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Dedicated to Professor Mitsuyoshi Kato for his 60th birthday

0. Introduction

In [**W**], Waldhausen considered a certain class of 3-manifolds called Haken manifolds, and classified it by their fundamental groups. Similarly, Takeuchi [**Ta1**] classified a certain class of very good 3-orbifolds, and in this paper we classify much larger class of 3-orbifolds by their orbifold fundamental groups.

The class considered here consists of compact and orientable 3-orbifolds which satisfy the following:
($\mathcal{W}1$) abad,
($\mathcal{W}2$) irreducible,
($\mathcal{W}3$) all boundary components are incompressible,
($\mathcal{W}4$) sufficiently large,
($\mathcal{W}5$) all turnovers with non-positive Euler numbers are boundary parallel.
Let us describe the improvements of this results comparing with that of [**Ta1**]:
(i) Orbifolds here are abad (i.e. there are no bad 2-suborbifolds), although they are required to be very good in [**Ta1**].
(ii) We can deform orbi-maps in the Main Theorem through orbi-homotopies, although the deformations in [**Ta1**] are merely through C-equivalent.
(iii) Orbifolds here are allowed to have ∂-parallel, non-spherical turnovers, although they have no non-spherical turnovers in [**Ta1**].
Since the goodness of the elements of $\mathcal{W}$ is immediately derived from [**D**] and [**Ta2**, Theorem A], we deal with good (not necessarily very good) orbifolds in this paper.

The improvement (i) was performed by the preparation in Section 1 and by examining details of the arguments in [**Ta1**].

For improvement (ii), we show an extension theorem of an orbi-map in Section 2. It is proved that an orbi-map from the double $\mathcal{D}B$ of a ballic 3-orbifold B to a 3-orbifold N can be extended to an orbi-map from the cone on $\mathcal{D}B$ to N.

For improvement (iii), we prepare the deformation theorem of orbi-maps on turnovers in Section 3. If an orbi-map between turnovers with infinite orbifold fundamental groups induces an isomorphism between their orbifold fundamental groups, then the orbi-map is orbi-homotopic to an orbi-isomorphism (Theorem 3.1). It is derived from Euclidean or Hyperbolic geometry.

By Theorem 3.1 we make two-dimensional analogies of the Main Theorem (Theorem 3.2 and Corollary 3.3).

Now we summarize the contents of the paper. Section 1 is devoted to the preparation of the basic tools to deal with 3-orbifolds in the combinatorial category. First we review the loop theorem, Dehn's lemma, and the sphere theorem for orbifolds from [**TY1**]. Using these theorems, we prove Propositions 1.4–1.8, which give us 'cut and paste methods' in the study of 3-orbifolds.

In Section 2, we consider some problems on extensions of orbi-maps defined on the boundary of orbifolds. It is well known that if $\pi_3(X) = 0$, a continuous map from S^3 to a space X is extendable to a continuous map from the cone on S^3 to X, and that C-equivalent maps to X are homotopic to each other. In the paper, we prove the orbifold versions of these facts in Theorem 2.2 and Proposition 2.3. Furthermore, Proposition 2.5 enables us to deform orbi-maps which appear in the last step of a hierarchy, to orbi-coverings fixing the boundaries. This is one of the core parts in the proof of the Main Theorem.

The topic in Section 3 is the deformations of orbi-maps between two-dimensional orbifolds. In [**Ta1**, Theorem 7.2], the non-turnover case is considered by decomposing the orbifolds to discal orbifolds along their essential curves. We study the turnover case in Theorem 3.1. Since there are no essential curves on turnovers, we cannot use the above method. Here we use the hyperbolic structures of the universal coverings of turnovers and modify the structure maps through an equivariant homotopy. These two theorems (Theorem 3.1 and [**Ta1**, Theorem 7.2]) yield the result of Theorem 3.2, which enables us to deform orbi-maps on boundaries to orbi-coverings at the first step of the proof of the Main Theorem.

The main results in Section 4 are theorems on I-bundles, which are Theorems 4.1 and 4.3. Roughly speaking, Theorem 4.1 states that if 3-orbifolds M, N and orbi-map $f : M \to N$ satisfy some conditions, then M should be an I-bundle over a closed 2-orbifold. Theorem 4.3 states that a certain orbi-map f from a product I-bundle over a closed 2-orbifold F to an appropriate 3-orbifold N retracts into a boundary component of N. The key point of the proof of Theorem 4.1 is to show that the fundamental group of a boundary component of M has a finite index in $\pi_1(M)$. Then the proof can be reduced to the case of [**Ta1**, 6.3]. The proof of Theorem 4.3 is performed by cutting open F into discal 2-orbifolds and skeleton-wise retracting the orbi-map on (each piece) $\times\ I$ into ∂N. The result in this section gives the 'breaking case' of the Main Theorem (i.e. Theorem 5.1(2)).

In Section 5, we prove Main Theorem which is as follows.

THEOREM 5.1. (Main Theorem) *Let $M, N \in \mathcal{W}$, and suppose $f : (M, \partial M) \to (N, \partial N)$ is an orbi-map such that $f_* : \pi_1(M) \to \pi_1(N)$ is monic. Then there exists an orbi-homotopy $f_t : (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$ and either*

(1) *$f_1 : M \to N$ is an orbi-covering, or*

(2) *M is a product I-bundle over a closed 2-orbifold and $f_1(M) \subset \partial N$.*

If, for a component B of ∂M, $(f|B) : B \to C$ is already an orbi-covering, we may assume $(f|B)_t = f|B$ for all t.

Here is an outline of the proof. By Theorem 3.2, we may assume that $f|\partial M$ is an orbi-covering. Take an orbi-covering $q : N' \to N$ associated with $f_*\pi_1(M)$ and a lift f' of f by q. There are two cases that $f'|\partial M$ is an orbi-embedding or not.

When $f'|\partial M$ is not an orbi-embedding, we apply the results in Section 4 to obtain conclusion (2).

When $f'|\partial M$ is an orbi-embedding, cut N open along an incompressible 2-suborbifold G. By the induction lemma, Lemma 5.2, we can modify (rel. ∂) f to f_1 so that each component of $f_1^{-1}(G)$ is an incompressible 2-suborbifold and, for each component of P (respectively Q) of $\mathrm{cl}(M - f_1^{-1}(G))$ (respectively $\mathrm{cl}(N - G)$), $(f_1|P) : P \to Q$ possesses the same property as that of $f : M \to N$. That is, the lift $(f_1|P)'|\partial P$ is an orbi-embedding.

Repeating the use of the induction lemma to the final stage of the hierarchy of N, we have a collection of orbi-maps into ballic orbifolds or (a non-spherical turnover) $\times\ I$.

With the results in Section 2, we deform the orbi-maps (rel. ∂) to orbi-coverings. Since all modifications are performed when fixed on boundaries, we can piece together these orbi-coverings to obtain the conclusion (1).

A part of the results in the paper are used in [**TY2**].

1. Preliminaries

Throughout this paper, all orbifolds are connected and locally orientable unless otherwise stated. For basic facts on orbifolds, see [**Th, BS, D, Ta1**]. Furthermore, see also [**Ta1, TY1**] for an orbi-map which plays an important role in this paper.

Roughly speaking, an orbi-map is a continuous map between orbifolds respecting their orbifold structures. In addition, there is a non-singular point mapped to a non-singular point by an orbi-map, which induces a homomorphism of their orbifold fundamental groups. When N is a suborbifold of M, the inclusion map $i : N \to M$ is naturally equipped with an orbi-map structure and induces a homomorphism $i_* : \pi_1(N) \to \pi_1(M)$. An orbifold covering also has a canonical orbi-map structure, so we use the terminology 'orbi-covering' in this sense. The elemental properties of the ordinal covering category are parallelly translated into those of the orbi-covering category (the existence of a covering associated with a subgroup, the equivalence of a regular covering and a normal subgroup, the lifting property of an orbi-map, and so on).

Theorems 1.1, 1.2, and 1.3 are derived from equivariant theorems [**JR1, JR2, MY2, MY3, TY1**].

THEOREM 1.1. (The loop theorem [**TY1**, Theorem 6.4]) *Let M be a good 3-orbifold with boundaries. Let F be a connected 2-suborbifold in ∂M. If* $\mathrm{Ker}(\pi_1(F) \to \pi_1(M)) \neq 1$, *then there exists a discal 2-suborbifold D properly embedded in M such that $\partial D \subset F$ and ∂D does not bound any discal 2-suborbifold in F.*

THEOREM 1.2. (Dehn's lemma [**TY1**, Theorem 6.5]) *Let M be a good 3-orbifold with boundaries. Let γ be a simple closed curve in $\partial M - \Sigma M$ such that the order of $[\gamma]$ is n in $\pi_1(M)$. Then there exists a discal suborbifold $D^2(n)$ properly embedded in M with $\partial D^2(n) = \gamma$.*

THEOREM 1.3. (The sphere theorem [**TY1**, Theorem 6.7]) *Let M be a good 3-orbifold. Let $p : \tilde{M} \to M$ be the universal cover of M. If $\pi_2(\tilde{M}) \neq 0$, then there exists a spherical suborbifold S in M such that $[\tilde{S}] \neq 0$ in $\pi_2(\tilde{M})$, where $\tilde{S}$ is any component of $p^{-1}(S)$.*

The next corollary is derived directly from Theorem 1.3.

COROLLARY 1.4. *Let M be a good 3-orbifold. If M is irreducible, then for any manifold covering $\tilde{M}$ of M, $\pi_2(\tilde{M}) = 0$.*

In the remaining part of this section, we describe several propositions derived from Theorems 1.1–1.3.

PROPOSITION 1.5. *Let M be a good 3-orbifold, F be a connected and incompressible 2-suborbifold which is 2-sided and properly embedded in M, and N be the orbifold derived from M by cutting open along F. Then, M is irreducible if and only if each component of N is irreducible.*

Proof. Suppose M is not irreducible. There is an incompressible spherical suborbifold S in M. By the innermost argument, we may assume that $S \cap F = \phi$. It is a contradiction. The converse is derived from the fact that a ballic 3-orbifold does not contain any incompressible 2-suborbifolds. □

PROPOSITION 1.6. *Let M be a good and compact 3-orbifold, F be a connected and incompressible 2-suborbifold which is 2-sided and properly embedded in M, and N be the orbifold derived from M by cutting open along F. Then, for any component N' of N,* $\mathrm{Ker}(\pi_1(N') \to \pi_1(M)) = 1$.

Proof. Otherwise there is an orbi-map $f : B^2 \to M$ which is transverse to $F - \Sigma F$ and satisfies $f^{-1}(f(B^2) \cap F) \subset \mathrm{Int}\, B^2$ and $[f|\partial B^2] \neq 1$ in $\pi_1(N')$. Then, by the innermost argument, we can find the subdisc E^2 of B^2 which satisfies $f(E) \cap (F - \Sigma F) = f(\partial E)$, $f(\mathrm{Int}\, E) \subset M - F$, and $[f|\partial E] \neq 1$ in $\pi_1(F)$. It is a contradiction. □

Let M be a good 3-orbifold and F a connected suborbifold which is 2-sided and properly embedded in M. It is clear that if $\mathrm{Ker}(\pi_1(F) \to \pi_1(M)) = 1$, then F is incompressible in M. By Theorem 1.1 and Proposition 1.6, the converse stands.

PROPOSITION 1.7. *Let M be a good 3-orbifold, F be a connected 2-suborbifold which is 2-sided and properly embedded in M. If F is incompressible, then* $\mathrm{Ker}(\pi_1(F) \to \pi_1(M)) = 1$.

We end up this section with describing the relation between orbi-coverings and incompressibility.

PROPOSITION 1.8. *Let M be a good 3-orbifold, and F a connected 2-suborbifold which is 2-sided and properly embedded in M. Let $p' : M' \to M$ be a covering and F' be a component of $p'^{-1}(F)$. Then, F is incompressible in M, if and only if F' is incompressible in M'.*

Proof. Suppose F' is incompressible in M'. When $F =$ (a discal, spherical orbifold), the conclusion is clear. We consider the case that $F \neq$ (a discal, spherical orbifold).

Suppose F is compressible. Then, there is a compressing discal suborbifold D in M. Let D' be a component of $p'^{-1}(D)$ such that $\partial D'$ is innermost one of $p'^{-1}(\partial D)$ in F'. Since D' is a discal orbifold and F' is incompressible in M', there is a discal suborbifold E' in F' such that $\partial E' = \partial D'$. Then, $p'(E')$ is a discal orbifold in F bounded by ∂D by the innermost property of $\partial D'$. It is a contradiction.

Suppose F is incompressible in M. When $F \neq$ (a discal, or a spherical orbifold), the conclusion is clear by the following commutative diagram:

$$\begin{array}{ccc} \pi_1(F') & \xrightarrow{i'_*} & \pi_1(M') \\ {\scriptstyle (p'|F')_*}\downarrow & & \downarrow{\scriptstyle p'_*} \\ \pi_1(F) & \xrightarrow[i_*]{} & \pi_1(M) \end{array}$$

When $F =$ (a discal orbifold). In this case F' is a discal orbifold. Suppose F' is compressible in M'. There is a ballic suborbifold B of cyclical type in M' such that $F' \subset \partial B$ and $\partial B - \operatorname{Int} F' \subset \partial M'$. Let F'' be the component of $p'^{-1}(F)$ such that $\partial F''$ is innermost one of $p'^{-1}(\partial F)$ in $\partial B - \operatorname{Int} F'$. Let D'' be the discal suborbifold in $\partial B - \operatorname{Int} F'$ bounded by $\partial F''$ and B' be the ballic suborbifold in B bounded by $D'' \cup F''$. Then, $p'(D'')$ is a discal suborbifold in ∂M bounded by ∂F and $p'(B')$ is a suborbifold in M bounded by $p'(D'')$ and F. The remaining is to show that $p'(B')$ is a ballic orbifold. Let $\tilde{p} : \tilde{M} \to M'$ be the universal covering and $\tilde{B}$ a component of $\tilde{p}^{-1}(B')$. Since B' is a ballic orbifold, $\tilde{B}$ is a ball. Put $p = p' \circ \tilde{p}$. $p|\tilde{B} : \tilde{B} \to p(\tilde{B})$ is the universal covering and $\operatorname{Aut}(\tilde{B}, p|\tilde{B})$ is a finite group acting on $\tilde{B}$. Hence by [**KS**, (5.6.2)] and [**MY1**], $p(\tilde{B})(= p'(B')) =$ (a ballic orbifold). It is a contradiction.

When $F =$ (a spherical orbifold), it is similar to the above. □

Remark 1.9. Theorems 1.1–1.3, Corollary 1.4, and Proposition 1.5 and 'only if' part of Proposition 1.8 hold without the local orientability of M.

2. Extensions of orbi-maps

We define the *double of M*, denoted by $\mathcal{D}M$, as follows. Let $\mathrm{id}_{\partial M} : \partial M \to \partial M$ be the identity orbi-map. $\mathcal{D}M$ is the orbifold obtained by identifying two copies of M with $\mathrm{id}_{\partial M}$. Note that if B be a ballic 3-orbifold and $S = \mathcal{D}B$, then $B \times I$ is orbi-isomorphic to the cone on S.

Let M be a 3-orbifold and X an orbifold. We say that two orbi-maps $f, g : M \to X$ are *C-equivalent* if there are orbi-maps $f = f_0, f_1, \ldots, f_n$ from M to X with either f_i is orbi-homotopic to f_{i-1} or f_i agrees with f_{i-1} on $M - B$ for some ballic

3-orbifold $B \subset M$ with $B \cap \partial M$ a discal orbifold or $|B| \cap |\partial M| = \phi$. Then, the following lemma is clear.

LEMMA 2.1. *Let M and N be good 3-orbifolds and B be a ballic 3-orbifold in M. Let $f, g : M \to N$ be orbi-maps which are C-equivalent with respect to B. Let S be the double of a ballic 3-orbifold. If any orbi-map from S to N is extendable to an orbi-map from the cone on S to N, then f and g are orbi-homotopic rel. $M - \operatorname{Int} B$.*

THEOREM 2.2. *Let S be the double of a ballic 3-orbifold and V be the cone on S. Let M be a good 3-orbifold such that the underlying space of the universal cover of $\operatorname{Int} M$ is homeomorphic to $\mathbb{R}^3$. Then, any orbi-map $f : S \to M$ is extendable to an orbi-map from V to M.*

Proof. Let $p : \tilde{S} \to S$ and $q : \tilde{M} \to M$ be the universal coverings. Let $\tilde{f} : \tilde{S} \to \tilde{M}$ be the structure map. Let e and e' be 3-balls in $|S|$ such that $|S| = e \cup e'$, $e \cap e' = \partial e = \partial e'$, and $\Sigma S \subseteq \partial e$. Let $\tilde{e}$ (respectively $\tilde{e}'$) be the closure of a connected component of $p^{-1}(e) - p^{-1}(\Sigma S)$ (respectively $p^{-1}(e') - p^{-1}(\Sigma S)$). We can describe $|\tilde{S}| = \cup_{\sigma_A \in \operatorname{Aut}(\tilde{S},p)} (\sigma_A \tilde{e} \cup \sigma_A \tilde{e}')$.

Let $\bar{p} : \tilde{V} \to V$ be the universal covering. Let c be the cone point of V and $\tilde{c} = \bar{p}^{-1}(c)$. We can describe $|\tilde{V}| = \cup_{\sigma_A \in \operatorname{Aut}(\tilde{S},p)} (\tilde{c} * \sigma_A \tilde{e} \cup \tilde{c} * \sigma_A \tilde{e}')$.

By the definition of $c * S$, a point in $|\tilde{V}|$ (respectively $|V|$) is described by $(1-t)\tilde{c} + t\tilde{x}$, $\tilde{x} \in |\tilde{S}|$, $0 \le t \le 1$ (respectively $(1-t)c + tx$, $x \in |S|$, $0 \le t \le 1$) and $\bar{p}((1-t)\tilde{c} + t\tilde{x}) = (1-t)c + tp(\tilde{x})$.

Since $\operatorname{Int}|\tilde{M}| \cong \mathbb{R}^3$, we may assume that $(f_*\pi_1(S))_A$ is a finite subgroup of $\operatorname{diff}_+(\mathbb{R}^3)$. By [**BKS**, Corollary I.1b], the fixed point set of $(f_*\pi_1(S))_A$ is homeomorphic to either $\mathbb{R}^0$ or $\mathbb{R}^1$. Let $\tilde{u}$, $\tilde{v}$ be the vertices of $p^{-1}(\Sigma S)$ and $\tilde{\ell}_1$, $\tilde{\ell}_2$, $\tilde{\ell}_3$ be the edges of $p^{-1}(\Sigma S)$ in $\tilde{e}$. Let $\tilde{E}_{12}$, $\tilde{E}_{23}$, $\tilde{E}_{31}$ be the faces of $\partial\tilde{e}$ which satisfy $\partial\tilde{E}_{12} = \tilde{\ell}_1 \cup \tilde{\ell}_2$, $\partial\tilde{E}_{23} = \tilde{\ell}_2 \cup \tilde{\ell}_3$, $\partial\tilde{E}_{31} = \tilde{\ell}_3 \cup \tilde{\ell}_1$.

Let σ_i be the normal loop around $\ell_i = p(\tilde{\ell}_i)$ in S and L_i be the fixed point set of $f_*(\sigma_i)_A$ in M. It follows that L_i is homeomorphic to either $\mathbb{R}^0$ or $\mathbb{R}^1$ and $\operatorname{Fix}((f_*\pi_1(S))_A) \subset L_1 \cap L_2 \cap L_3$.

Take any point $\tilde{d}$ in $\operatorname{Fix}((f_*\pi_1(S))_A)$. Define $\tilde{g}_{\tilde{c}}(\tilde{c}) = \tilde{d}$. Define $\tilde{g}_{\tilde{c}*\tilde{u}} : \tilde{c}*\tilde{u} \to \tilde{M}$ (respectively $\tilde{g}_{\tilde{c}*\tilde{v}}$) by a path (possibly a point in case $\operatorname{Fix}((f_*\pi_1(S))_A) \cong \mathbb{R}^0$) from $\tilde{d}$ to $\tilde{f}(\tilde{u})$ (respectively $\tilde{f}(\tilde{v})$) in $\operatorname{Fix}((f_*\pi_1(S))_A)$. Since $\tilde{f}(\tilde{\ell}_i) \subset L_i$, the map $(\tilde{f}|\tilde{\ell}_i) \cup \tilde{g}_{\tilde{c}*\tilde{u}} \cup \tilde{g}_{\tilde{c}*\tilde{v}}$ is one from $\tilde{\ell}_i \cup (\tilde{c} * \tilde{u}) \cup \tilde{c} * \tilde{v}$ to L_i. Note that $\partial(\tilde{c} * \tilde{\ell}_i) = \tilde{\ell}_i \cup (\tilde{c} * \tilde{u}) \cup (\tilde{c} * \tilde{v}) \cong S^1$. Since $\pi_1(L_i) = 1$, $(\tilde{f}|\tilde{\ell}_i) \cup \tilde{g}_{\tilde{c}*\tilde{u}} \cup \tilde{g}_{\tilde{c}*\tilde{v}}$ is extendable to a map from $\tilde{c} * \tilde{\ell}_i$ to L_i. Define $\tilde{g}_{\tilde{c}*\tilde{\ell}_i} : \tilde{c} * \tilde{\ell}_i \to |\tilde{M}|$ by the extension. Note that $\partial(\tilde{c} * \tilde{E}_{ij}) = \tilde{E}_{ij} \cup (\tilde{c} * \tilde{\ell}_i) \cup (\tilde{c} * \tilde{\ell}_j)$ $(\cong S^2)$. Since $\pi_2(|\tilde{M}|) = 0$,

$(\tilde{f}|\tilde{E}_{ij}) \cup \tilde{g}_{\tilde{c}*\tilde{\ell}_i} \cup \tilde{g}_{\tilde{c}*\tilde{\ell}_j} : \partial(\tilde{c} * \tilde{E}_{ij}) \to |\tilde{M}|$ is extendable to a map from $\tilde{c} * \tilde{E}_{ij}$ to $|\tilde{M}|$. Define $\tilde{g}_{\tilde{c}*\tilde{E}_{ij}} : \tilde{c} * \tilde{E}_{ij} \to |\tilde{M}|$ by the extension. Note that $\partial(\tilde{c} * \tilde{e}) = \tilde{e} \cup_{ij} \tilde{c} * E_{ij}$ ($\cong S^3$). Since $\pi_3(|\tilde{M}|) = 0$, $(\tilde{f}|\tilde{e}) \cup_{ij} \tilde{g}_{\tilde{c}*\tilde{E}_{ij}} : \partial(\tilde{c} * \tilde{e}) \to |\tilde{M}|$ is extendable to a map from $\tilde{c} * \tilde{e}$ to $|\tilde{M}|$.

We define $\tilde{g} : \tilde{V} \to \tilde{M}$, at first on $\tilde{c} * \tilde{e}$ by the extension, next on $\tilde{c} * p^{-1}(e)$ equivariantly, on $\tilde{c} * \tilde{e}'$ by cone extension, on $\tilde{c} * p^{-1}(e')$ equivariantly again, and finally piecing together $\tilde{g}_{\tilde{c}*p^{-1}(e)}$ and $\tilde{g}_{\tilde{c}*p^{-1}(e')}$. Then, $g = (\bar{g}, \tilde{g})$ is the desired orbi-map. □

The following, Propositions 2.3 and 2.4, are derived directly from Lemma 2.1 and Theorem 2.2.

PROPOSITION 2.3. *Let M and N be good 3-orbifolds. Let $f, g : M \to N$ be orbi-maps which are C-equivalent w.r.t. a ballic 3-orbifold B in M. If the universal cover of* Int N *is homeomorphic to $\mathbb{R}^3$, then f and g are orbi-homotopic rel. $M -$* Int B.

PROPOSITION 2.4. *Let P and Q be ballic 3-orbifolds. Suppose $f : (P, \partial P) \to (Q, \partial Q)$ is an orbi-map such that $(f|\partial P) : \partial P \to \partial Q$ is an orbi-isomorphism. Then, there is an orbi-isomorphism $g : P \to Q$ such that g and f are orbi-homotopic rel. ∂.*

PROPOSITION 2.5. *Let F and G be closed orientable 2-orbifolds which are orbi-isomorphic and have infinite orbifold fundamental groups. Suppose $f : (F \times I, \partial(F \times I)) \to (G \times I, \partial(G \times I))$ is an orbi-map such that $f|\partial(F \times I) : \partial(F \times I) \to \partial(G \times I)$ is an orbi-isomorphism and f_* is a monomorphism. Then, there is an orbi-isomorphism $g : F \times I \to G \times I$ such that g and f are orbi-homotopic rel. ∂.*

Proof. When $F \cong G \neq$ a turnover.

Let t be the genus of $|G|$ and s be the number of singular points of G. Let $a_1, \dots, a_t$ be mutually disjoint simple closed curves on $|G| - \Sigma G$ with which G is cut open into a 2-orbifold G_t whose underlying space is the planer surface with $2t$ boundaries, $a_{t+1}, \dots, a_{3t-1}$ be mutually disjoint simple arcs on $|G_t| - \Sigma G_t$ properly embedded in $|G_t|$ with which G_t is cut open into a 2-orbifold G_{3t-1} whose underlying space is a 2-disc, and $a_{3t}, \dots, a_{3t+s-1}$ be mutually disjoint simple arcs on $|G_{3t-1}| - \Sigma_{3t-1}$ properly embedded in $|G_{3t-1}|$ with which G_{3t-1} is cut open into a 2-orbifold G_{3t+s-1} which consists of discal 2-orbifolds $D_1, \dots, D_s$.

By [**Ta1**, 5.5] and Proposition 2.3, we may assume that $f^{-1}(a_1 \times I)$ is an incompressible 2-suborbifold in $F \times I$. Note that, by the proof of [**Ta1**, 5.5], the modification is invariant on $\partial(F \times I)$ when no component of $f^{-1}(a_1 \times I)$ is a compressible discal 2-orbifold. Since $f|\partial(F \times I)$ is an orbi-isomorphism, no component of $f^{-1}(a_1 \times I)$ is such a one. Hence the modification is invariant on $\partial(F \times I)$.

Let A_1 be a component of $f^{-1}(a_1 \times I)$. Since $(f|A_1)_* : \pi_1(A_1) \to \pi_1(a_1 \times I)$ must be monic, $\pi_1(A_1)$ is either infinite cyclic or a trivial group. By the orientability of A_1, it is orbi-isomorphic to a 2-sphere, a 2-disc, or an annulus. Furthermore, from the irreducibility of A_1, it must not be a 2-sphere or a 2-disc. Hence A_1 is an annulus. Since $\partial f^{-1}(a_1 \times I) \subset \partial(F \times I)$ and f is an orbi-isomorphism from $\partial(F \times I)$ to $\partial(G \times I)$, $\partial f^{-1}(a_1 \times I)$ consists of two simple closed curves on different components of $\partial(F \times I)$ each other. Thus, $f^{-1}(a_1 \times I) = A_1$ and does not separate M.

Since $f|\partial A_1$ is a homeomorphism from ∂A_1 to $\partial(a_1 \times I)$, we may assume that $f|A_1 : A_1 \to a_1 \times I$ is a homeomorphism under modifying an orbi-homotopy rel. ∂A_1.

Let $G_1 \times I$ (respectively M_1) be an orbifold derived from cutting $G \times I$ (respectively $F \times I$) open along $a_1 \times I$ (respectively A_1). We had an orbi-map $f_1 : M_1 \to G_1 \times I$ such that $f_1|\partial M_1 : \partial M_1 \to \partial(G_1 \times I)$ is an orbi-isomorphism, $f_1|\partial M_1 = f|\partial M_1$, and $(f_1)_*$ is monic.

By iterating the above procedure, we can get an orbi-map $f_i : M_i \to G_i \times I$ such that $(f_i|\partial M_i) : \partial M_i \to \partial(G_i \times I)$ is an orbi-isomorphism, $f_i|\partial M_i = f_{i-1}|\partial M_i$, and $(f_i)_*$ is monic, where $G_i \times I$ (respectively M_i) is an orbifold derived from cutting $G_{i-1} \times I$ (respectively M_{i-1}) open along $a_i \times I$ (respectively $f^{-1}(a_i \times I)$), $1 \le i \le 3t + s - 1$, where $f_0 = f$, $M_0 = M$, and $G_0 = G$.

Let Q be a component of $G_{3t-1} \times I$ which is orbi-isomorphic to (a discal 2-orbifold) $\times\, I$. Let P be the component of M_{3t+s-1} such that $f_{3t+s-1}(P) \subset Q$.

Since $f|\partial P : \partial P \to \partial Q$ is an orbi-isomorphism, ∂P is a spherical 2-orbifold of a cyclical type. Since $F \times I$ is irreducible, P is the cone on ∂P. Hence P is orbi-isomorphic to Q. Then, we can get an orbi-isomorphism $g_P : P \to Q$ by extending $f|\partial P : \partial P \to \partial Q$. By Theorem 2.2, $f|\partial P$ and g are orbi-homotopic (rel. ∂P).

Proceeding similarly on other components and piecing together the results, we get the desired orbi-isomorphism and orbi-covering.

When $F \cong G =$ a turnover.

Put $\Sigma(F \times 0) = \{v_1, v_2, v_3\}$. Let a_{12}, a_{23}, a_{31} be simple arcs on $F \times 0$ such that $\partial a_{ij} = v_i \cap v_j \cup v_k$ and $\mathrm{Int}\, a_i \cap \mathrm{Int}\, a_j = \phi$ $(i \ne j)$. Let e_0 and e_1 be 2-discs on $|F \times 0|$ bounded by $a_{12} \cup a_{23} \cup a_{31}$. Put $a_i' = (F \times 1) \cap (a_i \times I)$.

Since f_* is a monomorphism, and $(f|\partial(F \times I))$ is an orbi-isomorphism, $f(v_i \times I) = f(v_i) \times I$. Put $E_{ij} = a_{ij} \times I$. Let D_{ij} be the 2-disc bounded by $f(a_{ij}) \cup (f(v_i) \times I) \cup f(a_{ij}') \cup (f(v_j) \times I)$.

Since $f|\partial(F \times I)$ is an orbi-isomorphism, we may assume that $\mathrm{Int}\, D_{ij}$'s are mutually disjoint, $D_{ij} \cap \partial(G \times I) = f(a_{ij}) \cup f(a'_{ij})$, and $D_{ij} \cap D_{jk} = f(v_j) \times I$.

Let B_0 and B_1 be the 3-ball in $|G \times I|$ separated by $D_{12} \cup D_{23} \cup D_{31}$, where $B_i \cap f(e_i) = f(e_i)$.

We construct an orbi-isomorphism $g : F \times I \to G \times I$ as follows; Define $g|(v_i \times I) : v_i \times I \to f(v_i) \times I$ by extending $f|(v_i \times \partial I)$. Define $g|E_{ij} : E_{ij} \to D_{ij}$ by extending $g|\partial E_{ij}$. Define $g|(e_i \times I) : e_i \times I \to B_i$ by extending $g|\partial(e_i \times I)$. Furthermore, we construct an equivariant homotopy between the structure maps of f and g by the same way (i.e. skeleton-wise and equivariantly) as in the proof of Theorem 2.2 to get the desired orbi-homotopy. □

3. Orbi-maps between 2-orbifolds

A 2-orbifold T is called a *turnover* if $|T| =$ (the 2-sphere) and $\Sigma T =$ (three points).

THEOREM 3.1. *Let X and Y be turnovers with infinite π_1's. If $f : X \to Y$ is an orbi-map such that $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism, then f is orbi-homotopic to an orbi-isomorphism.*

Proof. At first, we show that X is orbi-isomorphic to Y. Put $\Sigma X = \{x_1, x_2, x_3\}$, $\Sigma Y = \{y_1, y_2, y_3\}$. Let m_i (respectively n_i) be the index of x_i (respectively y_i). Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be the universal coverings, and $\tilde{f} : \tilde{X} \to \tilde{Y}$ (respectively $\bar{f} : |X| \to |Y|$) be the structure map (respectively underlying map). Put $\tilde{\Sigma}X = p^{-1}(\Sigma X)$ and $\tilde{\Sigma}Y = q^{-1}(\Sigma Y)$. Since f_* is monic, $\bar{f}(\Sigma X) \subset \Sigma Y$, and hence $\tilde{f}(\tilde{\Sigma}X) \subset \tilde{\Sigma}Y$. We call the rotation of the angle $2\pi/m_i$ (respectively $2\pi/n_i$) around $\tilde{x}_i \in p^{-1}(x_i)$ (respectively $\tilde{y}_i \in q^{-1}(y_i)$) *the unit rotation around* $\tilde{x}_i$ (respectively $\tilde{y}_i$). Note that these unit rotations are elements of $\mathrm{Aut}(\tilde{X}, p)$ (respectively $\mathrm{Aut}(\tilde{Y}, q)) \subset \mathrm{Isom}(\mathbb{E}^2)$ or $\mathrm{Isom}(\mathbb{H}^2)$.

Fact 1. $(\tilde{f}|\tilde{\Sigma}X) : \tilde{\Sigma}X \to \tilde{\Sigma}Y$ is epic.

Take any $z \in \tilde{\Sigma}Y$. Let ρ_A be the unit rotation around z. Since f_* is epic, there is an element $\sigma_A \in \mathrm{Aut}(\tilde{X}, p)$ such that $f_*(\sigma)_A = \rho_A$. Since the oder of ρ_A is finite and f_* is monic, the order of σ_A is finite. Hence $\mathrm{Fix}(\sigma_A) \neq \phi$. Take $x \in \mathrm{Fix}(\sigma_A)$. Since $\rho_A \tilde{f}(x) = f_*(\sigma)_A \tilde{f}(x) = \tilde{f}(\sigma_A x) = \tilde{f}(x)$, $\tilde{f}(x) \in \mathrm{Fix}(\rho_A)$. Since $\mathrm{Fix}(\rho_A) = \{z\}$, $\tilde{f}(x) = z$.

Fact 2. $\tilde{f}|\tilde{\Sigma}X$ is monic.

Take any $y, y' \in \tilde{\Sigma}X$. Suppose $\tilde{f}(y) = \tilde{f}(y') = z$. Let ρ_A be the unit rotation around z. By the proof of Fact 1, there is $x \in \tilde{\Sigma}X$ and the unit rotation around x, σ_A, such that $f_*(\sigma)_A = \rho_A$ and $\tilde{f}(x) = z$. Let τ_A be the unit rotation around y. Since $f_*(\tau)_A = \rho_A^i$ and f_* is monic, it holds that $\sigma_A^i = \tau_A$. Hence, $\sigma_A^i = \tau_A = \mathrm{id}$ unless $x = y$. Then, it derived that $x = y$ from the fact τ_A is the unit rotation, and it holds that $x = y'$ by the parallel argument.

Fact 3. If, for $x, y \in \tilde{\Sigma} X$, $q\tilde{f}(x) = q\tilde{f}(y)$, then $p(x) = p(y)$.

Suppose $\tilde{f}(y) = \rho_A \tilde{f}(x)$, $\rho_A \in \mathrm{Aut}(\tilde{Y}, q)$. Since f_* is epic, there is an element $\sigma_A \in \mathrm{Aut}(\tilde{X}, p)$ such that $\rho_A = f_*(\sigma)_A$. Hence, $\tilde{f}(y) = f_*(\sigma)_A\tilde{f}(x) = \tilde{f}(\sigma_A x)$. Since $\sigma_A x \in \tilde{\Sigma} X$, by Fact 2, $y = \sigma_A x$.

By Fact 1 and the fact that $\#\Sigma X = \#\Sigma Y$, $(\bar{f}|\Sigma X) : \Sigma X \to \Sigma Y$ is one to one. Suppose $\bar{f}(x_i) = y_i$, $i = 1, 2, 3$.

Fact 4. $m_i = n_i$, $i = 1, 2, 3$.

Since f_* is monic, m_i divides n_i. Put $n_i = g_i m_i$ $(g_i \geq 1)$. Note $\chi(X) = \sum_{i=1}^{3}(1/m_i) - 1$ and $\chi(Y) = \sum_{i=1}^{3}(1/n_i) - 1 = \sum_{i=1}^{3}(1/g_i m_i) - 1$, where χ is the Euler number of orbifolds. Since the Euler numbers of 2-orbifolds with same fundamental groups are equal, $g_i = 1$, $i = 1, 2, 3$.

Thus far, it has shown that X is orbi-isomorphic to Y. Next, we construct an orbi-isomorphism which is orbi-homotopic to f.

Note that, since X and Y are orbi-isomorphic, $\tilde{X}$ and $\tilde{Y}$ are homeomorphic and have the same tesselations.

Let e, e' be the fundamental triangles of the tesselation of $\tilde{X}$ whose intersection is one edge. We construct a map $\tilde{g} : \tilde{X} \to \tilde{Y}$ as follows. Let v_1, v_2, v_3 be the vertices of e. Put $\tilde{g}(v_i) = \tilde{f}(v_i)$, $i = 1, 2, 3$. Let ℓ_{ij} be the geodesic segment between v_i and v_j. Note these coincide with the line of the tesselation (i.e. the edges of e). Define

$$\tilde{g}|\ell_{ij} : \ell_{ij} \to \tilde{Y} \quad \text{by} \quad \tilde{g}(tv_i + (1-t)v_j) = t\tilde{g}(v_i) + (1-t)\tilde{g}(v_j),$$

where $ta + (1-t)b$ means the point divides the geodesic segment between a and b by the ratio $(1-t) : t$, and $(i, j) = (1, 2), (2, 3), (3, 1)$. Define

$$\tilde{g}|e : e \to \tilde{Y} \quad \text{by} \quad \tilde{g}(sv_1 + tv_2 + uv_3) = s\tilde{g}(v_1) + t\tilde{g}(v_2) + u\tilde{g}(v_3).$$

Define also $\tilde{g}|e'$ similarly. Furthermore, define $\tilde{g}|\sigma_A e$ and $\tilde{g}|\sigma_A e'$ equivariantly. By piecing together $(\tilde{g}|\sigma_A e)$'s and $(\tilde{g}|\sigma_A e')$'s, we can get an equivariant map from $\tilde{X}$ to $\tilde{Y}$.

Similarly, we construct an equivariant map $\tilde{F} : \tilde{X} \times [0, 1] \to \tilde{Y}$, using the fact that $\pi_1(\tilde{Y}) = 0$.

The remainder of the proof is to show that $\tilde{g}$ is a homeomorphism.

Claim 1. $\tilde{g}$ is a local homeomorphism.

Step 1. $\tilde{g}$ is a local homeomorphism around $x \in \mathrm{Int}(p^{-1}(e))$ or $\mathrm{Int}(p^{-1}(e'))$.

Recall that $\tilde{f}(v_i) = \tilde{g}(v_i)$. Let σ_A (respectively τ_A) be the unit rotation around v_1 (respectively v_2). Since $f_*(\sigma)_A \tilde{f}(v_3) \neq \tilde{f}(v_3)$, $\tilde{f}(v_1)$, $\tilde{f}(v_3)$, and $f_*(\sigma)_A \tilde{f}(v_3)$ are the vertices of an isosceles triangle. Similarly, so are $\tilde{f}(v_2)$, $\tilde{f}(v_3)$, and $f_*(\tau)_A \tilde{f}(v_3)$. Then, it derived that there is no geodesic including $\tilde{f}(v_1)$, $\tilde{f}(v_2)$, and $\tilde{f}(v_3)$ by the triangle inequality.

Step 2. $\tilde{g}$ is a local homeomorphism around $p^{-1}(\ell_{12} \cup \ell_{23} \cup \ell_{31})$.

By the proof of Step 1, it holds that $\tilde{f}(v_3)$ and $f_*(\sigma)_A \tilde{f}(v_3)$ are included in different domains of $\tilde{Y}$ separated by the geodesic line extending $\tilde{g}(\ell_{12})$.

Step 3. $\tilde{g}$ is a local homeomorphism around $p^{-1}(v_i)$.

We only have to show that $f_*(\sigma)_A = \sigma_A'^{r_1}$, $|r_1| = 1$, where σ'_A is the unit rotation around $\tilde{g}(v_1)$.

Let τ'_A (respectively ρ'_A) be the unit rotation around $\tilde{g}(v_2)$ (respectively $\tilde{g}(v_3)$). Let α_i be the angle of ℓ_{ij} and ℓ_{ik}. Suppose $f_*(\tau)_A = \tau_A'^{r_2}$ and $f_*(\rho)_A = \rho_A'^{r_3}$. Then, $\alpha_i = \pi|r_i|/m_i$, $|r_i| \geq 1$.

When X is a Euclidean orbifold, by the fact that $\sum_{i=1}^{3} \pi|r_i|/m_i = \pi$, it holds that $|r_1| = 1$.

When X is a hyperbolic orbifold, let $\triangle$ be the fundamental triangle of the tesselation of $\tilde{Y}$ such that a vertex of $\triangle$ is $\tilde{g}(v_1)$. Put $v'_1 = \tilde{g}(v_1)$. Let v'_i be the vertex of $\triangle$ such that v'_i is conjugate to $\tilde{g}(v_i)$, $i = 2, 3$.

Let $\triangle'$ be the geodesic triangle derived from $\tilde{g}(e)$ by preserving the length of edges $\tilde{g}(\ell_{12})$ and $\tilde{g}(\ell_{13})$ and changing the angle $\pi|r_1|/m_1$ to π/m_1. Let ε_2 (respectively ε_3) be the remaining angle of $\triangle'$ corresponding to v'_2 (respectively v'_3).

It stands that either

$$\pi|r_i|/m_i \leq \varepsilon_i, \quad i = 2, 3 \quad \text{(the equality stands iff } |r_1| = 1\text{)}, \tag{3.1.1}$$

or

$$\mathrm{Area}(\tilde{g}(e)) \geq \mathrm{Area}(\triangle'). \tag{3.1.2}$$

Since there is no point u such that u is conjugate to v'_2 and $d(v'_1, u) < d(v'_1, v'_2)$, it holds that $d(v'_1, v'_2) \leq d(\tilde{g}(v_1), \tilde{g}(v_2))$. Similarly, $d(v'_1, v'_3) \leq d(\tilde{g}(v_1), \tilde{g}(v_3))$. Therefore, $\mathrm{Area}(\triangle') \geq \mathrm{Area}(\triangle)$. Then, suppose (3.1.1), it holds that $\pi - \pi/m_1 - \varepsilon_2 - \varepsilon_3 \geq \pi(1 - \sum_{i=1}^{3}(1/m_i))$. Hence

$$\varepsilon_2 + \varepsilon_3 \leq \pi/m_2 + \pi/m_3. \tag{3.1.3}$$

If $|r_1| > 1$, then, by (3.1.1), $\pi|r_i|/m_i < \varepsilon_i$, $i = 2, 3$. Then

$$\pi|r_2|/m_2 + \pi|r_3|/m_3 < \varepsilon_2 + \varepsilon_3.$$

Hence, by (3.1.3), $\pi|r_2|/m_2 + \pi|r_3|/m_3 < \pi/m_2 + \pi/m_3$. This contradicts the fact that $|r_i| \geq 1$, $i = 2, 3$. Therefore $|r_1| = 1$.

Suppose (3.1.2), it holds that $\text{Area}(\tilde{g}(e)) \geq \text{Area}(\Delta)$. Then, $\pi|r_1|/m_1 + \pi|r_2|/m_2 + \pi|r_3|/m_3 \leq \pi/m_1 + \pi/m_2 + \pi/m_3$. Hence, $|r_1| = 1$.

Claim 2. $\tilde{g}$ is surjective.

Take any point $y \in \tilde{Y}$. Let Δ_y be the fundamental triangle of the tesselation of $\tilde{Y}$ including y. We shall show that $\tilde{g}(\tilde{X}) \supset \Delta_y$. Let a, b, c be the vertices of Δ_y. We may assume that the geodesic segment from a to b is the longest edge of Δ_y.

Let x_1 be the point of $\tilde{\Sigma} X$ such that $\tilde{g}(x_1) = a$. Let D and D' be fundamental triangles of the tesselation of $\tilde{X}$ such that $x_1 \in D \cap D'$ and $D \cap D'$ is a segment. Let x_2 and x_3 be the other vertices of D. Note that the interior of the disc with the center a and the radius $d(a, c)$ does not include the vertices of the tesselation of $\tilde{Y}$ other than a. Hence,

$$d(\tilde{g}(x_1), \tilde{g}(x_2)) \geq d(a, c) \quad \text{and} \quad d(\tilde{g}(x_1), \tilde{g}(x_3)) \geq d(a, c). \tag{3.1.4}$$

Let ζ_A be the unit rotation around x_1 and n the order of it. Note that $\tilde{g}(\cup_{i=1}^{n} \zeta_A^i (D \cup D')) = \cup_{i=1}^{n} f_*(\zeta)_A^i (\tilde{g}(D) \cup \tilde{g}(D'))$ and $f_*(\zeta)_A \in \text{Isom}(\tilde{Y})$. By (3.1.4), the edges of $\tilde{g}(D)$ and $\tilde{g}(D')$ including a is longer than $d(a, c)$. By Claim 1, $\tilde{g}(\cup_{i=1}^{n} \zeta_A^i (D \cup D'))$ is homeomorphic to a 2-disc. Hence, $\tilde{g}(\cup_{i=1}^{n} \zeta_A^i (D \cup D')) \supset B_1$, where $B_1 = \{z \in \tilde{Y} | d(a, z) \leq d(a, c)\}$. Thus

$$B_1 \subset \tilde{g}(\tilde{X}). \tag{3.1.5}$$

Similarly, it holds that

$$B_2 \subset \tilde{g}(\tilde{X}), \tag{3.1.6}$$

where $B_2 = \{z \in \tilde{Y} | d(b, z) \leq d(b, c)\}$. Furthermore, it is clear that $\Delta_y \subset B_1 \cup B_2$. Hence, $\Delta_y \subset \tilde{g}(\tilde{X})$

Thus, $\tilde{g}$ is a local homeomorphism and surjective.

Furthermore, since $\tilde{X}$ is a complete metric space, the path lifting property stands. Hence $\tilde{g} : \tilde{X} \to \tilde{Y}$ is a covering. Since $\tilde{g}_*$ must be an isomorphism, $\tilde{g}$ is a homeomorphism. Define $\bar{g}(x) = q\tilde{g}(\tilde{x})$, $\tilde{x} \in p^{-1}(x)$. Then, $g = (\tilde{g}, \bar{g})$ is the desired orbi-map. □

Remark. On the other hand, when X and Y are orientable spherical orbifolds, there are infinitely many orbi-maps from X to Y which are not orbi-homotopic to any orbi-isomorphisms. We can construct such orbi-maps by using cyclic branched covering from $\tilde{X}$ to $\tilde{Y}$ branched over a pair of pre-image of ΣY.

THEOREM 3.2. *Let F and G be compact and orientable 2-orbifold such that $\#\pi_1(F) = \infty$. Suppose $f : (F, \partial F) \to (G, \partial G)$ is an orbi-map such that $f_* : \pi_1(F) \to \pi_1(G)$ is monic. Then there is an orbi-homotopy $f_t : (F, \partial F) \to (G, \partial G)$, $t \in |I|$, with $f_0 = f$ and either*
(1) *$f_1 : F \to G$ is an orbi-covering, or*
(2) *F is an annulus and $f_1(F) \subset \partial G$.*
If for some component J of ∂F, $(f|J) : J \to f(J)$ is an orbi-covering, it stands that $(f_t|J) = (f|J)$ for all t.

Proof. Let $p : \tilde{G} \to G$ be a covering associated with $f_*\pi_1(F)$ and $\tilde{f} : F \to \tilde{G}$ a lift of f by p. We only have to show that the statement stands for $\tilde{f}$.

When $\tilde{G} \neq$ (a turnover). By [**Ta1**, 7.2].

When $\tilde{G} =$ (a turnover). Then F is also a turnover. Otherwise, construct an orbi-map $g : \tilde{G} \to F$ which induces an isomorphism $g_* : \pi_1(\tilde{G}) \to \pi_1(F)$ (see [**Ta1**, 4.4]). By [**Ta1**, 7.2], F and $\tilde{G}$ are orbi-isomorphic. It is a contradiction. Hence, we can apply Theorem 3.1 to show that $\tilde{f}$ is orbi-homotopic to an orbi-isomorphism. □

COROLLARY 3.3. (The classification of closed 2-orbifolds) *Let X and Y be closed and orientable 2-orbifolds with infinite π_1's. If there is an isomorphism $\varphi : \pi_1(X) \to \pi_1(Y)$, then there is an orbi-isomorphism $f : X \to Y$ such that $f_* = \varphi$.*

Proof. By [**Ta1**, 4.4], we can construct an orbi-map $g : X \to Y$ with $g_* = \varphi$. Then Theorem 3.2 shows that g is orbi-homotopic to an orbi-isomorphism. □

4. I-bundles

THEOREM 4.1. *Let M be a good, compact, orientable, and irreducible 3-orbifold with boundaries. Let N be a good and orientable 3-orbifold with boundaries. Let $f : (M, \partial M) \to (N, \partial N)$ be an orbi-map such that f_* is monic. If there exist a path $\alpha : (I, \partial I) \to (|M| - \Sigma M, |\partial M|)$ and incompressible components B_0, B_1 of ∂M (possibly $B_0 = B_1$), C of ∂N which satisfy the following:*
(i) $\alpha(0) \neq \alpha(1)$;
(ii) $\bar{f}(\alpha(0)) = \bar{f}(\alpha(1)) \in |\partial N| - \Sigma N$;

(iii) $[\tilde{f} \circ \hat{\alpha}] = 1$ *in* $\pi_1(N)$,
where $\hat{\alpha}$ *is a lift of* α *to the universal cover of* M *and* $f = (\bar{f}, \tilde{f})$*;*

(iv) B_i *(respectively* C*) includes* $\alpha(i)$ *(respectively* $\bar{f}(\alpha(0))$*) and* $(f|B_i) : B_i \to C$ *is a covering,* $i = 0, 1$.

Then M *is an I-bundle over a closed 2-orbifold.*

Proof. Put $x_i = \alpha(i)$, $i = 0, 1$ and $y = \bar{f}(x_0)$. Let $\eta_0 : \pi_1(B_0, x_0) \to \pi_1(M, x_0)$ be the homomorphism induced by the inclusion orbi-map and $p : (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ the covering associated with $\eta_0\pi_1(B_0, x_0)$. Let $\tilde{\alpha}$ be the lift of α by p with $\tilde{\alpha}(0) = \tilde{x}_0$. Put $\tilde{x}_1 = \tilde{\alpha}(1)$. Let $\tilde{B}_i$ be the component of $p^{-1}(B_i)$ which includes $\tilde{x}_i$, $i = 0, 1$. By Lemma 4.2, we can conclude that $\tilde{B}_1$ is compact. (Note that $\tilde{B}_0$ is already compact, since $p|\tilde{B}_0$ is an orbi-isomorphism.) Let $\eta_1 : \pi_1(B_1, x_1) \to \pi_1(M, x_1)$, $\tilde{\eta}_i : \pi_1(\tilde{B}_i, \tilde{x}_i) \to \pi_1(\tilde{M}, \tilde{x}_i)$, $i = 0, 1$, be the homomorphisms induced by the inclusion orbi-maps. Note that $\tilde{\eta}_0$ is epic. Let $\Psi_\alpha : \pi_1(M, x_1) \to \pi_1(M, x_0)$ (respectively $\Psi_{\tilde{\alpha}} : \pi_1(\tilde{M}, \tilde{x}_1) \to \pi_1(\tilde{M}, \tilde{x}_0)$) be the change of base point map induced by α (respectively $\tilde{\alpha}$).

Claim 1. There is no orbi-homotopy which retracts α into $|\partial M| - \Sigma M$ (rel. x_0, x_1).

Otherwise, there is a path α_1 in $|B_0| - \bar{f}^{-1}(\Sigma C)$ such that $\alpha \sim \alpha_1$ (rel. x_0, x_1). Let $\hat{\alpha}_1$ be a lift of α_1 to the universal cover of M. Note that $\bar{f} \circ \alpha_1$ is a loop in C based at y. Since the loop $\bar{f} \circ \alpha_1$ lifts to the path α_1 under the covering $(\bar{f}|B_0) : B_0 \to C$, $[\bar{f} \circ \alpha_1] \neq 1$ in $\pi_1(C, y)$. This means $[\tilde{f} \circ \hat{\alpha}_1] \neq 1$ in $\pi_1(C, y)$. On the other hand,

$$\begin{aligned}
[\tilde{f} \circ \hat{\alpha}_1] &= [\tilde{f} \circ \hat{\alpha}]^{-1}[\tilde{f} \circ \hat{\alpha}_1] \\
&= [\tilde{f} \circ (\hat{\alpha}^{-1} \cdot \hat{\alpha}_1)] \\
&= f_*[\hat{\alpha}^{-1} \cdot \hat{\alpha}_1] \\
&= f_*[1] \\
&= 1 \quad \text{in } \pi_1(N, y).
\end{aligned}$$

By Theorem 1.1, this contradicts the fact that C is incompressible in N.

Claim 2. $\tilde{B}_0 \neq \tilde{B}_1$.

Suppose $\tilde{B}_0 = \tilde{B}_1$. Let $\tilde{\gamma}$ be any path in $|\tilde{B}_0| - p^{-1}(\Sigma B_0)$ from $\tilde{x}_1$ to $\tilde{x}_0$. Since $\tilde{\eta}_0 : \pi_1(\tilde{B}_0, \tilde{x}_0) \to \pi_1(\tilde{M}, \tilde{x}_0)$ is epic, there is a loop $\tilde{\gamma}'$ in $|\tilde{B}_0| - p^{-1}(\Sigma B_0)$ based on $\tilde{x}_0$ such that $[\tilde{\gamma}'] = [\tilde{\alpha} \cdot \tilde{\gamma}] \in \pi_1(\tilde{M}, \tilde{x}_0)$. Put $\tilde{\beta} = \tilde{\gamma} \cdot \tilde{\gamma}'^{-1}$. $\tilde{\beta}$ is a path in $|\tilde{B}_0| - p^{-1}(\Sigma B_0)$ from $\tilde{x}_1$ to $\tilde{x}_0$ which satisfies $[\tilde{\alpha} \cdot \tilde{\beta}] = 1$ in $\pi_1(\tilde{M}, \tilde{x}_0)$. Hence $p_*[\tilde{\alpha} \cdot \tilde{\beta}] = 1$ in $\pi_1(M, x_0)$. On the other hand, $p_*[\tilde{\alpha} \cdot \tilde{\beta}] = [(p \circ \tilde{\alpha}) \cdot (p \circ \tilde{\beta})]$. This means $p \circ \tilde{\alpha} = \alpha$ and $(p \circ \tilde{\beta})^{-1}$ ($\subset |B_0| - \Sigma M$) are orbi-homotopic (rel. x_0, x_1). This contradicts Claim 1.

By the incompressibility of $\tilde{B}_0$ and the surjectivity of $\tilde{\eta}_0$, $\tilde{\eta}_0 : \pi_1(\tilde{B}_0, \tilde{x}_0) \to \pi_1(\tilde{M}, \tilde{x}_0)$ is an isomorphism. Since $\tilde{B}_0$ is a closed 2-orbifold, there is a torsion free normal subgroup G with finite index in $\pi_1(\tilde{M}, \tilde{x}_0)$.

Let $p' : (M', x'_0) \to (\tilde{M}, \tilde{x}_0)$ be the covering associated with G. Since $p' : M' \to \tilde{M}$ is a finite regular orbi-covering and $\tilde{B}_i$ is compact, each component of $p'^{-1}(B_i)$ is compact, $i = 0, 1$.

Let B'_0 be a component of $p'^{-1}(\tilde{B}_0)$, x'_0 a lift of $\tilde{x}_0$ in B'_0, α' the lift of $\tilde{\alpha}$ with $\alpha'(0) = x'_0$, and B'_1 the component of $p'^{-1}(\tilde{B}_1)$ which includes $\alpha'(1)$. Put $x'_1 = \alpha'(1)$.

By Proposition 1.7, B'_i is incompressible, $i = 0, 1$. α' satisfies conditions (i)–(iii), where we replace α to α'. $((f \circ p \circ p')|B'_i) : B'_i \to C$ is a covering, $i = 0, 1$.

Let $\eta'_0 : \pi_1(B'_0, x'_0) \to \pi_1(M', x'_0)$ be the homomorphism induced by the inclusion orbi-map. Let $\tilde{p}' : (\tilde{M}', \tilde{x}'_0) \to (M', x'_0)$ be the covering associated with $\eta'_0\pi_1(B'_0, x'_0)$ and $\tilde{\alpha}'$ the lift of α' with $\tilde{\alpha}'(0) = \tilde{x}'_0$. Put $\tilde{x}'_1 = \tilde{\alpha}'(1)$. Let $\tilde{B}_i$ be the component of $\tilde{p}'^{-1}(B'_i)$ which includes $\tilde{x}_i$, $i = 0, 1$.

By Lemma 4.2, we can conclude that $(\tilde{p}'|\tilde{B}'_0) : \tilde{B}'_0 \to B'_0$ is an orbi-isomorphism and $\tilde{B}'_1$ is compact.

Since $\tilde{B}_0 \neq \tilde{B}_1$, $\tilde{B}'_0 \neq \tilde{B}'_1$. Then $H_2(\tilde{B}'_0 \cup \tilde{B}'_1; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\tilde{i}' : \tilde{B}'_0 \to \tilde{M}'$ be the inclusion orbi-map. $(\tilde{i}')_* : \pi_1(\tilde{B}'_0, \tilde{x}'_0) \to \pi_1(\tilde{M}', \tilde{x}'_0)$ is an isomorphism. Since M is irreducible, by Corollary 1.4, $\pi_2(\tilde{M}') = 0$. Hence $\tilde{i}' : \tilde{B}'_0 \to \tilde{M}'$ is a homotopy equivalence. Therefore, $H_2(\tilde{M}'; \mathbb{Z}_2) = \mathbb{Z}_2$ and $H_3(\tilde{M}'; \mathbb{Z}_2) = 0$. Hence, by the exact sequence

$$0 \to H_3(\tilde{M}', \tilde{B}'_0 \cup \tilde{B}'_1; \mathbb{Z}_2) \to H_2(\tilde{B}'_0 \cup \tilde{B}'_1; \mathbb{Z}_2) \to H_2(\tilde{M}'; \mathbb{Z}_2) \to \cdots,$$

we derive that

$$H_3(\tilde{M}', \tilde{B}'_0 \cup \tilde{B}'_1; \mathbb{Z}_2) \neq 0.$$

Thus, $\tilde{M}'$ is compact. Then $\tilde{M}$ is compact. Hence, $p : (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ is a finite covering. Therefore, $|\pi_1(M, x_0); \eta_0\pi_1(B_0, x_0)| < \infty$. By [**Ta1**, 6.3], M is an I-bundle over a closed 2-orbifold. □

The lemma used in the proof of Theorem 4.1 follows.

LEMMA 4.2. *Let M and N be good 3-orbifolds with boundaries. Let $f : (M, \partial M) \to (N, \partial N)$ be an orbi-map such that f_* is monic. Suppose that there exist a path $\alpha : (I, \partial I) \to (|M| - \Sigma M, |\partial M|)$, compact and incompressible components B_0, B_1 of ∂M (possibly $B_0 = B_1$), and incompressible component C*

of ∂N which satisfy Theorem 4.1(i)–(iv). Put $x_i = \alpha(i)$ and $y = \bar{f}(x_0), i = 0, 1$. Let $\eta_0 : \pi_1(B_0, x_0) \to \pi_1(M, x_0)$ be the homomorphism induced by the inclusion orbi-map and $p : (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ be the covering associated with $\eta_0\pi_1(B_0, x_0)$. Let $\tilde{\alpha}$ be the lift of α by p with $\tilde{\alpha}(0) = \tilde{x}_0$. Put $\tilde{x}_1 = \tilde{\alpha}(1)$. Let $\tilde{B}_i$ be the component of $p^{-1}(B_i)$ which includes $\tilde{x}_i$, $i = 0, 1$. Then, $(p|\tilde{B}_0) : \tilde{B}_0 \to B_0$ is an orbi-isomorphism (namely $\tilde{B}_0$ is compact) and $\tilde{B}_1$ is compact.

Proof. Let $i_0 : B_0 \to M$ be the inclusion orbi-map (that is, an orbi-embedding). Since $i_{0*}\pi_1(B_0, x_0) = \eta_0\pi_1(B_0, x_0) = p_*\pi_1(\tilde{M}, \tilde{x}_0)$, $i_0 : B_0 \to M$ can be lifted to $\tilde{M}$ by p. Since $\tilde{B}_0$ is one of such lift, $(p|\tilde{B}_0) : \tilde{B}_0 \to B_0$ is an orbi-isomorphism. Let $\eta_1 : \pi_1(B_1, x_1) \to \pi_1(M, x_1)$, $\tilde{\eta}_i : \pi_1(\tilde{B}_i, \tilde{x}_i) \to \pi_1(\tilde{M}, \tilde{x}_i)$, $i = 0, 1$, be the homomorphisms induced by the inclusion orbi-maps. Let $\Psi_\alpha : \pi_1(M, x_1) \to \pi_1(M, x_0)$ and $\Psi_{\tilde{\alpha}} : \pi_1(\tilde{M}, \tilde{x}_1) \to \pi_1(\tilde{M}, \tilde{x}_0)$ be the change of base point maps. We have the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_1(\tilde{B}_0, \tilde{x}_0) & \xrightarrow{\tilde{\eta}_0} & \pi_1(\tilde{M}, \tilde{x}_0) & \xleftarrow{\Psi_{\tilde{\alpha}}} & \pi_1(\tilde{M}, \tilde{x}_1) & \xleftarrow{\tilde{\eta}_1} & \pi_1(\tilde{B}_1, \tilde{x}_1) \\
{\scriptstyle (p|\tilde{B}_0)_*}\downarrow & & {\scriptstyle p_*}\downarrow & & {\scriptstyle p_*}\downarrow & & {\scriptstyle (p|\tilde{B}_1)_*}\downarrow \\
\pi_1(B_0, x_0) & \xrightarrow[\eta_0]{} & \pi_1(M, x_0) & \xleftarrow[\Psi_\alpha]{} & \pi_1(M, x_1) & \xleftarrow[\eta_1]{} & \pi_1(B_1, x_1)
\end{array}$$

Let γ be a loop in $|B_i| - \Sigma B_i$ (respectively $\tilde{\gamma}$ in $|\tilde{B}_i| - \Sigma\tilde{B}_i$). We shall use the symbol $\langle\gamma\rangle$ (respectively $\langle\tilde{\gamma}\rangle$) to mean the element of $\pi_1(B_i)$ (respectively $\pi_1(\tilde{B}_i)$) represented by γ (respectively $\tilde{\gamma}$) and $[\gamma]$ (respectively $[\tilde{\gamma}]$) to mean the element of $\pi_1(M)$ (respectively $\pi_1(\tilde{M})$) represented by γ (respectively $\tilde{\gamma}$).

Claim 1. $p_*\Psi_{\tilde{\alpha}}\tilde{\eta}_1\pi_1(\tilde{B}_1, \tilde{x}_1) = \Psi_\alpha\eta_1\pi_1(B_1, x_1) \cap \eta_0\pi_1(B_0, x_0)$.

By the commutative diagram, it is clear that

$$p_*\Psi_{\tilde{\alpha}}\tilde{\eta}_1\pi_1(\tilde{B}_1, \tilde{x}_1) \subset \Psi_\alpha\eta_1\pi_1(B_1, x_1) \cap \eta_0\pi_1(B_0, x_0).$$

Take any element g of $\Psi_\alpha\eta_1\pi_1(B_1, x_1) \cap \eta_0\pi_1(B_0, x_0)$. Since $g \in \Psi_\alpha\eta_1\pi_1(B_1, x_1)$, there is a loop γ in $|B_1| - \Sigma B_1$ based at x_1 such that $g = [\alpha \cdot \gamma \cdot \alpha^{-1}]$. Let $\widetilde{\alpha \cdot \gamma \cdot \alpha^{-1}}$ be the lift of $\alpha \cdot \gamma \cdot \alpha^{-1}$ to $(\tilde{M}, \tilde{x}_0)$ and $\tilde{\gamma}$ be the lift of γ to $(\tilde{M}, \tilde{x}_1)$. Since g is also an element of $\eta_0\pi_1(B_0, x_0) = p_*\pi_1(\tilde{M}, \tilde{x}_0)$, $(\widetilde{\alpha \cdot \gamma \cdot \alpha^{-1}})$ is a loop in $|\tilde{M}| - \Sigma\tilde{M}$ based at $\tilde{x}_0$. Hence, $\tilde{\gamma}$ is a loop based at $\tilde{x}_1$ and $(\widetilde{\alpha \cdot \gamma \cdot \alpha^{-1}}) = \tilde{\alpha} \cdot \tilde{\gamma} \cdot \tilde{\alpha}^{-1}$.

Therefore

$$\begin{aligned} g &= [\alpha\cdot\gamma\cdot\alpha^{-1}] \\ &= [p\circ(\widetilde{\alpha\cdot\gamma\cdot\alpha^{-1}})] \\ &= [p\circ(\tilde{\alpha}\cdot\tilde{\gamma}\cdot\tilde{\alpha}^{-1})] \\ &= p_*\Psi_{\tilde{\alpha}}[\tilde{\gamma}] \\ &= p_*\Psi_{\tilde{\alpha}}\tilde{\eta}_1\langle\tilde{\gamma}\rangle \\ &\in p_*\Psi_{\tilde{\alpha}}\tilde{\eta}_1\pi_1(\tilde{B}_1,\tilde{x}_1). \end{aligned}$$

It is easy to show that the following diagram commutes:

$$\begin{array}{ccccc} & & \pi_1(B_1,x_1) & & \\ & \swarrow^{\Psi_\alpha\eta_1} & & \searrow^{(f|B_1)_*} & \\ \pi_1(M,x_0) & \xrightarrow{f_*} & \pi_1(N,y) & \xleftarrow{i_*} & \pi_1(C,y) \\ & \nwarrow_{\eta_0} & & \nearrow_{(f|B_0)_*} & \\ & & \pi_1(B_0,x_0) & & \end{array}$$

where $i: C \to N$ is the inclusion orbi-map.

Since B_i is compact, $f|B_i : B_i \to C$ is a finite sheeted covering, $i = 0, 1$. Thus,

$$|\pi_1(C,y);\,(f|B_i)\pi_1(B_i,x)| < \infty, \quad i = 0, 1.$$

By using the commutativity of the diagram above, we calculate the index to prove the following claim.

Claim 2. $|\Psi_\alpha\eta_1\pi_1(B_1,x_1);\,\Psi_\alpha\eta_1\pi_1(B_1,x_1)\cap\eta_0\pi_1(B_0,x_0)| < \infty$.

It is derived that $(p|\tilde{B}_1) : \tilde{B}_1 \to B_1$ is a finite sheeted covering by using Claim 1 and Claim 2:

$$\begin{aligned} &|\pi_1(B_1,x_1);\,(p|\tilde{B}_1)_*\pi_1(\tilde{B}_1,\tilde{x}_1)| \\ &= |\Psi_\alpha\eta_1\pi_1(B_1,x_1);\,\Psi_\alpha\eta_1(p|\tilde{B}_1)_*\pi_1(\tilde{B}_1,\tilde{x}_1)| \\ &= |\Psi_\alpha\eta_1\pi_1(B_1,x_1);\,p_*\Psi_{\tilde{\alpha}}\tilde{\eta}_1\pi_1(\tilde{B}_1,\tilde{x}_1)| \\ &= |\Psi_\alpha\eta_1\pi_1(B_1,x_1);\,\Psi_\alpha\eta_1\pi_1(B_1,x_1)\cap\eta_0\pi_1(B_0,x_0)| \\ &< \infty. \end{aligned}$$

□

THEOREM 4.3. (Retraction theorem) *Let M be an orientable 3-orbifold which is orbi-isomorphic to an I-bundle over a closed 2-orbifold F. Let N be a good and orientable 3-orbifold such that the underlying space of the universal covering orbifold of* $\operatorname{Int} N$ *is homeomorphic to* $\mathbb{R}^3$. *Let $f : (M, \partial M) \to (N, C)$ be an orbi-map, where C is an incompressible component of ∂N. Suppose*

$$\begin{cases} \text{there is a point } x \in |F| - \Sigma F \text{ such that} \\ f|(\varphi^{-1}(x)) \text{ is orbi-homotopic to a path in } C \text{ rel. } \{x\} \times \partial I, \\ \text{where } \varphi : M \to F \text{ is the fibration.} \end{cases} \tag{4.3.1}$$

Then there is an orbi-homotopy $f_t : M \to N$ such that $f_0 = f$, $f_1(M) \subset C$, and $f_t|\partial M = f|\partial M$.

Proof. Let $s_1, \ldots, s_k$ be simple closed curves on $|F| - \Sigma F$ such that $s_i \cap s_j = x$ for any i, j, and such that discal orbifolds $D_1, \ldots, D_r$ are derived by cutting F open along $s_1, \ldots, s_k$. We construct the desired orbi-map $H : M \times J \to N$, $J = [0, 1]$, stepwise as follows:
Step 1: $H|\{\varphi^{-1}(x) \times J\}$,
Step 2: $H|\{\varphi^{-1}(s_i) \times J\}$,
Step 3: $H|\{\varphi^{-1}(D_i) \times J\}$.

Step 1. Note $\varphi^{-1}(x) = x \times I$. Hence we define $H|\{\varphi^{-1}(x) \times J\}$ by the orbi-homotopy given in the hypothesis.

Step 2. $H|\{\varphi^{-1}(s_i) \times 0\} := f|\{\varphi^{-1}(s_i) \times 0\}$. $H|\{\varphi^{-1}(x) \times J\}$ is defined in Step 1. $H|(s_i \times 0 \times t) := f|(s_i \times 0)$. $H|(s_i \times 1 \times t) := f|(s_i \times 1)$. Furthermore, we can extend it to $\varphi^{-1}(s_i) \times 1$ by the incompressibility of C and Theorem 1.1. Define $H|(\varphi^{-1}(s_i) \times 1)$ by the extension. Thus, we have defined $H|\partial(\varphi^{-1}(s_i) \times J)$. Since $H|\partial(\varphi^{-1}(s_i) \times J)$ is an orbi-map from the 2-sphere to N, it is extendable to an orbi-map from the cone on the 2-sphere to N by using [**Ta1**, 4.1]. Define $H|(\varphi^{-1}(s_i) \times J)$ by the extension.

Step 3. $H|(\varphi^{-1}(\partial D_i) \times J)$ is defined in Step 2. $H|(D_i \times 0 \times t) := f|(D_i \times 0)$. $H|(D_i \times 1 \times t) := f|(D_i \times 1)$. $H|(\varphi^{-1}(D_i) \times 0) := f|\varphi^{-1}(D_i)$.

Furthermore, we can extend it to $\varphi^{-1}(D_i) \times 1$ by the fact that C is not a spherical 2-orbifold and [**Ta1**, 4.3]. Define $H|(\varphi^{-1}(D_i) \times 1)$ by the extension. Thus, we have defined $H|\partial(\varphi^{-1}(D_i) \times J)$.

Since $H|\partial(\varphi^{-1}(D_i) \times J)$ is an orbi-map from the double of a ballic 3-orbifold to N, by Theorem 2.2, it is extendable to an orbi-map from the cone on it to N. Define $H|(\varphi^{-1}(D_i) \times J)$ by the extension. □

COROLLARY 4.4. *In Theorem 4.3, suppose there is a point* $x \in |F| - \Sigma F$ *such that* $\bar{f}(\{x\} \times \partial I) \subset |C| - \Sigma C$ *and suppose* $i_* : \pi_1(C) \to \pi_1(N)$ *is an isomorphism instead of* (4.3.1)*. Then, the conclusion still stands.*

Proof. By the surjectivity of i_*, it is derived that $f|(x \times I)$ retracts into C rel. $\{x\} \times \partial I$. That is, the condition (4.3.1) holds. □

LEMMA 4.5. *Let* M *and* N *be good and compact 3-orbifolds with boundaries. Let* B *and* C *be components of* ∂M *and* ∂N*, respectively. Let* $f : (M, B) \to (N, C)$ *be an orbi-map such that* f_* *is monic and* $(f|B) : B \to C$ *is a covering. Then* B *is incompressible if and only if* C *is incompressible.*

Proof. Let $i : B \to M$ and $j : C \to N$ be the inclusion orbi-maps. Suppose C is incompressible in N. Then $j_* \circ (f|B)_*$ is monic. Hence, i_* is monic. Thus, by Theorem 1.1, B is incompressible in M.

Suppose B is incompressible in M. If C is compressible in N, there is a compressing discal orbifold D for C. Let S be a component of $(f|B)^{-1}(\partial D)$. Since ∂D does not bound a discal orbifold in C, $(f|B)_*[S] = [\partial D]^k$ is not finite order in $\pi_1(C)$. Then $[S]$ is not finite order in $\pi_1(B)$. Hence, S does not bound any discal orbifold in B. That is, $[S]$ is not finite order in $\pi_1(B)$. Let $D = D^2(n)$. Then, $f_*(i_*[S])^n = (f_*i_*[S])^n = (j_*(f|B)_*[S])^n = (j_*[\partial D]^k)^n = (j_*[\partial D])^{kn} = (j_*[\partial D]^n)^k = 1$ in $\pi_1(N)$.

Since f_* is monic, $(i_*[S])^n = 1$ in $\pi_1(M)$. That is, $[S]$ is finite order in $\pi_1(M)$. These mean $i_* : \pi_1(B) \to \pi_1(M)$ is not monic. By Theorem 1.1, it is a contradiction. □

COROLLARY 4.6. *In Theorem 4.3, suppose* $f_* : \pi_1(M) \to \pi_1(N)$ *is an isomorphism instead of* (4.3.1)*. If* $f|\partial M$ *is not an orbi-embedding and, for each component of* B *of* ∂M, $f|B : B \to C$ *is an orbi-covering, then the conclusion still stands. Furthermore,* M *is orbi-isomorphic to the product I-bundle over a closed 2-orbifold* B_0 *and* B_0 *is orbi-isomorphic to* C*.*

Proof. Since $f|B : B \to C$ is an orbi-covering, it is easy to find a point $x \in |F| - \Sigma F$ that satisfies $\bar{f}(\{x\} \times \partial I) \subset |C| - \Sigma C$ and the fact that C is incompressible in N is derived from Lemma 4.5. Let B_0, B_1 be the components of ∂M (possibly $B_0 = B_1$). Let $\eta_0 : B_0 \to M$ and $i : C \to N$ be the inclusion orbi-maps. Since i_* and f_* are monic,

$$\begin{aligned}|\pi_1(M); (\eta_0)_*\pi_1(B_0)| &= |f_*\pi_1(M); f_*(\eta_0)_*\pi_1(B_0)| \\ &= |\pi_1(N); i_*(f|B_0)_*\pi_1(B_0)| \\ &= |\pi_1(N); i_*\pi_1(C)| \cdot |i_*\pi_1(C); i_*(f|B_0)_*\pi_1(B_0)| \\ &= |\pi_1(N); i_*\pi_1(C)| \cdot |\pi_1(C); (f|B_0)_*\pi_1(B_0)|.\end{aligned}$$

When $B_0 \neq B_1$. Since $|\pi_1(M); (\eta_0)_*\pi_1(B_0)| = 1$, $|\pi_*(N); i_*\pi_1(C)| = 1$.

When $B_0 = B_1$. Since $f|\partial M : \partial M \to C$ is not an orbi-embedding, $|\pi_1(C); (f|B_0)_*\pi_1(B_0)| \geq 2$. On the other hand, $|\pi_1(M); \eta_0\pi_1(B_0)| \leq 2$. Hence $|\pi_1(N); i_*\pi_1(C)| = 1$. Thus, in any case, $|\pi_1(N); i_*\pi_1(C)| = 1$. That is, $i_* : \pi_1(C) \to \pi_1(N)$ is an isomorphism. When F is orientable. Then $M = B_0 \times I$ and $\pi_1(B_0) \cong \pi_1(M) \cong \pi_1(N) \cong \pi_1(C)$. Hence, by Corollaries 3.3 and 4.4, we have the conclusion.

When F is non-orientable. Then, $\pi_1(F) \cong \pi_1(M) \cong \pi_1(N) \cong \pi_1(C)$. Since C is orientable and $\#\pi_1(C) = \infty$, this is a contradiction. □

5. Main Theorem

Let M be a compact 3-orbifold. A sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_n$$

of 3-orbifolds is called a *hierarchy* for M provided that M_{i+1} is obtained from M_i by cutting open along a properly embedded, 2-sided incompressible 2-suborbifold F_i and each component of M_n is either a ballic 3-orbifold or (a turnover with non-positive Euler number)$\times I$. If M has a hierarchy, then we can show that the underlying space of the universal covering orbifold of Int(M) is homeomorphic to $\mathbb{R}^3$, almost similarly to [**W**, 8.1] or [**H**, 13.4].

A 3-orbifold is *abad* if it includes no bad suborbifold. A 2-suborbifold F in a 3-orbifold M is said to be *boundary-parallel* if one of the components of cl$(M - F)$ is orbi-isomorphic to $F \times I$. A 3-orbifold M is *sufficiently large* if there is a 2-sided, properly embedded, and incompressible 2-suborbifold F which is not boundary parallel.

Let $\mathcal{W}$ be the class of all compact and orientable 3-orbifolds which are

($\mathcal{W}$1) abad,

($\mathcal{W}$2) irreducible,

($\mathcal{W}$3) all boundary components are incompressible,

($\mathcal{W}4$) sufficiently large,
($\mathcal{W}5$) every turnover in M with non-positive Euler number is boundary-parallel.
By [**D**], an orbifold $M \in \mathcal{W}$ has a hierarchy. Furthermore, by the following theorem, it should be a good orbifold.

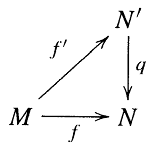
THEOREM. [**Ta2**, Theorem A] *Let M be an abad, compact, and orientable 3-orbifold. Let F be a compact and incompressible 2-suborbifold which is 2-sided and properly embedded in M. If each closed-up component of $M - F$ is good, then M is good.*

Now, we are at the place to state the main theorems.

THEOREM 5.1. (Main Theorem) *Let $M, N \in \mathcal{W}$, and suppose $f : (M, \partial M) \to (N, \partial N)$ is an orbi-map such that $f_* : \pi_1(M) \to \pi_1(N)$ is monic. Then there exists an orbi-homotopy $f_t : (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$ and either*
(1) *$f_1 : M \to N$ is an orbi-covering, or*
(2) *M is a product I-bundle over a closed 2-orbifold and $f_1(M) \subset \partial N$.*
If, for a component B of ∂M, $(f|B) : B \to C$ is already an orbi-covering, we may assume $(f|B)_t = f|B$ for all t.

Proof. By the hypothesis, for each component B of ∂M, there is a component C of ∂N such that $f(B) \subset C$ and $(f|B)_* : \pi_1(B) \to \pi_1(C)$ is monic. By Theorem 3.2, after changing f by an orbi-homotopy, we may assume that $(f|B) : B \to C$ is an orbi-covering. If this is already the case for some B, there is no need now nor in any future step to change $f|B$.

We construct a commutative diagram



where $q : N' \to N$ is an orbi-covering associated with $f_*\pi_1(M)$ and f' is the lift of f by q. Note that $f'_* : \pi_1(M) \to \pi_1(N')$ is an isomorphism. Let C' be the component of $q^{-1}(C)$ such that $f'(B) \subset C'$. It is derived that $f'|B : B \to C'$ is an orbi-covering from the fact that both $f|B$ and $q|C'$ are orbi-coverings and $f'|B$ is a lift of $f|B$ by q.

Case 1. $|\partial M| \neq \phi$ and $f'|\partial M$ is not an orbi-embedding.

By the hypothesis that $f'|\partial M$ is not an orbi-embedding and f'_* is epic, we can take a path $\alpha : (I, \partial I) \to (|M| - \Sigma M, |\partial M|)$ satisfying
(i) $\alpha(0) \neq \alpha(1)$,

(ii) $\bar{f}'(\alpha(0)) = \bar{f}'(\alpha(1)) \in |\partial N'| - \Sigma N'$,
(iii) $[\tilde{f}' \circ \hat{\alpha}] = 1$ in $\pi_1(N')$,
where $\bar{f}'$ and $\tilde{f}'$ are the underlying map and the structure map of f', respectively, and $\hat{\alpha}$ is the lift of α to the universal covering of M.

Let B_i be the component of ∂M including $\alpha(i)$, $i = 0, 1$, and C' be the component of $\partial N'$ including $\bar{f}'(\alpha(0))(= \bar{f}'(\alpha(1)))$.

Since B_i and C are incompressible and $f|B_i \to C'$'s are coverings, $i = 0, 1$, by Theorem 4.1, M is an I-bundle over a closed 2-orbifold. Furthermore, by Corollary 4.6, $M = B_0 \times I$ (B_0 is orbi-isomorphic to C') and there is an orbi-homotopy $f_t : M \to N'$ such that $f_0 = f$, $f_1(M) \subset C'$ and $f_t|\partial M = f|\partial M$. Then the conclusion (2) holds.

Case 2. $|\partial M| = \phi$ or $f'|\partial M$ is an orbi-embedding.

Let

$$\begin{array}{ccccccc} N = N_0 & \supset & N_1 & \supset & \cdots & \supset & N_n \\ & G_0 & & G_1 & & G_{n-1} & \end{array}$$

be a hierarchy for N. Since $|\partial M| = \phi$ or $f'|\partial M$ is an orbi-embedding, we can apply Lemma 5.2. Then there is an orbi-map $f_1 : (M, \partial M) \to (N, \partial N)$ such that

(1) f_1 and f are C-equivalent rel. ∂;
(2) each component of $f_1^{-1}(G_0)$ is an orientable and incompressible 2-suborbifold properly embedded in M;
(3) for each component Q of N_1 and P of $f_1^{-1}(Q)$, $(f_1|P) : P \to Q$ satisfies that $(f_1|P)_*$ is monic and either $|\partial P| = \phi$ or $(f_1|P)'|\partial P$ is an orbi-embedding, where $(f_1|P)'$ is the lift of $f_1|P$ by the covering $q_1 : Q'' \to Q$ associated with $(f_1|P)_*\pi_1(P)$.

Put $M_1 = f_1^{-1}(N_1)$. For each component P_1 of M_1, there is a component Q_1 of N_1 such that $f_1(P_1) \subset Q_1$. By the above paragraph, we can apply Lemma 5.2 for the orbi-map $f_1|P_1 : (P_1, \partial P_1) \to (Q_1, \partial Q_1)$ to have an orbi-map $g_2 : (P_1, \partial P_1) \to (Q_1, \partial Q_1)$ such that

(1) g_2 and $(f_1|P_1)$ are C-equivalent rel. ∂;
(2) each component of $g_2^{-1}(G_1)$ is an orientable and incompressible 2-suborbifold properly embedded in P_1;
(3) for each component Q of $N_2 \cap Q_1$ and P of $g_2^{-1}(Q)$, $g_2|P : P \to Q$ satisfies that $(g_2|P)_*$ is monic and either $|\partial P| = \phi$ or $(g_2|P)'|\partial P$ is an orbi-embedding, where $(g_2|P)'$ is the lift of $g_2|P$ by the covering $q_2 : Q'' \to Q$ associated with $(g_2|P)_*\pi_1(P)$.

Let $f_2 : M_1 \to N_1$ be the collection of the orbi-maps on every component of M_1 changed as above. Clearly, f_2 and $f_1|M_1$ are C-equivalent rel. ∂.

Put $M_2 = f_2^{-1}(N_2)$. Continuing in this manner, we have a sequence of maps:

$$\begin{array}{ccccccccccccc}
N & = & N & \supset & N_1 & = & N_1 & \supset\cdots\supset & N_{n-1} & = & N_{n-1} & \supset & N_n \\
\uparrow{\scriptstyle f_0} & & \uparrow{\scriptstyle f_1} & & \uparrow{\scriptstyle f_1|_{M_1}} & & \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle f_{n-1}|_{M_{n-1}}} & & \uparrow{\scriptstyle f_n} & & \uparrow{\scriptstyle f_n|_{M_n}} \\
M & = & M & \supset & M_1 & = & M_1 & \supset\cdots\supset & M_{n-1} & = & M_{n-1} & \supset & M_n
\end{array}$$

where $f_0 = f$, $M_i = f_i^{-1}(N_i)$, f_{i+1} and $f_i|M_i$ are C-equivalent rel. ∂, and for each component P_i of M_i, there is a component Q_i of N_i such that $f_i(P_i) \subset Q_i$ and the orbi-map $f_i|P_i : (P_i, \partial P_i) \to (Q_i, \partial Q_i)$ satisfies that $(f_i|P_i)_*$ is monic and either $|\partial P_i| = \phi$ or $(f_i|P_i)'|\partial P_i$ is an orbi-embedding, where $(f_i|P_i)'$ is the lift of $f_i|P_i$ by the orbi-covering $q_i : Q_i'' \to Q_i$ associated with $(f_i|P_i)_*\pi_1(P_i)$, $i = 1, 2, \ldots, n$.

Each piece of N_n is either a ballic orbifold or (a turnover T with non-positive Euler number) $\times I$.

Claim 1. $|\partial P_n| \neq \phi$.

Otherwise, $|\partial M| = \phi$ and $M = P_n$. Since $(f|P_n)_*$ is monic, $\pi_1(M)$ is a finite group or $\pi_1(M)$ is isomorphic to a subgroup of $\pi_1(T)$. Since M is sufficiently large, the first case must not occur. In the latter case, take the covering $r : \tilde{T} \to T$ associated with $f_*\pi_1(M)$.

Let $f' : M \to \tilde{T} \times I$ be the lift of f by $r \times \mathrm{id}$. Since f'_* is an isomorphism, $\pi_1(M)$ is isomorphic to $\pi_1(\tilde{T})$.

When $|\pi_1(T \times I); f_*\pi_1(M)| < \infty$, that is, $\tilde{T}$ is closed. Then, by [**Ta1**, 6.3], M is orbi-isomorphic to an I-bundle over a closed 2-orbifold. This contradicts $|\partial M| = \phi$.

When $|\pi_1(T \times I); f_*\pi_1(M)| = \infty$. Then, by [**LS**, 7.4, p. 137], $\pi_1(\tilde{T}) \cong \mathbb{Z} * \cdots * \mathbb{Z} * \mathbb{Z}_{n_1} * \cdots * \mathbb{Z}_{n_r}$, $n_j \geq 2$.

Let X be an orbifold which is a boundary connected sum of m solid tori and ballic orbifolds $B(n_j)$, $j = 1, 2, \ldots, r$. Then, by [**Ta1**, 4.2], we can construct an orbi-map $k : M \to X$ with k_* is an isomorphism. Since $\pi_1(M)$ is not a finite group, there is an incompressible disc D in X. By [**Ta1**, 5.5], we may assume that $k^{-1}(D)$ is an incompressible 2-suborbifold S in M. Since $(k|S)_*$ is monic, S is a sphere. This contradicts the irreducibility of M.

Hence, $|\partial P_n| \neq \phi$ and $(f_n|P_n)'|\partial P_n$ is an orbi-embedding.

Claim 2 . For each $(f_n|P_n) : (P_n, \partial P_n) \to (Q_n, \partial Q_n)$, there is an orbi-covering $g_n : P_n \to Q_n$ such that g_n and $(f_n|P_n)$ are orbi-homotopic rel. ∂.

Let B_n be a component of ∂P_n. Let C_n be the component of $\partial Q''_n$ such that $f_n(B_n) \subset C_n$. Denote $(f_n|P_n)'|B_n = h$. Since $\dim B_n = \dim C_n$, $\partial B_n = \partial C_n = \phi$, and $h : B_n \to C_n$ is an orbi-embedding, we have $h : B_n \to C_n$ is an orbi-isomorphism. When Q_n is a ballic 3-orbifold, by using Proposition 2.4, we can show the claim. When $Q_n = T \times I$, let $\eta : B_n \to P_n$ and $\xi : C_n \to Q''_n$ be the inclusion orbi-maps. Furthermore, since $Q''_n = C_n \times I$, $\xi_* : \pi_1(C_n) \to \pi_1(Q''_n)$ is an isomorphism. Since $(f_n|P_n)' \circ \eta = \xi \circ h$, $(f_n|P_n)'_* \circ \eta_* = \xi_* \circ h_*$. Hence $\{(f_n|P_n)'_*\}^{-1} \circ \xi_* \circ h_* = \eta_* : \pi_1(B_n) \to \pi_1(P_n)$ is an isomorphism. Then, by [**Ta1**, 6.1], $P_n = B_n \times I$. Hence, by using Proposition 2.5, we can show the claim.

Let $h_n : M_n \to N_n$ be the collection of the orbi-maps on every component of M_n changed as above. Since h_n and $f_n|M_n$ are C-equivalent rel. ∂, we can piece together $h_n : M_n \to N_n$ to have an orbi-covering $h_{n-1} : M_{n-1} \to N_{n-1}$ such that h_{n-1} and $f_{n-1}|M_{n-1}$ are C-equivalent rel. ∂.

Continuing in this manner, we have an orbi-covering $h_0 : M_0 \to N_0$ such that h_0 and $f_0 = f$ are C-equivalent rel. ∂. By Proposition 2.3, h_0 and f are orbi-homotopic. □

The lemma used in the proof of Theorem 5.1 follows.

LEMMA 5.2. (The induction lemma) *Let M and N be good, compact, orientable and irreducible 3-orbifolds. Let $f : (M, \partial M) \to (N, \partial N)$ be an orbi-map such that f_* is monic. Let $q : N' \to N$ be the covering associated with $f_*\pi_1(M)$. Suppose either $|\partial M| = \phi$ or $f'|\partial M$ is an orbi-embedding, where f' is the lift of f by q. Let G be an orientable and incompressible 2-suborbifold properly embedded in N. Then there exists an orbi-map $f_1 : (M, \partial M) \to (N, \partial N)$ such that*

(1) *f_1 and f are C-equivalent rel. ∂;*
(2) *each component of $f_1^{-1}(G)$ is an orientable and incompressible 2-suborbifold properly embedded in M;*
(3) *for each component Q of $\mathrm{cl}(N - G)$ and for each component P of $f_1^{-1}(Q)$, $(f_1|P) : P \to Q$ satisfies that $(f_1|P)_*$ is monic, and either $|\partial P| = \phi$ or $(f_1|P)'|\partial P$ is an orbi-embedding, where $(f_1|P)'$ is the lift of $f_1|P$ by the covering $q_1 : Q'' \to Q$ associated with $(f_1|P)_*\pi_1(P)$.*

Proof. Let $\hat{p} : \hat{M} \to M$ and $\hat{q} : \hat{N} \to N$ be the universal coverings. Let $\tilde{f}$ and $\bar{f}$ be the structure map and the underlying map of f, respectively. By observing the proof of [**Ta1**, 5.5], we see that f is modified C-equivalently without changing $f|\partial M$ to be that each component of $f^{-1}(G)$ is either an orientable and incompressible 2-suborbifold or a compressible discal suborbifold properly embedded in M.

Claim 1. No components of $f^{-1}(G)$ are compressible discal orbifolds.

Suppose a component D of $f^{-1}(G)$ is a compressible discal orbifold. There is a discal orbifold D' in ∂M such that $D \cap D' = \partial D = \partial D'$ and $D \cup D'$ bounds a ballic orbifold in M. By innermost arguments, we may assume that $D' \cap f^{-1}(G) = \partial D$. Hence, $f|D'$ is an orbi-covering and $f(D')$ is a discal orbifold. Put $S = \partial D$ and $S' = f(S)$. Let r be the number of the sheets of the covering $f|S : S \to S'$ and $D = D^2(n)$. Since $[S]^n = 1$ in $\pi_1(M)$, $f_*[S]^n = [f(S)]^n = [S']^{rn} = 1$ in $\pi_1(N)$. By the incompressiblity of G, it holds that $[S']^{rn} = 1$ in $\pi_1(G)$. Thus, G is a discal orbifold. By the irreducibility of N, it is derived that $G \cup f(D')$ bounds a ballic suborbifold in N. This contradicts the incompressibility of G.

Thus we may assume that each component of $f^{-1}(G)$ is an orientable and incompressible 2-suborbifold properly embedded in M.

At first, we consider the case $f^{-1}(G) = \phi$. Then $P = M$. We can take f as the desired f_1.

(1) and (2) are clear. Since $f_* : \pi_1(M) \to \pi_1(N)$ is monic, $f_* : \pi_1(M) \to \pi_1(Q)$ is monic. When $|\partial M| = \phi$, we are done. Suppose $|\partial M| \neq \phi$. Then, $f'|\partial M$ is an orbi-embedding. Let Q' be the component of $q^{-1}(Q)$ with $f'(M) \subset Q'$. Since $f'_*\pi_1(M) < \pi_1(Q')$ and $f'_*\pi_1(M) = \pi_1(N')$, it is derived that $f'_*\pi_1(M) = \pi_1(Q')$. Hence, $f_*\pi_1(M) = q_*f'_*\pi_1(M) = q_*\pi_1(Q')$. That is, $q|Q' : Q' \to Q$ is the covering with $q_*\pi_1(Q') = f_*\pi_1(M)$. This means that q_1 and $(f_1|P)'$ of (3) agree with $q|Q'$ and f', respectively. Then, (3) is derived from the fact that $f|\partial M$ is an orbi-embedding.

Next, we consider the case $f^{-1}(G) \neq \phi$. Let F be a component of $f^{-1}(G)$.

Claim 2. $\bar{f}(|F|) \cap (|G| - \Sigma G) \neq \phi$.

When $|\partial G| \neq \phi$. Then $|\partial F| \neq \phi$. Since $f|\partial M$ is an orbi-covering, $\bar{f}(|\partial F|) \subset |\partial G| \subset |G| - \Sigma G$.

When $|\partial G| = \phi$. Then $|\partial F| = \phi$. Suppose $\bar{f}(|F|) = z \in \Sigma G$. Take a point $\tilde{z} \in \hat{q}^{-1}(z)$. We may assume that $\tilde{\bar{f}}(|\tilde{F}|) = \tilde{z}$. Since F is not a spherical orbifold, there is an element $[\tilde{\alpha}] \in \pi_1(F)$ such that $[\tilde{\alpha}]$ is infinite order in $\pi_1(F)$. From the incompressibility of F, it is derived that $[\tilde{\alpha}]$ is infinite order in $\pi_1(M)$. On the other hand, since $[\tilde{f} \circ \tilde{\alpha}]_A(\tilde{z}) = \tilde{z}$, $[\tilde{f} \circ \tilde{\alpha}]_A$ is finite order in $\mathrm{Aut}(\hat{N}, \hat{q})$, that is, $f_*[\tilde{\alpha}]$ is finite order in $\pi_1(N)$. Then, from the injectivity of f_*, it is derived that $[\tilde{\alpha}]$ is finite order in $\pi_1(M)$. It is a contradiction.

Thus, we can define a restriction orbi-map $(f|F) : F \to G$. Since F is incompressible in M and $f_* : \pi_1(M) \to \pi_1(N)$ is monic, $(f|F)_* : \pi_1(F) \to \pi_1(G)$ is monic.

Claim 3. F is a discal orbifold if and only if G is a discal orbifold.

Suppose G is a discal orbifold. Since $(f|F)_*$ is monic, $\pi_1(F)$ is a finite cyclical group. Considering the incompressibility of F and the irreducibility of M, we derive that F is a discal orbifold.

Suppose F is a discal orbifold $D^2(n)$. There is a component K of ∂G such that $(f|\partial F) : \partial F \to K$ is a covering. Let r be the number of the sheets of the covering. Then, $[K]^{rn} = [f(\partial F)]^n = f_*[\partial F]^n = 1$ in $\pi_1(N)$. Since G is incompressible in N, $[K]^{rn} = 1$ in $\pi_1(G)$. Hence, G is a discal orbifold.

Claim 4. $(f|F) : F \to G$ is orbi-homotopic (rel. ∂) to an orbi-covering.

When F is a discal orbifold. Then, by Claim 3, G is also a discal orbifold. Let G' be a lift of G by q such that $f'(F) \subset G'$. Then, G' is a discal orbifold.

Since $f'|\partial M$ is an orbi-embedding, $(f'|\partial F) : \partial F \to \partial G'$ is an orbi-embedding. Furthermore, since $\pi_1(F) \to \pi_1(M)$ and $f'_* : \pi_1(M) \to \pi_1(N')$ is monic, it is derived that $(f'|F)_* : \pi_1(F) \to \pi_1(G')$ is monic. Then, by [**Ta1**, 7.1], $f'|F$ is orbi-homotopic (rel. ∂) to an orbi-isomorphism. Hence, $(f|F) = (q|G')\circ(f'|F) : F \to G$ is orbi-homotopic (rel. ∂) to an orbi-covering.

When F is not a discal orbifold. Then, by Claim 3, G is not a discal orbifold either. By the incompressibility of F and irreducibility of M, F is not a spherical orbifold. Hence $\#\pi_1(F) = \infty$. Then we can apply Theorem 3.2. We only have to show that the conclusion Theorem 3.2(2) must not occur in this situation. Suppose Theorem 3.2(2) occurs. That is, F is an annulus and there is an orbi-homotopy $\varphi_t : F \to G$ such that $\varphi_t|\partial F = f|\partial F$ and $\varphi_1(F) \subset \partial G$.

Let K be the component of ∂G such that $\varphi_1(F) \subset K$. Let y be a point of K. Let x_0,x_1 be points of $f^{-1}(y)$ which are included in different components of ∂F. Let α be a path in F with $\alpha(i) = x_i$, $i = 0, 1$.

Let $\tilde{F}$ (respectively $\tilde{G}$) be a component of $\hat{p}^{-1}(F)$ (respectively $\hat{q}^{-1}(G)$). Since F (respectively G) is incompressible in M (respectively N), $(\hat{p}|\tilde{F}) : \tilde{F} \to F$ (respectively $(\hat{q}|\tilde{G}) : \tilde{G} \to G$) is the universal covering.

Let $\tilde{\alpha}$ be a lift of α by $\hat{p}|\tilde{F}$. Note that $[\tilde{\varphi}_1 \circ \tilde{\alpha}] = [K]^r$ in $\pi_1(K, y)$ for some $r \in \mathbb{Z}$. Since $\tilde{f} \circ \tilde{\alpha} = \tilde{\varphi}_0 \circ \tilde{\alpha} \sim \tilde{\varphi}_1 \circ \tilde{\alpha}$ (rel. $\tilde{\alpha}(0), \tilde{\alpha}(1)$), $[\tilde{f} \circ \tilde{\alpha}] = [K]^r$ in $\pi_1(K, y)$.

Hence, by extending α in ∂F, we can take a path β in F which satisfies

$$\beta(0) \neq \beta(1), \tag{5.2.1}$$

$$[\tilde{f} \circ \tilde{\beta}] = 1 \quad \text{in } \pi_1(N) \tag{5.2.2}$$

where $\tilde{\beta}$ is a lift of β by $\hat{p}|\tilde{F}$.

Since (5.2.2) means $\tilde{f}(\tilde{\beta}(0)) = \tilde{f}(\tilde{\beta}(1))$, it holds that $f'(\beta(0)) = f'(\beta(1))$. Then, (5.2.1) contradicts the fact that $f'|\partial M$ is an orbi-embedding.

Hence, after changing f through an orbi-homotopy (the C-equivalent modification preceded), we may assume that for each component Q of $\mathrm{cl}(N - G)$ and for each component P of $f^{-1}(Q)$, $(f|\partial P) : \partial P \to \partial Q$ is an orbi-covering. By Proposition 1.6, it is derived that $(f|P)_*$ is monic.

If $(f|P)'|\partial P : \partial P \to \partial Q''$ is an orbi-embedding for each Q and P, we can take such f as the desired f_1, where $q_1 : Q'' \to Q$ is the covering associated with $f_*\pi_1(P)$ and $(f|P)'$ be the lift of $f|P$ by q_1.

Suppose $(f|P)'|\partial P : \partial P \to \partial Q''$ is not an orbi-embedding for some Q and P. By observing the proof of [**Ta1**, 5.5], we can see that since M is compact, the number of the components of $f^{-1}(G)$ is finite. Let n be the number of the components of $f^{-1}(G)$.

Claim 5. $|\partial G| = \phi$.

Let Q' be the component of $q^{-1}(Q)$ with $f'(P) \subset Q'$. Since $(f|P)_*\pi_1(P) < q_*\pi_1(Q')$, there is an orbi-covering $q'' : Q'' \to Q'$ such that $q_1 = (q|Q') \circ q''$. Hence, it is derived that $f'|P = q'' \circ (f|P)'$. Then, $(f'|\partial P) : \partial P \to \partial Q'$ is not an orbi-embedding.

Suppose $|\partial G| \neq \phi$. Let G' be the component of $q^{-1}(G)$ with $f'(F) \subset G'$. Since $f|F : F \to G$ and $q|G' : G' \to G$ are orbi-coverings, $f'|F : F \to G'$ is an orbi-covering. Furthermore, since $f'|\partial M : \partial M \to \partial N'$ is an orbi-embedding, $f'|\partial F : \partial F \to \partial G'$ is an orbi-embedding. Then $f'|F : F \to G'$ must be an orbi-embedding. Since this holds for each component of $f^{-1}(G)$, $f'|\partial P$ is an orbi-embedding. It is a contradiction.

Thus, each component of G is closed and so is each component of $f^{-1}(G)$. Considering $f'|\partial M$ is an orbi-embedding and $(f|P)'|\partial P$ is not an orbi-embedding, we derive that there are components F_i, F_j of $f^{-1}(G) \cap \partial P$ (possibly $F_i = F_j$) such that $(f|P)'|(F_i \cup F_j)$ is not an orbi-embedding. Let G'' be the component of $q_1^{-1}(G)$ such that $(f|P)'|(F_k) \subset G''$, $k = i, j$. Since $f|F_k : F_k \to G$ and $q_1|G'' : G'' \to G$ are orbi-coverings, $(f|P)'|(F_k) : F_k \to G''$ is an orbi-covering. Furthermore, since $(f|P)'_* : \pi_1(P) \to \pi_1(Q'')$ is an isomorphism and $(f|P)'|\partial P : \partial P \to \partial Q''$ is not an orbi-embedding, we can take a path $(I, \partial I) \to (|P| - \Sigma P, |\partial P|)$ which satisfies Theorem 4.1(i)–(iv). Then, by Theorem 4.1, it is derived that P is an I-bundle over a closed 2-orbifold.

We can also apply Corollary 4.6 to conclude that P is the product I-bundle over F_i (hence, $F_i \neq F_j$) and there is an orbi-homotopy $(f|P)'_t : P \to Q$ such that

$(f|P)'_0 = (f|P)'$, $(f|P)'_1(P) \subset G''$, and $(f|P)'_t|\partial P = (f|P)'|\partial P$.

Put $(f|P)_t = q_1 \circ (f|P)'_t$. Then, $(f|P)_t$ is an orbi-homotopy from P to Q such that $(f|P)_0 = f|P$, $(f|P)_1(P) \subset G$, and $(f|P)_t|\partial P = f|\partial P$. By using $(f|P)_t$ and the product structures of the neighborhoods of F_i and G, we can construct an orbi-homotopy $f_t : M \to N$ such that $f_0 = f$, $f_t|(M - \operatorname{Int} U(P)) = f|(M - \operatorname{Int} U(P))$ (namely $f_t|\partial M = f|\partial M$), and $f_1^{-1}(G) = f^{-1}(G) - (F_i \cup F_j)$.

Note that (the number of the components of $f_1^{-1}(G)$) $< n$ and that, for each component Q of $\mathrm{cl}(N - G)$ and P of $f^{-1}(Q)$, $(f_1|P)_* : \pi_1(P) \to \pi_1(Q)$ is monic. When $(f_1|P)'|\partial P$ is an orbi-embedding for each P and Q, we can take above f_1 as the desired one. When $(f_1|P)'|\partial P$ is not an orbi-embedding for some P and Q, we regard f_1 as the initial f and iterate above process. Since n is finite, by iterating this process, we can arrive at a point where $(f_1|P)'|\partial P$ is an orbi-embedding. □

Let M and N be 3-orbifolds. Let $\Psi : \pi_1(M, x) \to \pi_1(N, y)$ be a homomorphism. We say that Ψ *respects the peripheral structure*, if the following holds. For each boundary component F of M, there exists a boundary component G of N, such that $\Psi(i_*(\pi_1(F, x'))) \subset A$, and A is conjugate to $j_*(\pi_1(G, y'))$ in $\pi_1(N, y)$, where i and j are inclusions.

LEMMA 5.3. [**Ta1**, 7.5] *Let M and N be good, compact, and orientable 3-orbifolds, such that each component of ∂N is incompressible and that the underlying space of the universal covering orbifold of* Int(N) *is homeomorphic to $\mathbb{R}^3$. Let $\Psi : \pi_1(M, x) \to \pi_1(N, y)$ be a homomorphism which respects the peripheral structure. Then, there exists an orbi-map $f : (M, \partial M) \to (N, \partial N)$ which induces Ψ.*

We conclude this paper with describing the classification theorems of 3-orbifolds by their orbifold fundamental groups derived from Theorem 5.1 and Lemma 5.3.

THEOREM 5.4. *Let $M, N \in \mathcal{W}$. Let $\Psi : \pi_1(M) \to \pi_1(N)$ be a monomorphism which respects the peripheral structure, then there exists an orbi-map $f : (M, \partial M) \to (N, \partial N)$ which induces the monomorphism Ψ. Hence, the conclusion of Theorem 5.1 follows.*

COROLLARY 5.5. *Let $M, N \in \mathcal{W}$. Suppose M and N are closed. If there is a monomorphism $\Psi : \pi_1(M) \to \pi_1(N)$, then there exists an orbi-covering $f : M \to N$ which induces the monomorphism Ψ.*

THEOREM 5.6. *Let $M, N \in \mathcal{W}$. Let $\Psi : \pi_1(M) \to \pi_1(N)$ be an isomorphism which respects the peripheral structure, then there exists an orbi-isomorphism $f : M \to N$ which induces Ψ.*

Proof. Suppose M is not a product I-bundle over a closed 2-orbifold. Then we apply Theorem 5.4 to obtain a 1-sheeted covering (i.e. an orbi-isomorphism).

Suppose M is a product I-bundle over a closed 2-orbifold F ($M = F \times [0,1]$). By Theorem 5.4, there is an orbi-map $f : (M, \partial M) \to (N, \partial N)$ such that $f_* = \Psi$. By [**Ta1**, 6.2, 6.3], N is also a product I-bundle over a closed 2-orbifold G ($N = G \times [0, 1]$ and $f(F \times 0) \subset G \times 0$), where F and G are orbi-isomorphic. By Theorem 3.2, we may assume that $f|F \times 0 : F \times 0 \to G \times 0$ is a covering. Since we may assume that $(f|F \times 0)_* = f_*$, $f|F \times 0$ is an orbi-isomorphism from $F \times 0$ to $G \times 0$. The orbi-map $(f|F \times 0) \times \mathrm{id} : F \times [0, 1] \to G \times [0, 1]$ is the desired orbi-map. □

COROLLARY 5.7. *Let $M, N \in \mathcal{W}$. Suppose M and N are closed. If there is an isomorphism $\Psi : \pi_1(M) \cong \pi_1(N)$, then there is an orbi-isomorphism $f : M \to N$ which induces Ψ.*

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