準凸な汎函数に対応する勾配流方程式および作用積分のラグランジュ方程式の解析

課題番号 14540202

平成 14 年度～平成 15 年度科学研究費補助金基盤研究 (C)(2) 研究成果報告書

平成 16 年 6 月

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交付決定額（配分額）

	直接経費	間接経費	合計
平成 14 年度	1,900	0	1, 900
平成 15 年度	2,100	0	2, 100
総計	4,000	0	4,000

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他数件

(3) 出版物

特になし

研究成果による工業所有権の出願・取得状況

特になし

研究成果

本件は主に以下の点を解明するために計画された研究であった。

1. 典型的な準凸汎函数に対する勾配流の構成
2. 典型的な準凸汎函数に対応する作用積分のラグランジュ方程式の研究
3. 凸汎函数と準凸汎函数との違いを示す現象の発見。

本件の研究期間中，研究代表者菊地は他大学を訪問するなどし，北海道大学儀我教授，利根川助教授，慶應義塾大学菊池教授，愛媛大学坂口教授，金沢大学小俣助教授，東北大学堀畑助手他の関連する分野の専門家らと研究連絡を行った。また第２年目には中国上海市復旦大学において研究集会 Workshop on spectral theory and differential operators が開催され研究代表者菊地が参加し成果発表及び情報収集を行った。これらのほかにも研究代表者菊地及び各分担者が国内外の研究集会に参加し研究に必要な情報を収集した。その結果以下のような研究成果を得ることができた。

本件の研究でもっとも目覚しい進展がみられたのは研究目的の 2 に関してである。準凸な汎函数に関連する発展方程式は困難点が多くなかなか研究が進展しないが，線型方程式の場合には解の存在などの基本的な結果はすべて得られている。そこで u に対し F (D u (x)) の積分値を対応させる汎函数の作用積分のラグランジュ方程式を中心に線形近似についての研究を行い F が準凸で一次増大度を持つ場合について結果を得た。エネルギーが一次増大度である場合は汎函数をソボレフ空間上で取り扱うのは不十分であり，有界変動函数の空間で取り扱わなくてはならない。このことに応じて初期値も有界変動函数で考えるのが自然であるが，現時点では初期値がソボレフ空間に属する場合に限って線形近似が成立することがわかった。またこの結果に先立ち同じ方程式について，汎函数が凸の場合に得られている結果，即ち，近似解を Rothe の方法で求めその極限がエネルギー保存則を満たせば弱解になるという定理を準凸の場合にも拡張することに成功した。ここで研究目的の 3 とも関連するが，エネルギー不等式を得るのに従来は汎函数の凸性を用いていたためそのままではこの方法が適用できず，準凸独自の考察が必要となる。結果として近似解を成分ごとに構成することで結論を得ることができた。これは凸汎函数と準凸汎函数との違いをかなりはっきりと示す事実であると思われる。

このほか研究目的の 1 に関する研究では勾配流の構成を完遂することは出来なかったものの，その目的達成のための鍵となりうる恒等式を発見することに成功した。ここでは幾何学的測度論のテクニックが本質的に用いられている。

以上の結果の内，2 および 3 の両方に関係する Rothe 近似に関する結果については既に印刷公表されているので，本報告書では 2 の線形近似に関する結果および 1 に関する結果を報告する。

Linear approximation of a system of quasilinear hyperbolic equations having linear growth energy

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Tentative version, June 3, 2004

1 Introduction

In the sequel the set of all N by n matrices with real elements is simply denoted by $\boldsymbol{R}^{nN}$. Let f be a real valued function defined on $\boldsymbol{R}^{nN}$ and suppose that

(A1) f is asymptotically linear, i.e., there exist constants m and M such that

$$m|p| \le f(p) \le M(1+|p|).$$

and

(A2) f is quasiconvex, i.e.,

$$\frac{1}{\mathcal{L}^n(D)}\int_D f(p_0+\nabla\varphi(x))dx \ge f(p_0)$$

for each bounded domain $D \subset \boldsymbol{R}^n$, for each $p_0 \in \boldsymbol{R}^{nN}$, and for each $\varphi \in [W_0^{1,\infty}(D)]^N$.

In the author's previous work [4] he treated a system of second order quasilinear hyperbolic equations

$$\frac{\partial^2 u^i}{\partial t^2}(t,x) - \sum_{\alpha=1}^{n}\frac{\partial}{\partial x^\alpha}\{f_{p^i_\alpha}(\nabla u(t,x))\} = 0, \quad i=1,2,\dots,N$$

in a bounded domain $\Omega \subset \boldsymbol{R}^n$ and obtained that a sequence of approximate solutions to this equation constructed by Rothe's method converges to a function u in $L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))$, and that, if u satisfies the energy conservation law, it is a weak solution in the space of BV functions. In this article we are going to investigate linear apploximation to this system. Let ε be a positive number. Our purpose is to investigate the behavior as $\varepsilon \to 0$ of a weak solution to

$$\text{(1.1)} \qquad \frac{\partial^2 u^i}{\partial t^2}(t,x) - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\{\frac{1}{\varepsilon}f_{p^i_\alpha}(\varepsilon\nabla u(t,x))\} = 0, \quad x\in\Omega, \quad i=1,2,\dots,N$$

with initial and boundary conditions

$$\text{(1.2)} \qquad u(0,x)=u_0(x), \qquad \frac{\partial u}{\partial t}(0,x)=v_0(x), \qquad x\in\Omega,$$

This research was partially supported by Grant-in-Aid for Scientific Research (No. 14540202), Japan Society for the Promotion of Science

(1.3) $$u(t,x)=0, \quad x\in\partial\Omega.$$

Let us put $f_\varepsilon(p)=\dfrac{1}{\varepsilon^2}(f(\varepsilon p)-f(0))$. The relaxed functional of the functional $u\mapsto \int_\Omega f_\varepsilon(\nabla u(x))dx$ in the $L^1(\Omega)$ norm, which is denoted by J_ε, is finite for $u\in[BV(\Omega)]^N$ and is expressed as

(1.4) $$J_\varepsilon(u,\Omega)=\int_\Omega f_\varepsilon(\nabla u(x))dx+\frac{1}{\varepsilon}\int_\Omega f_\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)d|D^s u|,$$

where $Du=D^a u+D^s u$ (absolutely continuous part and singular part with respect to $\mathcal{L}^n$), $D^a u=\mathcal{L}^n \llcorner \nabla u$. Indeed $(f_\varepsilon)_\infty(p)$ is defined as, for $p\in \boldsymbol{R}^n$,

(1.5) $$(f_\varepsilon)_\infty(p)=\limsup_{\rho\to0} f_\varepsilon\Big(\frac{p}{\rho}\Big)\rho=\limsup_{\rho\to0}\frac{1}{\varepsilon^2}f\Big(\varepsilon\frac{p}{\rho}\Big)\rho=\frac{1}{\varepsilon}f_\infty(p).$$

Thus we find (1.4) (compare to [1, Theorem 5.47]). Similarly to the scalar case the most appropriate weak formulation of Dirichlet condition (1.3) is to replace $J_\varepsilon(u,\Omega)$ with $J_\varepsilon(u,\overline{\Omega})$. The functional $J_\varepsilon(u,\overline{\Omega})$ is expressed as

(1.6) $$J_\varepsilon(u,\overline{\Omega})=J_\varepsilon(u,\Omega)+\frac{1}{\varepsilon}\int_{\partial\Omega} f_\infty(\gamma u\times\vec{n})d\mathcal{H}^{n-1},$$

where $\vec{n}$ denotes the inward pointing unit normal to $\partial\Omega$ and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

We further suppose that

(A3) $f\in C^2(\boldsymbol{R}^{nN})$ and $f_{pp}\in W^{1,\infty}(\boldsymbol{R}^{nN})\cap L^\infty(\boldsymbol{R}^{nN})$

(A4) there exists a positive constant c_0 such that, for each $\varphi\in W_0^{1,\infty}(\Omega)$,

$$\int_\Omega f_p(\nabla\varphi):\nabla\varphi dx\ge c_0\int_\Omega\frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}+1}dx.$$

Suppose that $u_0\in[L^2(\Omega)\cap BV(\Omega)]^N$ and $v_0\in[L^2(\Omega)]^N$. Following to [4] we define a BV solution to (1.1)–(1.3) as

Definition 1 A function u is said to be a BV solution to (1.1)–(1.3) in $(0,T)\times\Omega$ if and only if

i) $u\in[L^\infty((0,T);BV(\Omega))]^N,\ \ u_t\in[L^2((0,T)\times\Omega)]^N$

ii) $u(0,x)=u_0(x)$

iii) for any $\phi\in[C_0^1([0,T)\times\Omega)]^N$,

$$\int_0^T\{-\int_\Omega u_t\phi_t(t,x)dx+\int_\Omega\frac{1}{\varepsilon}f_p(\varepsilon\nabla u):\nabla_x\phi(t,x)dx\}dt=\int_\Omega v_0(x)\phi(0,x)dx$$

iv) for any $\psi\in C_0^1([0,T))$,

$$\begin{aligned}&\int_0^T\{-\int_\Omega u_t(\psi'(t)u+\psi(t)u_t)dx+\psi(t)\int_\Omega\frac{1}{\varepsilon}f_p(\varepsilon\nabla u):\nabla u dx\\
&+\ \psi(t)\int_\Omega\frac{1}{\varepsilon}f_\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)d|D^s u|+\psi(t)\int_{\partial\Omega}\frac{1}{\varepsilon}f_\infty(\gamma u\otimes\vec{n})d\mathcal{H}^{n-1}\}dt\\
&=\ \psi(0)\int_\Omega v_0(x)u_0(x)dx.\end{aligned}$$

Let us put $f_{p^i_\alpha p^j_\beta}(0) = a^{\alpha\beta}_{ij}$ and write

$$Lu = {}^t(\sum_{j=1}^N \sum_{\alpha=1}^n \sum_{\beta=1}^n a^{\alpha\beta}_{ij} \frac{\partial^2 u^j}{\partial x^\alpha \partial x^\beta}; i = 1, 2, \ldots, N).$$

Our man theorem is as follows:

Theorem 1.1 *Suppose that $u_0 \in W_0^{1,2}(\Omega)$ and $v_0 \in L^2(\Omega)$ and that u^ε is a BV solution to (1.1)–(1.3) in $(0,T) \times \Omega$. We further suppose u^ε satisfies energy inequality: for $\mathcal{L}^1$-a.e. $t \in (0,T)$,*

$$\frac{1}{2}\int_\Omega |u_t^\varepsilon(t,x)|^2 dx + J_\varepsilon(u^\varepsilon(t,\cdot), \overline{\Omega}) \leq \frac{1}{2}\int_\Omega |v_0|^2 dx + J_\varepsilon(u_0, \overline{\Omega}). \tag{1.7}$$

Then there exists a function u such that

1). $\{\| u_t^\varepsilon \|_{L^\infty((0,T);L^2(\Omega))}\}$ *is uniformly bounded with respect to ε*

2). $\{\| u^\varepsilon \|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to ε*

3). *u^ε converges to u as $\varepsilon \to 0$ weakly star in $L^\infty((0,T); L^2(\Omega))$*

4). *u_t^ε converges to u_t as $\varepsilon \to 0$ weakly star in $L^\infty((0,T); L^2(\Omega))$*

5). *u^ε converges to u as $\varepsilon \to 0$ strongly in $L^p((0,T) \times \Omega)$ for each $1 \leq p < 1^*$*

6). *for $\mathcal{L}^1$-a.e. $t \in (0,T)$, $Du^\varepsilon(t,\cdot)$ converges to $Du(t,\cdot)$ as $\varepsilon \to 0$ in the sense of distributions*

7). *$u \in L^\infty((0,T); W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T) \times \Omega)$*

8). *u is a weak solution to*

$$\begin{cases} u_{tt} - Lu = 0, & (t,x) \in (0,T) \times \Omega) \\ u(0,x) = u_0(x), & x \in \Omega \\ u_t(0,x) = v_0(x), & x \in \Omega \\ u(t,x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

2 Proof of Theorem 1.1

First we show the following technical lemma.

Lemma 2.1 *For each $v \in [BV(\Omega)]^N$,*

$$J_\varepsilon(v, \Omega) \geq \frac{c_0}{2}(\int_\Omega \frac{|\nabla v|^2}{\sqrt{1 + \varepsilon^2 |\nabla v|^2} + 1} dx + \frac{1}{\varepsilon}|D^s v|(\Omega)).$$

Proof. For each $v \in [BV(\Omega)]^N$ there exists a sequence $\varphi_k \in C^1(\Omega)$ such that $\varphi_k \to v$ strongly in $[L^1(\Omega)]^N$ and

$$\lim_{k\to\infty} J_\varepsilon(\varphi_k, \overline{\Omega}) = J_\varepsilon(v, \overline{\Omega}). \tag{2.1}$$

While, we have

$$\begin{aligned}(2.2)\qquad & \liminf_{k\to\infty} \int_\Omega \frac{|\nabla\varphi_k|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}+1}dx \\ = & \frac{1}{\varepsilon^2}\liminf_{k\to\infty}\int_\Omega (\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}-1)dx \geq \frac{1}{\varepsilon^2}\int_\Omega (\sqrt{1+\varepsilon^2|Dv|^2}-1) \\ = & \int_\Omega \frac{|\nabla v|^2}{\sqrt{1+\varepsilon^2|\nabla v|^2}+1}dx + \frac{1}{\varepsilon}|D^s v|(\Omega).\end{aligned}$$

Noting that

$$f_\varepsilon(p) = \frac{1}{\varepsilon}\int_0^1 f_p(\varepsilon\theta p) : p d\theta,$$

we have by (A4)

$$J_\varepsilon(\varphi_k, \overline{\Omega}) \geq \int_0^1 c_0 \int_\Omega \frac{\varepsilon^2\theta^2|\nabla\varphi_k|^2}{\varepsilon^2\theta\sqrt{1+\varepsilon^2\theta^2|\nabla\varphi_k|^2}+1}dxd\theta \geq \frac{c_0}{2}\int_\Omega \frac{|\nabla\varphi_k|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}+1}dx \tag{2.3}$$

Combining (2.1), (2.2), (2.3), we obtain the assertion. Q.E.D.

Proposition 2.2 *There exists a function u such that*

1). $\{\| u_t^\varepsilon \|_{L^\infty((0,T);L^2(\Omega))}\}$ *is uniformly bounded with respect to* ε

2). $\{\| u^\varepsilon \|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to* ε

3). *Passing to a subsequence if necessary,* u^ε *converges to* u *as* $\varepsilon \to 0$ *weakly star in* $L^\infty((0,T); L^2(\Omega))$

4). *Passing to a subsequence if necessary,* u_t^ε *converges to* u_t *as* $\varepsilon \to 0$ *weakly star in* $L^\infty((0,T); L^2(\Omega))$

5). *Passing to a subsequence if necessary,* u^ε *converges to* u *as* $\varepsilon \to 0$ *strongly in* $L^p((0,T)\times\Omega)$ *for each* $1 \leq p < 1^*$

6). $u \in L^\infty((0,T); BV(\Omega) \cap L^2(\Omega))$

7). *Passing to a subsequence if necessary,* $Du^\varepsilon(t,\cdot)$ *converges weakly star to* $Du(t,\cdot)$ *for* $\mathcal{L}^1$*-a.e.* $t \in (0,T)$ *as* $\varepsilon \to 0$ *in the sense of Radon measures*

8). $\text{s-}\lim_{t\searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$

Proof. By (A3) we have

$$f_\varepsilon(p) = \frac{1}{\varepsilon} f_p(0) : p + \int_0^1 < f_{pp}(\theta\varepsilon p)p, p > d\theta \tag{2.4}$$

and furthermore there exists a constant C_1 such that $|f_{pp}(p)| \le C_1$. Since

$$\int_\Omega f_p(0)\nabla u_0(x)dx = 0,$$

we find

$$J_\varepsilon(u_0, \overline{\Omega}) \le C_1 \int_\Omega |\nabla u_0(x)|^2 dx. \tag{2.5}$$

Thus Assertion 1) immediately follows from (1.7). Since the function $\varepsilon \mapsto \varepsilon^{-2}(\sqrt{1+\varepsilon^2|p|^2} - 1)$ is decreasing, we have by Lemma 2.1

$$J_\varepsilon(u^\varepsilon, \overline{\Omega}) \ge \int_\Omega \frac{1}{\varepsilon^2}(\sqrt{1+|Du|^2} - 1) \ge \int_\Omega (\sqrt{1+|Du|^2} - 1)$$

Thus it also follows from (1.7) and (2.5) that $\{\| \, |Du|(\Omega) \, \|_{L^\infty(0,T)}\}$ is uniformly bounded with respect to ε. Then Assertion 2) follows from Assertion 1) because

$$u^\varepsilon(t,x) = u_0(x) + \int_0^t u_t^\varepsilon(s,x)ds.$$

Passing to a subsequence if necessary, we have Assertions 3) and 4) by Assertions 2) and 1), respectively. By Sobolev's theorem $BV(\Omega) \subset L^p(\Omega)$ compactly for each $1 \le p < 1^*$. Then in the same way as in the proof of [3] Proposition 5.1, passing to a subsequence if necessary, we obtain Assertion 5). Assertions 3) and 5) imply $u \in L^\infty((0,\infty); BV(\Omega) \cap L^2(\Omega))$. Assertion 7) follows from 5). Assertion 8) is obtained in the same way as in the proof of [5, Theorem 4.1].

Q.E.D.

Then, up to a subsequence, Assertions 1) $\sim$ 6) of Theorem 1.1 are proved in the above proposition. Rests are proofs of Asseertions 7) and 8).

Lemma 2.3 $\int_{\{|\nabla u^\varepsilon| \ge 1/\varepsilon\}} |\nabla u^\varepsilon(x)|dx + |D^s u^\varepsilon|(\overline{\Omega}) \to 0$ *as* $\varepsilon \to 0$.

Proof. By (1.7) and (2.5) we have

$$J_\varepsilon(u^\varepsilon, \overline{\Omega}) \le C_1 \int_\Omega |\nabla u_0(x)|^2 dx. \tag{2.6}$$

Assumption (A1) implies $f_\infty(p) \ge m|p|$, thus we have by Lemma 2.1 and (1.6)

$$J_\varepsilon(u^\varepsilon, \overline{\Omega}) \ge c_1\Big(\int_\Omega \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega})\Big), \tag{2.7}$$

where $c_1 = \min\{c_0/2, \ m\}$. Putting $C_2 = C_1 c_1^{-1}$, we have by (2.6) and (2.7)

$$\int_\Omega \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega}) \le C_2 \int_\Omega |\nabla u_0(x)|^2 dx. \tag{2.8}$$

Note that

$$\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega})$$
$$\geq \frac{1}{(\sqrt{2}+1)\varepsilon}\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} |\nabla u^\varepsilon| dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega}),$$

and thus we have by (2.8)

$$\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} |\nabla u^\varepsilon(x)|dx + |D^s u^\varepsilon|(\overline{\Omega}) \leq \text{Const. } \varepsilon$$

as $\varepsilon \to 0$. Q.E.D.

Next two propositions are proved by the use of Radon measures in $\overline{\Omega}\times\overline{S}_+$, and the proofs are postponed to the next section.

Proposition 2.4 $|D^s u|(\overline{\Omega}) = 0$

Remark that this proposition implies, in particular, $\gamma u = 0$.

Proposition 2.5 $u \in L^\infty((0,T); W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T)\times\Omega)$

Proof of Theorem 1.1. Assertion 7) is proved in Proposition 2.5. Now the rest is the proof of 8).

Noting

$$\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}dx \geq \frac{1}{\sqrt{2}+1}\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx,$$

we have by (2.8)

$$\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx \leq C_2(\sqrt{2}+1)\int_\Omega |\nabla u_0(x)|^2 dx. \tag{2.9}$$

Now, for each $\phi \in [C_0^1([0,T)\times\Omega)]^N$

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon} f_p(\varepsilon\nabla u^\varepsilon) : \nabla\phi dx &= \int_\Omega \Big(\frac{1}{\varepsilon} f_p(0) : \nabla\phi + \int_0^1 < f_{pp}(\varepsilon\theta\nabla u^\varepsilon)\nabla u^\varepsilon, \nabla\phi > d\theta\Big)dx \\ &= \int_\Omega\int_0^1 < f_{pp}(\varepsilon\theta\nabla u^\varepsilon)\nabla u^\varepsilon, \nabla\phi > d\theta dx.\end{aligned}$$

Now

$$\begin{aligned}&\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}}\int_0^1 < f_{pp}(\varepsilon\theta\nabla u^\varepsilon)\nabla u^\varepsilon, \nabla\phi > d\theta dx \\ =& \int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} < f_{pp}(0)\nabla u^\varepsilon, \nabla\phi > dx + \int_0^1 < [f_{pp}(\varepsilon\theta\nabla u^\varepsilon) - f_{pp}(0)]\nabla u^\varepsilon, \nabla\phi > d\theta dx \\ =:& \ I + II.\end{aligned}$$

First we have by Lemma 2.3

$$\begin{aligned} I &= \int_\Omega < f_{pp}(0) dDu^\varepsilon, \nabla\phi > \\ &\quad - (\int_{\{|\nabla u^\varepsilon| \geq 1/\varepsilon\}} < f_{pp}(0)\nabla u^\varepsilon, \nabla\phi > dx + \int_\Omega < f_{pp}(0) dD^s u^\varepsilon, \nabla\phi >) \\ &\to \int_\Omega < f_{pp}(0) dDu, \nabla\phi > (= \int_\Omega \sum_{i=1}^N \sum_{j=1}^N \sum_{\alpha=1}^n \sum_{\beta=1}^n a_{ij}^{\alpha\beta} \frac{\partial u^j}{\partial x^\beta} \frac{\partial \phi^i}{\partial x^\alpha} dx). \end{aligned}$$

Next, since (A3) (Lipschitz continuity of f_{pp}) implies

$$|[f_{pp}(\varepsilon\theta\nabla u^\varepsilon) - f_{pp}(0)]\nabla u^\varepsilon| \leq \text{Const. } \varepsilon|\nabla u^\varepsilon|^2,$$

we have by (2.9)

$$\begin{aligned} |II| &\leq \text{Const. } \varepsilon \sup|\nabla\phi| \int_{\{|\nabla u^\varepsilon| < 1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx \\ &\leq \text{Const. } C_2(\sqrt{2}+1)\varepsilon \sup|\nabla\phi| \int_\Omega |\nabla u_0(x)|^2 dx \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Hence, letting $\varepsilon \to 0$, we have by Definition 1 iii)

$$\int_0^T \{-\int_\Omega u_t \phi_t(t,x) dx + \int_\Omega \sum_{i=1}^N \sum_{j=1}^N \sum_{\alpha=1}^n \sum_{\beta=1}^n a_{ij}^{\alpha\beta} \frac{\partial u^j}{\partial x^\beta} \frac{\partial \phi^i}{\partial x^\alpha} dx\} dt = \int_\Omega v_0(x)\phi(0,x) dx,$$

which means u satisfies (1.8) in a weak sense.

Finally the uniqueness of a solution to the linear wave equation implies the rest of the subsequence has another subsequence that converges to the same function u. Thus we do not have to pass a subsequence. Q.E.D.

3 Radon measures in $\overline{\Omega} \times \overline{S}_+$

Let μ be a $\boldsymbol{R}^m$ valued Radon measure. Then we write its total variation as $|\mu|$ and the Radon-Nikodym derivative of μ with respect to $|\mu|$ as $\vec{\mu}$. In particular, $\mu = |\mu| \llcorner \vec{\mu}$.

For $v \in [BV(\Omega)]^N$ we define an $\boldsymbol{R}^{nN+1}$ valued Radon measure μ_v by

$$\mu_v = {}^t(-Dv, \mathcal{L}^n).$$

For an open set $A \subset \Omega$, total variation $|\mu_v|$ is given by

$$|\mu_v|(A) = \sup\{\int_\Omega (g_0 + v \operatorname{div} g) dx; (g_0, g) \in C^1(\Omega, \boldsymbol{R}^{nN+1}),\ |g_0|^2 + |g|^2 \leq 1\}$$

In this article, for the sake of simplicity, we write $S_+^{nN+1} = S_+$:

$$S_+ = \{\vec{s} = (s^1, \cdots, s^{nN+1}) \in S^{nN}; s^{nN+1} > 0\}.$$

We also write

$$S_0 = \{\vec{s} = (s^1, \cdots, s^{nN+1}) \in S^n; s^{nN+1} = 0\}.$$

Then $\overline{S}_+ = S_+ \cup S_0$. Given a Radon measure λ in $\overline{\Omega}\times\overline{S}_+$, we let $|\lambda|$ denote a Radon measure on $\overline{\Omega}$ defined by

$$|\lambda|(A) = \lambda(A \times \overline{S}_+) \qquad \text{for a Borel set } A \subset \overline{\Omega}.$$

Clearly this notation is an analogy with that of a total variations of a vector valued Radon measure. In particular, letting λ be a Radon measure in $\overline{\Omega}\times\overline{S}_+$ defined as, for a BV function $v \in [BV(\Omega)]^N$,

$$\int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})d\lambda = \int_{\overline{\Omega}} \beta(x,\vec{\mu_v}(x))d|\mu_v| \qquad (\beta \in C^0(\overline{\Omega}\times\overline{S}_+)), \tag{3.1}$$

then we have $|\lambda| = |\mu_v|$. For each Radon measure λ in $\overline{\Omega}\times\overline{S}_+$, there exists a probability Radon measure $\nu_{\lambda,x}$ on $\overline{S}_+$ for $|\lambda|$-a.e. $x \in \overline{\Omega}$ such that

$$\int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})d\lambda = \int_{\overline{\Omega}}(\int_{\overline{S}_+} \beta(x,\vec{s})d\nu_{\lambda,x})d|\lambda| \qquad (\beta \in C^0(\overline{\Omega}\times\overline{S}_+))$$

(for example, Theorem 10 of page 14 of [2]). Using these notations, we often write $\lambda = |\lambda| \otimes \nu_{\lambda,x}$. In particular, if λ is as in (3.1), then $\lambda = |\mu_v| \otimes \delta_{\vec{\mu}_v(x)}$.

Let u^ε and u be as in Proposition 2.2. In the sequel we simply write $\mu_t^\varepsilon = \mu_{u^\varepsilon(t,\cdot)}$ and $\mu_t = \mu_{u(t,\cdot)}$. Now we define a one parameter family of Radon measures in $\overline{\Omega}\times\overline{S}_+$ by $\lambda_t^\varepsilon = |\mu_t^\varepsilon| \otimes \delta_{\vec{\mu}_t^\varepsilon(x)}$. Proposition 2.2 2) implies

$$\operatorname*{ess.sup}_{t>0} |\int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})d\lambda_t^\varepsilon| \leq C \sup|\beta| \tag{3.2}$$

for some constant C independent of ε. Thus we obtain the following theorem by a standard compactness argument (compare to [3, Proposition 4.3]).

Lemma 3.1 *There exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a one parameter family of Radon measures λ_t in $\overline{\Omega}\times\overline{S}_+$, $t \in (0,\infty)$, such that, for each $\psi \in L^1(0,\infty)$ and $\beta \in C^0(\overline{\Omega}\times\overline{S}_+)$,*

$$\lim_{\varepsilon\to 0}\int_0^\infty \psi(t)\int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})d\lambda_t^\varepsilon dt = \int_0^\infty \psi(t)\int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})d\lambda_t dt.$$

Now we sum up properties of λ_t.

Lemma 3.2 *For $\mathcal{L}^1$-a.e. $t \in (0,\infty)$,*

1). $\mu_t = |\lambda_t| \llcorner \int_{\overline{S}_+} \vec{s}d\nu_{\lambda_t,x}$

2). $|\lambda_t|(A) \geq |\mu_t|(A)$ *for each Borel set* $A \subset \overline{\Omega}$

3). $|\lambda_t|(A) = \int_A D_{|\mu_t|}|\lambda_t|(x)d|\mu_t| + (|\lambda_t| \llcorner Z)(A)$ *for* $A \subset \overline{\Omega}$, *where* $D_{|\mu_t|}|\lambda_t|$ *is the derivative of* $|\lambda_t|$ *with respect to* $|\mu_t|$ *and* Z *is the* $|\mu_t|$*-null set defined by* $Z = \{x; D_{|\mu_t|}|\lambda_t|(x) = \infty\}$

4). $\int_{\overline{S}_+} \vec{s}d\nu_{\lambda_t,x} = 0$ *for* $|\lambda_t| \llcorner Z$*-a.e.* x

5). spt $\nu_{\lambda_t,x} \subset S_0$ *for* $|\lambda_t| \llcorner Z$*-a.e.* x.

6). $|D^s u|(\overline{\Omega}) \le \lambda_t(\overline{\Omega} \times S_0)$.

Proof. 1) For any $g \in C^0(\overline{\Omega}; \boldsymbol{R}^{nN+1})$ and $\psi \in L^1(0,\infty)$

$$\begin{aligned} &\int_0^\infty \psi(t) \int_{\overline{\Omega}} g(x) d\mu_t dt = \int_0^\infty \psi(t) [\int_{\overline{\Omega}} g^0(x) dx + \int_{\overline{\Omega}} g'(x) dDu] dt \\ = \ & \lim_{\varepsilon\to 0} \int_0^\infty \psi(t) [\int_{\overline{\Omega}} g^0(x) dx + \int_{\overline{\Omega}} g'(x) dD\overline{u}^h] dt = \lim_{\varepsilon \to 0} \int_0^\infty \psi(t) \int_{\overline{\Omega}} g(x) d\mu_t^\varepsilon dt \\ = \ & \lim_{\varepsilon\to 0} \int_0^\infty \psi(t) \int_{\overline{\Omega}} g(x) \vec{\mu}_t^\varepsilon d|\mu_t^\varepsilon| dt = \lim_{\varepsilon\to 0} \int_0^\infty \psi(t) \int_{\overline{\Omega}\times\overline{S}_+} g(x) \cdot \vec{s} d\lambda_t^\varepsilon dt \\ = \ & \int_0^\infty \psi(t) \int_{\overline{\Omega}\times\overline{S}_+} g(x) \cdot \vec{s} d\lambda_t(x,\vec{s}) dt = \int_0^\infty \psi(t) \int_{\overline{\Omega}} g(x) (\int_{\overline{S}_+} \vec{s} d\nu_{\lambda_t,x}) d|\lambda_t| dt, \end{aligned}$$

where $\lambda_t^\varepsilon = |\mu_t^\varepsilon| \otimes \delta_{\vec{\mu}_t^\varepsilon(x)}$. This shows assertion 1).

2) First we consider the case that A is the intersection of an open set and $\overline{\Omega}$. By assertion 1) we have, for any $g \in C^0(A; \boldsymbol{R}^{nN+1})$,

$$|\int_A g(x) d\mu_t| \le \int_A |g(x)| d|\lambda_t| \le \sup |g| |\lambda_t|(A).$$

Taking supremum with respect to $g \in C^0(A; \boldsymbol{R}^{nN+1})$ with $|g| \le 1$, we obtain $\mu_t(A) \le |\lambda_t|(A)$.

Let A be any Borel set. For each open set O with $A \subset O$, $\mu_t(A) \le \mu_t(O \cap \overline{\Omega}) \le |\lambda_t|(O \cap \overline{\Omega})$. Thus, since $\inf_{A \subset O \cap \overline{\Omega}} |\lambda_t|(O \cap \overline{\Omega}) = |\lambda_t|(A)$, we have $\mu_t(A) \le |\lambda_t|(A)$.

3) It is the direct consequence of the differentiation theory for Radon measures (see, for example, [6, Theorem 4.7]).

4) By assertions 1) and 3) we have, for any $g(x) \in C^0(\overline{\Omega}; \boldsymbol{R}^{nN+1})$,

$$0 = \int_Z g(x) d\mu_t = \int_Z g(x) (\int_{\overline{S}_+} \vec{s} d\nu_{\lambda_t,x}) d|\lambda_t|.$$

This shows assertion 4).

5) By 4), in particular, we have $\int_{\overline{S}_+} s^{nN+1} d\nu_{\lambda_t,x} = 0$ for $|\lambda_t| \llcorner Z$-a.e. x. Since $s^{nN+1} \ge 0$, we have $s^{nN+1} = 0$ for $\nu_{\lambda_t,x}$-a.e. for $|\lambda_t| \llcorner Z$-a.e. x. Thus assertion 5) holds.

6) By 1) and 3) we have, for μ_t-a.e. $x \in \overline{\Omega}$,

$$\mu_t^{nN+1}(x) = D_{|\mu_t|} |\lambda_t|(x) \int_{\overline{S}_+} s^{nN+1} d\nu_{\lambda_t,x}.$$

Then, since $D_{|\mu_t|}|\lambda_t|(x) \ge 1$ for $|\mu_t|$-a.e. $x \in \overline{\Omega}$, $\mu_t^{nN+1}(x) = 0$ implies

$$\int_{\overline{S}_+} s^{nN+1} d\nu_{\lambda_t,x} = 0,$$

which means spt $\nu_{\lambda_t,x} \subset S_0$. Thus, for $|\mu_t|$-a.e. $x \in \overline{\Omega}$,

$$S_u := \{x \in \overline{\Omega}; \mu_t^{nN+1}(x) = 0\} \subset \{x \in \overline{\Omega}; \text{spt } \nu_{\lambda_t,x} \subset S_0\} =: S_{\mu_t}.$$

More presicely we have $|\mu_t|(S_u \setminus S_{\mu_t}) = 0$, and hence

$$\begin{aligned}|D^s u|(\overline{\Omega}) &= |\mu_t|(S_u) \le |\lambda_t|(S_{\mu_t}) = \lambda_t(S_{\mu_t} \times \overline{S}_+) = \int_{S_{\mu_t}} \nu_{\lambda_t,x}(\overline{S}_+) d|\lambda_t| \\ &= \int_{S_{\mu_t}} \nu_{\lambda_t,x}(S_0) d|\lambda_t| \le \int_{\overline{\Omega}\times\overline{S}_+} \nu_{\lambda_t,x}(S_0) d|\lambda_t| = \lambda_t(\overline{\Omega} \times S_0).\end{aligned}$$

Q.E.D.

Lemma 3.3 $\lambda_t(\overline{\Omega} \times S_0) = 0$

Proof. By the definition of λ_t^ε we immediately obtain

$$(3.3)\quad \int_\Omega \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1} dx + \frac{1}{\varepsilon}|Du^\varepsilon|(\overline{\Omega}) = \int_{\overline{\Omega}\times\overline{S}_+} \frac{|\overline{s}'|^2}{\sqrt{(s^{nN+1})^2+\varepsilon^2|\overline{s}'|^2}+s^{nN+1}} d\lambda_t^\varepsilon,$$

the left hand side of which is estimated from above by $C_2 \int_\Omega |\nabla u_0(x)|^2 dx$ in (2.8). On the other hand, given $\sigma > 0$, we have

$$\int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}} \frac{|\overline{s}'|^2}{\sqrt{(s^{nN+1})^2+\varepsilon^2|\overline{s}'|^2}+s^{nN+1}} d\lambda_t^\varepsilon \ge \frac{1-\varepsilon^2}{\sqrt{\sigma^2+\varepsilon^2}+\sigma} \lambda_t^\varepsilon(\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}).$$

By the lower semicontinuity of Radon measure, we have

$$\begin{aligned}\lambda_t(\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}) &\le \liminf_{\varepsilon\to0} \lambda_t^\varepsilon(\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}) \\ &\le \liminf_{\varepsilon\to0} \frac{\sqrt{\sigma^2+\varepsilon^2}+\sigma}{1-\varepsilon^2} \int_\Omega |\nabla u_0(x)|^2 dx \\ &= 2\sigma \int_\Omega |\nabla u_0(x)|^2 dx.\end{aligned}$$

Letting $\sigma \to 0$, we have the conclusion. Q.E.D.

Proof of Proposition 2.4. The conclusion immediately follows from Lemma 3.2 6) and Lemma 3.3. Q.E.D.

Proof of Proposition 2.5. It is sufficient to show $u \in L^\infty((0,T); W^{1,2}(\Omega))$. Since

$$\begin{aligned}&\int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}>\varepsilon\}} \frac{|\overline{s}'|^2}{\sqrt{(s^{nN+1})^2+\varepsilon^2|\overline{s}'|^2}+s^{nN+1}} d\lambda_t^\varepsilon \\ &\ge \int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}>\varepsilon\}} \frac{|\overline{s}'|^2}{s^{nN+1}(\sqrt{1+|\overline{s}'|^2}+1)} d\lambda_t^\varepsilon \\ &\ge \frac{1}{\sqrt{2}+1} \int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}>\varepsilon\}} \frac{|\overline{s}'|^2}{s^{nN+1}} d\lambda_t^\varepsilon,\end{aligned}$$

we have by (3.3) and (2.8)

$$(3.4)\quad \int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}>\varepsilon\}} \frac{|\overline{s}'|^2}{s^{nN+1}} d\lambda_t^\varepsilon \le C_2(\sqrt{2}+1) \int_\Omega |\nabla u_0(x)|^2 dx.$$

For each $\sigma > \varepsilon$, since $\{s^{nN+1} > \sigma\} \subset \{s^{nN+1} > \varepsilon\}$, we have by (3.4)

$$\int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}>\sigma\}} \frac{|\vec{s}'|^2}{s^{nN+1}} d\lambda_t^\varepsilon \le C_2(\sqrt{2}+1)\int_\Omega |\nabla u_0(x)|^2 dx.$$

On $\{s^{nN+1} > \sigma\}$ the integrand of the left hand side of above is positive and continuous. Thus by the lower semicontinuity of Radon measures we have by letting $\varepsilon \to 0$

$$\int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}>\sigma\}} \frac{|\vec{s}'|^2}{s^{nN+1}} d\lambda_t \le C_2(\sqrt{2}+1)\int_\Omega |\nabla u_0(x)|^2 dx.$$

Hence, letting $\sigma \to 0$, we obtain with Lemma 3.3

$$\int_{\overline{\Omega}\times\overline{S}_+} \frac{|\vec{s}'|^2}{s^{nN+1}} d\lambda_t \le C_2(\sqrt{2}+1)\int_\Omega |\nabla u_0(x)|^2 dx.$$

Since Lemma 3.3 implies the support of $\nu_{\lambda_t,x}$ is contained in S_+ for $|\lambda_t|$-a.e. x, it induces a measure in $\boldsymbol{R}^{nN}$ by a mapping $\iota : S_+ \ni \vec{s} \mapsto \vec{s}'/s^{nN+1} \in \boldsymbol{R}^{nN}$. More precisely we have

$$\int_{\overline{\Omega}\times\overline{S}_+} \frac{|\vec{s}'|^2}{s^{nN+1}} d\lambda_t = \int_{\overline{\Omega}}\int_{\overline{S}_+} \frac{|\vec{s}'|^2}{s^{nN+1}} d\nu_{\lambda_t,x} d|\lambda_t| = \int_{\overline{\Omega}}\int_{\boldsymbol{R}^{nN}} |p|^2 d\iota_\#(\nu_{\lambda_t,x} \llcorner s^{nN+1}) d|\lambda_t|.$$

Let us define a measure Φ on $\boldsymbol{R}^{nN}$ by

$$\Phi = (\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x})^{-1} \iota_\#(\nu_{\lambda_t,x} \llcorner s^{nN+1}).$$

Then, noting

$$\begin{aligned}\Phi(\boldsymbol{R}^{nN}) &= (\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x})^{-1} \iota_\#(\nu_{\lambda_t,x} \llcorner s^{nN+1})(\boldsymbol{R}^{nN}) \\ &= (\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x})^{-1} \int_{S_+} s^{nN+1} d\nu_{\lambda_t,x} = 1,\end{aligned}$$

we have by Jensen's inequality

$$\int_{\boldsymbol{R}^{nN}} |p|^2 d\Phi \ge |\int_{\boldsymbol{R}^{nN}} p d\Phi|^2.$$

Here, note that

$$\int_{\boldsymbol{R}^{nN}} p d\Phi = (\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x})^{-1} \int_{\boldsymbol{R}^{nN}} p d\iota_\#(\nu_{\lambda_t,x} \llcorner s^{nN+1}) = (\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x})^{-1} \int_{S_+} \vec{s}' d\nu_{\lambda_t,x}.$$

On the other hand Lemma 3.2 1), 3) impliy

$$\int_{S_+} \vec{s}' d\nu_{\lambda_t,x} = D_{|\mu_t|}|\lambda_t|(x)\vec{\mu}_t(x).$$

Finally we have

$$\begin{aligned}\int_{\overline{\Omega}\times\overline{S}_+} \frac{|\vec{s}'|^2}{s^{nN+1}} d\lambda_t &= \int_{\overline{\Omega}}\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x} \int_{\boldsymbol{R}^{nN}} |p|^2 d\Phi d|\lambda_t|. \\ &\ge \int_{\overline{\Omega}} (\int_{S_+} s^{nN+1} d\nu_{\lambda_t,x})^{-1} |\int_{S_+} \vec{s}' d\nu_{\lambda_t,x}|^2 d|\lambda_t| \\ &= \int_{\overline{\Omega}} (D_{|\mu_t|}|\lambda_t|(x)\mu_t^{nN+1}(x))^{-1} |D_{|\mu_t|}|\lambda_t|(x)\vec{\mu}_t'(x)|^2 D_{|\mu_t|}|\lambda_t|(x) d|\mu_t| \\ &= \int_{\overline{\Omega}} \frac{|\vec{\mu}_t(x)|^2}{\mu_t^{nN+1}(x)} d|\mu_t| = \int_{\overline{\Omega}} |\nabla u(x)|^2 dx.\end{aligned}$$

This implies the assertion. Q.E.D.

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An identity in an approximating method of gradient flows for quasiconvex functionals

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Tentative version, June 3, 2004

1 Introduction

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with Lipschitz continuous boundary and let $F = F(p)$ be a function defined on $\boldsymbol{R}^{nN}$, the set of all N by n matrices with real elements. Now let us consider the functional

$$J(u) = \int_\Omega F(\nabla u)dx,$$

where $\nabla u = (\dfrac{\partial u^i}{\partial x^\alpha})$.

We suppose the following facts for the function F.

(A1) $F \in C^1(\boldsymbol{R}^{nN})$

(A2) F is *quasiconvex* with respect to p, that is,

$$\frac{1}{\mathcal{L}^n(D)}\int_D F(p_0 + \nabla\phi(x))dx \geq F(p_0)$$

for each bounded domain $D \subset \boldsymbol{R}^n$, for each $p_0 \in \boldsymbol{R}^{nN}$, and for each $\phi \in W_0^{1,\infty}(D; \boldsymbol{R}^N)$

(A3) There exist positive constants μ, λ and a constant $q > 1$ such that

$$\begin{cases} \lambda|p|^q \leq F(p) \leq \nu(1 + |p|^q) \\ |F_p| \leq \mu(1 + |p|^{q-1}) \end{cases}$$

The equation of gradient flow for J is given by

$$\frac{\partial u^i}{\partial t}(t,x) - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\{F_{p^i_\alpha}(\nabla u(x))\} = 0, \quad x \in \Omega. \tag{1.1}$$

We impose the initial and the boundary conditions

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(t,x) = w(x), \qquad x \in \partial\Omega. \tag{1.3}$$

This research was partially supported by Grant-in-Aid for Scientific Research (No. 14540202), Japan Society for the Promotion of Science

We suppose that u_0 and w belong to $W^{1,q}(\Omega, \boldsymbol{R}^N) \cap L^2(\Omega)$ and that $\gamma u_0 = \gamma w$ on $\partial\Omega$ (γ is the trace operator to $\partial\Omega$).

In this article we say that a function u is a *weak solution* to (1.1)–(1.3) in $(0,\infty) \times \Omega$ if u satisfies i) $u \in L^\infty((0,\infty); W^{1,q}(\Omega) \cap L^2(\Omega))$, $u_t \in L^2((0,T) \times \Omega)$ for any $T > 0$, ii) $\operatorname{s-lim}_{t\searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$, iii) $\gamma u(t) = \gamma w$ on Γ for $\mathcal{L}^1$-a.e. t, and iv) for any $\phi \in C_0^1((0,\infty) \times \Omega)$

$$\sum_{i=1}^{N} \int_0^\infty \int_\Omega \{u_t^i(t,x)\phi^i(t,x) + \sum_{\alpha=1}^{n} F_{p_\alpha^i}(\nabla u) \frac{\partial \phi^i}{\partial x^\alpha}(t,x)\} dx dt = 0. \tag{1.4}$$

If u is a weak solution to (1.1), then $J(u(t))$ is absolutely continuous and it holds that $dJ(u(t))/dt = -(u_t, u_t)_{L^2(\Omega)} \le 0$ for $\mathcal{L}^1$-a.e. t. Thus this defines a gradient flow for J.

We construct an approximate solution to (1.1)–(1.3) by the method of discretization in time and minimizing variational functionals. In recent several years this approximating way is widely applied to constructing weak solutions to nonlinear partial differential equations.

Let h be a positive number. A sequence $\{u_l\}$ in $W^{1,q}(\Omega, \boldsymbol{R}^N)$ is constructed as follows: we let u_0 be as in (1.2) and for $l \ge 1$ we define u_l as a minimizer of the functional

$$\mathcal{F}_l(v) = \frac{1}{2} \int_\Omega \frac{|v - u_{l-1}|^2}{h} dx + J(v)$$

in the class $w + W_0^{1,q}(\Omega, \boldsymbol{R}^N)$ (that is, among functions in $W^{1,q}(\Omega, \boldsymbol{R}^N)$ with $\gamma v = \gamma w$). The existence of a minimizer of $\mathcal{F}_l$ is assured by the quasiconvexity of F and (A3) (see, for example, [1, Chapter 4, Theorem 2.9]). Note also that (A3) assures $\mathcal{F}_l$ is Gâteaux differentiable. Approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ ($(t,x) \in (0,\infty) \times \Omega$) are defined as, for $(l-1)h < t \le lh$,

$$u^h(t,x) = \frac{t - (l-1)h}{h} u_l(x) + \frac{lh - t}{h} u_{l-1}(x)$$

and

$$\overline{u}^h(t,x) = u_l(x).$$

Then the following facts hold (see, for example, [6] or other references cited in [5]).

Proposition 1.1 *We have*

1) $\{\| u_t^h \|_{L^2((0,\infty)\times\Omega)}\}$ *is uniformly bounded with respect to* h

2) $\{\| \overline{u}^h \|_{L^\infty((0,\infty);W^{1,q}(\Omega)\cap L^2(\Omega))}\}$ *is uniformly bounded with respect to* h

3) $\{\| u^h \|_{L^\infty((0,\infty);W^{1,q}(\Omega)\cap L^2(\Omega))}\}$ *is uniformly bounded with respect to* h

4) *for any* $T > 0$, $\{\| u^h \|_{W^{1,\tilde{q}}((0,T)\times\Omega))}\}$ *is uniformly bounded with respect to* h, *where* $\tilde{q} = \min\{q, 2\}$.

Then there exist a function u *such that, passing to a subsequence if necessary,*

5) $\overline{u}^h$ *converges to* u *as* $h \to 0$ *weakly star in* $L^\infty((0,\infty); W^{1,q}(\Omega))$

6) *for any* $T > 0$, u^h *converges to* u *as* $h \to 0$ *weakly in* $W^{1,\tilde{q}}((0,T) \times \Omega))$

7) u^h *converges to* u *as* $h \to 0$ *strongly in* $L^{\tilde{q}}((0,T) \times \Omega)$

8) $\overline{u}^h$ *converges to* u *as* $h \to 0$ *strongly in* $L^{\tilde{q}}((0,T) \times \Omega)$

9) $s\text{-}\lim_{t\searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Proposition 1.1 9) means that u satisfies (1.2) in a weak sense. Proposition 1.1 5) implies that u satisfies (1.3) in a weak sense since $\overline{u}^h - w \in L^\infty((0,\infty); W_0^{1,q}(\Omega))$ for each h (note that $W_0^{1,q}(\Omega)$ is a closed subspace of $W^{1,q}(\Omega)$). Thus the problem is whether u satisfies (1.4). Since u_l is a minimizer of $\mathcal{F}_l(v)$, $d\mathcal{F}_l(u_l + \varepsilon\phi)/d\varepsilon|_{\varepsilon=0} = 0$ for any $\phi \in W_0^{1,q}(\Omega)$, and noting that, for $(l-1)h < t < lh$, $u_t^h(t,x) = (u_l(x) - u_{l-1}(x))/h$, we have

$$\sum_{i=1}^{N} \int_\Omega \{(u_t^h)^i(x)\phi^i(x) + \sum_{\alpha=1}^{n} F_{p_\alpha^i}(\nabla \overline{u}^h) \frac{\partial \phi^i}{\partial x^\alpha}(x)\} dx = 0 \tag{1.5}$$

for any $\phi \in W_0^{1,q}(\Omega) \cap L^2(\Omega)$ and any $t \in \bigcup_{\ell=0}^\infty ((\ell-1)h, \ell h)$. This equality leads us to expect that the limit u is a weak solution to (1.1)–(1.3). However we have not yet succeeded to show it. Instead in this article we show an identity, which should be a key estimate in our problem. Our main tool is varifold theory and it is discussed in Section 2. The main theorem is stated in Section 3 and it is proved in Section 4.

2 Varifold setting and first variation with respect to J

Let $U = \Omega \times \boldsymbol{R}^N$. Usually an n-varifold in U is a Radon measure on $U \times G$, where G is the collection of all n-dimensional vector subspaces of $\boldsymbol{R}^{n+N}$. However in this article we say V is an *n-varifold* in U if V is a Radon measure in $U \times \boldsymbol{R}^{nN}$. Let V be an n-varifold in U. The weight of V is defined by $\mu_V(B) = V(B \times \boldsymbol{R}^{nN})$ for each Borel set $B \subset U$. Clearly μ_V is a Radon measure on U. By Lemma 38.4 of [7] there is a probability Radon measure $\eta_V^{(z)}$ for μ_V-a.e. $z \in U$ such that

$$\int_{U \times \boldsymbol{R}^{nN}} \beta(z,p) dV_t(z,p) = \int_U \Big(\int_{\boldsymbol{R}^{nN}} \beta(z,p) d\eta_{V_t}^{(z)}(p)\Big) d\mu_{V_t}.$$

Suppose that $\mathcal{M}$ is a countably n-rectifiable set in U (refer to [7, Chapter 3] for the definition and basic properties of an n-rectifiable set) and that θ is a locally $\mathcal{H}^n$-integrable function on $\mathcal{M}$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. We further suppose that $\mathcal{H}^n$-a.e. $z \in \mathcal{M}$ the approximate tangent space $T_z\mathcal{M}$ is not transversal to $\boldsymbol{R}^n$. Then it is a linear space given by the equation $y = A_z x$ ($x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^N$) for an N by n matrix A_z. Then a varifold in U, i.e. a Radon measure on $U \times \boldsymbol{R}^{nN}$, is defined by a continuous linear functional $\varphi \mapsto \int_{\mathcal{M}} \varphi(z,A)\theta(z) d\mathcal{H}^n$. It is called an n-rectifiable varifold and denoted by $\boldsymbol{v}(\mathcal{M},\theta)$. We call θ a multiplicity function. For each N by n matrix A, let $M(A)$ denote the vector consists of all minor determinants of A including 0-th order determinant, i.e., 1. Then $\int_D |M(A)| dx$ is the area of the plane $y = Ax$ over the set $D \subset \boldsymbol{R}^n$. Given a countably n-rectifiable set such that $\mathcal{H}^n$-a.e. $z \in \mathcal{M}$ the approximate tangent space $T_z\mathcal{M}$ is not transversal to $\boldsymbol{R}^n$, we put $\theta_0(z) = 1/|M(A_z)|$ and define an n-rectifiable varifold $\boldsymbol{v}(\mathcal{M},\theta_0)$ when θ_0 is (locally) $\mathcal{H}^n$-integrable on $\mathcal{M}$. It is called a graph type n-rectifiable varifold.

The terminology "graph type" comes from the following fact.

Proposition 2.1 *Let v be a function in $W^{1,1}(\Omega, \boldsymbol{R}^N)$.*

1) *G_v is countably n-rectifiable*

2) $\int_{G_v} \theta_0(z) d\mathcal{H}^n = \mathcal{L}^n(\Omega)$, *where G_v is the graph of v*

3) *θ_0 is $\mathcal{H}^n$-integrable on G_v and thus there exists a graph type n-rectifiable varifold $V = \boldsymbol{v}(G_v, \theta_0)$*

4) *for each nonnegative continuous function f on $\Omega \times \mathbf{R}^N \times \mathbf{R}^{nN}$,*

$$\int_\Omega f(x, v(x), Dv(x))dx = \int_{U \times \mathbf{R}^{nN}} f(z,p)dV(z,p). \tag{2.1}$$

Proof. Assertion 1) is well-known (compare to, for example, Theorems 4 of [4, I Section 3.1.5.]).

For $\mathcal{H}^n$-a.e. $z \in G_v$, the approximate tangent space $T_z G_v$ exists and is expressed by the equation $y = Dv(\pi(z))x$. Hereby $\theta_0(z) = 1/|M(Dv(\pi(z)))|$ and hence

$$\int_{G_v} \theta_0(z) d\mathcal{H}^n = \int_\Omega \frac{1}{|M(Dv(x))|} |M(Dv(x))| dx = \int_\Omega dx.$$

Thus Assertion 2) follows and Assertion 3) is the immediate consequence of 2).

When spt f is compact, we have by the definition of graph type rectifiable varifold

$$\int_{U \times \mathbf{R}^{nN}} f(z,p)dV(z,p) = \int_{G_v} f(z, Dv(\pi(z)))\theta_0(z) d\mathcal{H}^n(z). \tag{2.2}$$

Replacing Ω with any open set in Assertion 2), we have $\pi_\sharp((\mathcal{H}^n \llcorner G_v) \llcorner \theta_0) = \mathcal{L}^n$. Thus the right hand side of (2.2) coincides with the left hand side of (2.1). Hereby we obtain the conclusion for a function f with a compact support.

Suppose that f is a general nonnegative continuous function. Then, approximating f with an increasing sequence of functions in $C_0^0(\Omega \times \mathbf{R}^N \times \mathbf{R}^{nN})$, we obtain the conclusion by the monotone convergence theorem. Q.E.D.

We say an n-varifold V is pre-graph type if spt μ_V is countably n-rectifiable and μ_V-a.e. z the approximate tangent space $T_z \mu_V$ is not transversal to $\mathbf{R}^n$. We say an pre-graph type varifold V is L^r if

$$\|V\|_{L^r} := \int_{U \times \mathbf{R}^{nN}} |p|^r dV < \infty.$$

Proposition 2.2 *Suppose that $q < r$. Let $f(z,p)$ be a continuous function on $U \times \mathbf{R}^n$. Suppose that for each $z = (x,y) \in U$ and each $p \in \mathbf{R}^{nN}$*

$$|f(z,p)| \leq \mu_1(1 + |p|^q) \tag{2.3}$$

holds with a constant μ_1. Let V_k and V be L^r pre-graph varifolds and suppose that $\{\|V_k\|_{L^r}\}$ is uniformly bounded and $V_k \rightharpoonup V$ in the sense of Radon measures in $U \times \mathbf{R}^{nN}$. Then we have

$$\lim_{k \to \infty} \int_{U \times \mathbf{R}^{nN}} f(z,p)dV_k(z,p) = \int_{U \times \mathbf{R}^{nN}} f(z,p)dV(z,p).$$

Proof. Let ι be an increasing continuous function on $\mathbf{R}$ with $\iota(r) = 1$ for $r > 2$, $\iota = 0$ for $r < 1$, and we put $f_\varepsilon(z,p) = f(z,p)(1 - \iota(\varepsilon(1 + |p|^q)))$. Note that $f_\varepsilon = f$ for $1 + |p|^q \leq \varepsilon^{-1}$, $f_\varepsilon = 0$ for $1 + |p|^q \geq 2\varepsilon^{-1}$. Thus $f_\varepsilon(z,p)$ converges to $f(z,p)$ for each $(z,p) \in U \times \mathbf{R}^{nN}$. Further we have $|f_\varepsilon(z,p)| \leq 2\mu_1 \varepsilon^{-1}$ by (2.3). It follows from Proposition 2.1 that

$$\int_{U \times \mathbf{R}^{nN}} f_\varepsilon(z,p)dV_k(z,p) - \int_{U \times \mathbf{R}^{nN}} f(z,p)dV_k(z,p)| \tag{2.4}$$

$$\leq \int_{U\times \mathbf{R}^{nN}} |f_\varepsilon(z,p) - f(z,p)| dV_k(z,p)$$
$$= \int_{U\times \mathbf{R}^{nN}} |f(z,p)||\iota(\varepsilon(1+|p|^q))| dV_k(z,p)$$
$$\leq \int_{F_\varepsilon} |f(z,p)| dV_k$$
$$\leq \mu_1 \int_{F_\varepsilon} (1+|p|^q) dV_k,$$

where $F_\varepsilon = U \times \{p; 1+|p|^q \geq \varepsilon^{-1}\}$. Since

$$\varepsilon V_k(F_\varepsilon) \leq \int_{F_\varepsilon} (1+|p|^q) dV_k \leq C \int_{U\times \mathbf{R}^{nN}} (1+|p|^r) dV_k \leq K_0 \tag{2.5}$$

for some constant K_0 independent of k, we have $V_k(F_\varepsilon) < K_0\varepsilon$. Furthermore by Hölder's inequality

$$\int_{F_\varepsilon} (1+|p|^q) dV_k \leq V_k(F_\varepsilon)^{1-q/r} \left(\int_{F_\varepsilon} (1+|p|^q)^{r/q} dV_k\right)^{q/r} \leq K_1 \varepsilon^{1-q/r}.$$

Hence we have that, as $\varepsilon \to 0$,

$$\int_{U\times \mathbf{R}^{nN}} f_\varepsilon(z,p) dV_k(z,p) \longrightarrow \int_{U\times \mathbf{R}^{nN}} f(z,p) dV_k(z,p)$$

uniformly with respect to k.

On the other hand, since $f_\varepsilon(z,p) \in C_0^0(U \times \mathbf{R}^{nN})$, we have

$$\lim_{k\to\infty} \int_{U\times \mathbf{R}^{nN}} f_\varepsilon(z,p) dV_k(z,p) = \int_{U\times \mathbf{R}^{nN}} f_\varepsilon(z,p) dV(z,p). \tag{2.6}$$

By Proposition 2.1 and (2.3) we have

$$\left|\int_{U\times \mathbf{R}^{nN}} f_\varepsilon(z,p) dV_k(z,p)\right| \leq \int_{U\times \mathbf{R}^{nN}} |f(z,p)| dV_k(z,p)$$
$$\leq \mu_1 \int_{U\times \mathbf{R}^{nN}} (1+|p|^q) dV_k \leq K_0 \mu_1,$$

where K_0 is as in (2.5). This and (2.6) imply

$$\left|\int_{U\times \mathbf{R}^{nN}} f_\varepsilon(z,p) dV(z,p)\right| \leq K_1 \mu_1.$$

Without loss of generality we may assume that f is nonnegative, and then we have $f_\varepsilon(z,p) < f_{\varepsilon'}(z,p)$ whenever $\varepsilon > \varepsilon'$. Hence by the monotone convergence theorem we have, as $\varepsilon \to 0$,

$$\int_{U\times \mathbf{R}^{nN}} f_\varepsilon(z,p) dV(z,p) \longrightarrow \int_{U\times \mathbf{R}^{nN}} f(z,p) dV(z,p).$$

Thus we have the conclusion by the use of a standard fact in iterated limits. Q.E.D.

Let V be an n-varifold in $\Omega \times \mathbf{R}^n$. The first variation of J for ν, which is denoted by $\delta J[V](\phi)$, is defined as

$$\delta J[V](\phi) = \int_{U\times \mathbf{R}^{nN}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} F_{p_\alpha^i}(p) \left(\frac{\partial \phi^i}{\partial x^\alpha}(z) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(z) p_\alpha^j\right) dV(z,p) \tag{2.7}$$

We say that V has locally bounded first variation of J in Ω if for each $W \subset\subset \Omega$ and each $\phi = (\phi^1, \dots, \phi^N) \in C_0^1(\Omega; \boldsymbol{R}^N)$ with $\mathrm{spt}\phi \subset W$ there exists a constant $C > 0$ such that $|\delta J[V](\phi)| \leq C \sup |\phi|$. Note that $\delta J[V]$ defines an $\boldsymbol{R}^N$-valued Radon measure. Its total variation is denoted by $\|\delta J[V]\|$.

Theorem 2.3 *Let V be a L^r pre-graph type n-varifold for some $r > q$. Suppose that V has locally bounded first variation of J in Ω. Then V satisfies*

$$\int_{\boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} F_{p_\alpha^i}(p)(p_\alpha^j - p_{0\alpha}^j) d\eta_V^{(z)}(p) = 0 \qquad (i,\ j = 1, \dots, N)$$

for $\mathcal{H}^n$-a.e. $z \in \mathrm{spt}\ \mu_V$, where p_0 denotes the matrix which describs the approximate tangent space of μ_V at z, namely, $T_z\mu_V = \{y = p_0 x\}$.

Proof. Since $\mathrm{spt}\ \mu_V$ is countably n-rectifiable, $\Theta^n(\mu_V, z) =: \theta(z)$ exists for μ_V-a.e. $z \in U$. For such a z we have

$$V_{z,\lambda}(B_\rho(z) \times \boldsymbol{R}^{nN}) := (\lambda^{-n}(\eta_{z,\lambda})_\sharp V \llcorner \eta_{z,\lambda}(U))(B_\rho(z) \times \boldsymbol{R}^{nN}) \to \rho^n \omega_n \theta(z)$$

as $\lambda \to 0$, where $\eta_{z,\lambda}(\zeta) = \lambda^{-1}(\zeta - z)$. Thus, passing to a subsequence if necessary, $V_{z,\lambda}$ converges to a varifold C as $\lambda \to 0$.

Taking a function depending only z as a test function in the limiting procedure, we have $\mu_C = \theta_0(z)\mathcal{H}^n \llcorner T_z\mu_V$.

Since V has locally bounded first variation of J in Ω, we have

$$\lim_{\rho \searrow 0} \rho^{1-n} \|\delta J[V]\|(B_\rho(z)) = 0. \tag{2.8}$$

Noting that $\sum_{\alpha=1}^n \sum_{i=1}^N F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(z) + \sum_{j=1}^N \frac{\partial \phi^i}{\partial y^j}(z) p_\alpha^j)$ is continuous in $\boldsymbol{R}^{n+N} \times \boldsymbol{R}^{nN}$ and q growth order with respect to p, we have by Proposition 2.2 and (2.8), for each $R > 0$ and $\phi \in C_0^1(B_R(0); \boldsymbol{R}^{n+N})$ with $|\phi| \leq 1$,

$$\begin{aligned}
&|\delta J[C](\phi)| \\
= &\ |\int_{\boldsymbol{R}^{n+N} \times \boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(\zeta) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(\zeta) p_\alpha^j) dC(\zeta, p)| \\
= &\ \lim_{\lambda \to 0} |\int_{\boldsymbol{R}^{n+N} \times \boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(\zeta) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(\zeta) p_\alpha^j) dV_{z,\lambda}(\zeta, p)| \\
= &\ \lim_{\lambda \to 0} \lambda^{-n} |\int_{U \times \boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(\lambda^{-1}(\zeta - z)) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(\lambda^{-1}(\zeta - z)) p_\alpha^j) dV(\zeta, p)| \\
= &\ \lim_{\lambda \to 0} \lambda^{-n} |\int_{U \times \boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} F_{p_\alpha^i}(p)\lambda(\frac{\partial}{\partial x^\alpha}(\phi^i(\lambda^{-1}(\zeta - z))) \\
&\qquad + \sum_{j=1}^{N} \lambda \frac{\partial}{\partial y^j}(\phi^i(\lambda^{-1}(\zeta - z))) p_\alpha^j) dV(\zeta, p)| \\
= &\ \lim_{\lambda \to 0} \lambda^{1-n} |\delta J[V](\phi(\lambda^{-1}(\cdot - z)))| \\
\leq &\ R^{n-1} \liminf_{\lambda \to 0} (\lambda R)^{1-n} \|\delta J[V]\|(B_{\lambda R}(z)) = 0.
\end{aligned} \tag{2.9}$$

Let φ be a function in $C_0^1(\boldsymbol{R}^{n+N})$, and we test a vector field $\varphi(\zeta)\Delta_{ij}(y-p_0x)$, where Δ_{ij} is the $NbyN$ matrix such that (i,j) element is 1 and others are 0. Then (2.9) implies

$$\begin{aligned}
0 &= \int_{\boldsymbol{R}^{n+N}\times\boldsymbol{R}^{nN}}[\sum_{\alpha=1}^{n}\sum_{k=1}^{N}F_{p_\alpha^k}(p)(\frac{\partial\varphi}{\partial x^\alpha}(\zeta)[\Delta_{ij}(y-p_0x)]^k+\varphi(\zeta)\delta^{ik}(-p_{0\alpha}^j)) \\
&\quad +\sum_{l=1}^{N}(\frac{\partial\varphi}{\partial y^l}(\zeta)[\Delta_{ij}(y-p_0x)]^k p_\alpha^l+\varphi(\zeta)\delta^{ik}\delta^{jl}p_\alpha^l)]dC(z,p) \\
&= \int_{\boldsymbol{R}^{n+N}}\int_{\boldsymbol{R}^{nN}}[\sum_{\alpha=1}^{n}\sum_{k=1}^{N}F_{p_\alpha^k}(p)(\frac{\partial\varphi}{\partial x^\alpha}(\zeta)[\Delta_{ij}(y-p_0x)]^k+\varphi(\zeta)\delta^{ik}(-p_{0\alpha}^j)) \\
&\quad +\sum_{l=1}^{N}(\frac{\partial\varphi}{\partial y^l}(\zeta)[\Delta_{ij}(y-p_0x)]^k p_\alpha^l+\varphi(\zeta)\delta^{ik}\delta^{jl}p_\alpha^l)]d\eta_C^{(\zeta)}d\mu_C.
\end{aligned}$$

Since spt $\mu_C=T_z\mu_V=\{y=p_0x\}$, the terms having $y-p_0x$ vanish and thus

$$\int_{\boldsymbol{R}^{n+N}}\varphi(\zeta)\int_{\boldsymbol{R}^{nN}}\sum_{\alpha=1}^{n}F_{p_\alpha^i}(p)(-p_{0\alpha}^j+p_\alpha^j)d\eta_C^\zeta d\mu_C=0.$$

Since φ is arbitrary, we have

$$\int_{\boldsymbol{R}^{nN}}\sum_{\alpha=1}^{n}F_{p_\alpha^i}(p)(p_\alpha^j-p_{0\alpha}^j)d\eta_C^{(\zeta)}=0.$$

It is not difficult to show $\eta_C^{(\zeta)}=\eta_V^{(z)}$ for μ_C-a.e. $\zeta\in\boldsymbol{R}^{n+N}$. Thus the proof is complete. Q.E.D.

3 Main Theorem

Suppose that $u\in L^\infty((0,\infty);W^{1,q}(\Omega))\cap\bigcup_{T>0}W^{1,\tilde q}((0,T)\times\Omega)$ is a weak solution to (1.1)–(1.3). There is a rectifiable varifold $\boldsymbol{v}(G_{u(t,\cdot)})$ in U for $\mathcal{L}^1$-a.e. t. By (1.4) and Proposition 2.1 we have, for each $\phi(z)=\phi(x,y)\in C_0^1((0,\infty)\times U)$ (we use notations x and $z=(x,y)$ for variables in Ω and $U=\Omega\times\boldsymbol{R}^N$, respectively),

$$(3.1)\quad \begin{aligned}
&\sum_{i=1}^{N}\int_0^\infty\{\int_\Omega u_t^i(t,x)\phi^i(t,x,u(t,x))dx \\
&+\int_{U\times\boldsymbol{R}^{nN}}\sum_{\alpha=1}^{n}\sum_{i=1}^{N}F_{p_\alpha^i}(p)(\frac{\partial\phi^i}{\partial x^\alpha}(t,z)+\sum_{j=1}^{N}\frac{\partial\phi^i}{\partial y^j}(t,z)p_\alpha^j)dV_t(z,p)\}dt=0,
\end{aligned}$$

where $V_t=\boldsymbol{v}(G_{u(t,\cdot)},\theta_0)$. Conversely suppose that a function u and a general varifold V_t with a parameter $t\in(0,\infty)$ satisfy (3.1). Then u is a weak solution to (1.1) if

$$(3.2)\qquad V_t=\boldsymbol{v}(G_{u(t,\cdot)},\theta_0)\quad\text{for}\quad\mathcal{L}^1\text{-a.e. }t.$$

Let $u^h(t,x)$ and $\overline{u}^h(t,x)$ be approximate solutions constructed in Section 1. By Proposition 2.1 there exists a one parameter family of graphic rectifiable varifolds

$$V_t^h=\boldsymbol{v}(G_{\overline{u}^h(t,\cdot)},\theta_0).$$

Further by Proposition 2.1 we can rewrite (1.5) as (3.1): for each $\psi(t) \in C_0^\infty(0,\infty)$ and $\phi(z) \in C_0^\infty(U)$

$$(3.3) \quad \sum_{i=1}^{N} \int_0^\infty \{ \int_\Omega (u_t^h)^i(t,x)\phi^i(t,x,\overline{u}^h(t,x))dx$$
$$+ \int_{U\times \boldsymbol{R}^{nN}} [\sum_{\alpha=1}^{n}\sum_{i=1}^{N} F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(t,z) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(t,z)p_\alpha^j)]dV_t^h(z,p)\}dt = 0.$$

We have by Proposition 2.1 2)

$$(3.4) \qquad \operatorname*{ess.\,sup}_{t>0} | \int_{U\times \boldsymbol{R}^{nN}} \beta(z,p)dV_t^h(z,p)| \le K \sup|\beta|$$

for any $\beta(z,p) \in C_0^0(U \times \boldsymbol{R}^{nN})$.

The following proposition can be obtained by the use of (3.4) in the standard compactness argument (compare to Proposition 4.3 of [3]).

Proposition 3.1 *There exists a subsequence of $\{V_t^h\}$ (still denoted by $\{V_t^h\}$) and a one parameter family of varifolds V_t in U, for $t \in (0,\infty)$, such that, for each $\psi(t) \in L^1(0,\infty)$ and $\beta(z,p) \in C_0^0(U \times \boldsymbol{R}^{nN})$,*

$$\lim_{h\to 0} \int_0^\infty \psi(t) \int_{U\times \boldsymbol{R}^{nN}} \beta(z,p)dV_t^h(z,p)dt = \int_0^T \psi(t) \int_{U\times \boldsymbol{R}^{nN}} \beta(z,p)dV_t(z,p)dt.$$

If we could show V_t as in Proposition 3.1 satisfies (3.2), the we would arrive our final destination. However we have not yet succeeded it. Instead in this article we show the following identity.

Theorem 3.2 *Suppose that $\{\| \overline{u}^h \|_{L^\infty((0,\infty);W^{1,r}(\Omega))}\}$ is uniformly bounded with respect to h for some $r > q$. Then, for $\mathcal{L}^1$-a.e. t, and for $\mathcal{H}^n$-a.e. $z \in G_{u(t,\cdot)}$,*

$$\int_{\boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} F_{p_\alpha^i}(z,p)(\frac{\partial u^j(z)}{\partial x^\alpha} - p_\alpha^j)d\eta_V^{(z)}(p) = 0 \qquad (i,\ j = 1,\dots,N).$$

4 Proof of the Main Theorem

Since $\{\| \overline{u}^h \|_{L^\infty((0,\infty);W^{1,r}(\Omega))}\}$ is uniformly bounded with respect to h, we obtain the following proposition in the same way as in the proof of Proposition 2.2.

Proposition 4.1 *Suppose that $q < r$. Let $f(z,p)$ be a continuous function on $U\times \boldsymbol{R}^n$ such that (2.3) holds. Let T be any positive number. Then, if $\{V_t^h\}$ and V_t are as in Proposition 3.1, for each $\psi(t) \in L^1(0,T)$ we have*

$$\lim_{k\to\infty} \int_0^T \psi(t) \int_{U\times \boldsymbol{R}^{nN}} f(z,p)dV_t^h(z,p)dt = \int_0^T \psi(t) \int_{U\times \boldsymbol{R}^{nN}} f(z,p)dV_t(z,p)dt.$$

Note that the following proposition holds.

Proposition 4.2 *The function* u *of Proposition* 1.1 *and* V_t *of Proposition* 3.1 *satisfy* (3.1).

Proof. Possibly passing to further subsequences, we have by Proposition 1.1 5) and 6) that the first integral of (3.3) converges to that of (3.1) as $h \to 0$. Since, in (3.3), spt $\psi \subset (0,T)$ for some T, we apply Proposition 4.1 to the case that

$$f(z,p) = \sum_{i=1}^{N}[\sum_{\alpha=1}^{n} F_{p^i_\alpha}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(z) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(z)p^j_\alpha)]. \tag{4.1}$$

Then the second integral of (3.3) converges to that of (3.1) as $h \to 0$. Q.E.D.

Lemma 4.3 *For each* $\psi \in L^\infty(0,T)$ *and* $\phi \in C^0_0(U)$ *we have*

$$\int_0^T \psi(t) \int_\Omega \phi(x, u(t,x))dxdt = \int_0^T \psi(t) \int_{U\times \mathbf{R}^{nN}} \phi(z)dV_t(z,p)dt.$$

Proof. It follows from Proposition 1.1 8) that, for any $\psi \in L^\infty(0,T)$,

$$\lim_{h\to 0} \int_0^T \psi(t) \int_\Omega \phi(x, \overline{u}^h(t,x))dxdt = \int_0^T \psi(t) \int_\Omega \phi(x, u(t,x))dxdt. \tag{4.2}$$

On the other hand, since Proposition 2.1 4) implies

$$\int_\Omega \phi(x, \overline{u}^h(t,x))dx = \int_{U\times \mathbf{R}^{nN}} \phi(z)dV^h_t(z,p),$$

we have by Proposition 2.2

$$\lim_{h\to 0} \int_0^T \psi(t) \int_\Omega \phi(x, \overline{u}^h(t,x))dxdt = \int_0^T \psi(t) \int_{U\times \mathbf{R}^{nN}} \phi(z)dV_t(z,p)dt. \tag{4.3}$$

Thus the conclusion follows from (4.2) and (4.3). Q.E.D.

Lemma 4.4 μ_{V_t} *and* $(\mathcal{H}^n \llcorner G_{u(t,\cdot)})\llcorner\theta_0$ *are mutually absolutely continuous for* $\mathcal{L}^1$*-a.e.* $t \in (0,\infty)$.

This lemma implies in particular that spt $\mu_{V_t} =$ spt $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$.

Proof. We put $\mu_t = (\mathcal{H}^n \llcorner G_{u(t,\cdot)})\llcorner\theta_0$. By (2.1) we have

$$\int_\Omega \phi(x,u(t,x))dx = \int_U \phi(z)|M(Du(\pi(z)))|^{-1}d(\mathcal{H}^n \llcorner G_{u(t,\cdot)}) = \int_U \phi(z)d\mu_t..$$

Then we have by Lemma 4.3 that, for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$,

$$\int_U \phi(z)d\mu_t = \int_U \phi(z)(\int_{\mathbf{R}^{nN}} d\eta^{(z)}_{V_t}(p))d\mu_{V_t}$$

for each $\phi \in C^0_0(U)$. This means, for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$,

$$\mu_t(A) = \int_A (\int_{\mathbf{R}^{nN}} d\eta^{(z)}_{V_t}(p))d\mu_{V_t} \tag{4.4}$$

for each Borel set $A \subset U$.

When a Borel set A satisfies $\mu_{V_t}(A) = 0$, we have $\mu_t(A) = 0$ by (4.4). Thus $\mu_t = (\mathcal{H}^n \llcorner G_{u(t,\cdot)}) \llcorner \theta_0$ is absolutely continuous with respect to μ_{V_t} for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$.

Conversely, when $\mu_t(A) = 0$, we have $V_t(A \times \boldsymbol{R}^{nN}) = 0$ for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$ by (4.4). Then μ_{V_t} is absolutely continuous with respect to $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$ for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$. Q.E.D.

By Theorem 2.3 the proof of Theorem 3.2 is complete if we show the following lemma.

Lemma 4.5 *V_t has locally finite first variation of J in Ω for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$.*

Proof. Let T be any positive number and let W be any open set in U such that $\Omega' := \pi(W) \subset\subset \Omega$. Suppose that $\phi \in C_0^1((0, \infty) \times U; \boldsymbol{R}^N)$ satisfies spt $\phi \subset (0, T) \times W$. By (3.3) and the definition of $\delta J[V_t^h]$ we have

$$\int_0^\infty \delta J[V_t^h](\phi) dt = -\sum_{i=1}^N \int_0^\infty \int_\Omega (u_t^h)^i(t, x) \phi^i(t, x, \overline{u}^h(t, x)) dx dt. \tag{4.5}$$

Thus by Proposition 1.1 1)

$$|\int_0^T \delta J[V_t^h](\phi) dt| \le C_{T,W} \sup |\phi|,$$

where $C_{T,W}$ is a constant independent of h. This inequality implies that the linear functional

$$C_0^1((0, \infty) \times U; \boldsymbol{R}^N) \ni \phi \longmapsto \int_0^\infty \delta J[V_t^h](\phi) dt \in \boldsymbol{R}$$

has a unique extension to a functional L_h on $C_0^0((0, \infty) \times U; \boldsymbol{R}^N)$ and that

$$|L_h \phi| \le C_{W,T} \sup |\phi|.$$

By the Banach-Alaoglu theorem and the Banach-Steinhaus theorem there exists a subsequence (still denoted by $\{L_h\}$) and a functional L such that $L_h \phi$ converges to $L\phi$ for each $\phi \in C_0^0((0, \infty) \times U; \boldsymbol{R}^N)$ and for ϕ with spt $\phi \subset (0, T) \times W$

$$|L\phi| \le C_{T,W} \sup |\phi|. \tag{4.6}$$

By the Riesz representation theorem there are a Radon measure ν on $(0, \infty) \times U$ and a ν measurable $\boldsymbol{R}^N$ valued function v with $|v| = 1$, ν-a.e., such that

$$L\phi = \int_{(0,\infty) \times U} v \cdot \phi d\nu.$$

Inequality (4.6) implies

$$\nu((0, T) \times W) = \sup\{\int_{(0,T) \times W} v \cdot \phi d\nu; \phi \in C_0^0((0, T) \times W; \boldsymbol{R}^N),\ |\phi| \le 1\} \le C_{T,W}.$$

We define $\rho(A) = \nu(A \times W)$ for a Borel set $A \subset (0, T)$. It is well-known that for ρ-a.e. $t \in (0, T)$ there exists a probability Radon measure m_t on W such that $d\nu = dm_t(z) d\rho(t)$ (refer to [2, Chapter 1, Theorem 10]). Then for $\phi \in C_0^0((0, T) \times W; \boldsymbol{R}^N)$

$$L\phi = \int_0^T \int_W v \cdot \phi dm_t d\rho. \tag{4.7}$$

Let ϕ be a vector field in $C_0^1(U;\boldsymbol{R}^N)$. Applying Proposition 4.1 for f as in (4.1), we have for each $T>0$ and $\psi \in L^1(0,T)$

$$\lim_{h\to 0}\int_0^T \psi(t)\delta J[V_t^h](\phi)dt = \int_0^T \psi(t)\delta J[V_t](\phi)dt. \tag{4.8}$$

Thus, since $L_h(\psi\phi) = \int_0^\infty \psi(t)\delta J[V_t^h](\phi)dt$ and $\lim_{h\to 0} L_h\phi = L\phi$, we have for $\psi \in C_0^0(0,\infty)$ and $\phi \in C_0^1(U;\boldsymbol{R}^N)$

$$L(\psi\phi) = \int_0^\infty \psi(t)\delta J[V_t](\phi)dt. \tag{4.9}$$

For ϕ with spt $\phi \subset W$ and for ψ with spt $\psi \subset (0,T)$ we have by (4.7) and (4.9) that

$$\int_0^\infty \psi(t)\delta J[V_t](\phi)dt = \int_0^T \psi(t)\int_W v\cdot\phi dm_t d\rho. \tag{4.10}$$

It follows from Proposition 1.1 1), (4.5), and (4.8) that there exists a constant $C'_{T,W}$ such that for each $\psi \in L^2(0,T)$

$$\left|\int_0^T \psi(t)\delta J[V_t](\phi)dt\right| \le C'_{T,W}\ \|\psi\|_{L^2(0,T)} \sup|\phi|. \tag{4.11}$$

Note that $m_t(W) = \sup\{\int_W v\cdot\phi dm_t; \phi \in C_0^0(W;\boldsymbol{R}^N),\ |\phi|\le 1\}$. Since m_t is a probability measure,

$$\sup\{\int_W v\cdot\phi dm_t; \phi \in C_0^0(W;\boldsymbol{R}^N),\ |\phi|\le 1\} = 1. \tag{4.12}$$

Combining (4.10), (4.11), and (4.12), we have

$$\left|\int_0^T \psi(t)d\rho(t)\right| \le C'_{T,W}\ \|\psi\|_{L^2(0,T)}\ .$$

Thus the functional $\psi \mapsto \int_0^T \psi(t)d\rho(t)$ is bounded in $L^2(0,T)$. Then there exists a function $\tilde\rho \in L^2(0,T)$ such that $\int_0^T \phi(t)d\rho(t) = \int_0^T \psi(t)\tilde\rho(t)dt$ for any $\psi \in L^2(0,T)$. Putting $\tilde m_t = \tilde\rho(t)m_t$, we have by (4.10) that

$$\int_0^T \psi(t)\delta J[V_t](\phi)dt = \int_0^T \psi(t)\int_W v\cdot\phi d\tilde m_t dt.$$

Thus for $\mathcal{L}^1$-a.e. $t\in(0,T)$

$$\delta J[V_t](\phi) = \int_W v\cdot\phi d\tilde m_t.$$

Since T is arbitrary, we have the conclusion. Q.E.D.

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