変分問題に関連する発展方程式の研究

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研究発表

(1) 学会誌等

1. 菊地光嗣, A remark on Dirichlet boundary condition for the nonlinear equation of motion of a vibrating membrane, Nonlinear Analysis, 47 巻, 2001 年
2. 今井仁司, 菊地光嗣, 中根和昭, 小俣正朗, 立川智美, A numerical approach to the asymptotic behavior of solutions of a one-dimensional free boundary problem of hyperbolic type, JJIAM, 18 巻, 2001 年
3. 菊地光嗣, An analysis of the nonlinear equation of motion of a vibrating membrane in the space of BV functions, J. Math. Soc. Japan, 52 巻, 2000 年
4. 柴田良弘，清水 扇丈，A decay property of the Fourier Transform and its application to Stokes problem, J. Math. Fluid. Mech., 3 巻 3 号, 2001 年
5. 清水扇丈，Inner estimate of singularities to solutions for elastic wave propagation problems in stratified media, Commun. Math. Phys., 208 巻, 2000 年
6. Piero D'Ancona，Vladimir Georgiev，久保 英夫，Weighted Strichartz estimates for the wave equation, C. R. Acad. Sci. Paris, t.330 Série I, 2000 年
7. 星賀彰，久保英夫，Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition, SIAM Journal on Mathematical Analysis, 31 巻, 2000 年

8. 久保英夫，太田雅人，Small data blowup for systems of semilinear wave equations with different propagation speeds in three space dimensions, J. Differential Equations, 163 巻2号, 2000年

9. 久保英夫，太田雅人，Global existence and blowup of the classical solutions to systems of semilinear wave equations in three space dimensions, Rend. Istit. Mat. Univ. Trieste, 31巻, 2000年

10. 星賀 彰，The lifespan of solutions to quasilinear hyperbolic systems in two space dimensions, Nonlinear Analysis, 42巻, 2000年

11. 中西賢次，太田雅人，On global existence of solutions to nonlinear wave equations of wave map type, Nonlinear Analysis, 42巻7号, 2000年

(2) 口頭発表

1. 菊地光嗣, カルテシアンカレント列のヴァリフォルド収束, 松山解析セミナー, 愛媛大学, 2002年2月
2. 菊地光嗣, A remark on Dirichlet boundary condition for the nonlinear equation of motion of a vibrating membrane (K. Kikuchi), PDE meeting, Prague, Czech Republic, 2001年9月
3. 菊地光嗣, Constructing weak solutions in a direct variational method and an application of varifold theory (a recent result) (K. Kikuchi), Czechoslovak International Conference on Differential Equations and Their Applications, Prague, Czech Republic, 2001年8月
4. 菊地光嗣, Constructing gradient flows for quasiconvex functionals (K. Kikuchi), The 3rd World Congress of Nonlinear Analysts, Catania, Italy, 2000年7月
5. 菊地光嗣, Constructing gradient flows for quasiconvex functionals, 共同研究集会「変分問題とその周辺」, 京都大学, 2000年6月

他数件

(3) 出版物

特になし

研究成果による工業所有権の出願・取得状況

特になし

研究成果

本件は主に以下の点を解明するために計画された研究であった。

1. 非線形弾性体の変形問題などに対する勾配流の構成
2. 勾配流の方程式の解の分岐現象
3. 弾性体の変形問題および極小局面の問題に対応する双曲型方程式
4. 離散的勾配流の方法のシュレディンガー方程式への応用
5. 離散的勾配流の方法と爆発解との関係。

本研究の第1年目は4年にに一度開催される世界非線形解析学者会議（World Congress of Nonlinear Analysts）の開催年であったので研究代表者菊地の他、分担者の太田もこの研究集会に参加し最先端の情報を収集した。第2年目は国際研究集会 Czechoslovak International Conference on Differential Equations and Their Applications の開催年であったので研究代表者菊地が参加し成果発表及び情報収集を行った。これらのほかにも研究代表者菊地及び各分担者が国内外の研究集会に参加し研究に必要な情報を収集した。その結果以下のような研究成果を得ることができた。

研究期間中に特に目覚ましい進展があったのは上の目的の内の 1 と 3 である。1 に関する成果としてはまず準凸な汎函数がある種の下からの評価式を満たせば勾配流が構成できることがわかった。この評価式は近似解の空間変数に関する2階微分の一様評価を得るために必要なものであるが、さらにこの評価がなくても勾配流が構成できる場合があることがわかった。ただこの結果では方程式の形に多くの制限が残されておりさらなる改良が望まれるところである。3 に関する成果としては極小曲面の問題に対応する双曲型方程式に対して対応するディリクレ境界条件が従来の弱定式化（トレースが0）よりもさらに弱くしないといけないことがわかった。当該研究の主眼は離散的勾配流の方法を発展方程式に応用する事であるが，離散的勾配流の方法は Rothe の方法に直接変分法の手法を組み合わせた方法であるので直接変分法の結果がそのまま発展方程式の研究に応用できるというメリットがある。上の結果は面積汎函数に対する直接変分法の結果を双曲型方程式に応用することにより得られたものであり、直接変分法の結果をそのまま発展方程式の研究に応用するという当該研究の特徴を最も良くふまえたものである。このほか 4 に関しても若干の所見を得ることができた。2 と 5 に関しては今後の発展の可能性は見いだせたが今回の研究期間内には理論と呼べる形まではまとめることができなかった。これらについては今後の研究に期待するところである。

以上の結果の内，3 に関する結果は既に印刷公表されているので、本報告書では 1 および 4 に関する結果を報告する。

Global existence of a gradient flow for a quasiconvex functional satisfying some coerciveness condition

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1 Introduction

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with Lipschitz continuous boundary and let $F = F(x,u,p)$ be a function defined on $\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$. We define a functional for $u : \Omega \to \boldsymbol{R}^N$ by

$$J(u) = \int_\Omega F(x,u,Du)dx, \tag{1.1}$$

where Du denotes the Jacobian matrix of u. Our purpose is to show global existence of a weak solution to the equation of gradient flow for J which is possibly not convex but quasiconvex. If J is convex, the existence follows from the monotonicity of grad J. However it seems that there are a few works on evolution equations related to quasiconvex functionals. Similarly to other nonlinear problems the difficulty lies in showing the convergence of nonlinear terms. In this article, assuming some coersiveness condition, we overcome this difficulty.

The equation of gradient flow for J is given by $u_t + \operatorname{grad} J(u) = 0$, that is,

$$\frac{\partial u^i}{\partial t}(t,x) - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\{F_{p^i_\alpha}(x,u,Du(x))\} + F_{u^i}(x,u,Du(x)) = 0, \quad x \in \Omega, \tag{1.2}$$

where $Du = (D_\alpha u^i) = (\dfrac{\partial u^i(t,x)}{\partial x^\alpha})$ (throughout this paper D are used for differentiations with respect to only x variables). The initial and the boundary conditions are imposed as follows:

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

$$u(t,x) = w(x), \qquad x \in \partial\Omega. \tag{1.4}$$

We suppose that u_0 and w belong to $W^{1,2}(\Omega, \boldsymbol{R}^N)$ and that $\gamma u_0 = \gamma w$ (γ is the trace operator to $\partial\Omega$).

We say that a function $u \in L^\infty((0,\infty); W^{1,2}(\Omega)) \cap \bigcup_{T>0} W^{1,2}((0,T)\times\Omega)$ is a *weak solution* to (1.2)–(1.4) if u satisfies $\text{s-}\lim_{t\searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$, $\gamma u(t) = \gamma w$ for $\mathcal{L}^1$-a.e. t, and for

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any $\varphi \in C_0^\infty((0,\infty)\times\Omega)$

$$(1.5)\qquad \sum_{i=1}^{N}\int_0^\infty\int_\Omega\{u_t^i(t,x)\varphi^i(t,x)+\sum_{\alpha=1}^{n}F_{p_\alpha^i}(x,u,Du)D_\alpha\varphi^i(t,x) \\ +F_{u^i}(x,u,Du)\varphi^i(t,x)\}dxdt=0.$$

If u is a weak solution to (1.2), then $J(u(t))$ is absolutely continuous and it holds that $dJ(u(t))/dt=-(u_t,u_t)_{L^2(\Omega)}\le 0$ for $\mathcal{L}^1$-a.e. t. This means u defines a gradient flow for J. Our purpose is to construct a weak solution to (1.2)–(1.4) in the above sense.

Naturally several assumptions should be added on the function F. First we require the regularity as follows:

(A1) $F\in C^2(\Omega\times\boldsymbol{R}^N\times\boldsymbol{R}^{nN})$.

The quasiconvexity is well-known as a necessary and sufficient condition for the sequential weak lower semicontinuity of J ([8, 9]):

(A2) F is *quasiconvex* with respect to p, that is,

$$\frac{1}{\mathcal{L}^n(D)}\int_D F(x_0,y_0,p_0+D\varphi(x))dx\ge F(x_0,y_0,p_0)$$

for each bounded domain $D\subset\boldsymbol{R}^n$, for each $(x_0,y_0,p_0)\in\Omega\times\boldsymbol{R}^N\times\boldsymbol{R}^{nN}$, and for each $\varphi\in W_0^{1,\infty}(D;\boldsymbol{R}^N)$.

Growth conditions are required as follows:

(A3) There exist positive constants μ, λ, and a constant γ with $1\le\gamma<2^*$ (the Sobolev exponents for 2) such that

$$\begin{cases} \lambda|p|^2\le F(x,u,p)\le\mu(1+|u|^\gamma+|p|^2)\\ |F_u|,\ |F_{xu}|\le\mu(1+|u|^{\gamma-1}+|p|)\\ |F_p|,\ |F_{xp}|\le\mu(1+|u|^{\gamma/2}+|p|)\\ |F_{uu}|\le\mu(1+|u|^{\gamma-2})\\ |F_{up}|\le\mu(1+|u|^{\gamma/2-1})\\ |F_{pp}|\le\mu. \end{cases}$$

Note that these estimates are somewhat controlled ones. For example, a function of the form $F(x,u,p)=\sum a_{ij}^{\alpha\beta}(x,u)p_\alpha^i p_\beta^j$ is not admitted. Functions having the form $F(x,u,p)=g(x,p)+h(x,u)$ are possibly admitted. Anyway conditions up to this one are standard ones. In this article we should further require the following coersiveness condition:

(A4) There exists a positive constant m such that

$$\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N}\int_\Omega F_{p^i_\alpha p^j_\beta}(x,\psi,D\psi)D_\alpha\varphi^i D_\beta\varphi^j dx \geq m\int_\Omega |D\varphi(x)|^2 dx$$

for any ψ, $\varphi \in W^{1,2}_0(\Omega, \boldsymbol{R}^N)$.

Remark. If a quadratic function, $F(p) = \sum a^{\alpha\beta}_{ij} p^i_\alpha p^j_\beta$, satisfies the strong Legendre-Hadamard condition

$$\sum a^{\alpha\beta}_{ij}\xi_\alpha\eta^i\xi_\beta\eta^j \geq \nu|\xi|^2|\eta|^2 \quad (\nu > 0,\ \xi\in\boldsymbol{R}^n,\ \eta\in\boldsymbol{R}^N),$$

then we easily find (A4) holds with $m = \nu$. Thus, if F has the form

$$F(x,u,p) = F_0(p) + G(x,u,p),$$

where F_0 is a quadratic function which satisfies the strong Legendre-Hadamard condition and where G satisfies $|G_{pp}| \leq c\nu$ with $c < 1$, then F satisfies (A4) with $m = (1-c)\nu$.

Example. Let $n = N = 2$. The function

$$\begin{aligned} F(p) = (p^1_1)^2 + (p^1_2)^2 + (p^2_1)^2 + (p^2_2)^2 + 2(1+\varepsilon)(p^1_1p^2_2 - p^1_2p^2_1) \\ +\varepsilon\sqrt{1 + (p^1_1)^4 + (p^1_2)^4 + (p^2_1)^4 + (p^2_2)^4 + 11(p^1_1p^2_2 - p^1_2p^2_1)^2} \end{aligned}$$

satisfies (A1)–(A4) if ε is sufficiently small. Indeed, (A1) is clear, it is easy to find that F is polyconvex, what implies (A2) (compare to [2, Section 4.1]), by Schwarz's inequality we have

$$\begin{aligned} F(p) &\geq |p|^2 + 2(1+\varepsilon)(p^1_1p^2_2 - p^1_2p^2_1) + \frac{1}{4}\varepsilon(1 + |p|^2 + 11|p^1_1p^2_2 - p^1_2p^2_1|) \\ &\geq (p^1_1 + p^2_2)^2 + (p^1_2 - p^2_1)^2 + \frac{3}{4}\varepsilon|p^1_1p^2_2 - p^1_2p^2_1| + \frac{1}{4}\varepsilon(1+|p|^2) \\ &\geq \frac{1}{4}\varepsilon(1+|p|^2), \end{aligned}$$

which shows the first inequality of (A3), other inequalities in (A3) are clear, and as has mentioned above (A4) holds. However, letting $p^{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $p^{(2)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, we have $F(p^{(1)}) = F(p^{(2)}) = 2 + \sqrt{3}\varepsilon$ and thus

$$\frac{1}{2}(F(p^{(1)}) + F(p^{(2)})) < F(\frac{p^{(1)} + p^{(2)}}{2}) = 2 + \frac{3\sqrt{5}+2}{4}\varepsilon.$$

Hence F is not convex.

Our main theorem is

Theorem 1.1 *There exists a weak solution to* (1.2)–(1.4).

We construct approximate solutions to (1.2)–(1.4) by the method of discretization in time and minimizing variational functionals. In recent several years this approximating way is widely applied to constructing weak solutions to nonlinear partial differential equations ([1, 3, 5, 6] and references cited therein). Thanks to this method and our assumptions, in particular Condition (A4), we are able to obtain the uniform estimate of second derivatives with respect to the space variables. In Section 2 we prove Theorem 1.1 accepting this estimate and in Section 3 we prove it.

2 Constructing a gradient flow

Let h be a positive number. A sequence $\{u_l\}$ in $W^{1,2}(\Omega, \boldsymbol{R}^N)$ is constructed as follows: we let u_0 be as in (1.3) and for $l \geq 1$ we define u_l as a minimizer of the functional

$$\mathcal{F}_l(v) = \frac{1}{2}\int_\Omega \frac{|v - u_{l-1}|^2}{h} dx + J(v) \qquad (J \text{ is as in } (1.1))$$

in the class $w + W_0^{1,2}(\Omega, \boldsymbol{R}^N)$ (that is, among functions in $W^{1,2}(\Omega, \boldsymbol{R}^N)$ with $\gamma v = \gamma w$). The existence of a minimizer of $\mathcal{F}_l$ is assured by the quasiconvexity of F and (A3) (see, for example, [2, Chapter 4, Theorem 2.9]). Note also that (A3) assures $\mathcal{F}_l$ is Gâteaux differentiable. Approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ ($(t,x) \in (0,\infty) \times \Omega$) are defined as, for $(l-1)h < t \leq lh$,

$$u^h(t,x) = \frac{t-(l-1)h}{h} u_l(x) + \frac{lh-t}{h} u_{l-1}(x)$$

and

$$\overline{u}^h(t,x) = u_l(x).$$

Then the following facts hold.

Lemma 2.1 *We have*
1) $\{\| u_t^h \|_{L^2((0,\infty)\times\Omega)}\}$ *is uniformly bounded with respect to* h
2) $\{\| \overline{u}^h \|_{L^\infty((0,\infty);W^{1,2}(\Omega))}\}$ *is uniformly bounded with respect to* h
3) $\{\| u^h \|_{L^\infty((0,\infty);W^{1,2}(\Omega))}\}$ *is uniformly bounded with respect to* h
4) *for any* $T > 0$, $\{\| u^h \|_{W^{1,2}((0,T)\times\Omega))}\}$ *is uniformly bounded with respect to* h.

Then there exist a function u *such that, passing to a subsequence if necessary,*
5) $\overline{u}^h$ *converges to* u *as* $h \to 0$ *weakly star in* $L^\infty((0,\infty); W^{1,2}(\Omega))$
6) *for any* $T > 0$, u^h *converges to* u *as* $h \to 0$ *weakly in* $W^{1,2}((0,T) \times \Omega))$
7) u^h *converges to* u *as* $h \to 0$ *strongly in* $L^2((0,T) \times \Omega)$
8) $\overline{u}^h$ *converges to* u *as* $h \to 0$ *strongly in* $L^2((0,T) \times \Omega)$
9) $s\text{-}\lim_{t \searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Sketch of the proof. The proof of this lemma can be found in formar works (for example [1] or [10]). But, taking account of its importance, we present its brief sketch.

Since u_l is a minimizer of $\mathcal{F}_l$, we have $\mathcal{F}_l(u_l) \leq \mathcal{F}_l(u_{l-1}) = J(u_{l-1})$. Then by iteration we have

$$\sum_{k=1}^{l} \int_\Omega \frac{|u_k - u_{k-1}|^2}{h} dx + J(u_l) \leq J(u_0).$$

Thereby, noting that, for $(l-1)h < t < lh$, $u_t^h(t,x) = (u_l(x) - u_{l-1}(x))/h$, we have the energy inequality

$$\int_0^\infty \int_\Omega |u_t^h|^2 dx dt + J(\overline{u}^h) \leq J(u_0).$$

This directly implies 1) $\sim$ 4) and hence 5) and 6). Combining with Sobolev's imbedding theorem, we have 7). By the definition of u^h and $\overline{u}^h$ we can obtain $\|u^h - \overline{u}^h\|_{L^2((0,T)\times\Omega)} \to 0$ as $h \to 0$. Hence 8) follows from 7). By definition, $u^h(0,x) = u_0(x)$ for each h, and thus 9) is obtained. Q.E.D.

Lemma 2.1 9) means that u satisfies (1.3) in a weak sense. Lemma 2.1 5) implies that u satisfies (1.4) in a weak sense since $\overline{u}^h - w \in L^\infty((0,\infty); W_0^{1,2}(\Omega))$ for each h. Thus the problem is whether u satisfies (1.5). Since u_l is a minimizer of $\mathcal{F}_l(v)$, $d\mathcal{F}_l(u_l + \varepsilon\varphi)/d\varepsilon|_{\varepsilon=0} = 0$ for any $\varphi \in W_0^{1,2}(\Omega)$, and hence we have

$$(2.1) \quad \sum_{i=1}^{N} \int_\Omega \{(u_t^h)^i(x)\varphi^i(x) + \sum_{\alpha=1}^{n} F_{p_\alpha^i}(x,\overline{u}^h, D\overline{u}^h) D_\alpha\varphi^i(x) + F_{u^i}(x,\overline{u}^h, D\overline{u}^h)\varphi^i(x)\} dx = 0$$

for any $\varphi \in W_0^{1,2}(\Omega)$ and any $t \in \bigcup_{\ell=0}^\infty ((\ell-1)h, \ell h)$.

Our key lemma is as in the following. The proof of this lemma is carried out in the next section and in this section we prove our main theorem, Theorem 1.1, accepting this lemma.

Lemma 2.2 *For any $\Omega' \subset\subset \Omega$ and for any $T > 0$,*

$$\{\| D_\alpha D_\beta \overline{u}^h \|_{L^2((0,T)\times\Omega')}; \alpha, \beta = 1, 2, \cdots, n\}$$

is uniformly bounded with respect to h.

Lemma 2.2 is just a result for the second derivatives with respect to only x variables, and hence we cannot apply Rellich's compactness theorem directly. However in the following proof the strong convergence of $\overline{u}^h$ (Lemma 2.1 8)) works instead of the regularity of $\overline{u}^h$ with respect to t (compare to the proof of Rellich's theorem, for example, Lemma 1.1 of [12, Section 1.1]).

Proof of Theorem 1.1. It is sufficient to show that $\{D\overline{u}^h\}$ converges strongly to Du in $L^2((0,T) \times \Omega')$ for any $\Omega' \subset\subset \Omega$ and for any $T > 0$.

Let $\rho_\sigma *$ be the standard mollifier with respect to x variables and let Ω'' be a domain with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Suppose σ is so small that $\operatorname{dist}(\partial\Omega'', \partial\Omega') > \sigma$. Then easily we have $D(\rho_\sigma * \overline{u}^h)(t,x) = (\rho_\sigma * D\overline{u}^h)(t,x)$ for each $x \in \Omega'$ and each t with $\overline{u}^h(t,\cdot) \in W^{1,2}(\Omega)$.

Claim 1. $D(\rho_\sigma * \overline{u}^h)$ converge to $D(\rho_\sigma * u)$ as $h \to 0$ strongly in $L^2((0,T)\times\Omega')$

Proof. By the Hausdorff-Young inequality we have

$$\begin{aligned}
&\| D(\rho_\sigma * \overline{u}^h) - D(\rho_\sigma * u) \|_{L^2((0,T)\times\Omega')} \\
= \ & \sigma^{-1}\Big(\int_0^T \int_{\Omega'} \Big| \int_{\boldsymbol{R}^n} (D\rho_\sigma)(x-y)(\overline{u}^h(t,y) - u(t,y))dy\Big|^2 dxdt\Big)^{1/2} \\
\le \ & \sigma^{-1} \int_{\boldsymbol{R}^n} |(D\rho_\sigma)(x-y)|dx \Big(\int_0^T \int_{\Omega'} |\overline{u}^h(t,y) - u(t,y)|^2 dydt\Big)^{1/2} \\
= \ & \sigma^{-1} \int_{\boldsymbol{R}^n} |(D\rho)(z)|dz \, \| \overline{u}^h - u \|_{L^2((0,T)\times\Omega')},
\end{aligned}$$

which shows Claim 1 by Lemma 2.1 8).

Claim 2. $\rho_\sigma * D\overline{u}^h$ converges to $D\overline{u}^h$ as $\sigma \to 0$ strongly in $L^2((0,T)\times\Omega')$ and uniformly with respect to h.

Proof. Let v^h denote $D\overline{u}^h$. Now

$$\begin{aligned}
&\| \rho_\sigma * v^h - v^h \|_{L^2((0,T)\times\Omega')} = \Big(\int_0^T \int_{\Omega'} |\rho_\sigma * v^h(t,x) - v^h(t,x)|^2 dxdt\Big)^{1/2} \\
= \ & \Big(\int_0^T \int_{\Omega'} \Big| \int_{B_\sigma(x)} \rho_\sigma(x-y)(v^h(t,y) - v^h(t,x))dy\Big|^2 dxdt\Big)^{1/2} \\
\le \ & \sigma\Big(\int_0^T \int_{\Omega'} \int_{B_\sigma(x)} \rho_\sigma(x-y) \int_0^1 |Dv^h(t, x+\theta(y-x))|^2 d\theta dydxdt\Big)^{1/2}.
\end{aligned}$$

Here we make a change of variables $(x,y) \mapsto (z,w)$ by

$$\begin{cases} z = x + \theta(y-x) \\ w = x - y. \end{cases}$$

The Jacobian coincides with $(-1)^n$ and $(z,w) \in \Omega'' \times B_\sigma(0)$. Hence we have

$$\| \rho_\sigma * v^h - v^h \|_{L^2((0,T)\times\Omega')} \le \sigma\Big(\int_0^T \int_{B_\sigma(0)} \rho_\sigma(w)dw \int_{\Omega''} |Dv^h(t,z)|^2 dzdt\Big)^{1/2} \le \sigma \, \| Dv^h \|_{L^2((0,T)\times\Omega'')},$$

the left hand side of which is uniformly bounded with respect to h by Lemma 2.2.

End of the proof of Theorem 1.1. Now we put $\overline{u}^h_\sigma = \rho_\sigma * \overline{u}^h$ and $u_\sigma = \rho_\sigma * u$. We write

$$\begin{aligned}
&\| D\overline{u}^h - Du \|_{L^2((0,T)\times\Omega')} \\
\le \ & \| D\overline{u}^h - D\overline{u}^h_\sigma \|_{L^2((0,T)\times\Omega')} + \| D\overline{u}^h_\sigma - Du_\sigma \|_{L^2((0,T)\times\Omega')} + \| Du_\sigma - Du \|_{L^2((0,T)\times\Omega')} \,.
\end{aligned}$$

As $\sigma \to 0$, the first term converges to zero uniformly with respect to h by Claim 2, and clearly the third term also converges to zero. Hence these are estimated from above by any given positive number ε if σ is sufficiently small. When such a small σ is fixed, the second term converges to zero as $h \to 0$ by Claim 1. Then we have

$$\limsup_{h\to 0} \| D\overline{u}^h - Du \|_{L^2((0,T)\times\Omega')} \leq 2\varepsilon.$$

Since ε is arbitrary, we have the conclusion. Q.E.D.

3 Uniform estimate for second derivatives

Readers should note that up to this step we do not require Condition (A4). It is required for the proof of Lemma 2.2.

Proof of Lemma 2.2. The proof is carried out by the use of difference quotient method (compare to the proof of Theorem 1.1 of [4, Chapter II]). For the sake of brevity we omit the dependence of t in each functions in this proof. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\boldsymbol{R}^n$ and let φ be a function with compact support in Ω. Let k be a sufficiently small number depending on the support of φ. We insert $\varphi(x-ke_s)$ in (2.1) instead of $\varphi(x)$, make the change of variables $y = x - ke_s$ in the second and third terms, and rewrite y as x again. Then we have

$$\begin{aligned}(3.1)\quad \sum_{i=1}^N \int_\Omega \{(u_t^h)^i(x)\varphi^i(x-ke_s) + \sum_{\alpha=1}^n F_{p_\alpha^i}(x+ke_s, \overline{u}^h(x+ke_s), D\overline{u}^h(x+ke_s))D_\alpha\varphi^i(x) \\ +F_{u^i}(x+ke_s, \overline{u}^h(x+ke_s), D\overline{u}^h(x+ke_s))\varphi^i(x)\}dx = 0.\end{aligned}$$

Subtracting (2.1) from (3.1) and dividing it by k, we have

$$\begin{aligned}(3.2)\quad & -\int_0^1 \{\sum_{i=1}^N \int_\Omega (u_t^h)^i(x) D_s\varphi^i(x-\tau k e_s)dx + \sum_{\alpha=1}^n\sum_{i=1}^N \int_\Omega F_{x^s p_\alpha^i}(\cdots)D_\alpha\varphi^i(x)dx \\ &+ \sum_{\alpha=1}^n\sum_{i=1}^N\sum_{j=1}^N \int_\Omega F_{u^j p_\alpha^i}(\cdots)\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k} D_\alpha\varphi^i(x)dx \\ &+ \sum_{\beta=1}^n\sum_{i=1}^N\sum_{j=1}^N \int_\Omega F_{p_\alpha^i p_\beta^j}(\cdots)D_\beta\left(\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k}\right)D_\alpha\varphi^i(x)dx \\ &+ \sum_{i=1}^N \int_\Omega F_{x^s u^i}(\cdots)\varphi^i(x)dx + \sum_{i=1}^N\sum_{j=1}^N \int_\Omega F_{u^i u^j}(\cdots)\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k}\varphi^i(x)dx \\ &+ \sum_{i=1}^N\sum_{j=1}^N\sum_{\beta=1}^n \int_\Omega F_{u^i p_\beta^j}(\cdots)D_\beta\left(\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k}\right)\varphi^i(x)dx\}d\tau = 0,\end{aligned}$$

where

$$(\cdots) = (x + \tau k e_s, \tau \overline{u}^h(x + k e_s) + (1 - \tau)\overline{u}^h(x), \tau D\overline{u}^h(x + k e_s) + (1 - \tau) D\overline{u}^h(x)).$$

Let B_R be a ball with radius R. We take $\eta \in C_0^\infty(B_{2R})$ and $\tilde{\eta} \in C_0^\infty(B_{3R})$ such that $B_{4R} \subset\subset \Omega$, $0 \leq \eta, \tilde{\eta} \leq 1$, $\eta \equiv 1$ on B_R, and $\tilde{\eta} \equiv 1$ on B_{2R}. Now we insert in (3.2)

$$\varphi(x) = \frac{\overline{u}^h(x + k e_s) - \overline{u}^h(x)}{k} \eta(x)^2.$$

By the use of Condition (A4) we can estimate the fourth term of (3.2) from below. Indeed change of variables in (A4) implies

$$\sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_\Omega F_{p^i_\alpha p^j_\beta}(x + \tau k e_s, \psi(x + \tau k e_s), D\psi(x + \tau k e_s))$$
$$D_\alpha \varphi^i(x + \tau k e_s) D_\beta \varphi^j(x + \tau k e_s) dx \geq m \int_\Omega |D\varphi(x + \tau k e_s)|^2 dx$$

for any ψ, $\varphi \in W_0^{1,2}(\Omega, \boldsymbol{R}^N)$ with dist(spt $\psi, \partial\Omega$), dist(spt $\varphi, \partial\Omega) > \tau k$, and hence applying this inequality in case that

$$\psi(x) = [\tau \overline{u}^h(x - \tau k e_s + k e_s) + (1 - \tau)\overline{u}^h(x - \tau k e_s)]\tilde{\eta}(x - \tau k e_s)$$

and

$$\varphi(x) = \frac{(\overline{u}^h)^i(x - \tau k e_s + k e_s) - (\overline{u}^h)^i(x - \tau k e_s)}{k} \eta(x - \tau k e_s),$$

we have

$$\sum_{\alpha,\beta,i,j} \int_\Omega F_{p^i_\alpha p^j_\beta}(\cdots) D_\beta\Big(\frac{(\overline{u}^h)^j(x + k e_s) - (\overline{u}^h)^j(x)}{k}\eta(x)\Big) D_\alpha\Big(\frac{(\overline{u}^h)^i(x + k e_s) - (\overline{u}^h)^i(x)}{k}\eta(x)\Big) dx$$

$$\geq m \int_\Omega |D(\frac{\overline{u}^h(x + k e_s) - \overline{u}^h(x)}{k}\eta(x))|^2 dx.$$

The first term of (3.2) is estimated by a simple calculus and we have, for each $\varepsilon > 0$,

$$|\sum_{i=1}^{N} \int_\Omega (u_t^h)^i(x) D(\frac{\overline{u}^h(x - \tau k e_s + k e_s) - \overline{u}^h(x - \tau k e_s)}{k} \eta(x - \tau k e_s)^2) dx|$$
$$\leq \varepsilon^{-1} \int_\Omega |u_t^h(x)|^2 dx + C + \varepsilon \int_\Omega |D(\frac{\overline{u}^h(x + k e_s) - \overline{u}^h(x)}{k})\eta(x)|^2 dx.$$

Absolute values of other terms in (3.2) are bounded from above by

$$C\{C(\varepsilon) + \varepsilon \int_{B_{2R}} |D(\frac{\overline{u}^h(x + k e_s) - \overline{u}^h(x)}{k}\eta(x))|^2 dx\}, \tag{3.3}$$

where $C(\varepsilon)$ denotes a constant depending on ε and C denotes a constant depending on μ, γ, R, η, and the uniform bound of $\{\| u^h \|_{L^\infty((0,\infty);W^{1,2}(\Omega))}\}$ (see Lemma 2.1 2)). For example, the last term of (3.2) is estimated as follows: by (A3) and Sobolev's imbedding theorem

$$
\begin{aligned}
&|\sum_{i,j=1}^{N}\sum_{\beta=1}^{n}\int_\Omega F_{u^j p^i_\beta}(\cdots)D_\beta(\frac{(\overline{u}^h)^j(x+ke_s)-(\overline{u}^h)^j(x)}{k})\frac{(\overline{u}^h)^i(x+ke_s)-(\overline{u}^h)^i(x)}{k}\eta(x)^2dx| \\
\leq\ &\mu\int_{B_{2R}}(1+|\tau\overline{u}^h(x+ke_s)+(1-\tau)\overline{u}^h(x)|^{\gamma/2-1})|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x)| \\
&\times[|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|+\sup|D\eta||\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|]dx \\
\leq\ &\mu\int_{B_{2R}}[A(\varepsilon)+\varepsilon|\tau\overline{u}^h(x+ke_s)+(1-\tau)\overline{u}^h(x)|^{2^*/2-1}]|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x)| \\
&\times[|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|+\sup|D\eta||\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|]dx \\
\leq\ &\mu\{\int_{B_{2R}}[(A(\varepsilon)^2\varepsilon^{-1}+A(\varepsilon)\sup|D\eta|)|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|^2 \\
&+\varepsilon|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|^2]dx \\
&+\varepsilon[\int_{B_{2R}}|\tau\overline{u}^h(x+ke_s)+(1-\tau)\overline{u}^h(x)|^{2^*}dx]^{(2^*-2)/2\cdot 2^*} \\
&\times[\int_{B_{2R}}|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x)|^{2^*}dx]^{1/2^*} \\
&\times[\int_{B_{2R}}(|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|+\sup|D\eta||\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|)^2dx]^{1/2}\} \\
\leq\ &\mu\{\int_{B_{2R}}[(A(\varepsilon)^2\varepsilon^{-1}+\sup|D\eta|)|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|^2 \\
&+\varepsilon|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|^2]dx \\
&+\sqrt{2}M^{2^*/2}\varepsilon[\int_{B_{2R}}|(\tau D\overline{u}^h(x+ke_s)+(1-\tau)D\overline{u}^h(x)|^2dx]^{(2^*-2)/4} \\
&\times[\int_{B_{2R}}|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|^2dx]^{1/2} \\
&\times[\int_{B_{2R}}|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}\eta(x))|^2dx \\
&+(\sup|D\eta|)^2\int_{B_{2R}}|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|^2dx]^{1/2}\},
\end{aligned}
$$

where $A(\varepsilon)$ denotes a constant such that $1+r-\varepsilon r^{(2^*-2)/(\gamma-2)}\leq A(\varepsilon)$ for each $r>0$ and M denotes a bound of Sobolev's imbedding $W^{1,2}(B_{2R})\subset L^{2^*}(B_{2R})$. Then there exist $C(\varepsilon)$ and

C such that (3.3) is an upper bound of the right hand side.

Thereby, taking ε sufficiently small and fixing it, we have

$$\int_{B_R} |D(\frac{\overline{u}^h(x+ke_s) - \overline{u}^h(x)}{k})|^2 dx \leq C(1 + \int_{\Omega} |u_t^h(x)|^2 dx),$$

where C is a constant depending on the fixed ε and the things specified before. This implies for any $T > 0$

$$\int_0^T \int_{B_R} |D_s D\overline{u}^h(x)|^2 dxdt \leq C(T + \int_0^T \int_{\Omega} |u_t^h(x)|^2 dxdt).$$

It follows from Lemma 2.1 4) that the right hand side of this inequality is less than a constant which is independent of h. This completes the proof. Q.E.D.

References

[1] F. Bethuel, J. M. Coron, J. M. Ghidallia and A. Soyeur, *Heat flow and relaxed energies for harmonic maps*, "Nonlinear Diffusion Equations and Their Equilibrium States, 3" eds. N. G. Lloyd, W. M. Ni, A. Peletier, J. Serrin, Progr. Nonlinear Differential Equations Appl. 7, Birkhäuser, Boston-Basel-Berlin, 1992, 99-109.

[2] B. Dacorogna, Direct Method in the Calculus of Variations, Springer-Verlag, 1988.

[3] G. Friesecke and G. Dolzmann, *Implicit time discretization and global existence for a quasi-linear evolution equation with nonconvex energy*, Siam. J. Math. Anal., **28**, (1997), 363-380.

[4] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Annal. Math. Studies, **105**, Princeton University Press.

[5] K. Kikuchi, *Constructing weak solutions in a direct variational method and an application of varifold theory*, J. Differential Equations **150** (1998), 1-23.

[6] K. Kikuchi, *An analysis of the nonlinear equation of motion of a vibrating membrane in the space of BV functions*, J. Math. Soc. Japan **52** (2000), 741-766.

[7] N. Kikuchi, *An approach to the construction of Morse flows for variational functionals*, in "Nematics Mathematical and Physical Aspects", ed.: J. M. Coron, J. M. Ghidaglia and F. Hélein, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **332**, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991, 195-199.

[8] C. B. Morrey, *Quasiconvexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25-53.

[9] C. B. Morrey, Multiple Integrals in the Calculus of Variations, Springer, Berlin, 1966.

[10] T. Nagasawa, *Discrete Morse semiflows and evolution equations* (in Japanese), Proceedings of the 16th Young Japanese Mathematicians' Seminar on Evolution Equations, 1994, 1 - 20.

[11] K. Rektorys, *On application of direct variational method to the solution of parabolic boundary value problems of arbitrary order in the space variables*, Czechoslovak Math. J. **21** (1971), 318-339.

[12] L. Simon, Theorems on regularity and singularity of energy minimizing maps, ETH Lectures on Mathematics, Birkhäuser, Basel, 1995.

Global existence of a weak solution to Semilinear beam equation

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1 Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ and let $F = F(x, y, p)$ be a real valued function in $C^1(\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n)$ and F_p and F_y denote the vectors whose elements are $\dfrac{\partial F}{\partial p_\alpha}$ and $\dfrac{\partial F}{\partial y}$, respectively. Partial derivatives are often denoted by the use of subindices. In [4] the author has constructed a weak solution to fourth order parabolic equation

$$\frac{\partial u}{\partial t}(t,x) + \triangle^2 u - \operatorname{div}\{F_p(x, u(t,x), \nabla u(t,x))\} \\ + F_y(x, u(t,x), \nabla u(t,x)) = 0.$$

The constructing way is the method of discretization in time and minimizing variational functionals. This approximating method is firstly applied to constructing weak solutions to linear parabolic equations ([11]). In [7] N. Kikuchi has independently rediscovered this method, and after [7] there are many works in applying this method to constructing weak solutions to nonlinear partial differential equations (see references cited in [4]), and it has turned out that this method is available for not only parabolic equations but also hyperbolic ones (compare to [2, 5, 12]). Our purpose here is to clarify that this method is still available for a class of Schrödinger type equations.

2 Semilinear beam equation

In this article we consider the following initial value problem for a fourth order equation:

(2.1) $$\frac{\partial^2 u}{\partial t^2}(t,x) + \triangle^2 u - \operatorname{div}\{F_p(x, u(t,x), \nabla u(t,x))\} \\ + F_y(x, u(t,x), \nabla u(t,x)) = 0, \quad x \in \Omega,$$

(2.2) $$u(0,x) = u_0(x), \quad , u_t(0,x) = v_0(x) \qquad x \in \Omega,$$

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$$(2.3) \qquad u(t,x)=w(x), \quad \nabla u(t,x)=\nabla w(x), \qquad x\in\partial\Omega,$$

where u_0 and w are functions in $W^{2,2}(\Omega)$ with $u_0-w\in W_0^{2,2}(\Omega)$ and $v_0\in L^2(\boldsymbol{R}^n)$. Throughout this paper ∇ and $\triangle$ are used for differentiations with respect to only x variables, that is, $\nabla={}^t(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\ldots,\frac{\partial}{\partial x^n})$ and $\triangle=(\frac{\partial}{\partial x^1})^2+(\frac{\partial}{\partial x^2})^2+\cdots+(\frac{\partial}{\partial x^n})^2$. Linear case of this equation is well-known as beam equation and is one of the Schrödinger type equations, while there are several works on semilinear beam equation ([8, 10]). This equation is derived as the Euler-Lagrange equation of the functional

$$\int_0^T\Big[\int_\Omega\frac{1}{2}|u_t(t,x)|^2dx-J(u(t,\cdot))\Big]dt,$$

where

$$(2.4) \qquad J(u)=\int_\Omega\{|\triangle u|^2/2+F(x,u,\nabla u)\}dx.$$

We assume that there exists a positive constant μ_0 such that

$$(2.5) \qquad \begin{cases} 0\le F(x,y,p)\le\mu_0(1+|y|^{q_0}+|p|^{r_0}) \\ |F_p|\le\mu_0(1+|y|^{q_1}+|p|^{r_1}) \\ |F_y|\le\mu_0(1+|y|^{q_2}+|p|^{r_2}), \end{cases}$$

where

$$0<q_0<\frac{2n}{n-4},\ 0<q_1<\frac{n+2}{n-4},\ 0<q_2<\frac{n}{n-4},$$
$$0<r_0<\frac{2n}{n-2},\ 0<r_1<\frac{n+2}{n-2},\ 0<r_2<\frac{n}{n-2}$$

when $n\ge5$,

$$q_0>0,\ q_1>0,\ q_2>0,\ 0<r_0<\frac{2n}{n-2},\ 0<r_1<\frac{n+2}{n-2},\ 0<r_2<\frac{n}{n-2}$$

when $n=3$, 4, and

$$q_0>0,\ q_1>0,\ q_2>0,\ r_0>0,\ r_1>0,\ r_2>0$$

when $n=1$, 2.

It follows from (2.5) and Sobolev's imbedding theorem that J is Gâteaux differentiable on $W^{2,2}(\Omega)$. Assumption (2.5) admits, for example, a function $F(x,y,p)=a(x,y)|p|^2$ when $n\le7$.

We say that a function u is a weak solution to (2.1)–(2.3) if u satisfies $u\in L^\infty((0,\infty);W^{2,2}(\Omega))$, $\frac{\partial u}{\partial t}\in\bigcup_{T>0}L^2((0,T)\times\Omega)$, $\text{s-}\lim_{t\searrow0}u(t,x)=u_0(x)$ in $L^2(\Omega)$, $u(t)-w\in W_0^{2,2}(\Omega)$ for $\mathcal{L}^1$-a.e. t, and

$$(2.6) \qquad \int_0^\infty\int_\Omega\{-\frac{\partial u}{\partial t}(t,x)\phi_t(t,x)+\triangle u(t,x)\triangle\phi(t,x)+F_p(x,u(t,x),\nabla u(t,x))\cdot\nabla\phi(t,x)$$
$$+F_y(x,u(t,x),\nabla u(t,x))\phi(t,x)\}dxdt=\int_\Omega v_0(x)\phi(0,x)dx$$

for any $\phi \in C_0^\infty([0,\infty) \times \Omega)$.

3 Constructing a weak solution

For a positive number h we construct a sequence $\{u_\ell\}_{\ell=-1}^\infty$ in $W^{2,2}(\Omega)$ in the following way. For $\ell = 0$ we let u_0 be as above and for $\ell = -1$ we set $u_{-1} = u_0 - hv_0$. For $\ell \geq 1$ u_ℓ is defined as the minimizer of the functional

$$\mathcal{F}_\ell(v) = \frac{1}{2}\int_\Omega \frac{|v - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}dx + J(v) \qquad (J \text{ is as in } (2.4))$$

in the class $\{v \in W^{2,2}(\Omega); v - w \in W_0^{2,2}(\Omega)\}$. By (2.5) and Poincaré's inequality we see that there exist constants c_0 and c_1 depending on Ω and w such that

$$J(v) \geq c_0 \parallel v \parallel_{W^{2,2}(\Omega)} - c_1$$

for each $v \in W^{2,2}(\Omega)$ with $v - w \in W_0^{2,2}(\Omega)$. It follows from (2.5) and Sobolev's imbedding theorem that J is weakly lower semicontinuous on $W^{2,2}(\Omega)$. These facts yield the existence of a minimizer of $\mathcal{F}_\ell$.

If J is convex, then the following energy inequality holds ([9]):

$$\frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) \leq \frac{1}{2}\int_\Omega |v_0|^2 dx + J(u_0).$$

In our case J is probably not convex, but we can obtain the weakened inequality.

Lemma 3.1

$$\frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) + 1 \leq \left(\frac{1 + C\mu_0 h}{1 - C\mu_0 h}\right)^\ell \left[\frac{1}{2}\int_\Omega |v_0|^2 + J(u_0) + 1\right],$$

where C denotes a constant.

Proof. Let us write $J = J_1 + J_2$, where $J_1(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx$ for $u \in W^{2,2}(\Omega)$ and $J_2(u) = \int_\Omega F(x, u, \nabla u)dx$.

By the minimality of $\mathcal{F}_\ell(u_\ell)$ we have

$$\begin{aligned} (3.1) \qquad \mathcal{F}_\ell(u_\ell) &= \frac{1}{2}\int_\Omega \frac{|u_\ell - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}dx + J(u_\ell) \leq \mathcal{F}(\theta u_\ell + (1-\theta)u_{\ell-1}) \\ &= \frac{1}{2}\int_\Omega \frac{|\theta(u_\ell - u_{\ell-1}) - u_{\ell-1} + u_{\ell-2}|^2}{h^2}dx + J(\theta u_\ell + (1-\theta)u_{\ell-1}) \end{aligned}$$

for $0 \leq \theta \leq 1$. By an easy calculas we obtain

$$\begin{aligned} &|u_\ell - 2u_{\ell-1} + u_{\ell-2}|^2 - |\theta(u_\ell - u_{\ell-1}) - u_{\ell-1} + u_{\ell-2}|^2 \\ &\qquad \leq (1-\theta)(\theta|u_\ell - u_{\ell-1}|^2 - |u_{\ell-1} - u_{\ell-2}|^2). \end{aligned}$$

This and (3.1) imply

$$(3.2) \qquad (1-\theta)\frac{1}{2}\int_\Omega \frac{\theta|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) \\ \leq (1-\theta)\frac{1}{2}\int_\Omega \frac{|u_{\ell-1} - u_{\ell-2}|^2}{h^2}dx + J(\theta u_\ell + (1-\theta)u_{\ell-1}).$$

By the convexity of J_1 we have

$$(3.3) \quad J(\theta u_\ell + (1-\theta)u_{\ell-1}) \leq \theta J_1(u_\ell) + (1-\theta)J_1(u_{\ell-1}) + J_2(\theta u_\ell + (1-\theta)u_{\ell-1}) \\ = \theta J(u_\ell) + (1-\theta)J(u_{\ell-1}) - \theta J_2(u_\ell) - (1-\theta)J_2(u_{\ell-1}) + J_2(\theta u_\ell + (1-\theta)u_{\ell-1}).$$

Since F is convex with respect to p, we have

$$(3.4) \quad J_2(\theta u_\ell + (1-\theta)u_{\ell-1}) \\ \leq \theta \int_\Omega F(x, \theta u_\ell + (1-\theta)u_{\ell-1}, \nabla u_\ell)dx + (1-\theta)\int_\Omega F(x, \theta u_\ell + (1-\theta)u_{\ell-1}, \nabla u_{\ell-1})dx.$$

Now we write

$$(3.5) \quad \theta \int_\Omega F(x, \theta u_\ell + (1-\theta)u_{\ell-1}, \nabla u_\ell)dx - \theta J_2(u_\ell) \\ = \theta(1-\theta)\int_\Omega \int_0^1 F_y(x, u_\ell - \sigma(1-\theta)(u_\ell - u_{\ell-1}), \nabla u_\ell)d\sigma(u_\ell - u_{\ell-1})dx.$$

Thus by (3.2), (3.3), (3.4), and (3.5) we have

$$(1-\theta)\frac{1}{2}\int_\Omega \frac{\theta|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) - \theta J(u_\ell) \\ \leq (1-\theta)\frac{1}{2}\int_\Omega \frac{|u_{\ell-1} - u_{\ell-2}|^2}{h^2}dx + (1-\theta)J(u_{\ell-1}) \\ - (1-\theta)J_2(u_{\ell-1}) + (1-\theta)\int_\Omega F(x, \theta u_\ell + (1-\theta)u_{\ell-1}, \nabla u_{\ell-1})dx \\ + \theta(1-\theta)\int_\Omega \int_0^1 F_y(x, u_\ell - \sigma(1-\theta)(u_\ell - u_{\ell-1}), \nabla u_\ell)d\sigma(u_\ell - u_{\ell-1})dx.$$

Multiplying $(1-\theta)^{-1}$ to the both side and letting $\theta \nearrow 1$, we have

$$(3.6) \quad \frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) \leq \frac{1}{2}\int_\Omega \frac{|u_{\ell-1} - u_{\ell-2}|^2}{h^2}dx + J(u_{\ell-1}) \\ - J_2(u_{\ell-1}) + \int_\Omega F(x, u_\ell, \nabla u_{\ell-1})dx + \int_\Omega F_y(x, u_\ell, \nabla u_\ell)(u_\ell - u_{\ell-1})dx.$$

Here

$$(3.7) \quad \int_\Omega F(x, u_\ell, \nabla u_{\ell-1})dx - J_2(u_{\ell-1}) \\ = \int_\Omega \int_0^1 F_y(x, u_{\ell-1} + \sigma(u_\ell - u_{\ell-1}), \nabla u_{\ell-1})d\sigma(u_\ell - u_{\ell-1})dx.$$

Using (2.5), we have by (3.6) and (3.7)

$$(3.8)\qquad \frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) \le \frac{1}{2}\int_\Omega \frac{|u_{\ell-1} - u_{\ell-2}|^2}{h^2}dx + J(u_{\ell-1})$$
$$+C\mu_0 \int_\Omega (1 + |u_{\ell-1}|^{q_2} + |u_\ell|^{q_2} + |\nabla u_{\ell-1}|^{r_2} + |\nabla u_\ell|^{r_2})|u_\ell - u_{\ell-1}|dx.$$

Now note the inequality

$$(3.9)\qquad \int_\Omega (1 + |u_{\ell-1}|^{q_2} + |u_\ell|^{q_2} + |\nabla u_{\ell-1}|^{r_2} + |\nabla u_\ell|^{r_2})|u_\ell - u_{\ell-1}|dx$$
$$\le Ch[\int_\Omega (1 + |u_{\ell-1}|^{2q_2} + |u_\ell|^{2q_2} + |\nabla u_{\ell-1}|^{2r_2} + |\nabla u_\ell|^{2r_2})dx + \int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx]$$
$$\le Ch[\int_\Omega (1 + |\Delta u_{\ell-1}|^2 + |\Delta u_\ell|^2 + |\Delta u_{\ell-1}|^2 + |\Delta u_\ell|^2)dx + \int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx]$$

since $0 < q_2 < \dfrac{n}{n-4}$ and $0 < r_2 < \dfrac{n}{n-2}$ when $n \ge 5$, $q_2 > 0$ and $0 < r_2 < \dfrac{n}{n-2}$ when $n = 3, 4$, and $q_2 > 0$ and $r_2 > 0$ when $n = 1, 2$. Here the constant C varies tacitly. Hence we have

$$(1-C\mu_0 h)[\frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell) + 1] \le (1+C\mu_0 h)[\frac{1}{2}\int_\Omega \frac{|u_{\ell-1} - u_{\ell-2}|^2}{h^2}dx + J(u_{\ell-1}) + 1].$$

Then we obtain the conclusion by induction on ℓ. Q.E.D.

Next we define approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ for $(t,x) \in (0,\infty) \times \Omega$ as follows: for $(\ell-1)h < t \le \ell h$

$$u^h(t,x) = \frac{t-(\ell-1)h}{h}u_\ell(x) + \frac{\ell h - t}{h}u_{\ell-1}(x)$$

and

$$\overline{u}^h(t,x) = u_\ell(x).$$

Then the following facts hold (see, for example, [1] or [9]).

Theorem 3.2 *For any* $T > 0$

1) $\{\| \dfrac{\partial u^h}{\partial t} \|_{L^2((0,T)\times\Omega)}\}$ *is uniformly bounded with respect to* h

2) $\{\| \overline{u}^h \|_{L^\infty((0,T);W^{2,2}(\Omega))}\}$ *is uniformly bounded with respect to* h

3) $\{\| u^h \|_{L^\infty((0,T);W^{2,2}(\Omega))}\}$ *is uniformly bounded with respect to* h

4) $\{\| u^h \|_{W^{1,2}((0,T)\times\Omega)}\}$ *is uniformly bounded with respect to* h.

Then there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j \to \infty$ *and a function* $u \in L^\infty((0,T);$ $W^{2,2}(\Omega)) \cap \bigcup_{T>0} W^{1,2}((0,T)\times\Omega)$ *such that*

5) $\overline{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *weakly star in* $L^\infty((0,T); W^{2,2}(\Omega))$
6) u^{h_j} *converges to* u *as* $j \to \infty$ *weakly in* $W^{1,2}((0,T) \times \Omega)$
7) u^{h_j} *converges to* u *as* $j \to \infty$ *strongly in* $L^2((0,T) \times \Omega)$
8) $\overline{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *strongly in* $L^2((0,T) \times \Omega)$
9) $s\text{-}\lim_{t \searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Proof. By Lemma 3.1 we obtain, for each $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega |u_t^h(t,x)|^2 + J(\overline{u}^h(t,\cdot)) + 1 \le \left(\frac{1+C\mu_0 h}{1-C\mu_0 h}\right)^{\frac{t}{h}}\left[\frac{1}{2}\int_\Omega |v_0(x)|^2 + J(u_0) + 1\right].$$

Since

$$\lim_{h\to 0}\left(\frac{1+C\mu_0 h}{1-C\mu_0 h}\right)^{\frac{t}{h}} = e^{2C\mu_0 t},$$

we find that there exists a constant $C_1 = C_1(T)$ such that

$$\frac{1}{2}\int_\Omega |u_t^h(t,x)|^2 + J(\overline{u}^h(t,\cdot)) + 1 \le C_1\left[\frac{1}{2}\int_\Omega |v_0(x)|^2 + J(u_0) + 1\right].$$

Hence every assertions are obtained as in the former works, for example compare to [1] or [9]. Q.E.D.

Remark. In the sequel $\{u^{h_j}\}$ and $\{\overline{u}^{h_j}\}$ are often denoted by $\{u^h\}$ and $\{\overline{u}^h\}$ for simplicity.

Since u_ℓ is a minimizer of $\mathcal{F}_\ell(v)$, we have

$$\begin{aligned}(3.10)\qquad 0 &= \frac{d}{d\varepsilon}\mathcal{F}_\ell(u_\ell + \varepsilon\phi)|_{\varepsilon=0} \\ &= \int_\Omega \Big\{\frac{u_\ell(x) - 2u_{\ell-1}(x) + u_{\ell-2}(x)}{h^2}\phi(x) + \triangle u_\ell(x)\triangle\phi(x) \\ &\qquad + F_p(x, u_\ell(x), \nabla u_\ell(x)) \cdot \nabla\phi(x) + F_y(x, u_\ell(x), \nabla u_\ell(x))\phi(x)\Big\}dx\end{aligned}$$

for any $\phi \in W_0^{2,2}(\Omega)$. Note that, for $(\ell-1)h < t < \ell h$, $\frac{\partial u^h}{\partial t}(t,x) = \frac{u_\ell(x) - u_{\ell-1}(x)}{h}$. Thus (3.10) implies

$$\begin{aligned}(3.11)\qquad &\int_0^\infty\int_\Omega \frac{u_t^h(t,x) - u_t^h(t-h,x)}{h}\phi(t,x)dxdt + \int_0^\infty\int_\Omega \{\triangle\overline{u}^h(t,x)\triangle\phi(t,x) \\ &+ F_p(x, \overline{u}^h(t,x), \nabla\overline{u}^h(t,x))\cdot\nabla\phi(t,x) + F_y(x, \overline{u}^h(t,x), \nabla\overline{u}^h(t,x))\phi(t,x)\}dxdt = 0\end{aligned}$$

for any $\phi \in C_0^1([0,\infty) \times \Omega)$. In the same way as in the proof of (4.5) of [5] (compare also to [12]) we have, as $h \to 0$, passing to a subsequence if necessary,

$$\int_0^T\int_\Omega \frac{u_t^h(t,x) - u_t^h(t-h,x)}{h}\phi(t,x)dxdt \longrightarrow -\int_0^T\int_\Omega u_t(t,x)\phi_t(t,x)dxdt - \int_\Omega v_0(x)\phi(0,x)dx.$$

Theorem 3.2 9) means that u satisfies $u(0,x) = u_0(x)$ in the weak sense. Theorem 3.2 5) implies that u satisfies (2.3) in the weak sense since $\overline{u}^h - w \in L^\infty((0,\infty); W_0^{2,2}(\Omega))$ for each h. Hence our purpose is achieved if we show

$$
\begin{aligned}
(3.12) \qquad & \int_0^\infty \int_\Omega \{\Delta \overline{u}^h(t,x)\Delta\phi(t,x) + F_p(x,\overline{u}^h(t,x),\nabla\overline{u}^h(t,x))\cdot\nabla\phi(t,x) \\
& \qquad + F_y(x,\overline{u}^h(t,x),\nabla\overline{u}^h(t,x))\phi(t,x)\}dxdt \\
\longrightarrow & \int_0^\infty \int_\Omega \{\Delta u(t,x)\Delta\phi(t,x) + F_p(x,u(t,x),\nabla u(t,x))\cdot\nabla\phi(t,x) \\
& \qquad + F_y(x,u(t,x),\nabla u(t,x))\phi(t,x)\}dxdt
\end{aligned}
$$

Theorem 3.2 provides just a result for the second derivatives with respect to only x variables, and hence we cannot apply Rellich's compactness theorem directly. However the strong convergence of $\overline{u}^h$ (Lemma 3.2 8)) works instead of the regularity of $\overline{u}^h$ with respect to t (see the proof of Theorem 1.1 of [6]). Hence (3.12) can be obtained and we have

Theorem 3.3 *u is a weak solution to* (2.1)–(2.3).

References

[1] F. Bethuel, J. M. Coron, J. M. Ghidallia and A. Soyeur, *Heat flow and relaxed energies for harmonic maps*, "Nonlinear Diffusion Equations and Their Equilibrium States, 3" eds. N. G. Lloyd, W. M. Ni, A. Peletier, J. Serrin, Progr. Nonlinear Differential Equations Appl. 7, Birkhäuser, Boston-Basel-Berlin, 1992, 99-109.

[2] G. Friesecke and G. Dolzmann, *Implicit time discretization and global existence for a quasi-linear evolution equation with nonconvex energy*, Siam. J. Math. Anal., **28**, (1997), 363-380.

[3] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, Boston-Basel-Stuttgart, 1984.

[4] K. Kikuchi, *Constructing weak solutions in a direct variational method and an application of varifold theory*, J. Differential Equations **150** (1998), 1-23.

[5] K. Kikuchi, *An analysis of the nonlinear equation of motion of a vibrating membrane in the space of BV functions*, J. Math. Soc. Japan **52** (2000), 741-766.

[6] K. Kikuchi, *Global existence of a gradient flow for a quasiconvex functional satisfying some coersiveness condition*, in this report.

[7] N. Kikuchi, *An approach to the construction of Morse flows for variational functionals*, in "Nematics Mathematical and Physical Aspects", ed.: J. M. Coron, J. M. Ghidaglia and F. Hélein, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **332**, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991, 195-199.

[8] S. P. Levandosky, *Decay estimates for fourth order wave equations*, J. Differential Equations **143** (1998), 360-413.

[9] T. Nagasawa, *Discrete Morse semiflows and evolution equations* (in Japanese), Proceedings of the 16th Young Japanese Mathematicians' Seminar on Evolution Equations, 1994, 1-20.

[10] T. Narazaki, *$H^{m(s+1)}$-solutions of semilinear beam equations*, Summer Seminar on Partial Differential Equations at Tokai University Yamanakako Seminar House (2000).

[11] K. Rektorys, *On application of direct variational method to the solution of parabolic boundary value problems of arbitrary order in the space variables*, Czechoslovak Math. J. **21** (1971), 318-339.

[12] A. Tachikawa, *A variational approach to constructing weak solutions of semilinear hyperbolic systems*, Adv. Math. Sci. Appl. **3** (1994), 93-103.