm{cf}(\mu)$ or $\rho < \mu$. If $\{X_\alpha\}_{\alpha<\rho}$ is a sequence of λ-additive spaces, then ${}^{<\mu}\square_{\alpha<\rho} X_\alpha$ is λ-additive.*

Proof. Let λ, ρ and μ be cardinals and $\{X_\beta\}_{\beta<\rho}$ be a sequence of λ-additive spaces. Generally, to prove the λ-additivity of a space, it suffices to verify that every intersection of less than λ-many basic open subsets is open in the space. Fix $\alpha < \lambda$ and let $\{\prod_{\beta<\rho} U^\xi_\beta\}_{\xi<\alpha}$ be a family of basic open subsets of ${}^{<\mu}\square_{\beta<\rho} X_\beta$. We must check that $\bigcap_{\xi<\alpha} \prod_{\beta<\rho} U^\xi_\beta$ is open in ${}^{<\mu}\square_{\beta<\rho} X_\beta$. Let $\Lambda_\xi := \{\beta < \rho \mid U^\xi_\beta \neq X_\beta\}$ for all $\xi < \alpha$. By the definition of the topology on ${}^{<\mu}\square_{\beta<\rho} X_\beta$, we have the following conditions.

- U^ξ_β is open in X_β for all $\langle \xi, \beta\rangle \in \alpha \times \rho$.
- $\overline{\overline{\Lambda_\xi}} < \mu$ for all $\xi < \alpha$.

Let $V_\beta := \bigcap_{\xi<\alpha} U^\xi_\beta$ for all $\beta < \rho$. Note that $\bigcap_{\xi<\alpha} \prod_{\beta<\rho} U^\xi_\beta = \prod_{\beta<\rho} V_\beta$ and each of V_β is open in X_β by λ-additivity. So, if μ satisfies either $\lambda \leq \mathrm{cf}(\mu)$ or $\rho < \mu$, then $\bigcap_{\xi<\alpha} \prod_{\beta<\rho} U^\xi_\beta$ is open in ${}^{<\mu}\square_{\beta<\rho} X_\beta$ because $\{\beta < \rho \mid V_\beta \neq X_\beta\} = \bigcup_{\xi<\alpha} \Lambda_\xi$. ■

Corollary 2.2.5. *An arbitrary finite product of λ-additive spaces is λ-additive.*

Lemma 2.2.6. *If λ is an infinite cardinal, then every regular λ-additive space of size $\leq \lambda$ is normal.*

Proof. Suppose that a regular space X is λ-additive and $\overline{\overline{X}} \leq \lambda$. Let $\langle F^0, F^1\rangle$ be a pair of disjoint closed subsets of X and $f^i : \lambda \to F^i$ be an enumeration of F^i for each $i \in 2$. For every $\xi < \lambda$ and $i \in 2$, since $f^i(\xi) \notin F^{1-i}$, using regularity we can find a neighborhood u^i_ξ of $f^i(\xi)$ such that $\mathrm{Cl}_X(u^i_\xi) \cap F^{1-i} = \varnothing$. Let $U^i_\xi := \bigcap\{X \setminus \mathrm{Cl}_X(u^{1-i}_\eta) \mid \eta \leq \xi\}$. Then for all $\xi < \lambda$ and $i \in 2$, $F^i \subseteq U^i_\xi$ and U^i_ξ is open by λ-additivity of X. Let $\langle W^i\rangle_{i\in 2} := \langle \bigcup\{u^i_\xi \cap U^i_\xi \mid \xi < \lambda\}\rangle_{i\in 2}$. Then $\langle W^i\rangle_{i\in 2}$ separates $\langle F^i\rangle_{i\in 2}$. ■

Remark. The lemma generalizes a basic fact on regular spaces: every countable regular space is normal (cf. p.44 in [11]). Lemma 2.2.6 for the case $\lambda = \omega_1$ is the assertion that 'every P-space of size $\leq \omega_1$ is normal', which is known as a corollary of theorem 7.7-(1) of the paper [12].

Chapter 3

A construction from Suslin trees

3.1 Introduction

The main subject of this chapter is Rudin's (truly) first Dowker space from Suslin trees. In 1955, Rudin constructed the first example (but not in ZFC) of a Dowker space on $R \times \omega$, here R is an ω_1-Suslin tree [23]. This construction is valid for the successor of an uncountable regular cardinal λ^+, but then a λ^+-Suslin tree R' is used to define a topology on $R' \times \omega$ [24]. On these spaces, we proved the following:

Main theorem A. *[17] Let λ be an uncountable regular cardinal. Suppose that $\{R^i\}_{i \in N}$ is a finite collection of λ^+-Suslin trees such that $\prod_{i \in N} R^i$ is λ^+-c.c. Then the product space $\prod_{i \in N} X^i$ of Rudin's Dowker spaces $\{\langle R^i \times \omega, \tau(R^i) \rangle\}_{i \in N}$ is also Dowker.*

Hence we present a new example of Dowker spaces with normal products. But as both sizes of Rudin's first ZFC Dowker space and Szeptycki's Dowker space are larger than $\aleph_1$, our result cannot stand when $\lambda = \omega_1$. Moreover this restriction cannot be dropped by the reason explained in chapter 5.

3.2 Definition of a Rudin's Dowker space from a Suslin tree.

Let us start this by recalling some basics on general trees.

Definition 3.2.1. *Suppose that $\langle R, \leq_R \rangle$ be a tree and $\alpha \in \mathrm{ORD}$.*

1. *For $x \in R$, $\mathrm{ht}_R(x)$ denotes the order type of $\{y \in R \mid y <_R x\}$.*

2. *For $\Delta \in \{\leq, \geq, <, >, =\}$, $R_{\Delta\alpha} := \{x \in R \mid \mathrm{ht}_R(x) \Delta \alpha\}$.*

3. *$R_\alpha := R_{=\alpha}$.*

4. height$(R) := \min\{\alpha \in \mathrm{ORD} \mid R_\alpha = \varnothing\}$.

5. $[x]_R := \{y \in R \mid x \leq_R y\}$, $(x)_R := [x]_R \setminus \{x\}$.

6. For $\alpha \leq \mathrm{ht}_R(x)$, $x \restriction \alpha$ *denotes the element* $y \in R_\alpha$ *such that* $y \leq_R x$.

Definition 3.2.2. *Let* κ *be an uncountable regular cardinal. A tree* $\langle R, \leq \rangle$ *is said to be*

1. *a* κ*-tree if and only if* height$(R) = \kappa$ *and* $(\forall \alpha < \kappa)[\overline{\overline{R_\alpha}} < \kappa]$.
2. κ*-Suslin if and only if it is a* κ*-tree whose chains and antichains have size less than* κ.

From here, let λ be an infinite regular cardinal, R be a tree of height λ^+ and $S := \{x \in R \mid \mathrm{cf}(\mathrm{ht}_R(x)) = \lambda\}$. Suppose that we can choose a family $\{f_\xi \mid \xi < \lambda\}$ of functions from S into S so that satisfies the properties below:

1. For all $x \in S$ and $\xi < \lambda$, $\mathrm{ht}_R(f_\xi(x)) = \mathrm{ht}_R(x)$.
2. For all $y \in R$ and $x \in S \cap (y)_R$, there exists $\xi < \lambda$ such that $\{f_\eta(x) \mid \eta \in \lambda \setminus (\xi + 1)\} \subseteq (y)_R$.
3. For all $x, y \in S$ and $\xi, \eta < \lambda$, if $x \neq y$ then $f_\xi(x) \neq f_\eta(y)$.

Families like the above are available for usual λ^+-Suslin trees. Throughout this chapter, we always consider Suslin trees with such usual properties. Now we're ready to define the topology $\tau(R)$.

1. For all $\langle x, n \rangle \in S \times \omega$, $\alpha < \lambda$ and $\beta < \lambda^+$,
   - $F_R(\langle x, 0 \rangle, \alpha) := \varnothing$,
   - $F_R(\langle x, n+1 \rangle, \alpha) := \{\langle f_\xi(x), n \rangle \mid \xi \in \lambda \setminus (\alpha + 1)\}$,
   - $G_R(\langle x, n \rangle, \beta) := \{\langle y, n \rangle \mid \beta < \mathrm{ht}_R(y) \wedge y \leq_R x\}$.
2. For all $\langle x, n \rangle \in (R \setminus S) \times \omega$, $\alpha < \lambda$ and $\beta < \lambda^+$,
   - $F_R(\langle x, n \rangle, \alpha) := \varnothing$,
   - $G_R(\langle x, n \rangle, \beta) := \{\langle x, n \rangle\}$.
3. $U \subseteq R \times \omega$ is $\tau(R)$-open if and only if for all $\langle x, n \rangle \in R \times \omega$ there is $\langle \alpha, \beta \rangle \in \lambda \times \mathrm{ht}_R(x)$ such that $F_R(\langle x, n \rangle, \alpha) \cup G_R(\langle x, n \rangle, \beta) \subseteq U$.

The next lemma is a list of properties of the topology $\tau(R)$ which will easily be proved using its definition or the properties of the tree R.

Lemma 3.2.3. *Let λ be an infinite cardinal and R be a λ^+-Suslin tree. Then the topological space $X = R \times \omega$ defined by $\tau(R)$ has the following properties:*

1. *X is λ-additive and Hausdorff.*

2. *$F_R(\langle x, n\rangle, \alpha) \cup G_R(\langle x, n\rangle, \beta)$ and $G_R(\langle x, n\rangle, \beta)$ are both closed for all $\alpha < \lambda$, $\beta < \mathrm{ht}_R(x)$ and $\langle x, n\rangle \in X$.*

3. *For all $x \in R$ and $k \leq \omega$, $[x]_R \times k$ is closed and $(x)_R \times k$ is open. In particular, $(x)_R \times \omega$ is clopen.*

4. *For all $\alpha < \lambda$, $R_{\leq\alpha} \times \omega$ is clopen. So is $R_{>\alpha} \times \omega$.*

5. *$R \times (\omega \setminus n)$ is closed for all $n \in \omega$.*

6. *Let $U \in \tau(R)$, $\langle x, m\rangle \in U$ and $\langle y, n\rangle \in X$. If $\mathrm{cf}(\mathrm{ht}_R(x)) = \lambda$, $y <_R x$ and $n < m$ then there is $z \in (y)_R$ such that $\mathrm{ht}_R(z) = \mathrm{ht}_R(x)$ and $\langle z, n\rangle \in U$.*

Proof. We only prove (6). Suppose that $\langle x, m\rangle \in U \in \tau(R)$ and $\mathrm{cf}(\mathrm{ht}_R(x)) = \lambda$. Then by the definition of $\tau(R)$, there is $\alpha_0 < \lambda$ such that $F_R(\langle x, n\rangle, \alpha_0) \subseteq U$. Next, let $y <_R x$, $n < m$ and $k := m - n$. Then there is $\xi_0 \in \lambda \setminus (\alpha_0 + 1)$ such that $y <_R f_{\xi_0}(x)$, where f_{ξ_0} is the function that is fixed to define $\tau(R)$. Let $x_0 := f_{\xi_0}(x)$. If $m - 1 = n$, then we can finish the proof since $\langle x_0, m - 1\rangle \in F_R(\langle x, m\rangle, \alpha_0) \subseteq U$. Otherwise, pick some $\alpha_1 < \lambda$ and $\xi_1 \in \lambda \setminus (\alpha_1 + 1)$ such that $F_R(\langle x_0, m-1\rangle, \alpha_1) \subseteq U$ and $y <_R f_{\xi_1}(x_0)$. Let $x_1 := f_{\xi_1}(x_0)$. Then it follows that $\langle x_1, m - 2\rangle \in U$. Repeating this argument, finally we have $\langle x_{k-1}, n\rangle \in U$. x_{k-1} is a desired z in the statement. ■

Theorem 3.2.4 (M. E. Rudin, [23, 24])**.** *Let κ be the successor of an infinite regular cardinal. If $\langle R, \leq_R\rangle$ is a κ-Suslin tree, then there is a topology $\tau(R)$ on $X = R \times \omega$ such that $\langle X, \tau(R)\rangle$ is Dowker.*

Based on this theorem, in the next section we prove the theorem A.

3.3 Proof of the main theorem.

Throughout this section, we fix an uncountable regular cardinal λ, $N \in \omega$ and a family $\{R^i\}_{i\in N}$ of λ^+-Suslin trees such that $\prod_{i\in N} R^i$ is λ^+-c.c. The existence of such Suslin trees is guaranteed by the following fact.

Fact 3.3.1 ([16], Theorem 27.7)**.** *Let λ be an infinite regular cardinal. If $\Diamond(E^{\lambda^+}_{\lambda}) + \lambda^{<\lambda} = \lambda$ holds, then for any $N \in \omega$ there are λ^+-Suslin trees $\{R^i\}_{i\in N}$ such that $\prod_{i\in N} R^i$ is λ^+-c.c.*

Notations and remarks.

1. For each $i \in N$, X^i denotes the corresponding Dowker space $\langle R^i \times \omega, \tau(R^i)\rangle$ and $X^i_{\Delta\alpha} := R^i_{\Delta\alpha} \times \omega$ where $\alpha < \lambda^+$ and $\Delta \in \{\leq, \geq, <, >, =\}$.
2. Sometimes, $\gamma \leq \lambda^+$ will be treated as the constant function $\langle \gamma \rangle_{i \in N}$.
3. We use bold-styled symbols to indicate functions on a subset of N. Namely, we shall use $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ for elements of $\prod_{i\in I} R^i$, $\boldsymbol{m}$ and $\boldsymbol{n}$ for $\prod_{i \in J} \omega$ and $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$ for $\prod_{i\in K} X^i$ (I, J and K are subsets of N).
4. When a subset I of N is clear by the context, we use $\mathrm{ht}(\boldsymbol{x})$ to describe $\max\{\mathrm{ht}_{R^i}(\boldsymbol{x}(i)) \mid i \in I\}$.
5. '$\leq$' will be used as the coordinate-wise order on $\prod_{i\in N} \omega$ and $\prod_{i \in N} R^i$.
6. 'Cl' denotes the closure operator of the space $\prod_{i\in N} R^i$ w.r.t. the topology generated by $\{\prod_{i \in N}[\boldsymbol{x}(i)]_{R^i} \mid \boldsymbol{x} \in \prod_{i\in N} R^i\}$ as a basis.
7. '$\mathfrak{Cl}_I$' denotes the closure operator of the product space $\prod_{i\in I} X^i$ where $I \subseteq N$.

Definition 3.3.2.

1. *Let $I \subseteq N$, $\boldsymbol{x} \in \prod_{i\in I} R^i$, $Q \subseteq \prod_{i \in I} R^i$ and $\alpha < \lambda^+$.*
   (a) *Q is $\boldsymbol{x}$-dense if and only if Q is dense in $\prod_{i\in I}[\boldsymbol{x}(i)]_{R^i}$ w.r.t. the coordinate-wise ordering of $\langle \leq_{R^i}\rangle_{i\in I}$ (i.e. $(\forall \boldsymbol{y} \geq \boldsymbol{x})[(\prod_{i\in I}[\boldsymbol{y}(i)]_{R^i}) \cap Q \neq \varnothing]$).*
   (b) *$L(\boldsymbol{x}, \alpha, Q)$ is a maximal antichain in $(\prod_{i\in I}([\boldsymbol{x}(i)]_{R^i} \cap R^i_{\geq\alpha})) \cap Q$.*
   (c) *When $L(\boldsymbol{x}, \alpha, Q) \neq \varnothing$, $\delta(\boldsymbol{x}, \alpha, Q)$ denotes the minimal $\beta < \lambda^+$ such that $(\forall \boldsymbol{y} \in L(\boldsymbol{x}, \alpha, Q))(\forall i \in I)[\mathrm{ht}_{R^i}(\boldsymbol{y}(i)) < \beta]$ (such β can be taken because of the λ^+-c.c. of $\prod_{i\in I} R^i$). If $L(\boldsymbol{x}, \alpha, Q) = \varnothing$, we set $\delta(\boldsymbol{x}, \alpha, Q) := \alpha + 1$.*
2. *Let $I \subseteq N$, $\boldsymbol{x} \in \prod_{i\in I} R^i$, $\boldsymbol{k} \in \prod_{i \in I} \omega$ and $\alpha < \lambda^+$.*
   (a) *$[\boldsymbol{x}, I] := \prod_{i\in I}[\boldsymbol{x}(i)]_{R^i}$ and $(\boldsymbol{x}, I) := \prod_{i \in I}(\boldsymbol{x}(i))_{R^i}$.*
   (b) *$[\boldsymbol{x}, I]_{\Delta\alpha} := \prod_{i\in I}([\boldsymbol{x}(i)]_{R^i} \cap R^i_{\Delta\alpha})$ for $\Delta \in \{\leq, \geq, <, >, =\}$.*
   (c) *$[\boldsymbol{x}, I]_\alpha := [\boldsymbol{x}, I]_{=\alpha}$.*
   (d) *$[\boldsymbol{x}, \boldsymbol{k}, I] := \prod_{i\in I}([\boldsymbol{x}(i)]_{R^i} \times \boldsymbol{k}(i))$.*
   (e) *$(\boldsymbol{x}, \boldsymbol{k}, I) := \prod_{i\in I}((\boldsymbol{x}(i))_{R^i} \times \boldsymbol{k}(i))$.*
3. *For $I \subseteq N$, $\boldsymbol{p} \in \prod_{i\in I} X^i$ and $\boldsymbol{\alpha} \in \prod_{i\in I} \lambda^+$,*

   $\mathfrak{G}(\boldsymbol{p}, \boldsymbol{\alpha}, I) := \prod_{i\in I} G_{R^i}(\boldsymbol{p}(i), \boldsymbol{\alpha}(i))$.

Lemma 3.3.3. *Suppose that $Q \subseteq \prod_{i\in N} R^i$ is upward-closed (i.e. $[\boldsymbol{x}, N] \subseteq Q$ for all $\boldsymbol{x} \in Q$). Then there is $\alpha < \lambda^+$ such that* $\mathrm{Cl}(Q) \cap (\prod_{i\in N} R^i_{\geq\alpha}) \subseteq Q$.

Proof. Recall that for any $\boldsymbol{x} \in \prod_{i\in N} R^i$ and $A \subseteq \prod_{i\in N} R^i$, $\boldsymbol{x}$ is a member of $\mathrm{Cl}(A)$ if and only if $[\boldsymbol{x}, N] \cap A \neq \varnothing$. Suppose that a subset Q of $\prod_{i\in N} R^i$ is upward closed. Let Q' be a maximal antichain in Q. Then $\overline{\overline{Q'}} < \lambda^+$ by λ^+-c.c. of $\prod_{i\in N} R^i$. Pick some $\alpha < \lambda^+$ such that $\sup\{\mathrm{ht}(\boldsymbol{x}) \mid \boldsymbol{x} \in Q'\} < \alpha$. Choose arbitrary $\boldsymbol{x} \in \mathrm{Cl}(Q) \cap (\prod_{i\in N} R^i_{\geq\alpha})$. Then there is $\boldsymbol{y} \in ([\boldsymbol{x}, N]) \cap Q$. Since Q' is a maximal antichain in Q, there is $\boldsymbol{z} \in Q'$ such that $(\forall i \in N)[\boldsymbol{z}(i) \not\perp_{R^i} \boldsymbol{y}(i)]$. Now $\mathrm{ht}_{R^i}(\boldsymbol{z}(i)) < \alpha \leq \mathrm{ht}_{R^i}(\boldsymbol{x}(i))$ and $\boldsymbol{x}(i) \leq_{R^i} \boldsymbol{y}(i)$ for all $i \in N$. It follows that $\boldsymbol{x} \geq \boldsymbol{z}$. Thus $\boldsymbol{x} \in Q$ because Q is upward-closed. ■

In lemma 3.3.4, we shall prove that the product space is not countably metacompact. Note that (countable) paracompactness implies (countable) metacompactness. So the next lemma 3.3.4 implies that the product space is not countably paracompact. We don't need any modifications to Rudin's original proof.

Lemma 3.3.4. *The product space $\prod_{i\in N} X^i$ is not countably metacompact.*

Proof. Let $C_n := \prod_{i\in N}(R^i \times (\omega \setminus (n+1)))$. $\vec{C} := \langle C_n \rangle_{n\in\omega}$ is a sequence for $\neg$CPN, so we shall prove $\vec{C}$ has no open expansion. Towards a contradiction, suppose that $\vec{D} := \langle D_n \rangle_{n\in\omega}$ is an open expansion of $\vec{C}$. For each $n \in \omega$, let $Q_n := \{\boldsymbol{x} \in \prod_{i\in N} R^i \mid [\boldsymbol{x}, 1, N] \subseteq D_n\}$. Clearly Q_n's are all upward-closed in $\prod_{i\in N} R^i$. So applying lemma 3.3.3, for each $n \in \omega$ there is $\alpha_n < \lambda^+$ such that $\mathrm{Cl}(Q_n) \cap \prod_{i\in N} R^i_{\geq\alpha_n} \subseteq Q_n$. Let $\alpha := \sup\{\alpha_n \mid n \in \omega\}$ and fix arbitrary $\boldsymbol{x} \in \prod_{i\in N} R^i_\alpha$.

Claim 3.3.5. *$P_n := \{\boldsymbol{y} \in \prod_{i\in N} R^i \mid \langle \boldsymbol{y}(i), 0\rangle_{i\in N} \notin D_n\}$ is $\boldsymbol{x}$-dense for some $n \in \omega$.*

Proof. Note that $\bigcap\{Q_n \mid n \in \omega\} = \varnothing$ because $\bigcap(\vec{D}) = \varnothing$. So there is $n \in \omega$ such that $\boldsymbol{x} \notin Q_n$. Since $\mathrm{Cl}(Q_n) \cap \prod_{i\in N} R^i_{\geq\alpha} \subseteq Q_n$ and $\boldsymbol{x} \in \prod_{i\in N} R^i_\alpha$, it follows that $\boldsymbol{x} \notin \mathrm{Cl}(Q_n)$ i.e. $[\boldsymbol{x}, N] \cap Q_n = \varnothing$. Then for every $\boldsymbol{y} \in [\boldsymbol{x}, N]$ we have $\boldsymbol{y} \notin Q_n$, so by the definition of Q_n, there is $\boldsymbol{z} \in [\boldsymbol{y}, N]$ such that $\langle \boldsymbol{z}(i), 0\rangle_{i\in N} \notin D_n$ i.e. $\boldsymbol{z} \in P_n$. Therefore P_n is $\boldsymbol{x}$-dense. □

Next define $\gamma < \lambda^+$ as follows.

$$\gamma_0 := \mathrm{ht}(\boldsymbol{x}),$$
$$\gamma_\beta := \sup\{\delta(\boldsymbol{x}, \gamma_{\beta'}, P_n) \mid \beta' < \beta\},$$
$$\gamma := \sup\{\gamma_\beta \mid \beta < \lambda\}.$$

$\{\gamma_\beta\}_{\beta<\lambda}$ is strictly increasing hence γ has cofinality λ. Pick some $\boldsymbol{y} \in [\boldsymbol{x}, N]_\gamma$ and let $\boldsymbol{p} := \langle \boldsymbol{y}(i), n+1\rangle_{i\in N}$. Since $\boldsymbol{p} \in C_n \subseteq D_n$ and D_n is open, there is $\langle U_i\rangle_{i\in N} \in \prod_{i\in N} \tau(R^i)$ such that $\boldsymbol{p} \in \prod_{i\in N} U_i \subseteq D_n$. Also applying lemma 3.2.3-(6), we can find $\boldsymbol{z} \in [\boldsymbol{x}, N]_\gamma$ such that $\boldsymbol{q} := \langle \boldsymbol{z}(i), 0\rangle_{i\in N} \in \prod_{i\in N} U_i$. By the definition of $\tau(R^i)$'s, $\mathfrak{G}(\boldsymbol{q}, \boldsymbol{\alpha}, N) \subseteq \prod_{i\in N} U_i$ for some $\boldsymbol{\alpha} \in \prod_{i\in N} \mathrm{ht}_{R^i}(\boldsymbol{z}(i)) = \prod_{i\in N} \gamma$. Pick $\beta < \lambda$ such that $\max\{\boldsymbol{\alpha}(i) \mid$

$i \in N\} < \gamma_\beta$. Since P_n is $\boldsymbol{x}$-dense and $\boldsymbol{z} \in [\boldsymbol{x}, N]$, there is $\boldsymbol{w} \in [\boldsymbol{z}, N] \cap P_n$. Now $\boldsymbol{w} \in [\boldsymbol{x}, N]_{\geq\gamma_\beta} \cap P_n$, so we can find some $\boldsymbol{w'} \in L(\boldsymbol{x}, \gamma_\beta, P_n)$ such that $(\forall i \in N)[\boldsymbol{w}(i) \not\leq_{R^i} \boldsymbol{w'}(i)]$. It follows that $\boldsymbol{w'} \leq \boldsymbol{z}$ because $\mathrm{ht}(\boldsymbol{w'}(i)) < \gamma_{\beta+1} < \gamma = \mathrm{ht}(\boldsymbol{z}(i))$ for all $i \in N$. Moreover $(\forall i \in N)[\boldsymbol{\alpha}(i) < \gamma_\beta \leq \mathrm{ht}_{R^i}(\boldsymbol{w'}(i))]$, hence $\langle \boldsymbol{w}'(i), 0\rangle_{i\in N} =: \boldsymbol{q'} \in \mathfrak{G}(\boldsymbol{q}, \boldsymbol{\alpha}, N)$. Since $\mathfrak{G}(\boldsymbol{q}, \boldsymbol{\alpha}, N) \subseteq D_n$, it follows that $\boldsymbol{q'} \in D_n$. On the other hand, by $\boldsymbol{w'} \in P_n$ and the definition of P_n we have $\boldsymbol{q'} \notin D_n$. This is a contradiction. Therefore $\vec{C}$ has no open expansion. ■

The rest of the paper devotes the proof of the normality of $\prod_{i\in N} X^i$. We already known each of $\{X^i\}_{i\in N}$ is normal by theorem 3.2.4, so in particular $\prod_{i\in N} X^i$ is at least regular. This is a key to prove subsequent lemmas together with λ-additivity.

Definition 3.3.6.

1. *For any sets A and B which consist of functions,*

$$A^\frown B := \{f \cup g \mid \langle f, g\rangle \in A \times B \wedge \mathrm{dom}(f) \cap \mathrm{dom}(g) = \varnothing\}.$$

2. *Let $I \subseteq N$.*

   (a) *For $A \subseteq \prod_{i\in N} X^i$, $B \subseteq \prod_{i\in N\setminus I} X^i$ and $\boldsymbol{m} \in \prod_{i\in I} \omega$, let*
   $A\restriction(B, \boldsymbol{m}) := \{\boldsymbol{x} \in \prod_{i\in I} R^i \mid (\exists \boldsymbol{q} \in \mathfrak{Cl}_{N\setminus I}(B))[\langle \boldsymbol{x}(i), \boldsymbol{m}(i)\rangle_{i\in I} \cup \boldsymbol{q} \in A]\}$.

   (b) *For $\boldsymbol{p} \in \prod_{i\in I} X^i$, let*
   $\mathfrak{N}(\boldsymbol{p}) := \{\prod_{i\in I}(U_i \cap X^i_{\leq \mathrm{ht}(\boldsymbol{p})}) \mid (\forall i \in I)[\boldsymbol{p}(i) \in U_i \in \tau(R^i)]\}$.

   (c) *For $A \subseteq \prod_{i\in N} X^i$, $\boldsymbol{x} \in \prod_{i\in I} R^i$, $\boldsymbol{m} \in \prod_{i\in I} \omega$ and $\boldsymbol{q} \in \prod_{i\in N\setminus I} X^i$,*

   $\Phi(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ holds if and only if $A\restriction(\mathfrak{U}, \boldsymbol{m})$ is $\boldsymbol{x}$-dense for all $\mathfrak{U} \in \mathfrak{N}(\boldsymbol{q})$,

   $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q}) := \{\gamma \geq \mathrm{ht}(\boldsymbol{x}) \mid [\boldsymbol{x}, I]_\gamma \subseteq \bigcap\{A\restriction(\{\boldsymbol{q}\}, \boldsymbol{l}) \mid \boldsymbol{m} \leq \boldsymbol{l}\}\}$.

Lemma 3.3.7. *Suppose that $A \subseteq \prod_{i\in N} X^i$ is closed, $I \subseteq N$, $\boldsymbol{x} \in \prod_{i\in I} R^i$, $\boldsymbol{m} \in \prod_{i\in I} \omega$ and $\boldsymbol{q} \in \prod_{i\in N\setminus I} X^i$. If $\Phi(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ holds, then $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ is a club in λ^+.*

Proof. Fix a closed subset A of $\prod_{i\in N} X^i$. Assume that $\Phi(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ where $\boldsymbol{x} \in \prod_{i\in I} R^i$, $\boldsymbol{m} \in \prod_{i\in I} \omega$ and $\boldsymbol{q} \in \prod_{i\in N\setminus I} X^i$. First to prove the closedness of $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$, fix a limit ordinal $\xi < \lambda^+$ and let $\{\delta_\alpha\}_{\alpha<\xi}$ be a strictly increasing sequence in $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$. We verify that $\delta := \sup\{\delta_\alpha \mid \alpha < \xi\}$ is a member of $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$. Assume that there are $\boldsymbol{y} \in [\boldsymbol{x}, I]_\delta$ and $\boldsymbol{l} \in \prod_{i\in I}(\omega \setminus \boldsymbol{m}(i))$ such that $\boldsymbol{y} \notin A\restriction(\{\boldsymbol{q}\}, \boldsymbol{l})$. Let $\boldsymbol{r} := \langle \boldsymbol{y}(i), \boldsymbol{l}(i)\rangle_{i\in I}$ then $\boldsymbol{r} \cup \boldsymbol{q} \notin A$. Since A is closed in $\prod_{i\in N} X^i$, there is $\boldsymbol{\alpha} \in \prod_{i\in I} \mathrm{ht}_{R^i}(\boldsymbol{y}(i)) = \prod_{i\in I} \delta$ such that $(\mathfrak{G}(\boldsymbol{r}, \boldsymbol{\alpha}, I)^\frown\{\boldsymbol{q}\}) \cap A = \varnothing$. Pick $\beta < \xi$ so that $\boldsymbol{\alpha}(i) < \delta_\beta$ for all $i \in I$. Also choose $\boldsymbol{z} < \boldsymbol{y}$ from $\prod_{i\in I} R^i_{\delta_\beta}$. Then $\langle \boldsymbol{z}(i), \boldsymbol{l}(i)\rangle_{i\in I} =: \boldsymbol{r'} \in \mathfrak{G}(\boldsymbol{r}, \boldsymbol{\alpha}, I)$ so $\boldsymbol{r'} \cup \boldsymbol{q} \notin A$. But since $\delta_\beta \in \Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ and $\boldsymbol{z} \geq \boldsymbol{x}$ (because $\mathrm{ht}(\boldsymbol{x}) \leq \delta_\beta$ and $\boldsymbol{x} \leq \boldsymbol{y}$) it follows that $\boldsymbol{z} \in A\restriction(\{\boldsymbol{q}\}, \boldsymbol{l})$ i.e. $\boldsymbol{r'} \cup \boldsymbol{q} \in A$. This is a contradiction. Thus $\boldsymbol{y} \in A\restriction(\{\boldsymbol{q}\}, \boldsymbol{l})$ for every

$\boldsymbol{y} \in [\boldsymbol{x}, I]_\delta$ and $\boldsymbol{l} \in \prod_{i\in I}(\omega \setminus \boldsymbol{m}(i))$. Therefore $\delta \in \Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ and we conclude that $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ is closed in λ^+.

Next, to prove the unboundedness of $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$, fix arbitrary $\delta < \lambda^+$ and we are going to find $\gamma \in \Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ such that $\gamma > \delta$. Let $\{\boldsymbol{q}_\alpha\}_{\alpha<\lambda}$ be an enumeration of $\prod_{i\in N\setminus I} X^i_{\le \mathrm{ht}(\boldsymbol{q})}$ such that each $\boldsymbol{q}_\alpha$ appears λ-many times. We define $\{\gamma_\alpha\}_{\alpha<\lambda}$ by induction, and define γ as follows.

$$\gamma_0 := \max(\delta, \mathrm{ht}(\boldsymbol{x})),$$
$$\gamma_\alpha := \sup\{\delta(\boldsymbol{x}, \gamma_\beta, A\restriction(\{\boldsymbol{q}_\beta\}, \boldsymbol{m})) \mid \beta < \alpha\},$$
$$\gamma := \sup\{\gamma_\alpha \mid \alpha < \lambda\}.$$

$\{\gamma_\alpha\}_{\alpha<\lambda}$ is strictly increasing, so γ has cofinality λ. We show that $\gamma \in \Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ by the following two claims.

Claim 3.3.8. *$\langle \boldsymbol{y}(i), \boldsymbol{m}(i)\rangle_{i\in I} \cup \boldsymbol{q} \in A$ for all $\boldsymbol{y} \in [\boldsymbol{x}, I]_\gamma$.*

Proof. Assume that there is $\boldsymbol{y} \in [\boldsymbol{x}, I]_\gamma$ such that $\boldsymbol{r} \cup \boldsymbol{q} \notin A$ where $\boldsymbol{r} = \langle \boldsymbol{y}(i), \boldsymbol{m}(i)\rangle_{i\in I}$. Since A is closed in $\prod_{i\in N} X^i$, using the regularity of $\prod_{i\in N\setminus I} X^i$ we can take a basic open neighborhood $\prod_{i\in N} U_i$ of $\boldsymbol{r}\cup\boldsymbol{q}$ such that $((\prod_{i\in I} U^i)^\frown \mathfrak{Cl}_{N\setminus I}(\prod_{i\in N\setminus I} U^i)) \cap A = \varnothing$. Let $\mathfrak{U} := \prod_{i\in N\setminus I}(U_i \cap X^i_{\le\mathrm{ht}(\boldsymbol{q})})$. Note that $\mathfrak{U} \in \mathfrak{N}(\boldsymbol{q})$. Also, by the definition of $\tau(R^i)$'s, there is a sequence $\boldsymbol{\alpha} \in \prod_{i\in I} \mathrm{ht}_{R^i}(\boldsymbol{y}(i)) = \prod_{i\in I}\gamma$ such that $(\mathfrak{G}(\boldsymbol{r}, \boldsymbol{\alpha}, I)^\frown \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})) \cap A = \varnothing$. Pick $\beta < \lambda$ such that $\max\{\boldsymbol{\alpha}(i) \mid i \in I\} < \gamma_\beta$. Now $A\restriction(\mathfrak{U}, \boldsymbol{m})$ is $\boldsymbol{x}$-dense because of $\Phi(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$. Hence there is $\boldsymbol{z} \ge \boldsymbol{y}$ such that $\boldsymbol{z} \in A\restriction(\mathfrak{U}, \boldsymbol{m})$ so $\langle \boldsymbol{z}(i), \boldsymbol{m}(i)\rangle_{i\in I} \cup \boldsymbol{q'} \in A$ for some $\boldsymbol{q'} \in \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})$. Since $\mathfrak{Cl}_{N\setminus I}(\mathfrak{U}) \subseteq \prod_{i\in N\setminus I} X^i_{\le\mathrm{ht}(\boldsymbol{q})} = \{\boldsymbol{q}_\alpha\}_{\alpha<\lambda}$ and each $\boldsymbol{q}_\alpha$ appears λ-many times, we can choose $\beta' > \beta$ so that $\boldsymbol{q'} = \boldsymbol{q}_{\beta'}$. Then $\boldsymbol{z} \in [\boldsymbol{x}, I]_{\ge\gamma_{\beta'}} \cap (A\restriction(\{\boldsymbol{q}_{\beta'}\}, \boldsymbol{m}))$. So there is $\boldsymbol{w} \in L(\boldsymbol{x}, \gamma_{\beta'}, A\restriction(\{\boldsymbol{q}_{\beta'}\}, \boldsymbol{m}))$ such that $(\forall i \in I)[\boldsymbol{w}(i) \not\perp_{R^i} \boldsymbol{z}(i)]$. Put $\gamma' := \delta(\boldsymbol{x}, \gamma_{\beta'}, A\restriction(\{\boldsymbol{q}_{\beta'}\}, \boldsymbol{m}))$. Now for all $i \in I$,

$$\boldsymbol{\alpha}(i) < \gamma_\beta < \gamma_{\beta'} \le \mathrm{ht}_{R^i}(\boldsymbol{w}(i)) < \gamma' < \gamma = \mathrm{ht}_{R^i}(\boldsymbol{y}(i)) \text{ and } \boldsymbol{y}(i) \le_{R^i} \boldsymbol{z}(i).$$

Thus $\boldsymbol{w} < \boldsymbol{y}$. So it follows that

$$\boldsymbol{r'} := \langle \boldsymbol{w}(i), \boldsymbol{m}(i)\rangle_{i\in I} \in \mathfrak{G}(\langle \boldsymbol{y}(i), \boldsymbol{m}(i)\rangle_{i\in I}, \boldsymbol{\alpha}, I) = \mathfrak{G}(\boldsymbol{r}, \boldsymbol{\alpha}, I).$$

Since $\boldsymbol{q'} \in \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})$, we have $\boldsymbol{r'} \cup \boldsymbol{q'} \in \mathfrak{G}(\boldsymbol{r}, \boldsymbol{\alpha}, I)^\frown \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})$ so $\boldsymbol{r'} \cup \boldsymbol{q'} \notin A$. On the other hand, $\boldsymbol{w} \in L(\boldsymbol{x}, \gamma_{\beta'}, A\restriction(\{\boldsymbol{q}_{\beta'}\}, \boldsymbol{m})) \subseteq A\restriction(\{\boldsymbol{q}_{\beta'}\}, \boldsymbol{m})$ so $\boldsymbol{r'} \cup \boldsymbol{q}_{\beta'} \in A$. But since $\boldsymbol{q'} = \boldsymbol{q}_{\beta'}$, it leads to a contradiction. Therefore there is no $\boldsymbol{r} \in [\boldsymbol{x}, \boldsymbol{m}, I]_\gamma$ such that $\boldsymbol{r} \cup \boldsymbol{q} \notin A$. □

Claim 3.3.9. *$\langle \boldsymbol{y}(i), \boldsymbol{l}(i)\rangle_{i\in I} \cup \boldsymbol{q} \in A$ for all $\boldsymbol{y} \in [\boldsymbol{x}, I]_\gamma$ and $\boldsymbol{l} \in \prod_{i\in I}(\omega \setminus \boldsymbol{m}(i))$.*

Proof. This will be proved by a modification of the proof of claim 3.3.8. Assume that there are $\boldsymbol{y} \in [\boldsymbol{x}, I]_\gamma$ and $\boldsymbol{l} \in \prod_{i\in I}(\omega \setminus \boldsymbol{m}(i))$ such that $\boldsymbol{r}\cup\boldsymbol{q} \notin A$ where $\boldsymbol{r} = \langle \boldsymbol{y}(i), \boldsymbol{l}(i)\rangle_{i\in I}$. Using the regularity of $\prod_{i\in N\setminus I} X^i$, since A is closed in $\prod_{i\in N} X^i$ there is a basic open neighborhood $\prod_{i\in N} U_i$ of $\boldsymbol{r}\cup\boldsymbol{q}$ such that $((\prod_{i\in I} U_i)^\frown \mathfrak{Cl}_{N\setminus I}(\prod_{i\in N\setminus I} U^i)) \cap A = \varnothing$. For each $i \in I$ with $\boldsymbol{m}(i) < \boldsymbol{l}(i)$, since $\boldsymbol{x}(i) <_{R^i} \boldsymbol{y}(i) \in R^i_\gamma$ applying (6) of lemma 3.2.3 we can find $y'_i \in [\boldsymbol{x}, I]_\gamma$ such that $\langle y'_i, \boldsymbol{m}(i)\rangle_{i\in I} \in \prod_{i\in I} U_i$. Set $y'_i = \boldsymbol{y}(i)$ for all $i \in I$ with $\boldsymbol{m}(i) = \boldsymbol{l}(i)$. Let $\boldsymbol{r'} := \langle y'_i, \boldsymbol{m}(i)\rangle_{i\in I}$ and $\mathfrak{U} := \prod_{i\in N\setminus I}(U_i \cap X^i_{\leq \mathrm{ht}(\boldsymbol{q})})$. Since each U_i is open in X^i, there is $\boldsymbol{\alpha} \in \prod_{i\in I} \mathrm{ht}_{R^i}(y'_i) = \prod_{i\in I}\gamma$ such that $\mathfrak{G}(\boldsymbol{r'}, \boldsymbol{\alpha}, I) \subseteq \prod_{i\in I} U_i$ so $(\mathfrak{G}(\boldsymbol{r'}, \boldsymbol{\alpha}, I)^\frown \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})) \cap A = \varnothing$. The rest of the proof is similar to of claim 3.3.8's. □

By claim 3.3.9, $\delta < \gamma \in \Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ hence $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ is unbounded in λ^+. Therefore $\Gamma(A, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})$ is club in λ^+. ■

Corollary 3.3.10. *If A^0 and A^1 are disjoint closed sets of $\prod_{k\in N} X^k$ and $I \subseteq N$, then*

$$(\forall \boldsymbol{x} \in \textstyle\prod_{k\in I} R^k)(\forall \boldsymbol{q} \in \prod_{k\in N\setminus I} X^k)(\exists i \in 2)(\forall \boldsymbol{m} \in \prod_{k\in I}\omega)[\neg\Phi(A^i, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})].$$

Proof. Let $\langle A^0, A^1\rangle$ be a pair of disjoint closed subsets of $\prod_{k\in N} X^k$, $\boldsymbol{x} \in \prod_{k\in I} R^k$ and $\boldsymbol{q} \in \prod_{k\in N\setminus I} X^k$. Suppose that there exists $\boldsymbol{m}^0$ and $\boldsymbol{m}^1 \in \prod_{k\in I}\omega$ such that $\Phi(A^i, \boldsymbol{x}, \boldsymbol{m}^i, \boldsymbol{q})$ for each $i \in 2$. Then by lemma 3.3.7, $\Gamma(A^i, \boldsymbol{x}, \boldsymbol{m}^i, \boldsymbol{q})$'s are both club hence there is $\gamma \in \Gamma(A^0, \boldsymbol{x}, \boldsymbol{m}^0, \boldsymbol{q}) \cap \Gamma(A^1, \boldsymbol{x}, \boldsymbol{m}^1, \boldsymbol{q})$. Pick some $\boldsymbol{y} \in [\boldsymbol{x}, I]_\gamma$ and $\boldsymbol{l} := \langle \max(\boldsymbol{m}^0(k), \boldsymbol{m}^1(k))\rangle_{k\in I}$, then $\langle \boldsymbol{y}(k), \boldsymbol{l}(k)\rangle_{k\in I} \cup \boldsymbol{q} \in A^0 \cap A^1$. This is a contradiction. ■

Lemma 3.3.11. *For every $\mu < \lambda^+$, $I \subseteq N$ and pair $\langle A^0, A^1\rangle$ of disjoint closed subsets of $\prod_{k\in N} X^k$, there is $\rho = \rho(I, \mu, A^0, A^1) \in \lambda^+ \setminus \mu$ such that*

$$(\forall \boldsymbol{x} \in \textstyle\prod_{k\in I} R^k_\rho)(\forall \boldsymbol{q} \in \prod_{k\in N\setminus I} X^k_{\leq\mu})$$
$$(\exists i \in 2)(\exists \mathfrak{U} \in \mathfrak{N}(\boldsymbol{q}))[([\boldsymbol{x}, \omega, I]^\frown \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})) \cap A^i = \varnothing].$$

Proof. For $i \in 2$, $\boldsymbol{m} \in \prod_{k\in I}\omega$ and $\boldsymbol{q} \in \prod_{k\in N\setminus I} X^k_{\leq\mu}$ we define $P'_i(\boldsymbol{m}, \boldsymbol{q})$ as follows.

$$P'_i(\boldsymbol{m}, \boldsymbol{q}) := \{\boldsymbol{x} \in \textstyle\prod_{k\in I} R^i \mid (\exists \mathfrak{U} \in \mathfrak{N}(\boldsymbol{q}))[[\boldsymbol{x}, I] \cap A^i \restriction (\mathfrak{U}, \boldsymbol{m}) = \varnothing]\}.$$

Let $P_i(\boldsymbol{m}, \boldsymbol{q})$ be a maximal antichain in $P'_i(\boldsymbol{m}, \boldsymbol{q})$. Using λ^+-c.c., we can choose ρ so that $\max(\mathrm{ht}(\boldsymbol{x}), \mu) < \rho$ for all $\boldsymbol{x} \in \bigcup\{P_i(\boldsymbol{m}, \boldsymbol{q}) \mid \boldsymbol{m} \in \prod_{k\in I}\omega \wedge \boldsymbol{q} \in \prod_{k\in N\setminus I} X^k_{\leq\mu}\}$.

Fix arbitrary $\boldsymbol{x} \in \prod_{k\in I} R^k_\rho$ and $\boldsymbol{q} \in \prod_{k\in N\setminus I} X^k_{\leq\mu}$. By corollary 3.3.10, there is $i \in 2$ such that $(\forall \boldsymbol{m} \in \prod_{k\in I}\omega)[\neg\Phi(A^i, \boldsymbol{x}, \boldsymbol{m}, \boldsymbol{q})]$. Hence for all $\boldsymbol{m} \in \prod_{k\in I}\omega$, there is $\boldsymbol{y_m} \geq \boldsymbol{x}$ such that $(\exists \mathfrak{U} \in \mathfrak{N}(\boldsymbol{q}))[[\boldsymbol{y_m}, I] \cap A^i \restriction (\mathfrak{U}, \boldsymbol{m}) = \varnothing]$ i.e. $\boldsymbol{y_m} \in P'_i(\boldsymbol{m}, \boldsymbol{q})$, so we can find $\boldsymbol{z_m} \in P_i(\boldsymbol{m}, \boldsymbol{q})$ such that $(\forall k \in I)[\boldsymbol{z_m}(k) \not\perp_{R^k} \boldsymbol{y_m}(k)]$. Then $\boldsymbol{z_m} < \boldsymbol{x}$ because $(\forall k \in I)[\mathrm{ht}_{R^k}(\boldsymbol{z_m}(k)) < \rho = \mathrm{ht}_{R^k}(\boldsymbol{x}(k))]$. By the definition of $P'_i(\boldsymbol{m}, \boldsymbol{q})$'s, for each $\boldsymbol{m} \in \prod_{k\in I}\omega$ there is $\mathfrak{U}_{\boldsymbol{m}} \in \mathfrak{N}(\boldsymbol{q})$ such that $[\boldsymbol{z_m}, I] \cap A^i \restriction (\mathfrak{U}_{\boldsymbol{m}}, \boldsymbol{m}) = \varnothing$. Since the space $\prod_{k\in N\setminus I} X^k_{\leq\mu}$ is λ-additive and $\omega_1 \leq \lambda$, $\mathfrak{U} := \bigcap\{\mathfrak{U}_{\boldsymbol{m}} \mid \boldsymbol{m} \in \prod_{k\in I}\omega\}$ is open. So $\mathfrak{U} \in \mathfrak{N}(\boldsymbol{q})$. From this and $(\forall \boldsymbol{m} \in \prod_{k\in I}\omega)[\boldsymbol{z_m} \leq \boldsymbol{x}]$, it follows that $[\boldsymbol{x}, I] \cap A^i \restriction (\mathfrak{U}, \boldsymbol{m}) = \varnothing$ for all $\boldsymbol{m} \in \prod_{k\in I}\omega$. This exactly means that $([\boldsymbol{x}, \omega, I]^\frown \mathfrak{Cl}_{N\setminus I}(\mathfrak{U})) \cap A^i = \varnothing$. ■

Definition 3.3.12.

1. *For each subspace Y of $\prod_{k\in N} X^k$ and each pair $\langle A^0, A^1\rangle$ of subsets of $\prod_{k\in N} X^k$, we write $\langle A^0, A^1\rangle \lhd Y$ if there is a pair $\langle U^0, U^1\rangle$ of disjoint open subsets of Y such that $(\forall i \in 2)[A^i \cap Y \subseteq U^i]$.*

2. *For* $I \subseteq N$, $\mu < \lambda^+$, $\{A^0, A^1\} \subseteq \mathcal{P}(\prod_{k\in N} X^k)$ *and* $\boldsymbol{x} \in \prod_{k\in I} R^k$,
   - $X(\boldsymbol{x}, I, \mu) := (\boldsymbol{x}, \omega, I)^\frown \prod_{k\in N\setminus I} X^k_{\le\mu}$.
   - $X(I, \mu) := (\prod_{k\in I} X^k_{>\rho(I,\mu,A^0,A^1)})^\frown(\prod_{k\in N\setminus I} X^k_{\le\mu})$.

Lemma 3.3.13. $\langle A^0, A^1\rangle \lhd X(I,\mu)$ *for every pair* $\langle A^0, A^1\rangle$ *of disjoint closed subsets of* $\prod_{i\in N} X^i$, $I \subseteq N$ *and* $\mu < \lambda^+$.

Proof. Fix $\mu < \lambda^+$ and $\langle A^0, A^1\rangle$ as above. Let $\rho := \rho(I, \mu, A^0, A^1)$. Since $\{X(\boldsymbol{x}, I, \mu) \mid \boldsymbol{x} \in \prod_{k\in I} R^k_\rho\}$ is a partition of $X(I,\mu)$ into open sets, it is sufficient to prove $\langle A^0, A^1\rangle \lhd X(\boldsymbol{x}, I, \mu)$ for every $\boldsymbol{x} \in \prod_{k\in I} R^k_\rho$. Let $\boldsymbol{x} \in \prod_{k\in I} R^k_\rho$ be fixed and $\{\boldsymbol{q}_\alpha\}_{\alpha<\lambda}$ be an enumeration of all members of $\prod_{k\in N\setminus I} X^k_{\le\mu}$. By lemma 3.3.11, we can assign functions $V \in \prod_{\alpha<\lambda} \mathfrak{N}(\boldsymbol{q}_\alpha)$ and $\iota \in {}^\lambda 2$ such that

$$(\forall\alpha < \lambda)[([\boldsymbol{x}, \omega, I]^\frown \mathfrak{Cl}_{N\setminus I}(V(\alpha))) \cap A^{\iota(\alpha)} = \varnothing].$$

For each $i \in 2$ and $\alpha < \lambda$, let $W^i(\alpha) := \bigcup\{V(\beta) \mid \iota(\beta) = i \wedge \beta < \alpha\}$. Since the space is λ-additive, $\mathfrak{Cl}_{N\setminus I}(W^i(\alpha)) \subseteq \bigcup\{\mathfrak{Cl}_{N\setminus I}(V(\beta)) \mid \iota(\beta) = i \wedge \beta < \alpha\}$ so $([\boldsymbol{x}, \omega, I]^\frown \mathfrak{Cl}_{N\setminus I}(W^i(\alpha))) \cap A^i = \varnothing$ for every $i \in 2$ and $\alpha < \lambda$. Define

$$U^i_{\boldsymbol{x}} := (\boldsymbol{x}, \omega, I)^\frown \bigcup\{V(\alpha) \setminus \mathfrak{Cl}_{N\setminus I}(W^i(\alpha)) \mid \iota(\alpha) = 1 - i\}.$$

Clearly each $U^i_{\boldsymbol{x}}$ is open. The following two claims complete the proof.

Claim 3.3.14. $U^0_{\boldsymbol{x}} \cap U^1_{\boldsymbol{x}} = \varnothing$.

Proof. Fix arbitrary $\alpha \in \iota^{-1}``\{0\}$ and $\beta \in \iota^{-1}``\{1\}$. Clearly $\alpha \ne \beta$. If $\beta < \alpha$, then $V(\beta) \subseteq W^1(\alpha)$ so $V(\beta) \setminus \mathfrak{Cl}_{N\setminus I}(W^1(\alpha)) = \varnothing$. Also, $\alpha < \beta$ implies that $V(\alpha) \setminus \mathfrak{Cl}_{N\setminus I}(W^0(\beta)) = \varnothing$. Thus $(V(\alpha) \setminus \mathfrak{Cl}_{N\setminus I}(W^1(\alpha))) \cap (V(\beta) \setminus \mathfrak{Cl}_{N\setminus I}(W^0(\beta))) = \varnothing$ for all $\alpha \in \iota^{-1}``\{0\}$ and $\beta \in \iota^{-1}``\{1\}$. It follows that $U^0_{\boldsymbol{x}} \cap U^1_{\boldsymbol{x}} = \varnothing$. □

Claim 3.3.15. $(\forall i \in 2)[A^i \cap X(\boldsymbol{x}, I, \mu) \subseteq U^i_{\boldsymbol{x}}]$.

Proof. Let $i \in 2$ and $\boldsymbol{p} \in A^i \cap X(\boldsymbol{x}, I, \mu)$. Pick $\boldsymbol{y} \in (\boldsymbol{x}, I)$, $\boldsymbol{m} \in \prod_{k\in I} \omega$ and $\alpha < \lambda$ such that $\boldsymbol{p} = \langle \boldsymbol{y}(k), \boldsymbol{m}(k)\rangle_{k\in I} \cup \boldsymbol{q}_\alpha$. Then $([\boldsymbol{x}, \omega, I]^\frown\{\boldsymbol{q}_\alpha\}) \cap A^i \ne \varnothing$ so $\iota(\alpha) = 1 - i$ and $\boldsymbol{q}_\alpha \in V(\alpha)$. Moreover, $\boldsymbol{q}_\alpha \notin \mathfrak{Cl}_{N\setminus I}(W^i(\alpha))$ because $([\boldsymbol{x}, \omega, I]^\frown \mathfrak{Cl}_{N\setminus I}(W^i(\alpha))) \cap A^i = \varnothing$. Thus $\boldsymbol{p} \in U^i_{\boldsymbol{x}}$. □ ■

Lemma 3.3.16. $\langle A^0, A^1\rangle \lhd \prod_{k\in N} X^k$.

Proof. Let Ψ be as follows:

$$\Psi := \{n \in N + 1 \mid (\forall I \in [N]^n)(\forall\mu < \lambda^+)[\langle A^0, A^1\rangle \lhd (\textstyle\prod_{k\in I} X^k)^\frown(\prod_{k\in N\setminus I} X^k_{\le\mu})]\}.$$

The lemma will be proved via proving $\Psi = N + 1$ by induction. Since every λ-additive space of size $\le \lambda$ is normal (cf. lemma 2.2.6), $\prod_{k\in N} X^k_{\le\mu}$ is normal for all $\mu < \lambda^+$. So $0 \in \Psi$. Next assume that $n \le N$ and $n \subseteq \Psi$. We must check $n \in \Psi$. Fix $I \in [N]^n$ and $\mu < \lambda^+$. Let $\rho := \rho(I, \mu, A^0, A^1)$ and $Y(J) := (\prod_{k \in J} X^k_{>\rho})^\frown(\prod_{k \in I \setminus J} X^k_{\leq\rho})^\frown(\prod_{k \in N \setminus I} X^k_{\leq\mu})$ for $J \subseteq I$. Since $\{Y(J) \mid J \subseteq I\}$ is a partition of $(\prod_{k \in I} X^k)^\frown(\prod_{k \in N \setminus I} X^k_{\leq\mu})$ into open subspaces of X, we show that $\langle A^0, A^1 \rangle \lhd Y(J)$ for all $J \subseteq I$ and that completes the inductive step. Let $J \subseteq I$ be fixed.

Case 1. Suppose that $J \subsetneq I$. Then $J \in [N]^{<n}$ because N is finite. Let $Y := (\prod_{k \in J} X^k)^\frown(\prod_{k \in N \setminus J} X^k_{\leq\rho})$. By the inductive hypothesis, $\langle A^0, A^1 \rangle \lhd Y$. Moreover, $Y(J)$ is an open subspace of Y since $\mu \leq \rho$. Hence $\langle A^0, A^1 \rangle \lhd Y(J)$.

Case 2. Suppose that $J = I$. Then

$$Y(J) = (\textstyle\prod_{k \in I} X^k_{>\rho})^\frown(\prod_{k \in N \setminus I} X^k_{\leq\mu}) = X(I, \mu).$$

We have already proved this in lemma 3.3.13.

Therefore $n \in \Psi$.

After the induction, we have $N \in \Psi$. That is, $\langle A^0, A^1 \rangle \lhd \prod_{k \in N} X^k$. ■

Remark.

(1) Proofs of foregoing lemmas are founded by the fact that each of $\{X^i\}_{i \in N}$ is normal and so $\prod_{i \in N} X^i$ is regular. But notice that proofs of lemma 3.3.7-3.3.13 only require the regularity of $\prod_{i \in N \setminus I} X^i$. So putting $N = I = \{i\}$ and reproducing the entire arguments, finally we can prove that the normality of X^i's self-containedly.

(2) We proved that arbitrary finite families of Rudin's Dowker spaces can have normal products. It is natural to ask about the normality of infinite products. But this may not be straightforwardly because infinite products of λ-additive spaces need not be λ-additive.

Chapter 4

A construction from the principle $\clubsuit_{\mathrm{AD}}$

4.1 Introduction

In this chapter, we present another result on normality of the square of a Dowker space, which is constructed from $\clubsuit_{\mathrm{AD}}$-principle, a weakening of $\clubsuit$ developed by A. Rinot and R. Shalev [21]. $\clubsuit_{\mathrm{AD}}$-principle is proposed in a way of considering the *dual* of Juhasz's question: "Does the existence of an ω_1-Suslin tree imply $\clubsuit$?" Rinot and Shalev gave "No" to this question, but partially affirmative in the sense of $\clubsuit_{\mathrm{AD}}$ by the following facts.

Fact 4.1.1.

1. *(Lemma 2.10 of [21]) If $\clubsuit(S)$ holds for a stationary subset S of $E^{\omega_1}_{\omega}$, then $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \omega, < \omega)$ holds for any partition $\mathcal{S}$ of S into stationary subsets.*

2. *(Corollary 2.26 of [21]) The existence of an ω_1-Suslin tree implies $\clubsuit_{\mathrm{AD}}(\{\omega_1\}, \omega, < \omega)$.*

3. *(Corollary 2.27 of [21]) It is consistent with CH that $\clubsuit_{\mathrm{AD}}(\{\omega_1\}, \omega, < \omega)$ holds but $\clubsuit$ fails.*

On the other hand, $\clubsuit_{\mathrm{AD}}$ is enough strong to construct a Dowker space of size $\aleph_1$. From an instance of $\clubsuit_{\mathrm{AD}}$, Rinot and Shalev have proved that there is a Dowker space which is collectionwise normal and hereditary separable. Their topology is so called de Caux type. Originally, P. de Caux has constructed a collectionwise normal Dowker space of size $\aleph_1$ using $\clubsuit$ [8]. In fact, Rudin has also proved that de Caux type spaces from $\clubsuit$ are hereditarily separable [25]. Note that every paracompact space is collectionwise normal. In section 4.4, we present one more $\clubsuit_{\mathrm{AD}}$-based construction of a collectionwise normal space. Our example doesn't have small density, but the square is normal moreover Dowker.

In this chapter, for every cardinal κ we use $\mathrm{Stat}(\kappa)$ and $\mathrm{Club}(\kappa)$ to describe the set of all stationary subsets of κ and the set of club subsets of κ, respectively.

4.2 Basics of $\clubsuit_{\mathrm{AD}}$

Definition 4.2.1. *For every ordinal α and a set S of ordinals,*

1. $[\alpha]^+ := \{A \subseteq \alpha \mid \sup(A) = \alpha\}$,
2. *A guessing sequence on S is a member from $\prod_{\alpha \in S}[\alpha]^+$.*
3. *A ρ-multi-guessing sequence on S is a member from $\prod_{\alpha \in S}[[\alpha]^+]^\rho$.*

Note. $[\alpha]^+$ is the set of all cofinal subsets of α, and $[\kappa]^+ = [\kappa]^\kappa$ for regular κ. We use symbols such as $\vec{A}$ to indicate a function on the specified subset of ordinals, and we write its value by $\vec{A}(\alpha) = A_\alpha$. In particular, we use fraktur-styled symbol $\vec{\mathfrak{A}}$ to indicate a member of $\prod_{\alpha \in \mathrm{dom}(\vec{\mathfrak{A}})} \mathcal{P}(\mathcal{P}(\alpha))$. We use the term *multi-guessing sequence* when ρ in 3 of definition 4.2.1 is not specified.

Definition 4.2.2.

1. *For each multi-guessing-sequence $\vec{\mathfrak{A}}$ and each set X, we define*

   $G(X, \vec{\mathfrak{A}}) := \{\alpha \in \mathrm{dom}(\vec{\mathfrak{A}}) \mid (\forall A \in \mathfrak{A}_\alpha)[B \cap A \in [\alpha]^+]\}$.

2. *Suppose that κ is a regular cardinal and $\mathcal{S} \subseteq \mathcal{P}(\kappa)$. $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \rho, \mu)$-sequence is a ρ-multi-guessing sequence $\vec{\mathfrak{A}}$ on $\bigcup(\mathcal{S})$ which satisfies each of the followings:*

   (a) $(\forall \alpha \in \bigcup(\mathcal{S}))(\forall \{A, A'\} \in [\mathfrak{A}_\alpha]^2)[A \cap A' = \varnothing]$;
   (b) $(\forall \{\alpha, \beta\} \in [\bigcup(\mathcal{S})]^2)(\forall \langle A, B\rangle \in \mathfrak{A}_\alpha \times \mathfrak{A}_\beta)[A \cap B \notin [\alpha]^+]$;
   (c) $(\forall \mathfrak{X} \in [[\kappa]^\kappa]^{\leq\mu})[\bigcap\{G(X, \vec{\mathfrak{A}}) \mid X \in \mathfrak{X}\} \in \mathrm{Stat}(\kappa)]$.

Note. A $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \rho, <\mu)$-sequence is obviously defined by replacing "$\mathfrak{X} \in [[\kappa]^\kappa]^{\leq\mu}$" by "$\mathfrak{X} \in [[\kappa]^\kappa]^{<\mu}$" in the clause 2c of the definition 4.2.2.

Definition 4.2.3. *Suppose that λ is a regular cardinal and $S \subseteq \lambda^+$.*

1. *$\clubsuit(S)$ asserts the existence of a guessing sequence $\vec{A}$ on $S \cap \mathrm{LIM}(\lambda^+)$ such that $(\forall X \in [\lambda^+]^\lambda)(\exists \alpha \in S \cap \mathrm{LIM}(\lambda^+))[A_\alpha \subseteq X]$.*
2. *$\P(\lambda^+)$ asserts the existence of a family $\{A_\alpha\}_{\alpha<\lambda^+} \subseteq [\lambda^+]^\lambda$ such that $(\forall X \in [\lambda^+]^{\lambda^+})(\exists \alpha < \lambda^+)[A_\alpha \subseteq X]$.*

Fact 4.2.4 ([21], lemma 2.10)**.** *Let χ be a regular cardinal and κ be a cardinal with $\chi < \kappa$. If $\clubsuit(S)$ holds for a stationary $S \subseteq E^\kappa_\chi$, then there is a partition $\{S_i\}_{i<\kappa} \subseteq \mathrm{Stat}(\kappa)$ of S for which $\clubsuit_{\mathrm{AD}}(\{S_i\}_{i<\kappa}, \chi, <\omega)$ holds.*

Clearly ⫯(λ^+) is obtained from $\clubsuit(\lambda^+)$ and $2^\lambda = \lambda^+$ implies ⫯(λ^+). Furthermore, it is known that ⫯(λ^+) implies $\clubsuit_{\mathrm{AD}}$ by the next fact.

Fact 4.2.5 ([22], theorem 4.3)**.** *If ⫯(λ^+) holds for a regular cardinal λ, then $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \lambda, \lambda)$ holds for every partition $\mathcal{S} \subseteq \mathrm{Stat}(\lambda^+)$ of $E^{\lambda^+}_\lambda$.*

Remark. Rinot, Shalev and Todorcĕvić proved a stronger form of this theorem. Their resulting sequence $\langle \mathfrak{A}_\alpha \rangle_{\alpha \in \bigcup(\mathcal{S})}$ is a $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \lambda, \lambda)$ such that $\{\alpha \in \bigcup(\mathcal{S}) \mid (\forall i < \alpha)(\forall A \in \mathfrak{A}_\alpha)[Y_i \cap A \in [\alpha]^+]\}$ is stationary for all $\{Y_i\}_{i<\lambda^+} \subseteq [\lambda^+]^{\lambda^+}$. Such sequence is called $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \lambda, \lambda^+)$.

Lemma 4.2.6 ([21], Proposition 2.8)**.** *Let $\vec{\mathfrak{A}}$ be a $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \kappa)$ and $\{A_\xi\}_{\xi<\lambda} \subseteq \bigcup(\vec{\mathfrak{A}})$. If $(\forall \alpha \in \mathrm{dom}(\vec{A}))[\lambda \leq \mathrm{cf}(\alpha)]$, then there is a function $f \in \prod_{\xi<\lambda} A_\xi$ such that $\{A_\xi \setminus f(\xi) \mid \xi < \lambda\}$ is pairwise disjoint.*

Proof: Fix $\vec{\mathfrak{A}}$, $\{A_\xi\}_{\xi<\lambda}$ as the above. Define a function f on λ as follows.

- $f(0) := \min(A_0)$.
- Since $\lambda \leq \mathrm{cf}(\sup(A_\xi))$ and $(\forall \eta < \xi)[\sup(A_\eta \cap A_\xi) < \sup(A_\xi)]$ by 2b of definition 4.2.2, we set $f(\xi) := \min\{\epsilon \in A_\xi \mid \sup\{\sup(A_\eta \cap A_\xi) \mid \eta < \xi\} < \epsilon\}$.

Clearly f is a desired function. ■

4.3 $\clubsuit^2_{\mathrm{AD}}$, a variation of $\clubsuit_{\mathrm{AD}}$

Here we newly introduce a variation of the $\clubsuit_{\mathrm{AD}}$-principle on products of cardinals.

Definition 4.3.1.

- $[\alpha \times \alpha]^+ := \{A \subseteq \alpha \times \alpha \mid (\forall \langle \beta, \beta' \rangle \in \alpha \times \alpha)[A \cap ((\alpha \setminus \beta) \times (\alpha \setminus \beta')) \neq \varnothing]\}$.
- $G^2(X, \vec{\mathfrak{A}}) := \{\alpha \in \mathrm{dom}(\vec{\mathfrak{A}}) \mid (\forall \{A, A'\} \subseteq \mathfrak{A}_\alpha)[B \cap (A \times A') \in [\alpha \times \alpha]^+]\}$.
- *Let $\mathcal{S} \subseteq \mathcal{P}(\lambda^+)$. A $\clubsuit^2_{\mathrm{AD}}(\mathcal{S}, \rho, \mu; \theta)$-sequence is a $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \rho, \mu)$ sequence $\vec{\mathfrak{A}}$ such that $(\forall \mathfrak{X} \in [[\kappa \times \kappa]^+]^{\leq \theta})[\bigcap\{G^2(X, \vec{\mathfrak{A}}) \mid X \in \mathfrak{X}\} \in \mathrm{Stat}(\kappa)]$.*

Note. $[\alpha \times \alpha]^+$ is the set of all *dominating* (a term from definition 2.1.6) subsets of $\alpha \times \alpha$.

Lemma 4.3.2. *If λ is an uncountable regular cardinal, then $\Diamond^*(S)$ implies $\clubsuit^2_{\mathrm{AD}}(\{S\}, \omega, \lambda; \lambda)$ for every stationary $S \subseteq E^{\lambda^+}_\lambda$.*

Proof: Fix $S \subseteq E^{\lambda^+}_\lambda$ and a $\diamondsuit^*(S)$ sequence $\vec{\mathfrak{D}} := \langle \mathfrak{D}_\alpha \rangle_{\alpha \in S}$, that is

- $(\forall \alpha \in S)[\mathfrak{D}_\alpha \in [\mathcal{P}(\alpha) \cup \mathcal{P}(\alpha \times \alpha)]^{\leq \lambda}]$.
- $(\forall B \subseteq \lambda^+)(\exists C \in \mathrm{Club}(\lambda^+))[C \cap S \subseteq \{\alpha \in S \mid B \cap \alpha \in \mathfrak{D}_\alpha\}]$.

We can add the following property to $\vec{\mathfrak{D}}$:

$(\forall B \subseteq \lambda^+ \times \lambda^+)(\exists C \in \mathrm{Club}(\lambda^+))[C \cap S \subseteq \{\alpha \in S \mid B \cap (\alpha \times \alpha) \in \mathfrak{D}_\alpha\}]$.

From $\vec{\mathcal{D}}$, we construct a $\clubsuit^2_{\mathrm{AD}}(\{S\}, \omega, \lambda; \lambda)$ sequence $\vec{\mathfrak{A}}$ that satisfies the following.

- $(\forall \{B^0, B^1\} \subseteq [\lambda^+ \times \lambda^+]^+)$
  $[\{\alpha \in S \mid (\forall i \in 2)(\forall \{m, n\} \subseteq \omega)[(A^m_\alpha \times A^n_\alpha) \cap B^i \in [\alpha \times \alpha]^+]\} \in \mathrm{Stat}(\lambda^+)]$.
- $(\forall \alpha \in S)(\forall n \in \omega)(\forall B \in [\alpha]^+ \cap \mathfrak{D}_\alpha)[A^n_\alpha \cap B \in [\alpha]^+]$.
- $(\forall \alpha \in S)(\forall \{m, n\} \subseteq \omega)(\forall B \in [\alpha \times \alpha]^+ \cap \mathfrak{D}_\alpha)[(A^m_\alpha \times A^n_\alpha) \cap B \in [\alpha \times \alpha]^+]$.

To define a desired sequence, fix $\alpha \in S$ and a cofinal function $\phi_\alpha : \lambda \to \alpha$. Let $\{D_{\xi,\alpha}\}_{\xi<\lambda}$ be a λ-times enumeration of $\mathfrak{D}_{\xi,\alpha} \cap ([\alpha]^+ \cup [\alpha \times \alpha]^+)$. Put functions $\{f^i_{\xi,\alpha}\}_{i \in 2 \wedge \xi < \lambda}$ as follows: For every $p = \langle \beta, \beta' \rangle \in \alpha \times \alpha$, $\max(\beta, \beta', \phi_\alpha(\xi)) < \min(f^0_{\xi,\alpha}(p), f^1_{\xi,\alpha}(p))$ and $(D_{\xi,\alpha} \subseteq \alpha \times \alpha \wedge \langle f^0_{\xi,\alpha}(p), f^1_{\xi,\alpha}(p) \rangle \in D_{\xi,\alpha}) \vee (D_{\xi,\alpha} \subseteq \alpha \wedge \langle f^0_{\xi,\alpha}(p), f^1_{\xi,\alpha}(p) \rangle \in D_{\xi,\alpha} \times D_{\xi,\alpha})$. For every $\langle m, i, \xi \rangle \in \omega \times 2 \times \lambda$, define $\Phi^{m,i}_{\xi,\alpha}$ as follows:

- $\Phi^{0,i}_{\xi,\alpha} := f^i_{\xi,\alpha}(\langle \sup\{\Phi^{n,j}_{\eta,\alpha} \mid \langle n, j, \eta \rangle \in \omega \times 2 \times \xi\} \rangle_{i \in 2})$;
- $\Phi^{m+1,i}_{\xi,\alpha} := f^i_{\xi,\alpha}(\langle \Phi^{m,0}_{\xi,\alpha}, \Phi^{m,1}_{\xi,\alpha} \rangle)$.

Let $A^m_\alpha := \{\Phi^{m,i}_{\xi,\alpha} \mid \langle m, i, \xi \rangle \in \omega \times 2 \times \lambda\}$ for each $\alpha \in S$. We're going to verify $\vec{\mathfrak{A}} := \langle A^m_\alpha \rangle_{\langle \alpha, m \rangle \in S \times \omega}$ is a $\clubsuit^2_{\mathrm{AD}}(\{S\}, \omega, \lambda)$.

Claim 4.3.3. *For all $\{m, n\} \subseteq \omega$ and $\alpha \in S$:*

1. $(\forall \beta < \alpha)[\sup(A^m_\alpha \cap A^n_\beta) < \beta]$;
2. $m \neq n \longrightarrow A^m_\alpha \cap A^n_\alpha = \varnothing$;
3. $(\forall D \in \mathfrak{D}_\alpha \cap [\alpha]^+)[A^m_\alpha \cap D \in [\alpha]^+]$;
4. $(\forall D \in \mathfrak{D}_\alpha \cap [\alpha \times \alpha]^+)[(A^m_\alpha \times A^n_\alpha) \cap D \in [\alpha \times \alpha]^+]$.

Proof. (1) holds because each A^m_α has order type λ. (2) Fix arbitrary $\langle \Phi^{m,i}_{\xi,\alpha}, \Phi^{n,j}_{\eta,\alpha}\rangle \in A^m_\alpha \times A^n_\alpha$. Suppose that $n < m$, then $(\eta \le \xi \wedge \Phi^{n,j}_{\eta,\alpha} < \Phi^{m,i}_{\xi,\alpha}) \vee (\xi < \eta \wedge \Phi^{m,i}_{\xi,\alpha} < \Phi^{n,j}_{\eta,\alpha})$. (3) Similar to of (4). (4) Let $D \in \mathfrak{D}_\alpha \cap [\alpha \times \alpha]^+$ and $\langle \beta^0, \beta^1\rangle \in \alpha \times \alpha$. Choose $\xi < \lambda$ such that $(\forall i \in 2)[\beta^i \le \phi_\alpha(\xi)]$. Since $\mathfrak{D}_\alpha \cap [\alpha \times \alpha]^+$ is enumerated by $\{D_{\eta,\alpha}\}_{\eta<\lambda}$ with λ times, we can find $\xi' \in \lambda \setminus \xi$ such that $D = D_{\xi',\alpha}$. Then $\langle \Phi^{m,0}_{\xi',\alpha}, \Phi^{n,1}_{\xi',\alpha}\rangle \in D \cap ((A^m_\alpha \setminus \phi(\xi')) \times (A^n_\alpha \setminus \phi(\xi')))$.

□

Thus $\vec{\mathfrak{A}}$ satisfies clauses (2a) and (2b) in the definition 4.2.2. We finish the proof of the lemma by proving the next claim.

Claim 4.3.4. *Let $\{\mathfrak{B}_\xi\}_{\xi<\lambda} \subseteq [\lambda^+]^{\lambda^+}$ and $\{\mathfrak{B}'_\xi\}_{\xi<\lambda} \subseteq [\lambda^+ \times \lambda^+]^+$. Then both of $\bigcap_{\xi<\lambda} G(B_\xi, \vec{\mathfrak{A}})$ and $\bigcap_{\xi<\lambda} G^2(B'_\xi, \vec{\mathfrak{A}})$ are stationary.*

Proof. Fix $\{\mathfrak{B}_\xi\}_{\xi<\lambda} \subseteq [\lambda^+]^{\lambda^+}$. For every $\xi < \lambda$ define a club $C_\xi := \{\alpha \in \mathrm{Lim}(\lambda^+) \mid B_\xi \cap \alpha \in [\alpha]^+\}$. By $\diamondsuit^*(S)$, we can find another club C'_ξ such that $C'_\xi \cap S \subseteq \{\alpha \in S \mid B_\xi \cap \alpha \in \mathfrak{D}_\alpha\}$. Evidently $\bigcap_{\xi<\lambda}(C_\xi \cap C'_\xi \cap S)$ is a stationary that is contained in $G(B_\xi, \vec{\mathfrak{A}})$ for all $\xi < \lambda$ by (3) in the claim 4.3.3. The latter part of the claim is similarly proved using club's $C_\xi := \{\alpha \in \mathrm{Lim}(\lambda^+) \mid B'_\xi \cap (\alpha \times \alpha) \in [\alpha \times \alpha]^+\}$ and C'_ξ such that $C'_\xi \cap S \subseteq \{\alpha \in S \mid B'_\xi \cap (\alpha \times \alpha) \in \mathfrak{D}_\alpha\}$. □

■

4.4 A Dowker space from $\clubsuit^2_{\mathrm{AD}}$ which has a normal square

Main theorem B. *Suppose that $\clubsuit^2_{\mathrm{AD}}(\{E^{\lambda^+}_\lambda\}, \omega, \lambda; 2)$ holds for an uncountable regular cardinal λ. Then there is a collection-wise normal Dowker space on $\lambda^+ \times \omega$ whose square is Dowker.*

Throughout this section, let λ be an uncountable regular cardinal and we fix a $\clubsuit^2_{\mathrm{AD}}(\{E^{\lambda^+}_\lambda\}, \omega, \lambda; 2)$-sequence $\vec{\mathfrak{A}} = \langle \mathfrak{A}_\alpha\rangle_{\alpha \in E^{\lambda^+}_\lambda}$. Also we fix an injective enumeration $\{A^{n,m}_\alpha \mid n \le m \in \omega\}$ of $\mathfrak{A}_\alpha$ for each $\alpha \in E^{\lambda^+}_\lambda$. A topology τ on $X = \lambda^+ \times \omega$ is defined by:

$$U \in \tau \;\Leftrightarrow\; (\forall \langle \alpha, m\rangle \in U \cap (E^{\lambda^+}_\lambda \times \omega))(\exists \epsilon < \alpha)[\textstyle\bigcup_{n \le m}(A^{n,m}_\alpha \setminus \epsilon) \times \{m\} \subseteq U].$$

To avoid frequent mentions about cofinalities, we set $\mathfrak{A}_\alpha = \{\varnothing\}$ for all α with $\mathrm{cf}(\alpha) < \lambda$. Clearly this assumption does not affect any properties of the original $\vec{\mathfrak{A}}$.

Notation.

1. τ^2 denotes the square topology of τ.

2. $\mathrm{Cl}(\bullet)$ and $\mathrm{Cl}^2(\bullet)$ indicate the closure operator of the space $\langle X, \tau\rangle$ and $\langle X \times X, \tau^2\rangle$, respectively.

3. $\mathrm{CL}(\mathcal{T})$ denotes the set of all $\mathcal{T}$-closed subsets, where $\mathcal{T} \in \{\tau, \tau^2\}$.

Definition 4.4.1.

- *For all* $\alpha < \lambda^+$, *let* $X_{\leq\alpha} := (\alpha+1) \times \omega$ *and* $X_{>\alpha} := X \setminus X_{\leq\alpha}$.
- *For all* $\{m, n\} \subseteq \omega$, $Y \subseteq X$ *and* $Z \subseteq X \times X$,
  - $Y \restriction_m := \{\alpha < \lambda^+ \mid \langle \alpha, m \rangle \in Y\}$,
  - $Z \restriction_{m,n} := \{\langle \alpha, \beta \rangle \in \lambda^+ \times \lambda \mid \langle \langle \alpha, m \rangle, \langle \beta, n \rangle \rangle \in Z\}$.
- $G(B) := G(B, \vec{\mathfrak{A}})$ *and* $G^2(Y) := G^2(Y, \vec{\mathfrak{A}})$ *for each* $B \subseteq \lambda^+$ *and* $Y \subseteq \lambda^+ \times \lambda^+$.

Lemma 4.4.2.

1. *The space* X *is* T_1 *and* λ*-additive.*
2. *For every* $\alpha < \lambda^+$ *and* $n \in \omega$,
   - $\alpha \times \omega$ *and* $\lambda^+ \times n$ *are open.*
   - $X_{\leq\alpha}$ *is clopen.*
3. $(\forall B \subseteq \lambda^+)(\forall n \in \omega)[G(B) \times (\omega \setminus n) \subseteq \mathrm{Cl}(B \times \{n\})]$
4. $(\forall \{m, n\} \subseteq \omega)(\forall k \in \omega \setminus (m \cup n))$
$$(\forall K \subseteq X \times X)(\forall \alpha \in G^2(K \restriction_{m,n}))[\langle \langle \alpha, k \rangle, \langle \alpha, k \rangle \rangle \in \mathrm{Cl}^2(K)].$$

Proof. (1) and (2) are obvious.
(3): Fix $B \subseteq \lambda^+$ and $n \in \omega$. Let $\langle \alpha, k \rangle \in G(B) \times (\omega \setminus n)$ and $U \subseteq X$ be an open neighborhood of $\langle \alpha, k \rangle$. Pick $\epsilon < \alpha$ such that $\bigcup_{m \leq k} (A_\alpha^{m,k} \setminus \epsilon) \times \{m\} \subseteq U$. By the definition of $G(B)$, there is $\epsilon' \in A_\alpha^{n,k} \setminus \epsilon$ such that $\epsilon' \in B$. Then $\langle \epsilon', n \rangle \in U \cap (B \times \{n\})$. Thus $\langle \alpha, k \rangle \in \mathrm{Cl}(B \times \{n\})$.
(4): Fix $K \subseteq X \times X$, $\{m, n\} \subseteq \omega$ and $k \in \omega \setminus \max(m, n)$. Let $\alpha \in G^2(K \restriction_{m,n}$) and $V \subseteq X \times X$ be an open neighborhood of $\langle \langle \alpha, k \rangle, \langle \alpha, k \rangle \rangle$. Pick $\epsilon < \alpha$ such that $((A_\alpha^{m,k} \setminus \epsilon) \times \{m\}) \times ((A_\alpha^{n,k} \setminus \epsilon) \times \{n\}) \subseteq V$. By the definition of $G^2(K \restriction_{m,n})$, $(A_\alpha^{m,k} \times A_\alpha^{n,k}) \cap K_{m,n} \in [\alpha \times \alpha]^+$. So there are $\epsilon' \in A_\alpha^{m,k} \setminus \epsilon$ and $\epsilon'' \in A_\alpha^{n,k} \setminus \epsilon$ such that $\langle \epsilon', \epsilon'' \rangle \in K \restriction_{m,n}$. Thus $\langle \langle \epsilon', m \rangle, \langle \epsilon'', n \rangle \rangle \in K \cap V$. This proves that $\langle \langle \alpha, k \rangle, \langle \alpha, k \rangle \rangle \in \mathrm{Cl}^2(K)$. ■

Remark. Let $C_n := \lambda^+ \times (\omega \setminus n+1)$. Then $\langle C_n \rangle_{n \in \omega}$ witnesses $\neg$CPN of X by lemma 2.1.5. Moreover $\langle C_n \times C_n \rangle_{n \in \omega}$ is a countable decreasing sequence of dominating closed subsets of $X \times X$ with empty intersection, so this also becomes a witness of $\neg$CPN of $X \times X$ by lemma 2.1.7. Hence after we showed that $X \times X$ is normal, combining with lemma 4.4.4 it follows that both of X and $X \times X$ are Dowker.

Definition 4.4.3.

1. $[X]^+ := \{Y \subseteq X \mid \neg(\exists \alpha < \lambda^+)[Y \subseteq X_{\leq\alpha}]\}$.
2. $[X \times X]^+ := \{Y \subseteq X \times X \mid \neg(\exists \alpha < \lambda^+)[Y \subseteq (X_{\leq\alpha} \times X) \cup (X \times X_{\leq\alpha})]\}$.

Remark. As λ is uncountable regular, $Y \in [X]^+$ if and only if there is $m \in \omega$ such that $Y \restriction_m \in [\lambda^+]^{\lambda^+}$. Similarly, $Y \in [X \times X]^+$ if and only if there are $m, n \in \omega$ such that $Y \restriction_{m,n} \in [\lambda^+ \times \lambda^+]^+$.

Lemma 4.4.4.

1. $(\forall \{F^0, F^1\} \subseteq \mathrm{CL}(\tau) \cap [X]^+)[F^0 \cap F^1 \neq \varnothing]$.
2. $(\forall \{K^0, K^1\} \subseteq \mathrm{CL}(\tau^2) \cap [X \times X]^+)[K^0 \cap K^1 \neq \varnothing]$.

Proof. (1): Fix a pair $\langle F^0, F^1\rangle$ of τ-closed subsets of size λ^+. Pick $\{m, n\} \subseteq \omega$ such that $\overline{\overline{F^0 \restriction_m}} = \overline{\overline{F^1 \restriction_n}} = \lambda^+$. Since $G(F^0 \restriction_m) \cap G(F^1 \restriction_n)$ is stationary and by lemma 4.4.2-(3), $(G(F^0 \restriction_m) \times (\omega \setminus m)) \cap (G(F^1 \restriction_n) \times (\omega \setminus n)) \subseteq \mathrm{Cl}_\tau(F^0 \restriction_m \times \{m\}) \cap \mathrm{Cl}_\tau(F^1 \restriction_n \times \{n\}) \subseteq F^0 \cap F^1$. Thus $F^0 \cap F^1 \neq \varnothing$.
(2): Fix a pair $\langle K^0, K^1\rangle$ of dominating τ^2-closed subsets. Then there are $\{m_0, n_0, m_1, n_1\} \subseteq \omega$ such that $K^i \restriction_{m_i,n_i} \in [\lambda^+ \times \lambda^+]^+$ for each $i \in 2$. $G^2(K^0 \restriction_{m_0,n_0}) \cap G^2(K^1 \restriction_{m_1,n_1})$ is stationary. Pick arbitrary $\alpha \in G^2(K^0 \restriction_{m_0,n_0}) \cap G^2(K^1 \restriction_{m_1,n_1})$ and let $k := \max(m_0, n_0, m_1, n_1)$. Then by lemma 4.4.2-(4), it follows that $\langle\langle\alpha, k\rangle, \langle\alpha, k\rangle\rangle \in K^0 \cap K^1$. ■

From now on, we're going to prove that our space X is collectionwise normal. Let us recall some definitions.

Definition 4.4.5. *For every topological space $\langle Y, \mathcal{T}\rangle$,*

- *A sequence $\langle F^i\rangle_{i<\theta}$ of subsets of Y is called discrete if and only if, for every $p \in Y$ we can find its neighborhood $U \in \mathcal{T}$ such that $\{i \in \theta \mid F^i \cap U \neq \varnothing\} \in [\theta]^{\leq 1}$. In particular, every discrete family is pairwise disjoint.*
- *$\langle Y, \mathcal{T}\rangle$ is collectionwise normal if and only if every discrete sequence of $\mathcal{T}$-closed subsets is separated.*

Lemma 4.4.6. *Every discrete sequence of subsets of X has length at most λ.*

Proof. Fix a discrete sequence $\vec{F} := \langle F^i\rangle_{i\in\theta}$ of τ-closed subsets and assume that $\overline{\overline{\theta}} = \lambda^+$. Pick a sequence $\langle \alpha^i, m^i\rangle_{i\in\theta} \in \prod_{i\in\theta} F^i$. For each $n \in \omega$, let $I_n := \{i \in \theta \mid m^i = n\}$. Since $\overline{\overline{\theta}} = \lambda^+$, there is $n_0 \in \omega$ such that $\overline{\overline{I_{n_0}}} = \lambda^+$. Note that $\langle \alpha^i\rangle_{i\in I_{n_0}}$ is injective. Let $B := \{\alpha^i\}_{i\in I_{n_0}}$. Then $B \in [\lambda^+]^{\lambda^+}$ and $G(B)$ is stationary. Pick some $\alpha \in G(B)$. By discreteness of $\vec{F}$, we can find a neighborhood $U \in \tau$ of $\langle \alpha, n_0\rangle$ and $i_0 \in \theta$ such that $\{i \in \theta \mid U \cap F^i \neq \varnothing\} \subseteq \{i_0\}$. There is $\epsilon < \alpha$ such that $(A_\alpha^{n_0,n_0} \setminus \epsilon) \times \{n_0\} \subseteq U$. Moreover, by $\alpha \in G(B)$ there are $\{\epsilon', \epsilon''\} \subseteq A_\alpha^{n_0,n_0} \cap B \setminus \epsilon$ such that $\epsilon' < \epsilon''$. Suppose that $\epsilon' = \alpha^i$ and $\epsilon'' = \alpha^j$ where $\{i,j\} \subseteq I_{n_0}$. Then $i \neq j$ because $\langle \alpha^i \rangle_{i \in I_{n_0}}$ is injective. But since $\langle \epsilon', n_0 \rangle \in U \cap F^i$ and $\langle \epsilon'', n_0 \rangle \in U \cap F^j$, we have $i = j = i_0$. This is a contradiction. Therefore $\overline{\overline{\theta}} < \lambda^+$. ■

Lemma 4.4.7. *X is collectionwise normal.*

Proof. Before starting the proof, for every subset F of X and a function $h \in \prod_{\langle \alpha,k \rangle \in X} \alpha$ we inductively define $\Phi(F,h)$ as follows:

- $\Phi_0(F,h) := F$;
- $\Phi_{n+1}(F,h) := \bigcup\{(\bigcup_{m \leq k}(A_\alpha^{m,k} \setminus h(\alpha,k)) \times \{m\}) \cup \{\langle \alpha,k \rangle\} \mid \langle \alpha,k \rangle \in \Phi_n(F,h)\}$;
- $\Phi(F,h) := \bigcup\{\Phi_n(F,g) \mid n \in \omega\}$.

It is easy to see that $\Phi(F,h)$ is open and $(\forall n \in \omega)[F \subseteq \Phi_n(F,h) \subseteq \Phi_{n+1}(F,h)]$. Moreover, if F is contained in $X_{\leq \gamma}$ then so is $\Phi(F,h)$.

Next fix a discrete sequence $\vec{F} = \langle F^i \rangle_{i<\theta}$ of τ-closed subsets. By lemma 4.4.6, we can set $\theta = \lambda$. Also by lemma 4.4.4, there are $\gamma < \lambda^+$ and $i_0 < \lambda$ such that $(\forall i \in \lambda \setminus \{i_0\})[F^i \subseteq X_{\leq \gamma}]$. Since $\vec{F}$ is discrete, we can assign functions $f \in \prod_{\langle \alpha,k \rangle \in X} \alpha$, $\tilde{U} \in \prod_{\langle \alpha,k \rangle \in X} \tau$ and $\iota \in \prod_{\langle \alpha,k \rangle \in X} \lambda$ such that $(\forall \langle \alpha,k \rangle \in X)[(\bigcup_{n \leq k}(A_\alpha^{n,k} \setminus f(\alpha,k)) \times \{n\}) \cup \{\langle \alpha,k \rangle\} \subseteq \tilde{U}(\alpha,k) \wedge \{i < \lambda \mid \tilde{U}(\alpha,k) \cap F_i \neq \varnothing\} \subseteq \{\iota(\alpha,k)\}]$. Also pick a function $f' \in \prod_{\langle \alpha,k \rangle \in X} \alpha$ such that $(\bigcup(A_\alpha^{n,k} \setminus f'(\alpha,k)) \times \{n\}) \cap F^{\iota(\alpha,k)} = \varnothing$ if $\alpha \notin F^{\iota(\alpha,k)}$ and otherwise $f'(\alpha,k) = 0$. And by lemma 4.2.6, there is $g \in \prod_{\alpha \leq \gamma} \alpha$ such that $\{A_\alpha^{n,k} \setminus g(\alpha) \mid \alpha \leq \gamma \wedge \{n,k\} \subseteq \omega\}$ is pairwise disjoint. Let $h := \{\langle \langle \alpha,k \rangle, f(\alpha,k) \cup f'(\alpha,k) \cup g(\alpha) \rangle \mid \langle \alpha,k \rangle \in X_{\leq \gamma}\}$ and $F'^i := F^i \cap X_{\leq \gamma}$ for each $i < \lambda$.

Claim 4.4.8. *$\langle \Phi(F'^i,h) \rangle_{i<\lambda}$ is pairwise disjoint.*

Proof. We use 2-step induction to prove this. Let $P^i := \{n \in \omega \mid (\forall j \in \lambda \setminus \{i\})[F'^i \cap \Phi_n(F'^j,h) = \varnothing]\}$ and fix $i \in \lambda$. Clearly $0 \in P^i$. Next let $n \in P^i$ and $j \in \lambda \setminus \{i\}$. Towards a contradiction, pick some $\langle \alpha,k \rangle \in \Phi_{n+1}(F'^j,h) \setminus \Phi_n(F'^j,h)$ that is $\alpha \in A_\beta^{k,l} \setminus h(\beta,l)$ for some $\langle \beta,l \rangle \in \Phi_n(F'^j,h,\gamma)$. If $\langle \alpha,k \rangle \in F'^i$, then $\iota(\alpha,k) = i$ and $i = \iota(\beta,l)$ because $\langle \alpha,k \rangle \in (((A_\beta^{k,l} \setminus f(\beta,l))) \times \{k\}) \cap F'^i \subseteq \tilde{U}(\beta,l) \cap F'^i$. By $n \in P^i$, $\langle \beta,l \rangle \in X_{\leq \gamma} \setminus F'^{\iota(\beta,l)}$ so $((A_\beta^{k,l} \setminus h(\beta,l)) \times \{k\}) \cap F'^i = \varnothing$. This is a contradiction. Thus $F'^i \cap \Phi_{n+1}(F'^j,h) \setminus \Phi_n(F'^j,h) = \varnothing$ and hence $n+1 \in P^i$. Therefore $P^i = \omega$ that is $F'^i \cap \Phi(F'^j,h) = \varnothing$.

Next, re-fix $i \in \lambda$ and let $Q^i := \{n \in \omega \mid (\forall j \in \lambda \setminus \{i\})[\Phi_n(F'^i,h) \cap \Phi(F'^j,h) = \varnothing]\}$. $0 \in Q^i$ because $P^i = \omega$. Suppose that $n \in Q^i$. Again, towards a contradiction let $\langle \alpha,k \rangle \in (\Phi_{n+1}(F'^i,h) \setminus \setminus\Phi_n(F'^i,h)) \cap \Phi(F'^j,h)$. Now $\alpha \in A_\beta^{k,l} \setminus h(\beta,l)$ for some

$\langle \beta, l\rangle \in \Phi_n(F'^i, h)$. And by $\langle \alpha, k\rangle \in \Phi(F'^j, h)$, there is $\bar{n} := \min\{n' \in \omega \mid \langle \alpha, k\rangle \in \Phi_{n'}(F'^j, h)\}$. Note that $\bar{n} \neq 0$ because $P^j = \omega$. Pick $\langle \beta', l'\rangle \in \Phi_{\bar{n}-1}(F'^j, h)$ such that $\alpha \in A^{k,l'}_{\beta'} \setminus h(\beta', l')$. Then $(A^{k,l}_{\beta} \setminus h(\beta, l)) \cap (A^{k,l'}_{\beta'} \setminus h(\beta', l')) \neq \varnothing$, so $\beta = \beta'$ by $\beta \cup \beta' \leq \gamma$ and the choice of h. Moreover $l = l'$ because $\mathfrak{A}_\beta$ is pairwise disjoint. Thus $\langle \beta, k\rangle = \langle \beta', l'\rangle \in \Phi_n(F'^i, h) \cap \Phi_{\bar{n}}(F'^j, h) \neq \varnothing$. This contradicts to $n \in Q^i$, so we conclude $\Phi_{n+1}(F'^i, h) \cap \Phi(F'^j, h) = \varnothing$ for every $j \in \lambda \setminus \{i\}$. That means $n+1 \in Q^i$. By the induction, finally we have $Q^i = \omega$ that is $\Phi(F'^i, h) \in \Phi(F'^j, h)$ for every $i \in \lambda$. □

Now we have $\langle F'^i \cap X_{\leq\gamma}\rangle_{i<\lambda} \lhd_X \langle \Phi(F'^i, h)\rangle_{i<\lambda}$, and since $(\forall i \in \lambda \setminus \{i_0\})[F^i \subseteq X_\gamma]$, it follows that $\langle F^i, F^{i_0}\rangle_{i\in\lambda\setminus\{i_0\}} \lhd_X \langle \Phi(F'^i, h), \Phi(F'^{i_0}, h) \cup X_{>\gamma}\rangle_{i\in\lambda\setminus\{i_0\}}$. $X_{>\gamma}$ is open and all of $\langle \Phi(F'^i, h)\rangle_{i\in\lambda\setminus\{i_0\}}$ are contained in $X_{\leq\gamma}$. This completes the proof. ■

Lemma 4.4.9. *For every $\alpha \in \lambda^+$, both subspaces $X_{\leq\alpha} \times X$ and $X \times X_{\leq\alpha}$ are normal.*

Proof. It is enough to prove this for $X_{\leq\alpha} \times X$. Fix $\alpha \in \lambda^+$ and a pair $\langle K^0, K^1\rangle$ of disjoint closed subsets of $X_{\leq\alpha} \times X$. Let $K^i\restriction_{p,0} := \{q \in X \mid \langle p, q\rangle \in K^i\}$ for every $\langle i, p\rangle \in 2 \times X$. $K^i\restriction_{p,1}$ is also defined in this way, but such restrictions only work in the proof of the normality in $X \times X_{\leq\alpha}$. We define the following sets.

- $H^i_n := \{p \in X_{\leq\alpha} \mid \overline{\overline{(K^i\restriction_{p,0})\restriction_n}} = \lambda^+\}$.
- $H^i := \bigcup\{H^i_n \mid n \in \omega\}$.
- $G := \bigcap\{G(K^i\restriction_{p,0}\restriction_n) \mid i \in 2 \wedge n \in \omega \wedge p \in H^i_n\}$.

Note that $H^0 \cap H^1 = \varnothing$ because $K^0\restriction_{p,0}$ and $K^1\restriction_{p,0}$ are disjoint closed subsets of X for every $p \in X_{\leq\alpha}$. Moreover, G is stationary since the size of $\bigcup\{H^i_n \mid i \in 2 \wedge n \in \omega\}$ is less than λ^+.

Claim 4.4.10. *H^i is closed in $X_{\leq\alpha}$ for every $i \in 2$.*

Proof. Let $\langle \beta, m\rangle \in \mathrm{Cl}(H^i)$. We verity that $\langle \beta, m\rangle \in H^i$. Since X is λ-additive, $\mathrm{Cl}(H^i) = \bigcup\{\mathrm{Cl}(H^i_n) \mid n \in \omega\}$. Pick $n \in \omega$ such that $\langle \beta, m\rangle \in \mathrm{Cl}(H^i_n)$. Towards a contradiction, assume that $\langle \beta, m\rangle \notin H^i$. That is, $\overline{\overline{K^i \cap (\{\langle \beta, m\rangle\} \times X)}} < \lambda^+$. Pick $\gamma \in G$ such that $K^i \cap (\{\langle \beta, m\rangle\} \times X) \subseteq \gamma \times \omega$. Since $\langle\langle \beta, m\rangle, \langle \gamma, n\rangle\rangle \notin K^i$ and K^i is a closed subset of a clopen subspace $X_{\leq\alpha} \times X$, there are $\{U, U'\} \subseteq \tau$ such that $\langle\langle \beta, m\rangle, \langle \gamma, n\rangle\rangle \in U \times U'$ and $(U \times U') \cap K^i = \varnothing$ ...(*). Also there is $\langle \delta, k\rangle \in H^i_n \cap U$. Then $\langle\langle \delta, k\rangle, \langle \gamma, n\rangle\rangle \in K^i$ because $\gamma \in G(K^i\restriction_{\langle\delta,k\rangle,0}\restriction_n)$ and $G(K^i\restriction_{\langle\delta,k\rangle,0}\restriction_n) \times (\omega \setminus n) \subseteq \mathrm{Cl}(K^i\restriction_{\langle\delta,k\rangle,0}\restriction_n \times\{n\}) \subseteq K^i\restriction_{\langle\delta,k\rangle,0}$. Now $\langle\langle \delta, k\rangle, \langle \gamma, n\rangle\rangle \in (U \times U') \cap K^i$, but this contradicts to (*). □

Using normality of X, we can find G^0 and G^1 such that $\langle H^i\rangle_{i\in 2} \lhd_{X_{\leq\alpha}} \langle G^i\rangle_{i\in 2}$.

Claim 4.4.11. *There is $\gamma < \lambda^+$ such that $(\forall i \in 2)[K^i \cap ((X_{\leq\alpha} \setminus H^i) \times X) \subseteq X_{\leq\alpha} \times X_{\leq\gamma}]$.*

Proof. For each $p \notin H^i$, this means $K^i \restriction_{p,0} \in [X]^{\leq\lambda}$ so we pick $\gamma^i(p)$ such that $K^i \restriction_{p,0} \subseteq X_{\leq\gamma^i(p)}$. $\gamma := \sup\{\gamma^i(p) \mid i \in 2 \wedge p \in X_{\leq\alpha}\}$ is a desired one because $\bigcup\{\{p\} \times K^i \restriction_{p,0} \mid p \in X_{\leq\alpha} \setminus H^i\} = K^i \cap ((X_{\leq\alpha} \setminus H^i) \times X)$. □

Since $X_{\leq\alpha} \times X_{\leq\gamma}$ is an λ-additive regular space of size λ, this is normal by lemma 2.2.6. So we can find a pair $\langle V^0, V^1\rangle$ such that $\langle K^i \cap (X_{\leq\alpha} \times X_{\leq\gamma})\rangle_{i\in 2} \lhd_{X_{\leq\alpha}\times X_{\leq\gamma}} \langle V^i\rangle_{i\in 2}$. Let $W^i := (G^i \times X_{>\gamma}) \cup V^i$. Then $\langle K^i\rangle_{i\in 2} \lhd_{X_{\leq\alpha}\times X} \langle W^i\rangle_{i\in 2}$. ■

Lemma 4.4.12. *$X \times X$ is normal.*

Proof. Let $\langle K^0, K^1\rangle$ be disjoint closed subsets of $X \times X$. By lemma 4.4.4, one of those does not dominate $X \times X$. So without loss of generality, fix $\alpha \in \lambda^+$ and we assume $K^0 \subseteq (X_{\leq\alpha} \times X) \cup (X \times X_{\leq\alpha})$. Since both of $X_{\leq\alpha} \times X$ and $X \times X_{\leq\alpha}$ are clopen and by lemma 4.4.9, there are open subsets $\{W_j^0, W_j^1\}_{j\in 2}$ if $X \times X$ such that $\langle K^i \cap (X_{\leq\alpha} \times X)\rangle \lhd_{X_{\leq\alpha}\times X} \langle W_0^i\rangle_{i\in 2}$ and $\langle K^i \cap (X \times X_{\leq\alpha})\rangle \lhd_{X\times X_{\leq\alpha}} \langle W_1^i\rangle_{i\in 2}$. Define W^0 and W^1 as follows.

- $W^0 := W_0^0 \cup (W_1^0 \cap (X_{>\alpha} \times X_{\leq\alpha}))$
- $W^1 := W_0^1 \cup (W_1^1 \cap (X_{>\alpha} \times X)) \cup (X_{>\alpha \times X_{>\alpha}})$.

Then $\langle K^0, K^1\rangle \lhd_{X\times X} \langle W^0, W^1\rangle$. ■

Chapter 5

Final remarks

First, we leave the next question.

Question 5.0.1. *Is it consistent that there is a Dowker space X of size $\aleph_1$ such that X has at least one point without countable neighborhoods?*

We have developed proofs for the normality in products of two kind of Dowker spaces: Those constructed by Suslin trees and $\clubsuit_{\mathrm{AD}}$-principle. Both are required to have size greater than $\aleph_1$ to have normal products, so it is natural to ask how about when those spaces have size $\aleph_1$. The following lemma points out the difficulties of this question.

Fact 5.0.1 (lemma 1 in [29])**.** *If X is Dowker and a space Y contains a countable non-discrete subspace, then $X \times Y$ is not normal.*

As a small variation of this, we have the next.

Corollary 5.0.2. *Suppose that X is Dowker and Y is a non-discrete space. Then $X \times Y$ is not normal if all points of Y have a countable open neighborhood.*

Thus question 5.0.1 asks the existence of a Dowker space of size $\aleph_1$ which satisfies a necessary condition to have normal square. It may require a quite unnatural way, because typically we define a topology on some products of ordered structure so that all initial segments (or the products with ω) is open. Anyway, most (or, all) examples of Dowker spaces of size $\aleph_1$ are locally countable. For example, for all countable ordinal α, $\{\langle x, m\rangle \in R \times \omega \mid \mathrm{ht}(x) \leq \alpha\}$ is open in a Rudin's Dowker space from an ω_1-Suslin tree R, and $\alpha \times \omega$ is open in de Caux type spaces. Similar situations for all Dowker spaces of size $\aleph_1$ which are shown in chapter 1.

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