非干渉計測法を用いたフレネル回折強度からの物体再生の研究

（課題番号　14550038）

平成１４年度～平成１６年度科学研究費補助金（基盤研究（Ｃ）（２））
研究成果報告書

平成１７年５月

研究代表者　　中　島　伸　治
（静岡大学工学部教授）

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はしがき

この報告書は、平成１４年度～平成１６年度に実行された科学研究費基盤研究（C）（２）「非干渉計測法を用いたフレネル回折強度からの物体再生の研究」の成果をまとめたものである。光のように高い振動数をもつ波動によって物体構造・形状測定を行う際、通常、参照波を用いた干渉縞測定法が使われるが、本研究は、ガウスフィルタ関数を物体波に掛けて物体の構造・形状をより簡便に再生する新しい方法の確立を目的としていた。この報告書では、研究成果として平成１４年度～平成１６年度の３年間に発表した論文３編と現在投稿中の論文１編をまとめてある。

最初の論文は、レーザ光を用いて実際に位相物体を再生する実験結果であり、レーザ光のガウスビームをガウスフィルタの代わりに使用することで実際の物体位相を波長の約 1/10 の精度で再生可能であることを示している。２番目の論文は、従来、ガウスフィルタによる位相回復では２次元物体に対して必要であった３回の２次元回折強度分布測定を、２回に減らすことができる新しい計算アルゴリズムを提案している。しかし、このアルゴリズムは使用できる画像の分点数に制限があるので、さらなる改善が必要である。３番目の論文は、最近、研究が盛んな X 線による回折強度からの物体再生の分野へ、本方法を適応するシステムの提案を行っている。X 線の領域ではレーザ光のように適当なガウスビームを作ることは難しいので、ここでは微小円形開口に X 線を照射し、その遠方回折分布であるエアリーディスクパターンの中心部分をガウスビームとして近似的に用いることを提案し、計算機シミュレーションでその有効性を示した。４番目の論文では、物体からのフレネル回折面上でスリット開口を走査し、その回折光強度を測定することで物体に間接的にガウス関数をかけたデータを得る新しいシステムを提案している。さらに、この論文ではフーリエ変換を用いた新しい位相計算法も提案し、２次元複素物体の計算機シミュレーションによってその有効性を示している。

本研究においては、干渉測定を用いない新しい物体再生システムをいくつか提案した。しかし、実際に実験を行うことは、ガウスビームを用いた光の実験だけに留まっており、新たに提案した円形開口照明システムや、走査スリットを用いたシステムについては、時間的および予算的問題で実験を行えなかった。今後これらの問題が解決した段階で、実証実験を行う予定である。

平成１７年５月

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研究組織

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交付決定額（配分額）　　　　　　　　　　　　　　　　　（金額単位：千円）

	直接経費	間接経費	合　計
平成１４年度	１，３００	０	１，３００
平成１５年度	１，３００	０	１，３００
平成１６年度	５００	０	５００
総　計	３，１００	０	３，１００

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CONTENTS

Phase retrieval from experimental far-field intensities using a Gaussian beam

Abstract

The noniterative phase retrieval method by use of Gaussian filtering is applied to the reconstruction of phase objects from experimental far-field intensities. In this method, the complex amplitude of transmitted light through an object is reconstructed from three far-field intensities, which are measured with the modulation of the object by laterally shifted and unshifted Gaussian filters. In the experiment, the amplitude of a Gaussian beam illuminating objects is utilized as a Gaussian filter, and, as the phase objects, a converging lens with a small exit pupil and a plastic fiber immersed in optical adhesive are used. The experimental results show that the Gaussian beam of a laser is capable of retrieving the phases of those objects with the accuracy of the range from $\sim 1/10$ to $1/4$ of the laser's wavelength.

I. INTRODUCTION

Phase measurements of wave fields are of importance in various areas of science and engineering. In optics, interferometric techniques are generally used to determine the phase. For high-frequency waves such as x-rays, electrons, and atomic waves, the phase measurement using interferometric techniques is difficult because, in such waves, a coherent reference wave is barely obtained and some severe conditions on measurement system are required when observing interference fringes. Thus there have been studies of noniterferometric methods of retrieving a phase distribution from the intensities of a wave field.[1–8] One approach to the phase retrieval from intensities is the use of iterative algorithms, in which the phase is determined as an iterative solution under some constraints, that is, measured intensities and *a priori* information of a wave field. In iterative algorithms, there are several types for using some constraints, for example, image and far-field intensities of an object,[9] *a priori* information (e.g, non-negativity and/or support of an object) and far-field intensity,[10,11] two far-field intensities of an object modulated and unmodulated by an exponential filter,[12] far-field intensities of an object and the object multiplied by a window function[13], and far-field intensities of an object and its small-phase solution used as a starting point for an iterative algorithm.[14] The demonstration of iterative algorithms has been shown in the phase measurement, for example, in electron microscopy[2] and the x-ray imaging for noncrystalline samples.[15] The other approach to the phase retrieval is the use of analytic (noniterative) methods, for example, phase retrieval based on the properties of wave functions pertaining to entire functions,[3,7,8] and phase recovery by solving the transport-of-intensity equation.[16–19] The intensity propagation-based method has been applied to phase measurements using x-rays,[20] electrons,[21] and neutrons.[22]

Recently we proposed a phase retrieval method by use of Gaussian filtering,[23,24] which is based on the properties of entire functions of a wave field. In this method, the phase of a wave field in a plane is retrieved by solving simultaneous equations consisting of unknown coefficients in the Fourier-series of the phase and intensity data in the far-field plane for an object measured by modulation of the object with a known Gaussian function and with its shifted functions along horizontal and vertical directions in rectangular coordinates.

In this paper we demonstrate the application of the method using the modulation with Gaussian filtering to the reconstruction of complex amplitude in the object plane from its Fourier intensity data collected in a laboratory experiment. Two types of objects are treated here: one is composed of a converging lens and a small circular aperture, and the other is a plastic fiber immersed in optical adhesive. In the present optical system, the amplitude of a Gaussian beam illuminating objects is utilized for modulating the objects as a Gaussian filter. From the experimental results, it is found that the Gaussian beam from a laser is capable of retrieving the phases of the objects with the averaged root-mean-squared errors of the range from $\sim 2\pi/10$ to $2\pi/4$ rad. A brief outline of the phase retrieval method used here is given in Sec. II. Experimental results and discussion are presented in Sec. III, and Sec. IV is the conclusions.

2. Outline of the Phase Retrieval Method

In this section, a summary of the phase retrieval method using a Gaussian beam[24] is described for the sake of readability. An object of two-dimensional complex amplitude transmittance (or reflectance) $f(u, v)$ (which is assumed to be of finite extent) is illuminated by a Gaussian beam with a wavelength λ, of which the complex amplitude is expressed in the form

$$b(u,v) = A\frac{W_0}{W}\exp\left(-\frac{u^2+v^2}{W^2}\right)\exp\left[i\frac{\pi(u^2+v^2)}{\lambda R}\right], \tag{1}$$

where A is a constant complex amplitude, W_0 is the waist radius, and W and R denote the beam radius and the radius of curvature, respectively, on the object plane and are assumed to be known constants. The Gaussian amplitude of the beam is utilized as a Gaussian filter that is necessary for the present phase retrieval. The object reconstruction is based on two far-field intensity measurements. One of the measured data terms is the intensity distribution of the Fourier transform $F_1(x,y)$ of the object illuminated by the Gaussian beam, and the other is the intensity distribution of the Fourier transform $F_2(x,y)$ of the object illuminated by its shifted beam with a horizontal displacement τ (which is assumed to be another known constant). The complex amplitude $F_1(x,y)$ is given by

$$F_1(x,y) = \iint_\sigma f(u,v)b(u,v)\exp\left[-i\frac{2\pi}{d\lambda}(xu+yv)\right]dudv, \tag{2}$$

where unimportant multiplicative constants associated with the diffraction integrals are ignored, d is the normal distance between the $u-v$ and $x-y$ planes, and σ is the extent of the object. With the definition in Eq.(1), the shifted Gaussian beam is represented by

$$b(u-\tau,v) = b(u,v)\exp\left(\frac{2\tau u}{W^2}\right)\exp\left(-i\frac{2\pi\tau u}{\lambda R}\right)\exp\left(-\frac{\tau^2}{W^2}+i\frac{\pi\tau^2}{\lambda R}\right). \tag{3}$$

Equation (3) indicates that the shifted Gaussian beam consists of the original beam, an exponential function, a linear phase factor, and a complex constant. The effect of the exponential function on the Fourier intensity of the object is utilized for phase retrieval. Substituting Eq.(3) into Eq.(2) instead of $b(u,v)$, we obtain

$$F_2(x,y) = \exp\left(-\frac{\tau^2}{W^2}+i\frac{\pi\tau^2}{\lambda R}\right)\iint_\sigma f(u,v)b(u,v)$$

$$\times \exp\left\{-i\frac{2\pi}{d\lambda}[(x+ic+s)u+yv]\right\}dudv, \tag{4}$$

where

$$c=\frac{d\lambda\tau}{\pi W^2}, \tag{5}$$

and

$$s=\frac{d\tau}{R}. \tag{6}$$

The values of c and s can be evaluated from the known constants τ, d, λ, W, and R. Then the relation between $F_1(x,y)$ and $F_2(x,y)$ is written from Eqs.(2) and (4)

$$F_2(x,y)=\exp\left(-\frac{\tau^2}{W^2}+i\frac{\pi\tau^2}{\lambda R}\right)F_1(x+ic+s,y). \tag{7}$$

The function $F_2(x,y)$ corresponds to the function $F_1(x,y)$ displaced along the imaginary axis of the complex plane of its argument x simultaneously with the displacement s along the x axis, except for the complex constant term. Then, let $M(x,y)$ and $\phi(x,y)$ be the modulus and the phase, respectively, of $F_1(x,y)$, i.e.,

$$F_1(x,y)=M(x,y)\exp[i\phi(x,y)]. \tag{8}$$

Expanding the real variable x in Eq.(8) into the complex one, $x+ic$, and using Eqs.(7) and (8), we can obtain a relationship

$$\ln\frac{|F_2(x-s,y)|}{|M\left(x+ic,y\right)|}+\frac{\tau^2}{W^2}=-\mathrm{Im}\,\phi\left(x+ic,y\right), \tag{9}$$

where Im denotes the imaginary part of a complex function. On the left-hand side of Eq.(9) the second term consists of the known constants, and the first term can be calculated from the observed intensity data in the Fourier plane because $|F_2(x-s,y)|$ is the modulus obtained through compensation of the modulus $|F_2(x,y)|$ for its displacement s along the x axis and because $|M\left(x+ic,y\right)|$ is the modulus of the

Fourier transform of the product of the inverse Fourier transform of $M\,(x,y)$ and the exponential function $\exp(2\pi cu)$, as described in the previous paper.[23] We can retrieve the phase distributions along lines parallel to the x axis by solving Eq.(9) with a Fourier series expansion. The phase $\phi(x,y)$ on a set of 1-D lines parallel to the x axis is represented in terms of a Fourier series basis:[24]

$$\phi(x,y) \cong \sum_{n=1}^{N} \Big[a_n(y)\cos\frac{n\pi}{l}x + b_n(y)\sin\frac{n\pi}{l}x\Big], \tag{10}$$

where the observational region of the function $F_1(x,y)$ is designated as $-l < x < l$, and N is sufficiently large. Thus the unknown function $\phi(x,y)$ on a set of 1-D lines parallel to the x axis is represented by the unknown coefficients $a_n(y)$ and $b_n(y)$, $(n = 1, ..., N)$. Substituting relation (10) into Eq.(9), we obtain

$$\ln\frac{|F_2(x-s,y)|}{|M\,(x+ic,y)|} + \frac{\tau^2}{W^2} \cong \sum_{n=1}^{N} \Big[a_n(y)\sin\frac{n\pi}{l}x - b_n(y)\cos\frac{n\pi}{l}x\Big]\sinh\left(\frac{n\pi c}{l}\right). \tag{11}$$

By calculating the left hand side of relation (11) at $2N$ values of x, we obtain $2N$ simultaneous equations from which the unknown coefficients $a_n(y)$ and $b_n(y)$, $(n = 1, ..., N)$ for the set of lines parallel to the x axis can be determined. However, there are unknown constant phase differences among the retrieved phase distributions along those lines. To resolve this ambiguity, we take an additional intensity measurement of the Fourier transform $F_3(x,y)$ with the Gaussian beam shifted in the v axis's direction. From the moduli $|F_1(x,y)|$ and $|F_3(x,y)|$, we determine the constant phase differences. Then the overall phase distribution $\phi(x,y)$ can be determined. Finally, we reconstruct the object function $f(u,v)$ by eliminating the effects of the known Gaussian beam from the inverse Fourier transform of the function consisting of the measured modulus $|F_1(x,y)|$ and the retrieved phase $\phi(x,y)$.

3. Experimental Results and Discussion

We have done the phase retrieval from experimental intensity data for two types of object: (1) a converging lens with a small circular aperture and (2) a plastic fiber immersed in optical adhesive. We first show the reconstruction of the former object. The optical system used in performing the experiment is shown in Fig.1. A He-Ne laser beam of wavelength $\lambda = 0.6328\mu$m is used to illuminate an object consisting of a converging lens L_1 and a 1-mm diam circular aperture situated on the center of the lens. Lens L_2 of focal length $f_2 = 405$mm produces a Fourier transform of the complex amplitude in the plane of the circular aperture that is defined as the object plane. The Fourier intensity data are collected by a charge coupled device (CCD) camera. The video signal is converted to a 256×256 8-bit digital image by using a digital image memory. Calculating the phase retrieval and reconstructing the complex amplitude in the object plane are carried out by a personal computer. The Gaussian beam of the laser used as a Gaussian filter has the radius $W = 0.546$mm and the radius of curvature $R = 455$mm in the place of the object plane. The examples of the object reconstruction are shown in Figs. 2 and 3, which correspond to the cases of the converging lens objects L_1 with focal lengths $f_1 = 202$mm and $f_1 = 253$mm, respectively. Figure 2(a) (or 3(a)) shows the measured Fourier intensity corresponding to the intensity of the complex amplitude $F_1(x, y)$ in Sec.2. Figures 2(b) and (c) (or 3(b) and (c)) show the Fourier intensities of the object shifted with displacement 0.05mm in each direction of u and v axes on the object plane, respectively, instead of shifting the Gaussian beam, because the measurement with shifting the object is easier in this optical system than that with shifting the beam. The amount of the displacement determines the inclination of the exponential function $\exp(2\tau u/W^2)$ in Eq.(3), which is a significant function for the phase retrieval method. Studies of numerical experiments on phase retrieval using

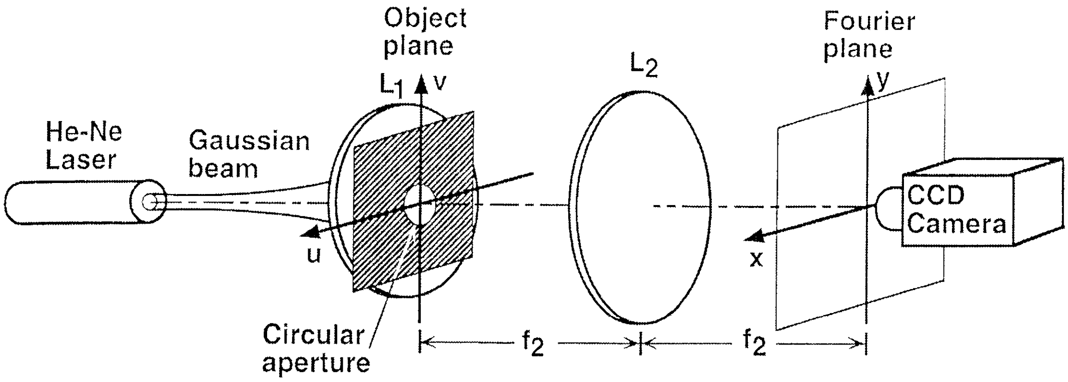


Fig.1. Schematic arrangement of the experiment: f_2 is a focal length of lens L_2.

exponential filtering indicate that suitable values of the term $2\tau/W^2$ are about in the range $0.02\pi/D < (2\tau/W^2) < 0.2\pi/D$, where D denotes the width of an object. In the present experiment, the value of $2\tau/W^2(=0.335)$ is in that range.

The Fourier phase of the object was calculated from three intensities in Figs.2(a), (b), and (c) (or 3(a), (b), and (c)). Then the object was reconstructed by eliminating the effect of the Gaussian beam from the inverse Fourier transform of the complex function with the retrieved phase and the modulus of the intensity in Fig.2(a) (or Fig.3(a)). Figures 2(d) and (e) (or 3(d) and (e)) represent the modulus and phase of the reconstructed object, where these figures show only the central 64×64 points for display purpose. In Fig.4, the cross-sectional profiles of the reconstructed phases along the lines that passes through the center of the reconstructed objects are compared with the phase distributions in the object plane calculated from the focal length f_1 of the lens L_1. The dotted curves in Figs.4(a) and (b) show the calculated phase distributions from the focal lengths $f_1 = 202$mm and $f_1 = 253$mm, respectively. The solid and dashed curves in Fig.4(a) (or Fig.4(b)) show the cross-sectional profiles at the center of the reconstructed phases along the lines parallel to the u and v axes, respectively. In Figs.4(a) and (b), the averaged root-mean-squared (RMS) errors for the cross-sectional profiles of the reconstructed phases are 0.557 rad and 0.168 rad, respectively, within the width of the object. Since the ideal form of the object modulus is a cylindrical shape, it can be seen from Figs.2, 3, and 4 that the reconstruction quality of the object modulus appears to be less accurate than that of the object phase. This fact is due to the following two reasons. One is that it is difficult to reconstruct the edge part of the cylindrical modulus from far-field data with very noisy components for high spatial-frequency. The other is that the Fourier modulus are less influenced by a change of object modulus than by a change of object phase. Conversely, the reconstruction of object modulus is very sensitive to noise in the Fourier modulus. This is a property in the Fourier transform relationship.[4]

Next we present the phase retrieval for the object of a plastic fiber. The transversal cross-section of the object for an optical axis is illustrated in Fig.5. The fiber consists of the core of a refractive index $n_1 = 1.492$ and the cladding of a refractive index $n_2 = 1.402$. The diameters of the fiber and its core are 250μm and 240μm, respectively. The fiber is put between slide glasses, and optical adhesive is poured into an opening surrounding the fiber. The refractive index of the adhesive after hardening is $n_3 = 1.519$. In order to remove unwanted light from measured data, the extent of illuminating light on the fiber is restricted by a slit on the slide glass as shown in Fig.5. The object plane is defined as the plane immediately behind the slide glass on the right hand side in Fig.5. Then the complex amplitude in the object plane is reconstructed by using the same optical system as shown in Fig.1 except for the use of the Fourier transforming lens L_2 of focal length $f_2 = 100$mm in order to obtain higher spatial-frequency components for the fiber object. Since the phase distribution of the fiber in the axial direction can be assumed to be constant, the fiber object can be regarded approximately as a one-dimensional (1-D) object. Thus we here treat the reconstruction of a 1-D complex amplitude corresponding to a two-dimensional

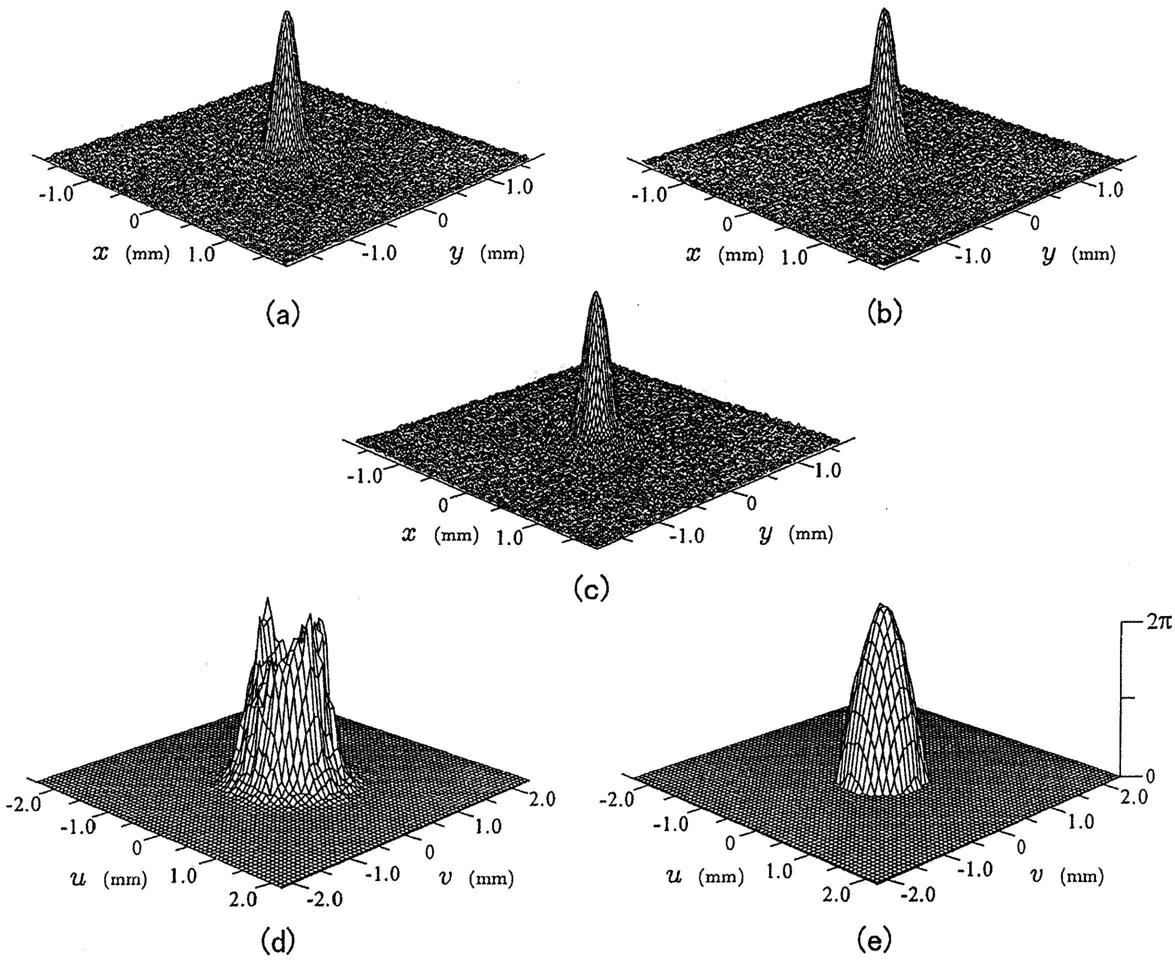


Fig.2. Reconstruction of the object consisting of the converging lens with the focal length $f_1 = 202$ mm and the circular aperture of 1-mm diameter: (a) is the Fourier intensity of the object. (b) and (c) are the Fourier intensities of the object shifted with displacement 0.05 mm in each direction of u and v axes in Fig.1, respectively. (d) and (e) are the modulus and the phase, respectively, of the reconstructed object from the data of (a), (b), and (c).

complex amplitude integrated in the direction of the v axis on the object plane. The 1-D complex amplitude was reconstructed by using a 1-D phase retrieval method (i.e., the same procedure as in Sec.2 except for substituting $y = 0$ into the equations) from 1-D intensity distributions along the x axis of two Fourier intensities of the object unshifted and shifted in the u axis's direction with displacement $\tau = 0.06$mm. Figures 6(a) and 6(b) show the modulus and the phase of the reconstructed complex amplitude in the object plane, respectively. In Fig.6(b), the reconstructed phase is represented only within the extent of the fiber's diameter for display purpose, and the dotted curve shows the phase distribution calculated from the known refractive indexes n_1, n_2, and n_3 in the assumption that the deviation of the light is negligible.[25] The averaged RMS error for the reconstructed phase is 1.76 rad within the extent of the fiber. Although the reconstructed modulus is contaminated by noise, the retrieved phase is very close to the calculated one except for the abrupt variations at the both ends. The abrupt variations appeared at both ends of the calculated phase are due to the influence of the fiber's cladding layer with 5.0μm. Those variations cannot be retrieved in this experiment, because the spatial resolution on the object plane is limited to 9.1μm by apparatuses used here for the measurement of Fourier intensities.

IV. CONCLUSIONS

Phase retrieval from Fourier intensities of objects by use of Gaussian filtering has been demonstrated in a laboratory experiment. The phase retrieval method used here is one of noninterferometric techniques, and is based on measurements of two Fourier intensities of an object modulated with shifted and unshifted Gaussian filters along the transverse direction for the optical axis. Although the Gaussian amplitude distribution

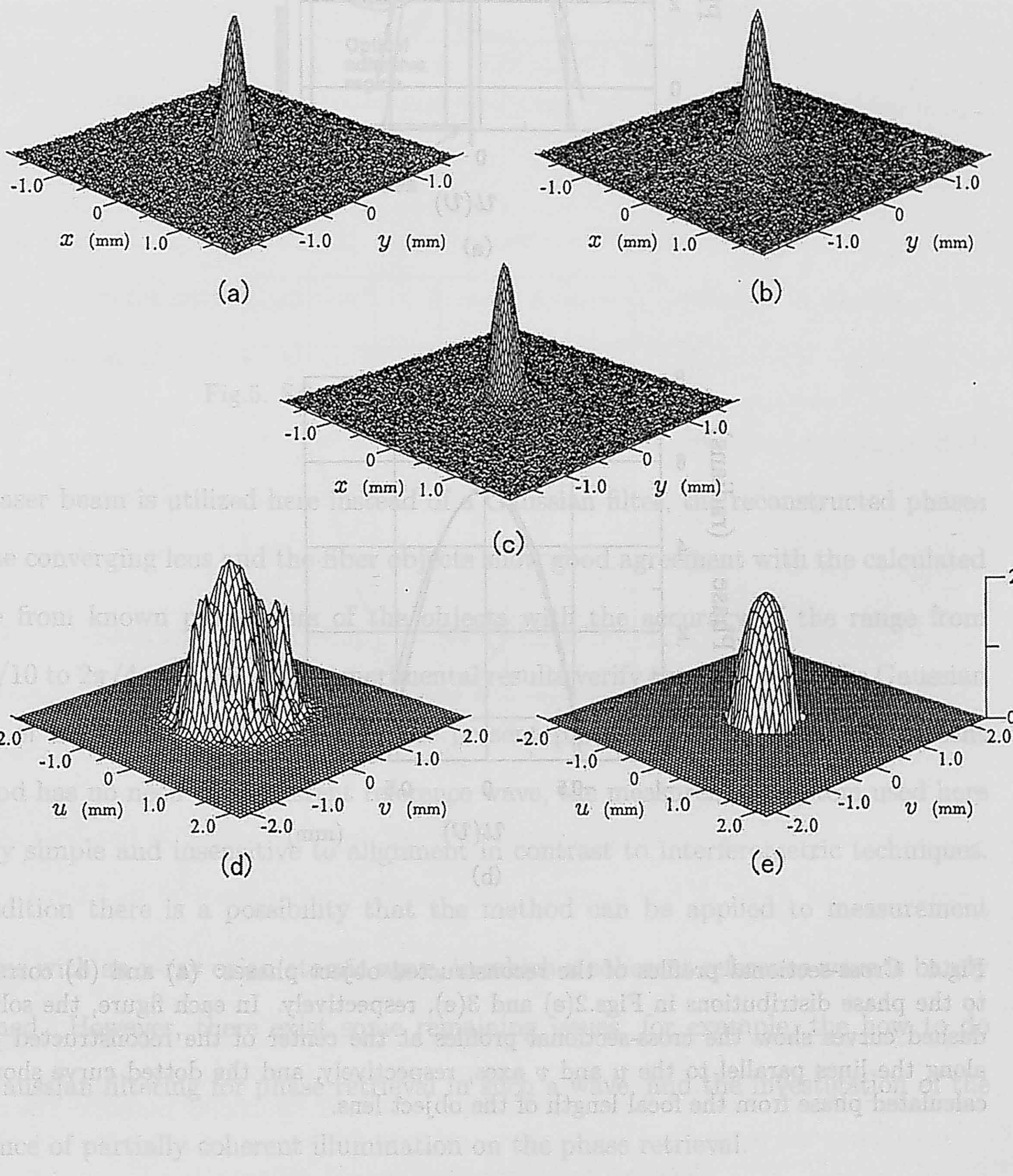
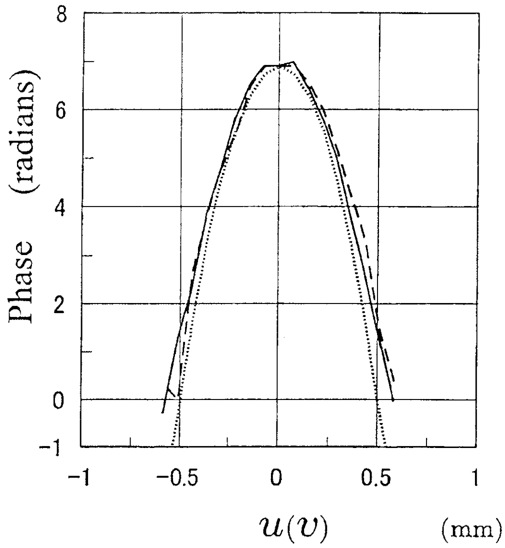
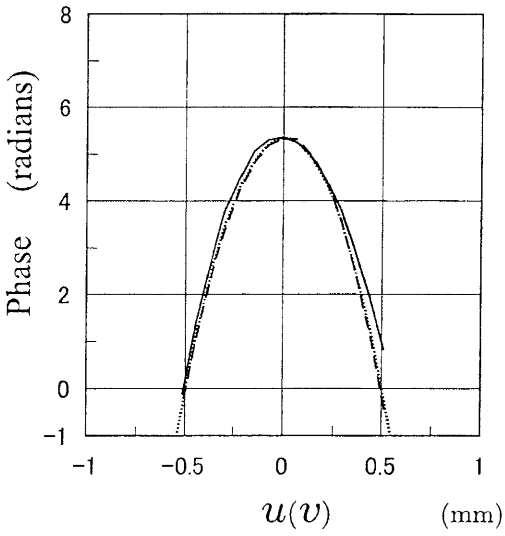


Fig.3. Same as in Fig.2 except that the focal length of the object lens is $f_1 = 253$ mm.



(a)



(b)

Fig.4. Cross-sectional profiles of the reconstructed object phases: (a) and (b) correspond to the phase distributions in Figs.2(e) and 3(e), respectively. In each figure, the solid and dashed curves show the cross-sectional profiles at the center of the reconstructed phases along the lines parallel to the u and v axes, respectively, and the dotted curve shows the calculated phase from the focal length of the object lens.

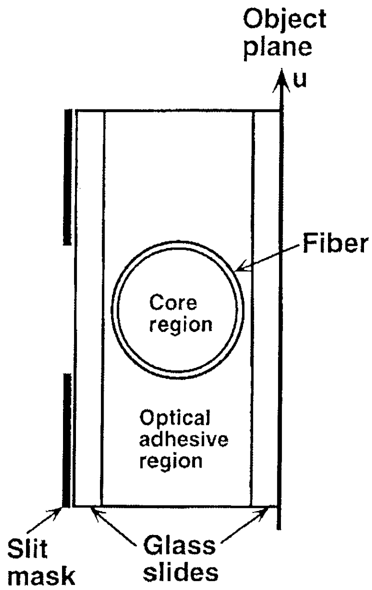
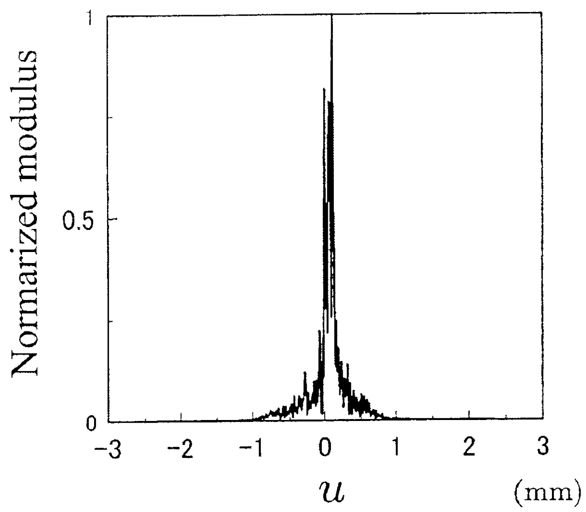
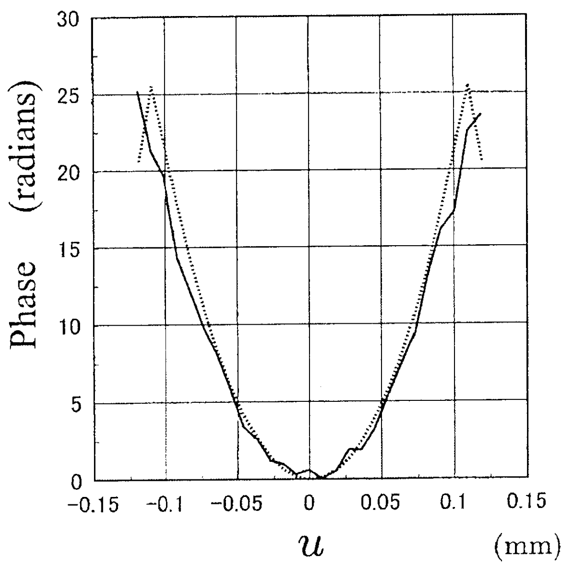


Fig.5. Schematic diagram of the fiber object.

of a laser beam is utilized here instead of a Gaussian filter, the reconstructed phases for the converging lens and the fiber objects show good agreement with the calculated phase from known parameters of the objects with the accuracy of the range from $\sim 2\pi/10$ to $2\pi/4$ rad. Thus the experimental results verify that the use of the Gaussian beam of a laser is very effective in the present phase retrieval. Since the present method has no need of a coherent reference wave, the measurement system used here is very simple and insensitive to alignment in contrast to interferometric techniques. In addition there is a possibility that the method can be applied to measurement systems with an x-ray or an atomic wave, in which a coherent reference wave is barely obtained. However, there exist some remaining issues, for example, the how to do the Gaussian filtering for phase retrieval in such a wave, and the investigation of the influence of partially coherent illumination on the phase retrieval.



(a)



(b)

Fig.6. Reconstruction of the complex amplitude in the object plane of the fiber object: (a) and (b) are the reconstructed modulus and phase, respectively. The reconstructed phase is represented only within the extent of the fiber's diameter for display purpose. The dotted curve shows the phase distribution calculated from the known refractive indexes of the fiber and the surrounding optical adhesive.

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Noniterative two-dimensional phase-retrieval method from two Fourier intensities using an exponential filter

Abstract

A noniterative method of retrieving the two-dimensional phase of a wave field from two intensity measurements is proposed. In the measurements, one records two far-field intensities of the wave field modulated and unmodulated with an exponential filter. The phase retrieval method is based on the solution of the simultaneous equations with unknown coefficients of the two-dimensional discrete Fourier transform for the phase. Then there is no need for the information about the wave field, which is used in iterative phase retrieval methods. The usefulness of this method is shown in computer simulated examples of the reconstruction of two-dimensional complex amplitude objects.

I. INTRODUCTION

Non-interferometric reconstruction of the phase of wave fields is a topic of current interest in a number of areas for object imaging or structure determination by use of various waves such as optical,[1] x-ray,[2] electron,[3] and atomic[4] waves. It is generally difficult that the determination of the phase of a wave field from one measurement of its intensity distribution. To retrieve the phase requires more information about the wave field. Many methods have been developed to solve the phase retrieval problem.[5–12] Iterative methods are widely used, of which there are several types for using some information, for example, two intensities of a wave field in object and far-field planes,[13] *a priori* information (e.g., non-negativity and/or extent of a wave field in an object plane) and far-field intensity,[14,15] and two far-field intensities of a wave field modulated and unmodulated by an exponential filter.[16] However, the use of iterative methods is accompanied by convergence problems, and hence those methods sometimes stagnate in a local minimum solution different from a true one.

On the other hand, there is the other approach to the phase retrieval problem, which is the use of noniterative (analytic) methods, for example, phase retrieval based on the properties of wave functions pertaining to entire functions,[7,11,12] and phase recovery by solving the transport-of-intensity equation.[1,17–20] The main difference between two types of the analytic methods is that the former methods are able to cope with the existence of vortices (i.e., screw dislocations) in the phase distribution in contrast to the latter ones. As one of the former methods, a linear, one-dimensional (1-D) method of retrieving the phase from two far-field intensities of a wave field modulated and unmodulated by an exponential filter in an object plane was proposed.[21,22] In this method, the phase in far-field plane of the wave field is retrieved by solving simultaneous equations from the two far-field intensities. When applying the 1-D method to the two-dimensional (2-D) case, three 2-D intensities measurements are required:[23] one far-field intensity of a wave field in an object plane and two far-field intensities of the wave field modulated with exponential filters along the horizontal and the vertical directions in 2-D coordinates. However, it has been shown that the iterative method[16] is able to retrieve the 2-D phase from only two far-field intensities of a wave field modulated and unmodulated with an exponential filter. So far, there has not been any noniterative method for the phase retrieval from such two far-field intensities.

In this paper, to the best of my knowledge, it is first shown that 2-D phase retrieval can be solved by using a noniterative method from only two far-field intensities obtained with and without an exponential filter. In the noniterative method, the 2-D analytic properties of these two intensity functions are utilized for the solution of the simultaneous equations with unknown coefficients of the 2-D discrete Fourier transform for the phase distribution in the far-field plane. Hence there is no need for the information about the wave field in an object plane (e.g., the extent of the object) , which is used in the iterative method[16] to increase the speed of convergence.

In Section 2, I formulate the 2-D noniterative phase retrieval method. Tests of the method by computer simulation of 2-D complex amplitude objects are represented in Section 3, and Section 4 presents the conclusions.

2. Formulation of the 2-D Phase Retrieval Method

We here consider the phase retrieval from intensities measured in the Fourier transform plane for an object illuminated by a quasi-monochromatic fully spatially coherent light. Although the theoretical development in this section is restricted to the optical diffraction phenomenon, there is a possibility that the present method can be applied

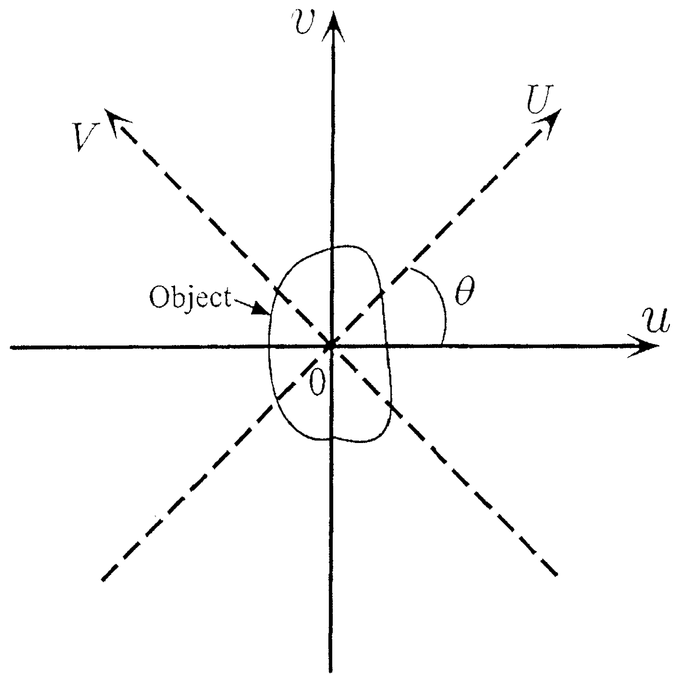

Fig.1. Axes of coordinates used in the formulation of the method: the rotational angle between the object function $f(u, v)$ and the exponential filter $\exp(-2\pi cU)$ is represented by θ.

to an arbitrary phenomenon with a Fourier transform relationship.

We assume that the object has a 2-D complex amplitude transmittance (or reflectance) $f(u, v)$ with a finite extent. The reconstruction of the object function $f(u, v)$ is based on two far-field intensity measurements. One of the measured data terms is the intensity distribution of the Fourier transform $F_1(x, y)$ of the object illuminated by a plane wave, and the other is the intensity distribution of the Fourier transform $F_2(x, y)$ of the object modulated by an exponential filter $\exp(-2\pi cU)$, where c is a known constant and U is the axis of coordinates rotated at an angle θ to the x axis as shown in Fig.1: $U = u\cos\theta + v\sin\theta$. Then the rotated axis for the y axis is given by $V = -u\sin\theta + v\cos\theta$. The angle θ is assumed to be another known constant for the filter. Then the complex amplitudes $F_1(x, y)$ and $F_2(x, y)$ are given by, respectively,

$$F_1(x, y) = \iint_\sigma f(u, v) \exp\left[-i\frac{2\pi}{d\lambda}(xu + yv)\right] du dv, \tag{12}$$

and

$$F_2(x,y) = \iint_\sigma f(u,v)\exp[-2\pi c(u\cos\theta + v\sin\theta)]\exp\left[-i\frac{2\pi}{d\lambda}(xu+yv)\right]dudv, \quad (13)$$

where unimportant multiplicative constants associated with the diffraction integrals are ignored, d is the normal distance between the $u-v$ and $x-y$ planes, λ is the wavelength of the illuminating light, and σ is the extent of the object. Then the relationship between $F_1(x,y)$ and $F_2(x,y)$ is written from Eqs.(1) and (2) as

$$F_2(x,y) = F_1(x - ic_1, y - ic_2) \quad (14)$$

where $c_1[= (cd\lambda)\cos\theta]$ and $c_2[= (cd\lambda)\sin\theta]$ are evaluated from the known constants c, θ, d, and λ. The function $F_2(x,y)$ corresponds to the function $F_1(x,y)$ displaced simultaneously along the imaginary axes of the Four-dimensional complex plane of its arguments x and y. In the previous paper,[23] we regarded the function $F_2(x,y)[= F_1(X - ic, Y)$, where (X,Y) are the rotated coordinates from (x,y) by the angle $\theta]$ as the function $F_1(x,y)[= F_1(X,Y)]$ displaced along the imaginary axis of the complex plane of the argument X, and then the 1-D phase retrieval procedure has been applied to the lines parallel to the X axis. In contrast to that, we here utilize the following 2-D phase retrieval procedure.

Let $M(x,y)$ and $\phi(x,y)$ be the modulus and the phase, respectively, of $F_1(x,y)$, i.e.,

$$F_1(x,y) = M(x,y)\exp[i\phi(x,y)]. \quad (15)$$

Expanding the real variables x and y of $F_1(x,y)$ into the complex ones, $x - ic_1$ and $y - ic_2$, and taking the modulus of the resultant function, the Fourier modulus of the object modulated with the exponential filter can be represented from Eq.(4) as

$$|F_1(x - ic_1, y - ic_2)| = |M(x - ic_1, y - ic_2)|\exp[-\mathrm{Im}\phi(x - ic_1, y - ic_2)], \quad (16)$$

where Im denotes the imaginary part of a complex function. Substitution of Eq.(3) into Eq.(5) yields

$$\ln \frac{|F_2(x,y)|}{|M\,(x - ic_1, y - ic_2)|} = -\mathrm{Im}\,\phi\,(x - ic_1, y - ic_2)\,, \tag{17}$$

where the values of the moduli on the left-hand side are assumed to be not zero. The term on the left-hand side of Eq.(6) can be calculated from the observed data because $|F_2(x,y)|$ is the measured modulus when using the exponential filter and because $|M\,(x - ic_1, y - ic_2)\,|$ is evaluated as the modulus of the discrete Fourier transform of the product of the inverse Fourier transform of the measured modulus $M\,(x,y)$ and the exponential function $\exp[-2\pi(c_1 u + c_2 v)/d\lambda]$ in the same way as in Eqs.(1)-(3).

Next we consider a method of computing the 2-D phase distribution directly from Eq.(6). One approach to solving Eq.(6) is to represent $\phi(x,y)$ in terms of a 2-D discrete Fourier transform:

$$\phi(x,y) \cong \sum_{n=-N/2+1}^{N/2-1} \sum_{m=-M/2+1}^{M/2-1} a_{nm} \exp\left[2\pi i \left(\frac{n}{K}x + \frac{m}{L}y\right)\right], \tag{18}$$

where the observational region of the function $F_1(x,y)$ is designated as $-K/2 \le x \le K/2$ and $-L/2 \le y \le L/2$, and both of N and M are even numbers more than 2 and sufficiently large to enable the phase distribution to be reconstructed. Thus the unknown function $\phi(x,y)$ is represented by the unknown complex coefficients a_{nm}. Since the Fourier transform of a real function is a Hermitian function, the complex Fourier coefficient a_{nm} of the real function $\phi(x,y)$ has the properties of Hermitian functions, namely the real and the imaginary parts of a_{nm} are even and odd functions, respectively. Thus, in the case of expression (7), a total of unknown coefficients amounts to $(N-1)(M-1)$, which are composed of $\dfrac{(N-1)(M-1)+1}{2}$ and $\dfrac{(N-1)(M-1)-1}{2}$ unknowns, respectively, for the real and the imaginary parts of a_{nm}. The imaginary part of $\phi(x - ic_1, y - ic_2)$ is written from expression (7) as

$$\mathrm{Im}\phi(x - ic_1, y - ic_2) \cong \sum_{n=-N/2+1}^{N/2-1} \sum_{m=-M/2+1}^{M/2-1} \left\{ \mathrm{Im}\, a_{nm} \cos\left[2\pi\left(\frac{n}{K}x + \frac{m}{L}y\right)\right] \right.$$
$$\left. + \mathrm{Re}\, a_{nm} \sin\left[2\pi\left(\frac{n}{K}x + \frac{m}{L}y\right)\right] \right\} \exp\left[2\pi\left(c_1\frac{n}{K} + c_2\frac{m}{L}\right)\right] (19)$$

where Re denotes the real part of a complex function, and the term $\mathrm{Im}\, a_{00}$ is always zero because of the Hermitian character of a_{nm}. By substituting expression (8) into Eq.(6), we have one equation in the $(N-1)(M-1)$ unknowns. Thus the $(N-1)(M-1)$ known values in the left-hand side of Eq.(6) generate $(N-1)(M-1)$ simultaneous equations in the $(N-1)(M-1)$ unknowns. When solving the simultaneous equations, however, we have to pay attention to the treatment of the constant and linear phases of $\phi(x, y)$. First, it is noted that the constant phase corresponding to the term $\mathrm{Re}\, a_{00}$ cannot be determined from the simultaneous equations, because the sinusoidal function for $n = m = 0$ in the right-hand side of expression (8) becomes zero for all values of x and y. However, the constant phase is a trivial ambiguity for the reconstruction of the object function and then it can be disregarded for phase retrieval. Thus the term $\mathrm{Re}\, a_{00}$ as well as $\mathrm{Im}\, a_{00}$ is excluded from expression (8). Secondly, we consider the influence of a linear phase $2\pi(ax + by)$ on the simultaneous equations, where a and b are relevant to the unknown coordinate components of the object's position in the $u - v$ plane. The imaginary part of the linear phase that is extended into the complex variables, $x - ic_1$ and $y - ic_2$, in the same procedure as in expressions (7) and (8) becomes to be a constant $-2\pi(ac_1 + bc_2)$. Therefore, we cannot determine two unknown parameters a and b from the constant value by solving the simultaneous equations. Thus we separate the linear phase from the phase in expression (7) and rewrite the imaginary part of the linear phase as $-2\pi c_1 a'$ in order to determine the one unknown value a' instead of a and b. Although the true position of the object cannot be obtained then, such ambiguity is unimportant in many applications. By rewriting the right hand side of expression (8) according to the above consideration and substituting this expression into Eq.(6), we obtain

$$\ln \frac{|F_2(x,y)|}{|M(x-ic_1, y-ic_2)|} \cong 2\pi c_1 a' - \sum_{n=-N/2+1}^{N/2-1} \sum_{m=-M/2+1}^{M/2-1} \left\{ \mathrm{Im}\, a_{nm} \cos\left[2\pi\left(\frac{n}{K}x + \frac{m}{L}y\right)\right] + \mathrm{Re}\, a_{nm} \sin\left[2\pi\left(\frac{n}{K}x + \frac{m}{L}y\right)\right]\right\} \exp\left[2\pi\left(c_1\frac{n}{K} + c_2\frac{m}{L}\right)\right] \quad (20)$$

where the two terms of $\mathrm{Re}\, a_{00}$ and $\mathrm{Im}\, a_{00}$ are not included on the right-hand side. By calculating the left-hand side of expression (9) at $(N-1)(M-1)$ values of x and y, we obtain $(N-1)(M-1)$ simultaneous equations from which the $(N-1)(M-1)$ unknown coefficients of $\mathrm{Re}\, a_{nm}$, $\mathrm{Im}\, a_{nm}$, and a' can be determined. The 2-D phase is retrieved by substituting the solutions for $\mathrm{Re}\, a_{nm}$ and $\mathrm{Im}\, a_{nm}$ into expression (7) and adding a deduced linear phase $2\pi a' x$ to the result. Consequently, the object function is reconstructed by an inverse Fourier transform of the function with the observed modulus and the retrieved phase. However, when an object function is a Hermitian function [i.e., $f(u,v) = f^*(-u,-v)$, where the asterisk denotes the complex conjugate], it is necessary to use the logarithmic Hilbert transform along with the present method in the same way as in the previous method,[23] because in that case the values of the left-hand side term in expression (9) become zero at all values of x and y.

Note that, if a sampling point (n', m') in the $n-m$ plane satisfies the condition of $c_1 n'/K + c_2 m'/L = 0$ (i.e., then the exponential function in the right-hand side of expression (9) is unity), the $(N-1)(M-1)$ simultaneous equations becomes rank-deficient because in expression (9) the sum of the terms at two points (n', m') and $(-n', -m')$ (which are symmetrical with respect to the origin) is zero for all values of x and y owing to the Hermitian property of a_{nm}. Thus it is necessary to select the angle θ between the (u, v) axes and the exponential filter (i.e., $\tan\theta = c_2/c_1$ in Fig.1) to satisfy the condition of $c_2/c_1 \neq -nL/mK$ for two integers of n and m.

3. Computer Simulation

The phase retrieval method presented in this paper has been tested by computer simulation of the reconstruction of 2-D complex-valued objects. Figures 2 and 3 show two examples of reconstructing 2-D complex-valued objects. Data processing by computer was performed with 32×32 sampling points. The duration for the fast Fourier transform algorithm used here was assumed to have the value of 10 for arbitrary units in each direction of the u and v axes in the object plane. Hence the region $(K \times L)$ of the Fourier phase shown in expression (7) becomes to be 3.2×3.2, and the sampling points $N \times M$ for the Fourier phase is 32×32. The parameters for the inclination of the exponential filter $\exp[-2\pi(c_1 u + c_2 v)/d\lambda]$ used in this simulation are set to be $c_1 = 0.0145$ and $c_2 = 0.0198$, which are chosen in order to prevent the rank-deficient of the simultaneous equations using expression (9) (i.e., $c_2/c_1 \neq -nL/mK$ for two integers of n and m). In addition, the value of $d\lambda$ in Eqs.(1) and (2) is set to be unity for simplicity.

Figures 2(a) and 2(b) [or Figs.3(a) and 3(b)] show the modulus and the phase of the original object function, respectively. Figures 2(c) and 2(d) [or Figs.3(c) and 3(d)] show the modulus and the phase of the object reconstructed from two noiseless Fourier intensities of the original object modulated and unmodulated with the exponential filter. The example of the reconstruction from noisy Fourier intensities is shown in Figs.2(e) and 2(f) [or Figs.3(e) and 3(f)]. To simulate the noisy intensities, complex normal random noises $n_1(x, y)$ and $n_2(x, y)$, which are independent of each other, are added to the Fourier transform functions $F_1(x, y)$ and $F_2(x, y)$ [which are shown in

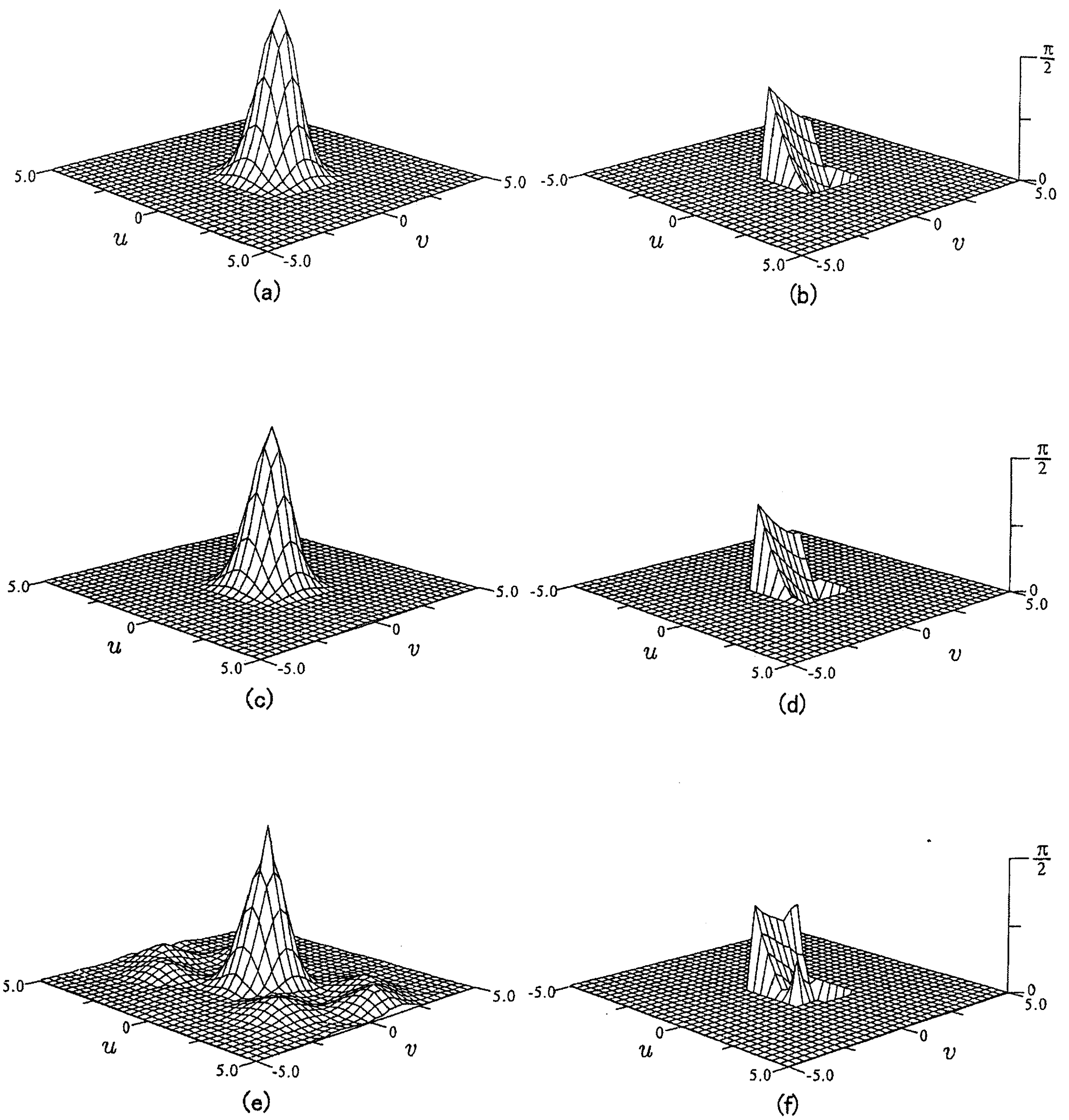

Fig.2. Examples of the reconstruction by the present method: (a) modulus and (b) phase of the original object function. (c) and (d) [or (e) and (f)] are the modulus and the phase, respectively, of the reconstructed object function from noiseless [or noisy (SNR=210)] Fourier intensities. The units of the coordinates u and v are arbitrary.

Eqs.(1) and (2)], respectively. A factor of the signal-to-noise ratio (SNR), defined by $\mathrm{SNR} = \sum_{x,y} |F_1(x,y)|^2 / \sum_{x,y} |n_1(x,y)|^2$ is now introduced, and then the SNR for the reconstruction in the noisy cases of Figs.2 and 3 become 210 and 690, respectively. It is found from comparison of the reconstructed objects with the original ones in Figs.2 and 3 that the errors surrounding the object's region in Fig.3 are larger than that in Fig.2. The reason of this fact is that the density of the sampling points, which is needed to encode the change of the Fourier phase of the object in Fig.3, is slightly insufficient for precise reconstruction like that in Fig.2. Hence, even in the high SNR case, the reconstructed object in Figs.3(e) and 3(f) suffers more errors from the influence of noise than that in Fig.2. However, it can be seen that the profiles within the object's extent is relatively well reconstructed in both cases of Figs.2 and 3.

4. Conclusions

A noniterative phase retrieval method from two Fourier intensities of a 2-D object modulated and unmodulated by an exponential filter has been presented. So far, only the iterative method[16] has been utilized for the solution of the 2-D phase retrieval from two Fourier intensities measured by use of an exponential filter. Since the present method is based on the properties of analytic functions, there is no need for the information about the object (e.g., the extent of the object), which is used in the iterative method to increase the speed of convergence. On the other hand, the previous noniterative-method[23] using an exponential filter requires three Fourier intensities in 2-D cases: one Fourier intensity of an object and two Fourier intensities of the object modulated with exponential filters along the horizontal and the vertical directions in 2-D coordinates. Although the procedure of calculating the phase in the present method

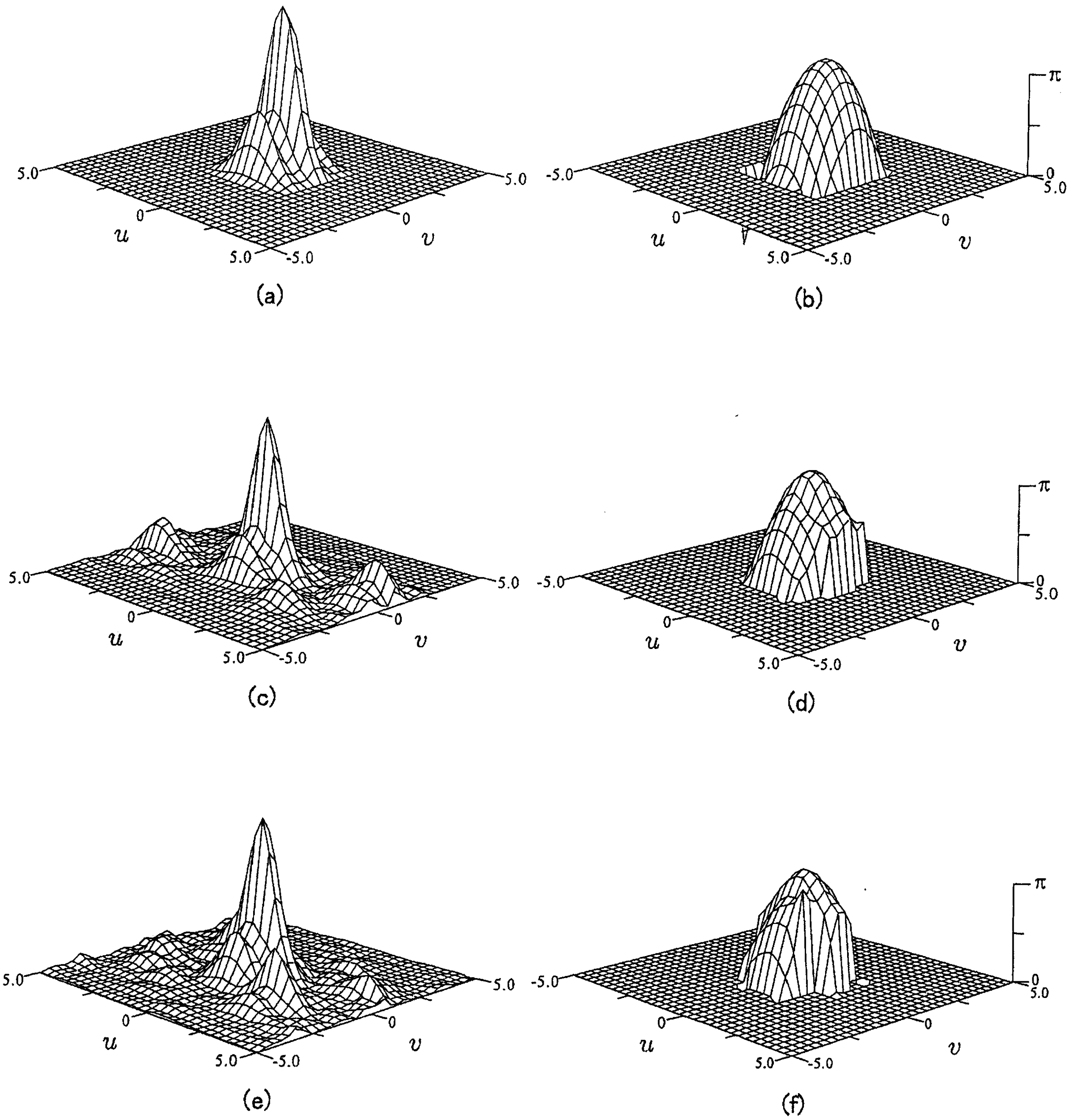

Fig.3. Examples of the reconstruction by the present method: (a) modulus and (b) phase of the original object function. (c) and (d) [or (e) and (f)] are the modulus and the phase, respectively, of the reconstructed object function from noiseless [or noisy (SNR=690)] Fourier intensities. The units of the coordinates u and v are arbitrary.

is similar to that in the previous noniterative-method, the former differs form the latter in respect that the analyticity of 2-D functions is used directly for the theoretical basis.

Computer-simulated examples of this method presented the good reconstructions of the complex objects. However, the main difficulty with practical use of this method arises from the huge size of the matrix treated in the phase calculation. For example, in the phase retrieval with $N \times N$ sampling points, it is necessary to solve the simultaneous equations with the matrix of N^4 elements. In the previous noniterative-method, the phase is gotten by simply repeating the solution of simultaneous equations (with the matrix of N^2 elements) N times over. Thus, in order to overcome this difficulty, it may be effective to decompose the huge matrix into a set of small matrices, from which each spatial part of the phase distribution is retrieved by the present method and then the phase is synthesized from the spatial parts. The details of such improvement are currently being studied.

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Lensless imaging from diffraction intensity measurements by use of a noniterative phase-retrieval method

Abstract

A method of reconstructing the complex amplitude of an object illuminated by a coherent wave from its Fresnel diffraction patterns is proposed for high-frequency wave phenomena such as x-rays and electron waves. A noniterative phase retrieval method by use of a Gaussian filter is utilized here, and it is shown that the object's illumination with amplitude distribution in the Fraunhofer diffraction pattern of a circular aperture can be used as a substitute for the Gaussian filter. This method has the advantage of retrieving phase vortices in compared with any other noniterative phase retrieval methods.

I. INTRODUCTION

The imaging technique utilizing high frequency wave phenomena such as x-rays and electron waves has been developed with the aim of understanding the structure of smaller objects. In such imaging techniques, however, we have to solve some problems relevant to lenses in order to achieve better resolution of images. For example, in electron microscopy, lens aberration is the major barrier limiting the resolution. Some software[1,2] and hardware[3] methods of correcting the images for the effects of spherical aberration have been proposed. However, spherically corrected images are still limited in resolution by chromatic aberration.[4] In x-ray microscopy, the efficiency of Fresnel zone plates used for imaging is worse than electron lenses. One approach to overcome these problems is to reconstruct an object from measurements of diffraction fields without using lenses. For this purpose, we have to measure the modulus and phase of the diffraction field of an object illuminated by a beam of coherent waves. However, the directly measurable quantity in the illumination with high frequency waves such as x-rays and electron waves is only the intensity, which is proportional to the square of the modulus of the wave field, and the phase is lost on an intensity recording. Determination of the phase from the observed intensity is referred to as a phase problem, and many method have been developed to solve this problem. Holography [5] is well-known as a standard approach to this problem in electron microscopy, in which a hologram is obtained from known reference waves and unknown object waves, and converted to an image by using either light-optical or digital reconstruction. For such high-frequency waves, however, the phase measurement of a diffraction field by a holographic or interferometric technique is difficult in practice because a coherent reference wave is barely obtained and some severe conditions on the measurement system are required when observing interference fringes. In another well-known diffraction methodology such as x-ray crystallography, the phase problem is solved by using a numerical algorithm [6] based on the properties of object structures with periodic repeats (i.e., crystals). However, many samples in materials sciences, structural biology, and other areas are very often noncrystalline or nonrepetitive, and thus cannot be accessed by this algorithm.

Recently, another approaches have been developed to solve the phase problem under the more general condition that an object is noncrystalline and its diffraction intensities are measured directly without using lenses and reference waves. One of them is an iterative method using *a priori* information (e.g., nonnegativity and/or extent of a wave field in an object plane) with the diffraction intensities, which is proposed by Fienup[7] as a modification of the original Gerchberg-Saxton algorithm.[8] The demonstration of iterative algorithms has been recently shown in the x-ray phase retrieval experiments for noncrystalline samples,[9,10] and the computer simulations of phase retrieval from diffraction patterns for complex large biomolecules with coherent x-ray illumination [11] and for a single noncrystalline object illuminated with a coherent electron beam.[12] However, the use of iterative methods is accompanied by convergence problems, and hence those methods sometimes exhibit slow convergence and stagnation in a local minimum solution that is different from a true one. Thus there are some studies on noniterative methods for phase retrieval from diffraction intensities, for example, phase recovery by solving the transport-of-intensity equation from intensity distributions in two or three transverse planes,[13–16] and phase retrieval by solving simultaneous equations for unknown phase from three diffraction intensities of an object filtered with known Gaussian functions,[17] which is based on the properties of entire functions (i.e., wave functions belong to entire functions).[18,19] The former method has been used for phase retrieval in the Fresnel region in electron [20] and optical microscopy,[21] x-ray imaging,[22] and so on. The reconstruction by the latter method was demonstrated in the optical experiments with a Gaussian beam instead of the Gaussian filter.[23] The latter method has the advantage of retrieving the phase distribution with the vortices (i.e., screw dislocations) in contrast with the former one, because entire functions, of which the properties is used in the method, intrinsically include zero points (which indicate the possible presence of vortices in the phase). In the former method, it has been shown[24] that the solution of the transport-of-intensity equation does not yield unique results when phase vortices are present. The presence of wave-field vortices is ubiquitous rather than exotic. It is the known fact that vortex-free propagating wave fields will almost always develop vortices after evolving through a certain critical length of space.[25,26] Thus, in the phase measurements for optical, x-ray, and electron wave fields, it is possible that phase vortices may arise. Therefore the retrieval of phase vortices is significant in the phase problem. Iterative methods have been used for the phase retireval in the presence of vortices.[24,26,27] In noniterative methods, to the best of my knowledge, the latter method is the only direct one of phase retrieval which can retrieve the phase when vortices are present. In the latter method, however, it is a subject of inquiry how to do Gaussian filtering for phase retrieval of x-rays or electron waves.

Thus, an extension of the latter method is presented in this paper. In the extended method, the Fraunhofer diffraction pattern of a circular aperture (i.e., a besinc function) is used as the object's illumination instead of a Gaussian beam, and such an illumination enables the object reconstruction by the latter method from the diffraction intensities for x-rays or electron waves, provided that the object's extent is confined within $\sim 1/3$ of the first zero's radius of the besinc function. The validity of this method is demonstrated here in computer simulations of the reconstructions of a

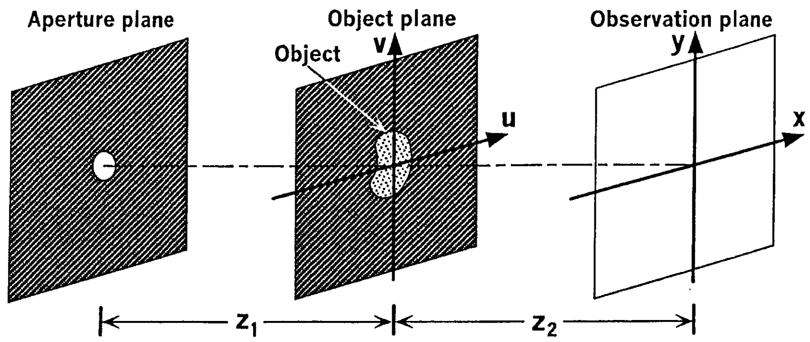


Fig.1. Schematic diagram of the object-reconstruction system by use of the phase retrieval. The object function is reconstructed from Fresnel-zone intensities of the object unshifted and shifted in each direction of u and v axes in the illumination of the Fraunhofer diffraction pattern of a circular aperture.

general complex object and a particular object with phase vortices.

2. Formulation of the Reconstruction Method

We consider the object reconstruction in the two-dimensional (2-D) arrangement of Fig.1. This layout consists of three planes perpendicular to an optical axis, which are the aperture, the object, and the observation planes. We assume that an object of complex-amplitude transmittance $f(u,v)$ (which is assumed to be of finite extent) in the object plane is illuminated by the amplitude distribution in the Fraunhofer diffraction pattern of a circular aperture with a known radius w in the aperture plane. If a unit-amplitude, coherent monochromatic plane wave of the wavelength λ is incident normally on that aperture, then the field distribution $b(u,v)$ in the object plane is given by [28]

$$b(u,v) = \frac{\pi w^2 e^{ikz_1}}{i\lambda z_1} \exp\left[i\frac{k}{2z_1}(u^2+v^2)\right] \frac{2J_1(kw\sqrt{u^2+v^2}/z_1)}{kw\sqrt{u^2+v^2}/z_1}, \tag{21}$$

where J_1 is a Bessel function of the first kind, order 1, z_1 is the normal distance between the aperture and object planes, and k is the wave number. Then, by use of the Fresnel approximation, the complex amplitude in the observation plane is written as

$$F_1(x,y) = \frac{e^{ikz_2}}{i\lambda z_2} \exp\left[i\frac{k}{2z_2}(x^2+y^2)\right] U_1(x,y), \tag{22}$$

where

$$U_1(x,y) \;=\; \iint_\sigma f(u,v)b(u,v) \exp\left[i\frac{k}{2z_2}(u^2+v^2)\right] \exp\left[-i\frac{2\pi}{\lambda z_2}(xu+yv)\right] dudv, \tag{23}$$

in which σ denotes the extent of the object. If the complex amplitude $U_1(x,y)$ is obtained in an experiment, the object function can be reconstructed from $U_1(x,y)$ by using the inverse Fourier transform and eliminating the effect of the illuminating field $b(u,v)$ (which is calculated from the known constants λ, w, and z_1) from the resultant function. Unfortunately, the only physical quantity that can be directly observed is the intensity for high frequency phenomena such as light, x-rays, and electron waves. The phase information is lost on an intensity recording. Thus we consider here the object reconstruction method based on the phase retrieval from three Fresnel-zone intensity measurements. One of the measured data terms is the intensity distribution of $F_1(x,y)$, and the others are the intensity distributions of the Fresnel-zone transforms of the object shifted along the u and v axes. First, we consider the Fresnel-zone transform $F_2(x,y)$ of the shifted object with horizontal displacement τ in the u axis's direction (which is assumed to be another known constant). The complex amplitude $F_2(x,y)$ is given by

$$F_2(x,y) = A \iint_\sigma f(u-\tau, v) b(u,v) \exp\left[i\frac{k}{2z_2}(u^2+v^2)\right] \exp\left[-i\frac{2\pi}{\lambda z_2}(xu+yv)\right] dudv$$

$$= A' \iint_\sigma f(u', v) b(u'+\tau, v) \exp\left[i\frac{k}{2z_2}(u'^2+v^2)\right]$$

$$\times \exp\left\{-i\frac{2\pi}{\lambda z_2}[(x-\tau)u' + yv]\right\} du'dv, \quad (24)$$

where $A = (e^{ikz_2}/i\lambda z_2) \exp\left[ik(x^2+y^2)/2z_2\right]$, $A' = A\exp[i\pi(\tau^2 - 2x\tau)/\lambda z_2]$, and $u' = u - \tau$. Equation (4) indicates that the object's shift corresponds to the both shifts of the illuminating field $b(u,v)$ in the object plane and the complex amplitude in the observation plane. Secondly, we assume that the extent of the central lobe of the illuminating field $b(u,v)$ is larger than the object's extent σ enough to represent approximately the besinc function of $b(u,v)$ as a Gaussian function in Eqs.(3) and (4). This approximation can provide the use of the noniterative phase retrieval method[17] in the following way.

The besinc function in Eq.(1) can be expanded by a polynomial approximation as

$$\frac{2J_1(r)}{r} = 1 - \frac{r^2}{8} + \frac{r^4}{192} + \cdots, \quad (25)$$

where

$$r = \frac{kw\sqrt{u^2+v^2}}{z_1}. \quad (26)$$

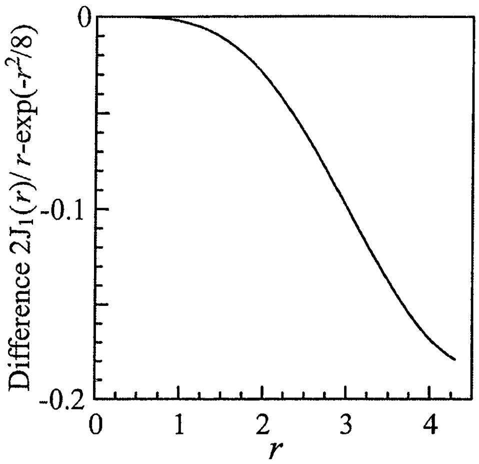


Fig.2. Difference between the functions $2J_1(r)/r$ and $\exp(-r^2/8)$ as a function of r.

Thus we try to approximate the besinc function as a Gaussian function $\exp(-r^2/8)$, of which the Maclaurin expansion becomes

$$\exp\left(-\frac{r^2}{8}\right) = 1 - \frac{r^2}{8} + \frac{r^4}{128} + \cdots. \tag{27}$$

The deviation of $2J_1(r)/r$ from $\exp(-r^2/8)$ is shown in Fig.2 as a function of r. From studies of numerical experiments, this approximation was found to be effective for the phase retrieval provided that the radius r is smaller than $\sim 1/3$ of the first zero's radius ($r_0 \cong 3.83$) of the besinc function. Under the condition that the object's extent is confined within the radius $r = r_0/3$, we can substitute $\exp(-r^2/8)$ into Eq.(1) instead of the besinc function:

$$b(u,v) \cong \frac{\pi w^2 e^{ikz_1}}{i\lambda z_1} \exp\left[i\frac{k}{2z_1}(u^2+v^2)\right] \exp\left[-\frac{k^2 w^2 (u^2+v^2)}{8z_1^2}\right], \tag{28}$$

and then the function $b(u'+\tau, v)$ in Eq.(4) can be rewritten as

$$b(u'+\tau, v) = b(u',v) \exp\left[-\left(\frac{\pi w}{\lambda z_1}\right)^2 \tau u'\right] \exp\left(i\frac{k\tau u'}{z_1}\right) \exp\left[i\frac{k\tau^2}{2z_1} - \frac{1}{2}\left(\frac{\pi w \tau}{\lambda z_1}\right)^2\right]. \tag{29}$$

Equation (9) indicates that the shifted function consists of the original function, an exponential function, a linear phase factor, and a complex constant. The present phase retrieval relies on the effect of the exponential function on the Fresnel-zone intensity of the object. Substituting Eq.(9) into Eq.(4) and using Eq.(3), we can obtain the relationship between the Fresnel-zone moduli of the shifted and unshifted objects:

$$|F_2(x,y)| = \exp\left[-\frac{1}{2}\left(\frac{\pi w \tau}{\lambda z_1}\right)^2\right] |U_1(x - s - ic, y)|, \tag{30}$$

where

$$s = \left(1 + \frac{z_2}{z_1}\right)\tau, \tag{31}$$

and

$$c = \frac{\lambda z_2}{2\pi}\left(\frac{\pi w}{\lambda z_1}\right)^2 \tau. \tag{32}$$

Equation (10) indicates that the modulus of $F_2(x,y)$ corresponds to one of $U_1(x,y)$ displaced along the imaginary axis of the complex plane of its argument x, simultaneously with the displacement s along the x axis, except for the exponential term with the known constant parameters. As we shall see, the displacement along the imaginary axis renders the modulus of $F_2(x,y)$ dependent on one-dimensional (1-D) phases of $U_1(x,y)$ along lines parallel to the x axis.

Let $M(x,y)$ and $\phi(x,y)$ be the modulus and the phase, respectively, of $U_1(x,y)$, i.e.,

$$U_1(x, y) = M(x, y) \exp[i\phi(x, y)]. \tag{33}$$

Note that $M(x, y)$ is equivalent to the observable modulus $|F_1(x, y)|$ of Eq.(2). When the real variable x of $U_1(x, y)$ is expanded into the complex one, $x - ic$, the modulus of the function $U_1(x, y)$ is written from Eq.(13) as

$$|U_1(x - ic, y)| = |M(x - ic, y)| \exp[-\mathrm{Im}\phi(x - ic, y)], \tag{34}$$

where Im denotes the imaginary part of a complex function. If the values of the moduli $|M(x - ic, y)|$ and $|F_2(x, y)|$ are not zero, the relationship between these moduli can be written from Eqs.(10) and (14) as

$$\ln \frac{|F_2(x + s, y)|}{|M\,(x - ic, y)|} + \frac{1}{2}\left(\frac{\pi w \tau}{\lambda z_1}\right)^2 = -\mathrm{Im}\,\phi\,(x - ic, y)\,. \tag{35}$$

On the left-hand side of Eq.(15) the second term consists of the known constants, and the first term can be calculated from the observed data because $|F_2\,(x + s, y)\,|$ is obtained through compensation of the modulus $|F_2\,(x, y)\,|$ in Eq.(10) for its displacement s along the x axis and because $|M\,(x - ic, y)\,|$ is given by the modulus of the Fourier transform of the product of the inverse Fourier transform of $M\,(x, y)(= |F_1(x, y)|)$ and the exponential function $\exp(-2\pi cu/\lambda z_1)$.[19]

By representing a 1-D phase along the line parallel to the x axis in terms of a Fourier series with unknown coefficients, we can use Eq.(15) to construct a set of simultaneous equations from which the unknown coefficients can be computed.[17] The 1-D phases along lines parallel to the x axis can be retrieved by this procedure. Note that, even if the moduli $|M(x-ic,y)|$ and $|F_2(x+s,y)|$ have some zeros, the unknown coefficients can be determined from the data of the moduli at the points except at the zeros. However, there are unknown constant phase differences among these lines because their phase calculations are independent of one another. To solve this ambiguity, we take a third intensity measurement of the Fresnel-zone transform $F_3(x,y)$ of the object $f(u,v-\nu)$ that is shifted with displacement ν (which is also assumed to be a known constant) in the v axis's direction. Using the same derivation as in Eqs.(4)-(10), we can obtain the relationship

$$|F_3(x,y)| = \exp\left[-\frac{1}{2}\left(\frac{\pi w \tau}{\lambda z_1}\right)^2\right] |U_1(x, y-s'-ic')|, \tag{36}$$

where

$$s' = \left(1 + \frac{z_2}{z_1}\right)\nu, \tag{37}$$

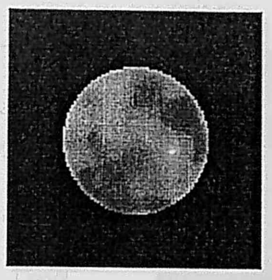
and

$$c' = \frac{\lambda z_2}{2\pi}\left(\frac{\pi w}{\lambda z_1}\right)^2 \nu. \tag{38}$$

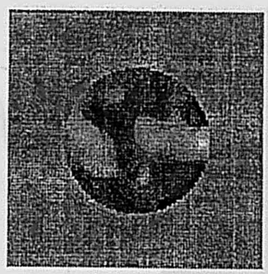
By using the relation of Eq.(16) and the same procedure as the phase calculations along lines parallel to the x axis, we can determine the phases of $U_1(x,y)$ along lines parallel to the y axis. A phase distribution along one line in the y axis's direction can be regarded as the constant phase difference among the 1-D phases along the lines parallel to the x axis. Then the overall 2-D phase $\phi(x, y)$ is determined by addition of the constant phase to the 1-D phase on each line parallel to the x axis. Note that the orthogonality between two displacements (τ and ν) of the object is not necessary for phase retrieval, because the constant phase difference among the 1-D phases can be also determined from the data for the object's displacement along an inclined line. It is found from Eq.(3) that the object function $f(u, v)$ is finally reconstructed by eliminating the effects of the known illuminating beam $b(u, v)$ and the quadratic phase factor $\exp[ik(u^2 + v^2)/2z_2]$ from the inverse Fourier transform of the complex amplitude function with the measured modulus $|U_1(x, y)|(= |F_1(x, y)|)$ and the retrieved phase $\phi(x, y)$. The present phase retrieval method is applicable for all kinds of functions except the Fourier transforms of Hermitian functions, in which the equations of Eq.(15) cannot be solved because the modulus $|F_2(x + s, y)|$ becomes equivalent to $|M(x - ic, y)|$ for all value of x. However, even if an object function is a Hermitian function [i.e.,$f(u, v) = f^*(-u, -v)$], such an object function can be also reconstructed by using the present method because the product of the object function and the quadratic phase factor becomes to be a non-Hermitian function generally. Hence, the reconstruction for all kinds of object functions can be expected in practice.

3. Numerical Examples

The reconstruction method presented in the previous section has been tested by computer simulation of examples of 2-D complex-valued objects. Data processing by computer was performed with 256×256 sampling points. The modulus and the phase of the original object function used in the first simulation are shown in Figs.3(a) and 3(b), respectively, where the figures show only the central 128×128 points out of



(a)



(b)

Fig.3. Original object function used in the computer simulation: (a) modulus and (b) phase of an object with the circular extent of radius $\sim 14\mu$m.

256×256 points in an assumed square with sides of 100μm on the object plane. The modulus and phase are constructed from normal random numbers by a computer. The distribution of the modulus is in the range between 1 and 3 for arbitrary units within circle of radius $\sim 14\mu$m, and the modulus outside the circle is zero. The distribution of the phase is in the range between -1.57 rad and 1.25 rad. We also assume that a monochromatic plane wave x-ray with wavelength $\lambda = 0.682$nm (1.83 keV) is normally incident on a circular aperture with the radius $5\mu m$, and an object is illuminated by the Fraunhofer diffraction amplitude distribution of the aperture. The distances between the aperture and object planes and between the object and Fresnel-zone planes in Fig.1 are assumed to be $z_1 = 600$mm and $z_2 = 80$mm, respectively. Since the first zero's radius of the diffraction pattern of the aperture is $\sim 50\mu$m in the object plane, an effective area for the illumination of objects in the present phase retrieval becomes to be within a circle of radius $\sim 17\mu$m (i.e., about one third of 50μm) as mentioned in Sec.2. The object's extent in Fig.3 is set to be smaller than this effective area. It was seen from another simulations that the reconstruction error gradually increases with

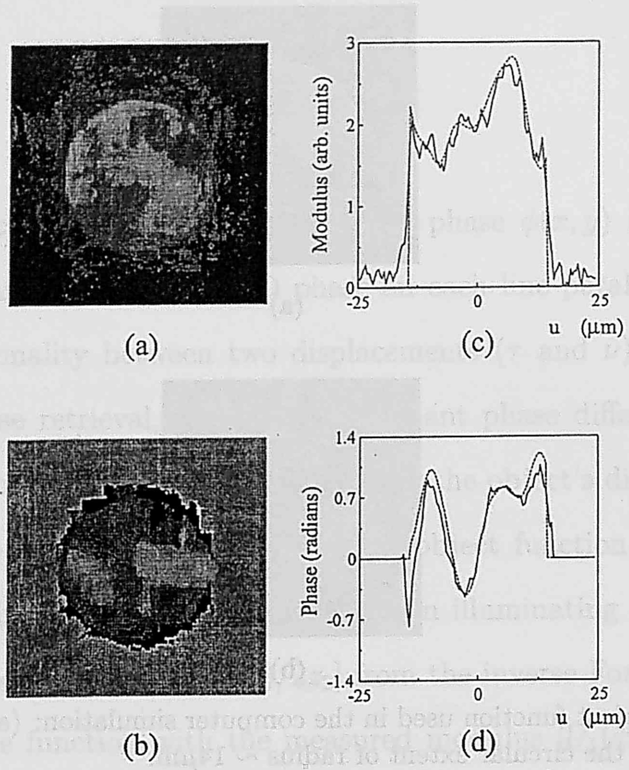
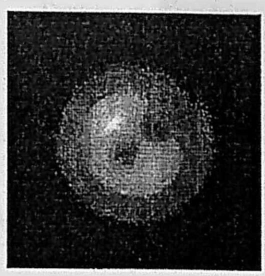


Fig.4. Reconstruction of the object shown in Fig.3 from noisy Fresnel-zone intensities when in the system of Fig.1 the distances $z_1 = 600$mm, $z_2 = 80$mm, the aperture of the radius 5μm, and the coherent illumination with wavelength $\lambda = 0.682$nm are used. (a)Modulus and (b) phase of the reconstructed object, and (c) and (d) are cross-sectional profiles of the functions in (a) and (b), respectively, taken along horizontal lines passing through the center. The dotted and solid curves represent the original and the reconstructed objects, respectively.

the increase of the object's extent from the effective area.

Figures 4(a) and 4(b) show the modulus and the phase of the object reconstructed from noisy Fresnel-zone intensities of the object and the shifted objects with a horizontal displacement of $\tau = 6.25\mu$m and a vertical displacement of $\nu = 6.25\mu$m. As shown in Eq.(9), the object's shift under the illumination of the Gaussian beam corresponds to the product of the object and the exponential function $\exp\left[-(\pi w/\lambda z_1)^2 \tau u'\right]$, which is the key to the phase retrieval in this paper. From studies of numerical experiments,[19] it was found that the suitable values of the inclination factor $\alpha = (\pi w/\lambda z_1)^2 \tau$ of the exponential function are approximately in the range $0.04\pi/\sigma_u < \alpha < 0.5\pi/\sigma_u$, where σ_u denotes the width of the object in the direction of u axis. In the present simulation using $\sigma_u = 28\mu$m, and $\tau = 6.25\mu$m, the value $\alpha (= 9.24 \times 10^{-3})$ is in that range. Cross-



(a)

(b)

Fig.5. Original object function used in the computer simulation: (a) modulus and (b) phase of an object with two first-order vortices of Laguerre-Gaussian mode.

sectional profiles of the original and the reconstructed modulus and phase along the horizontal line that passes through the center of the object are compared in Figs.4(c) and 4(d), respectively. For simulating the noisy intensities, three complex normal random noises $n_i(x, y)$ $(i = 1, 2, 3)$ are produced by a computer and added to the complex amplitudes $F_i(x, y)$ $(i = 1, 2, 3)$ in the Fresnel-zone plane, respectively. A factor of the signal-to-noise ratio (SNR), defined by $\mathrm{SNR} = \sum_{x,y} |F_1(x, y)|^2 / \sum_{x,y} |n_1(x, y)|^2$, is now introduced, and then the SNR in Fig.4 becomes 499.

A measure of the quality of reconstruction is defined by the normalized root-mean-squared (rms) error,

$$\mathrm{ER} = \left[\frac{\sum_{u,v} |f(u, v) - f_r(u, v)|^2}{\sum_{u,v} |f(u, v)|^2} \right]^{1/2}, \tag{39}$$

where $f(u, v)$ and $f_r(u, v)$ are the original and the reconstructed object functions, respectively. The rms error for the reconstructed object in Fig.4 is 0.463. The advantage of the present method is that one can retrieve the phase distribution with vortices (i.e., screw dislocations). To demonstrate the phase retrieval of vortices, an

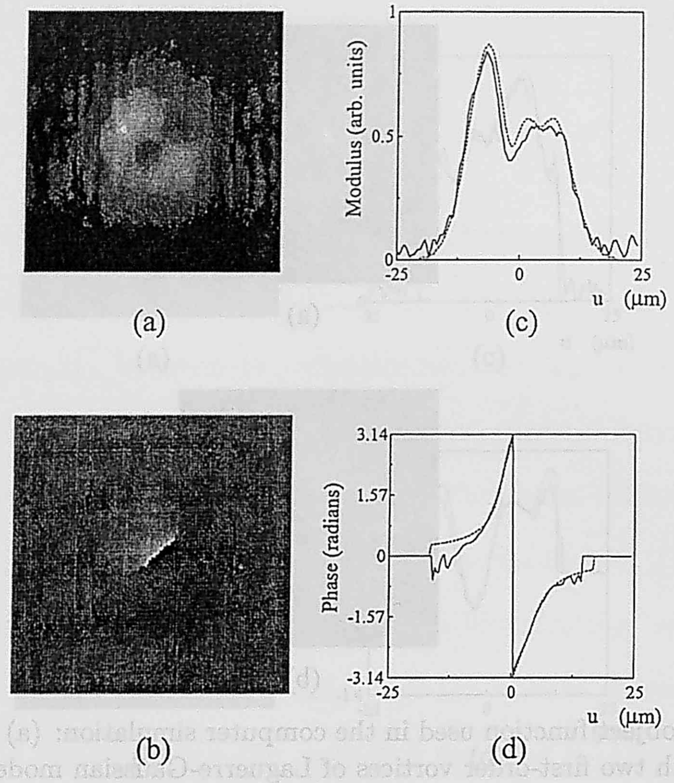


Fig.6. Reconstruction of the object shown in Fig.5 from noisy Fresnel-zone intensities when the same parameters as in Fig.4 are used in the system of Fig.1. (a)Modulus and (b) phase of the reconstructed object, and (c) and (d) are cross-sectional profiles of the functions in (a) and (b), respectively, taken along horizontal lines passing through the center. The dotted and solid curves represent the original and the reconstructed objects, respectively.

object with vortices of Laguerre-Gaussian mode was assumed. Figures 5(a) and 5(b) show the modulus and phase of the original object having two first-order vortices of opposite charge, where the figures are represented with the central 128×128 points on the same scale as in Fig.3. Figures 6(a) and 6(b) show the modulus and phase of the object reconstructed in noisy case (SNR= 444) by using the same object's displacement as in Fig.4. Figures 6(c) and 6(d) indicate comparison between cross-sectional profiles of the original and the reconstructed objects along the horizontal line through the center. The rms error for the reconstructed object in Fig.6 is 0.401. It is found from the results in Figs.4 and 6 that the reconstructed objects are somewhat degraded owing to the presence of noise, but they represent the main features of the original objects in the general case and in particular in the presence of vortices.

4. Conclusions

A noniterative method for reconstructing an object function from its diffraction intensities has been presented. In this method, the Fraunhofer diffraction pattern of a circular aperture is used as the illumination for an object, and the complex amplitude of the object is reconstructed by inverse Fourier transforming the complex amplitude in the Fresnel-zone plane, of which the phase is retrieved from three intensity measurements in the Fresnel-zone region: the Fresnel intensity of the object and the two intensities of the object shifted along two orthogonal directions in the object plane. The phase retrieval method in this paper is based on the approximation of the amplitude function of the object's illuminating beam to a Gaussian function. In the case of the illumination with the Fraunhofer diffraction pattern (i.e., besinc function) of a circular aperture, it was found from computer simulations that this approximation is effective for phase retrieval provided that the extent of the object is smaller than $\sim 1/3$ of the first zero's radius of the besinc function. This effective extent of the illuminating beam can be easily adjusted by changing the distance between the aperture and the object planes. Although the present method requires a shaping of the illuminating beam, the system of making the Gaussian-like beam with a circular aperture is very simple and implemented easily in measuring with x-rays or electron waves. However, there exists no high-power source like a laser in such wave phenomena, and so it may be necessary that a beam of x-rays or electron waves is focused to a small spot on the aperture plane by a zone plate or a electron lens in order to increase the signal-to-noise ratio of measurements.

Computer-simulated examples of this method have presented good reconstructions of the complex objects. Since the phase retrieval method used here is noniterative and analytic, there are not convergence problems such as the stagnation and the slow speed convergence. In addition, the present simulations have shown that this method is able to cope with the existence of vortices in the phase distribution in contrast with all the noniterative phase retrieval methods developed until now. Consequently, these results exhibit that the present method provides potentially very useful means for object's imaging in a wide class of wave fields that obey linear phenomena.

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Phase retrieval from diffraction intensities by use of a scanning slit aperture

Abstract

A noniterative method of retrieving the phase of a wave field from intensities measured while scanning a slit aperture is proposed. In the measurements, one records the diffraction intensities of wave fields transmitted through the slit which is scanned along two directions in the Fresnel-zone plane of an object's field. From these intensities, the phase in the Fresnel-zone plane can be retrieved by the method, in which a novel phase calculation technique using Fourier transforms is included. Since the method does not require lens systems, it provides potentially useful means for coherent imaging by use of x-rays, electrons, or nuclear particles.

I. INTRODUCTION

Non-interferometric reconstruction of the phase of wave fields in coherent scattering phenomena is a topic of current interest in a number of areas for object imaging or structure determination by use of various waves such as optical, x-ray, electron, and atomic waves.[1] The iterative method using *a priori* information (e.g., nonnegativity and/or extent of a wave field in an object plane) with the diffraction intensities are widely used, which was proposed by Fienup[2] as a modification of the original Gerchberg-Saxton algorithm.[3] For example, the demonstration of the object reconstruction by use of the iterative method has been shown in the x-ray phase retrieval experiments for noncrystalline samples,[4–6], the coherent electron imaging of a carbon nanotube from its diffraction intensity,[7] and the computer simulation of phase retrieval from diffraction patterns for complex large biomolecules with coherent x-ray illumination[8] or for a single noncrystalline object illuminated with a coherent electron beam.[9] However, the use of iterative methods is accompanied by convergence problems, and hence those methods sometimes stagnate in a local minimum solution different from a true one. Recently, it was shown that an iterative method from multi-diffraction patterns measured by use of overlapping aperture positions in the object plane is useful to eliminate such ambiguities in the retrieved phase.[10]

On the other hand, there are some studies on noniterative methods for phase retrieval from diffraction intensities, for example, phase recovery by solving the transport-of-intensity equation from intensity distributions in two or three transverse planes,[11–14] and phase retrieval by solving simultaneous equations for unknown phase from three diffraction intensities of an object filtered with known Gaussian functions,[15,16] which is based on the properties of entire functions. It is well-known that wave functions belong to entire functions.[17,18] The former method has been used for phase retrieval in electron microscopy,[19] x-ray imaging,[20] and so forth. The reconstruction by the latter method was demonstrated in the optical experiments with a Gaussian beam instead of the Gaussian filter.[21] The latter method has the advantage of retrieving the phase distribution with the vortices (i.e., screw dislocations[22]) in contrast with the former one,[23] because entire functions intrinsically include zero points, which indicate the possible presence of vortices in the phase. In the latter method, however, there exists the difficulty of filtering a small object with Gaussian functions.

Thus an extension of the latter method is presented in this paper. In the extended method, the Gaussian filtering for phase retrieval is done indirectly by taking the correlation of a slit aperture with the Fresnel diffraction amplitude of an object. The intensity data of such a correlation are obtained by measuring the diffraction intensities of a wave field transmitted through the slit aperture which is scanned on the Fresnel-zone plane. Then the one-dimensional phases in the Fresnel-zone plane of the object can be retrieved from two series of the intensities measured while scanning the slit by a new phase calculation technique using Fourier transforms. Combining these phases and another retrieved phase from the data obtained by a 90° rotation of the above slit, the two-dimensional phase in the Fresnel-zone plane can be determined, and then the object function is reconstructed by transforming the inverse Fresnel diffraction of the complex function consisting of the measured modulus and the retrieved phase.

The reconstruction method is developed in Section 2. In Section 3, the resolution attainable by the system with a scanning slit is discussed. The validity of this method is demonstrated in Section 4 with a computer simulation of the reconstruction of an object with phase vortices. Concluding remarks are given in Section 5.

2. Formulation of the Reconstruction Method

Figure 1 shows a schematic diagram of the measurement system with a scanning slit aperture. We assume that an object of complex amplitude transmittance $f(u,v)$ (which is assumed to be of finite extent) in the object plane is illuminated by a coherent monochromatic plane wave of wavelength λ. In the Fresnel-zone plane with coordinates x and y, a scanning slit aperture is inserted to take a correlation of the slit function and the field distribution $F(x,y)$ of the wave diffracted from the object. By use of the Fresnel approximation, $F(x,y)$ is written as

$$F(x,y)=\iint_{\sigma} f(u,v)\exp\left\{i\frac{\pi}{\lambda z}[(x-u)^2+(y-v)^2]\right\}dudv, \tag{40}$$

where σ denotes the extent of the object, z is the normal distance between the u-v and x-y planes, unimportant multiplicative constants associated with the diffraction integrals are ignored. Assuming the Fresnel diffraction of the wave transmitted through the slit aperture, the complex amplitude $H(\xi,\eta)$ in the detector plane is given by

$$\begin{aligned}H(\xi,\eta)=q\iint_{-\infty}^{\infty} F(x,y)R(x-s)\exp\left\{i\frac{\pi}{\lambda l}[(x-s)^2+y^2]\right\}\\ \times\exp\left\{-i\frac{2\pi}{\lambda l}[x(\xi-s)+y\eta]\right\}dxdy,\end{aligned} \tag{41}$$

where l is the normal distance between the x-y and ξ-η planes, $R(x)$ is the aperture function of unit-amplitude transmittance within its extent, s is the position of the center line of the slit on the x axis, and $q=\exp[i\pi(\xi^2-s^2+\eta^2)/\lambda l]$. First, in the present method, the one-dimensional phases of the function $H(\xi,\eta)$ along the lines

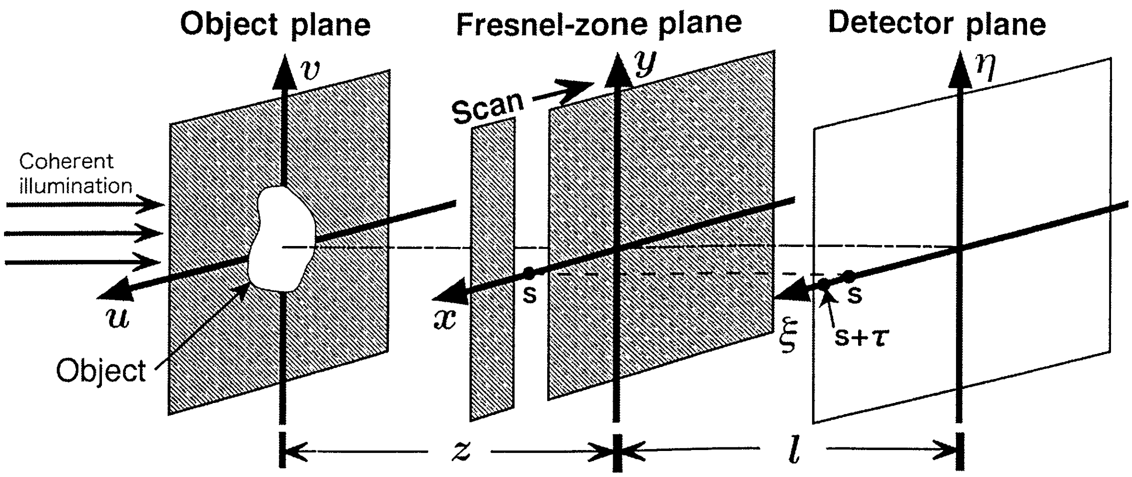

Fig.1. Schematic diagram of the object-reconstruction system that uses phase retrieval. The object reconstruction is based on the measurement of two series of intensities recorded along the lines parallel to the η axis at two coordinates of s and $s+\tau$ in the detector plane as a function of the slit position s.

parallel to the ξ axis are retrieved from two series of intensities, which are recorded at two coordinates $\xi = s$ and $\xi = s+\tau$ (where τ is a constant) in the detector plane as a function of the slit position s. Secondly, the phases of $H(\xi,\eta)$ along lines parallel to the η axis are also retrieved from data obtained by a 90° rotation of the slit. Then the two-dimensional phase of $F(x,y)$ is reconstructed by eliminating the effect of the slit function $R(x)$ from slices of $H(\xi,\eta)$ parallel to the ξ and η axes and combining one-dimensional phases of the resultant functions in two dimensions.

Thus, substituting Eq.(1) into Eq.(2) and integrating for y coordinate, the two intensities of $H(\xi,\eta)$ at $\xi = s, s+\tau$ can be written as

$$|H(s,\eta)|^2 = \left| \int_{s-a}^{s+a} F'(x,\eta)\, R(x-s) dx \right|^2 \tag{42}$$

and

$$|H(s+\tau,\eta)|^2 = \left| \int_{s-a}^{s+a} F'(x,\eta)\, R(x-s) \exp\left(-i2\pi x\tau/\lambda l\right) dx \right|^2, \tag{43}$$

where a denotes half a width of the slit,

$$F'(x,\eta)=\iint_{\sigma} f(u,v)\exp\left[i\frac{\pi}{\lambda z}(x-u)^2+i\frac{\pi}{\lambda(z+l)}v^2\right]\exp\left[-i\frac{2\pi}{\lambda(z+l)}\eta v\right]dudv \tag{44}$$

and unimportant multiplicative constant is ignored. In Eqs.(3) and (4), the Fraunhofer approximation was assumed to be satisfied for the diffraction intensities along the direction of ξ axis (i.e., the quadratic phase factor $\exp\left[i\frac{\pi}{\lambda l}(x-s)^2\right]$ in Eq.(2) is approximately unity over the slit function in the x direction).

To utilize the phase retrieve method based on the properties of entire functions,[15,16] we assume that the extent of the inverse Fourier transform of $R(x)$ [i.e., a sinc function] has large extent enough to approximate it into a Gaussian function within the extent of the inverse Fourier transform of $F'(x,\eta)$:

$$\begin{aligned}\int_{-\infty}^{\infty} R(x)\exp\left(i2\pi ux/\lambda z\right)dx &= 2a\ \mathrm{sinc}(2\pi au/\lambda z)\\ &= 2a\left[1-\frac{(2\pi au/\lambda z)^2}{6}+\cdots\right],\\ &\cong 2a\exp[-(2\pi au/\lambda z)^2/6],\end{aligned} \tag{45}$$

where the Maclaurin expansion of the sinc function is used for the approximation. It can be seen from Eq.(5) that the one-dimensional inverse Fourier transform of $F'(x,\eta)$ for x coordinate corresponds to the product of the spatial-frequency spectrum of the object function along the direction of u axis with a quadratic phase factor. From studies of numerical experiments, the approximation in Eq.(6) was found to be effective provided that the main part of the object's frequency spectrum is smaller than $\sim\frac{1}{2}$ of the first zero's extent of the sinc function. Thus substituting the Fourier transform of Eq.(6) into Eqs.(3) and (4) instead of $R(x)$, we obtain the relation between the two intensities of $H(\xi, \eta)$ at $\xi = s, s + \tau$:

$$|H(s+\tau,\eta)|^2 = \exp(-3c'^2/2a^2)|H(s-ic',\eta)|^2, \tag{46}$$

where $c' = 2\pi a^2 \tau / 3\lambda l$. Then substitution of $H(s,\eta) = M(s,\eta)\exp[i\phi(s,\eta)]$ into Eq.(7) yields

$$\ln\left[\frac{|H(s+\tau,\eta)|}{|M(s-ic',\eta)|}\right] + 3c'^2/4a^2 = -\mathrm{Im}\phi(s-ic',\eta), \tag{47}$$

where $M(s,\eta)$ and $\phi(s,\eta)$ are the modulus and phase of $H(s,\eta)$, and Im denotes the imaginary part of the complex function.

In the previous papers,[15,16] the phase $\phi(s,\eta)$ along the direction of ξ axis was solved from Eq.(8) by representing the phase in terms of a Fourier series basis. In this paper, a new method of solving Eq.(8) by Fourier transforms is presented. Thus we rewrite Eq.(8) in the form

$$D(s,\eta) = -\mathrm{Im}\phi(s-ic',\eta), \tag{48}$$

where $D(s,\eta)$ is a measurable function consisting of two terms on the left-hand side in Eq.(8). The Fourier transform of Eq.(9) for s coordinate can be written as

$$\mathcal{F}[D(s,\eta)] = -\frac{1}{2i}\int_{-\infty}^{\infty}[\phi(s-ic',\eta) - \phi^*(s+ic',\eta)]\exp(-i2\pi\alpha s)ds, \tag{49}$$

where $\mathcal{F}[\cdot]$ denotes the Fourier transform, which is assumed to be a function of variable α, and the asterisk means a complex conjugate. Integration by substitution for $s \pm ic'$ in the right-hand side of Eq.(10) yields

$$
\begin{aligned}
\mathcal{F}[D(s,\eta)] &= -\frac{1}{2i}\int_{-\infty}^{\infty}[\phi(s,\eta)\exp(2\pi\alpha c') - \phi^*(s,\eta)\exp(-2\pi\alpha c')] \\
&\qquad\qquad\qquad\qquad \times\exp(-i2\pi\alpha s)ds, \\
&= i\sinh(2\pi\alpha c')\int_{-\infty}^{\infty}\phi(s,\eta)\exp(-i2\pi\alpha s)ds, \qquad (50)
\end{aligned}
$$

where the property of the real function of the phase [i.e., $\phi(s,\eta) = \phi^*(s,\eta)$] was used. Hence the phase $\phi(s,\eta)$ along the direction of ξ axis can be obtained from the measurable function $D(s,\eta)$ by

$$
\phi(s,\eta) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[D(s,\eta)]}{i\sinh(2\pi\alpha c')}\right], \qquad (51)
$$

where $\mathcal{F}^{-1}[\cdot]$ denotes the inverse Fourier transform.

The complex function $H(s,\eta)$ on one-dimensional lines parallel to the ξ axis can be reconstructed from the measured modulus in Eq.(3) and the retrieved phase $\phi(s,\eta)$. Then we can calculate the deconvolution of $F'(x,\eta)$ from the reconstructed function $H(s,\eta)$ by using the known slit function $R(x)$. However, there are unknown constant phases among these lines that are independent of each other. To retrieve this ambiguity, we make a second measurement for two series of intensities of $H(\xi,\eta)$ at $\eta = s, s+\tau'$ (where τ' is also a constant) by scanning a slit aperture that is parallel to the x axis (obtained by a 90° rotation of the above slit). Then the function $F'(\xi,y)$ along the direction of y axis can be reconstructed in the same way as $F'(x,\eta)$. The two functions $F'(x,\eta)$ and $F'(\xi,y)$ can be combined in the following way. After canceling the known quadratic phase $\exp(i\pi x^2/\lambda z)$ of $F'(x,\eta)$, we calculate a set of one-dimensional inverse Fourier transforms of the resultant function along lines parallel to the x axis. Then one-dimensional Fourier transforms of the products of the set of the functions and the quadratic phase $\exp[-i\pi u^2/\lambda z + i\pi u^2/\lambda(z+l)]$ is given by

$$F'_\xi(\xi,\eta)=\iint_\sigma f(u,v)\exp\left[i\frac{\pi}{\lambda(z+l)}(u^2+v^2)\right]\exp\left[-i\frac{2\pi}{\lambda(z+l)}(\xi u+\eta v)\right]dudv, \quad (52)$$

on condition that the phase information among slices of $F'_\xi(\xi,\eta)$ parallel to the ξ axis is unknown. In the same way, the function $F'_\eta(\xi,\eta)$ having the phase information in the η direction can be obtained from $F'(\xi,y)$. The phase information of $F'_\eta(\xi,\eta)$ permits us to determine the differences among the one-dimensional phases associated with slices of $F'_\xi(\xi,\eta)$ parallel to the ξ axis. By taking the two-dimensional inverse Fourier transform of the resultant combined function and compensating for the quadratic phase $\exp\left[i\frac{\pi}{\lambda(z+l)}(u^2+v^2)\right]$, we can finally obtain the object function $f(u,v)$.

3. Resolution of the phase retrieval using a scanning slit

In this section we discuss the resolution of the object reconstruction using the present phase retrieval method. As described in Section 2, the approximation of Eq.(6) is effective provided that the main part of the object's spatial-frequency spectrum is smaller than $\sim \frac{1}{2}$ of the first zero's extent of the inverse Fourier transform of the slit function. Hence it is found that the resolution for the object reconstruction from the intensities of Eqs.(3) and (4) is limited by two times the width of the slit. However, the resolution can be improved by changing the coordinates of the measurement points in the detector plane. To explain this fact, we first rewrite Eq.(1) as

$$F(x,y)=\exp\left[i\frac{\pi}{\lambda z}(x^2+y^2)\right]F_f(x,y). \quad (53)$$

where

$$F_f(x,y)=\iint_\sigma f(u,v)\exp\left[i\frac{\pi}{\lambda z}(u^2+v^2)\right]\exp\left[-i\frac{2\pi}{\lambda z}(xu+yv)\right]dudv. \qquad (54)$$

Thus we consider the amplitude $H(\xi,\eta)$ at the coordinates $\xi = s(1+l/z)$ in the detector plane. Substitution of Eq.(14) and $\xi = s\,(1+l/z)$ into Eq.(2) yields

$$H\left[s\left(1+\frac{l}{z}\right),\eta\right]=q'\iint_{-\infty}^{\infty}\exp\left\{i\frac{\pi}{\lambda l}\left[\left(1+\frac{l}{z}\right)(x-s)^2\right]\right\}\exp\left[i\frac{\pi}{\lambda}\left(\frac{1}{z}+\frac{1}{l}\right)y^2\right]$$
$$\times F_f(x,y)R(x-s)\exp\left(-i\frac{2\pi}{\lambda l}y\eta\right)dxdy. \qquad (55)$$

where $q' = \exp[i\pi s^2(1+l/z)/\lambda z + i\pi\eta^2/\lambda l]$. In this equation, if the distances z and l satisfy the conditions required for validity of the Fraunhofer approximation for a slit of width D $(= 2a)$ in the direction of x axis, the quadratic phase factor $\exp\left\{i\,(\pi/\lambda l)\left[(1+l/z)\,(x-s)^2\right]\right\}$ is approximately unity over the slit function $R(x-s)$. Using this approximation, substituting Eq.(15) into Eq.(16), and integrating for y coordinate, the intensity of Eq.(16) can be written as

$$\left|H\left[s\left(1+\frac{l}{z}\right),\eta\right]\right|^2=\left|\int_{s-a}^{s+a}F'_f(x,\eta)\,R(x-s)dx\right|^2 \qquad (56)$$

where

$$F'_f(x,\eta)=\iint_\sigma f(u,v)\exp\left[i\frac{\pi}{\lambda}\left(\frac{u^2}{z}+\frac{v^2}{z+l}\right)\right]\exp\left[-i\frac{2\pi}{\lambda z}\left(xu+\frac{z\eta}{z+l}v\right)\right]dudv. \qquad (57)$$

where unimportant multiplicative constant is ignored. In the same way, the intensity of $H(\xi, \eta)$ at the coordinate $\xi = s(1 + l/z) + \tau$ is given by

$$\left| H\left[s\left(1 + \frac{l}{z}\right) + \tau, \eta \right] \right|^2 = \left| \int_{s-a}^{s+a} F'_f(x, \eta)\, R(x - s) \exp(-2\pi i x \tau / \lambda l) dx \right|^2 . \tag{58}$$

The resolution of the object reconstruction based on Eqs.(17) and (19) is made clear by investigating the inverse Fourier transform of the correlation integral in Eq.(17), which corresponds to the product of two inverse Fourier transforms of $R(x)$ and $F'_f(x, \eta)$. It can be seen form Eq.(18) that the inverse Fourier transform of $F'_f(x, \eta)$ for x coordinate becomes the product of the object function $f(u, v)$ in the u direction and the quadratic phase $\exp(i\pi u^2/\lambda z)$ in contrast to that of $F'(x, \eta)$ in Eq.(5). This reason is that the function $F'_f(x, \eta)$ does not include the quadratic phase factor $\exp[i\pi x^2/\lambda z]$ of $F'(x, \eta)$ in the Fresnel-zone plane. Consequently, we find that in this case the resolution of the object reconstruction is limited by the width of the scanning area with the slit on the Fresnel-zone plane, provided that the object's extent in the u (or v) direction is smaller than $\sim \frac{1}{2}$ of the first zero's extent of the inverse Fourier transform (i.e., the sinc function) of the slit function.

3. Computer simulation

The performance of the present method is demonstrated by computer simulation for the reconstruction of a two-dimensional complex-valued object. Data processing by computer was carried out with 256×256 sampling points, of which the physical size was assigned to 1.2×1.2 mm. The wavelength was set to 0.6328 μm. The distances z

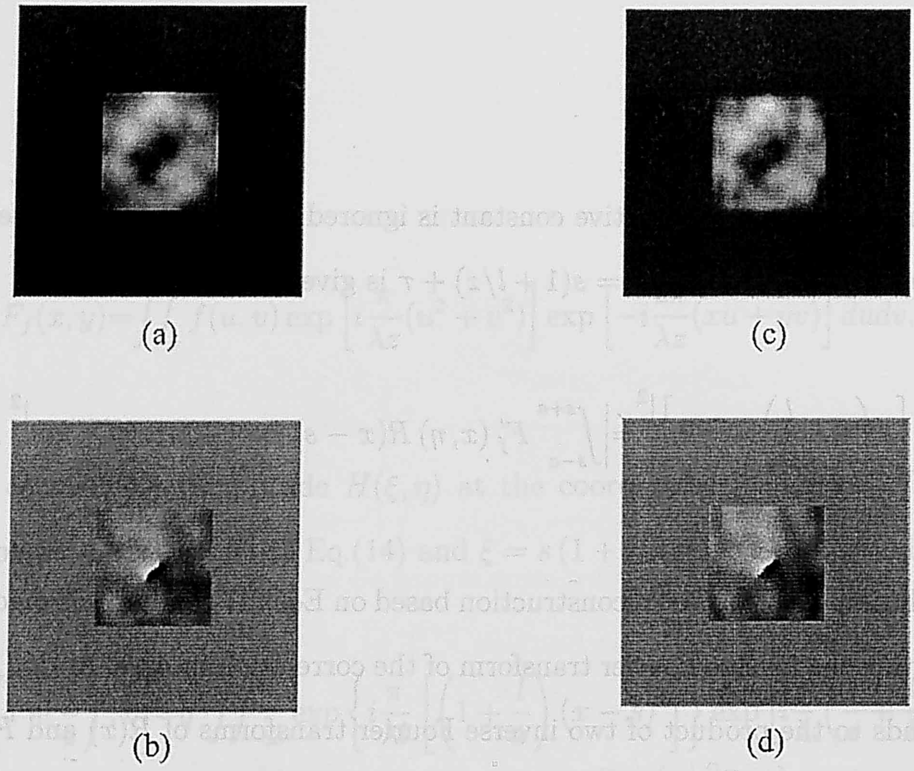

Fig.2. Reconstruction of a complex-valued object with phase vortices in noise-free case:(a) modulus and (b) phase of an original object, and (c) modulus and (d) phase of a reconstructed one.

and l were fixed to 6.67 mm and 20.0 mm, respectively, and the slit aperture of width 63.3 μm was assumed. The advantage of the present method is that one can retrieve the phase distribution with vortices in contrast to the other noniterative methods.[11–14] To demonstrate the phase retrieval of vortices, an object with vortices of Laguerre-Gaussian mode was assumed. Figures 2(a) an 2(b) show the modulus and phase, respectively, of the original object, which was produced by adding a function with two first-order vortices of opposite charge to a random complex function generated by a computer. The extent of the object was the square with sides of 0.48 mm. Figures 2(c) and 2(d) show the modulus and the phase of the object reconstructed from series of noise-free intensities observed at the coordinates $\xi = s, s + \tau$ and $\eta = s, s + \tau'$ in the detector plane as a function of the position s for the slit apertures parallel to η and ξ axes, respectively, where $\tau = \tau' = 28.1\mu$m was assumed.

The example of the reconstruction from noisy intensities is shown in Fig.3, where the distances τ and τ' were increased to 56.2μm in order to raise the signal level of the first term on the left-hand side in Eq.(8). For simulating the noisy intensities, complex normal random noises $n(\xi, \eta)$ are produced by a computer and are added to the complex amplitudes $H(\xi, \eta)$ in the detector plane. A factor of the signal-to-noise ratio (SNR), defined by SNR $= \sum_{s,\eta} |H(s,\eta)|^2 / \sum_{s,\eta} |n(s,\eta)|^2$, is now introduced, where $H(s, \eta)$ and $n(s, \eta)$ are the complex amplitude and the noise, respectively, at the coordinate of the position s of the slit parallel to the ξ axis. Then the SNR in Fig.3 becomes 347. Figure 3(a) and 3(b) show the modulus and the phase of the reconstructed object, and cross-sectional profiles of the original and the reconstructed moduli and phases along the horizontal line that passes through the center of the object are compared in Figs.3(c) and 3(d), respectively. It can be seen from Fig.3, that the reconstruction quality in the neighborhood of the edge of the object's extent appears to be less accurate than that of the central part of the object. The reason of this fact is that the high spatial-frequency components are generally sensitive to noise in the Fresnel intensities, though these components are necessary for reconstructing the edge. However, the result faithfully represents the main features of the original object, particularly, the vortices of the phase.

5. Conclusions

A noniterative phase retrieval method by use of a slit aperture has been presented that is capable of reconstructing a complex-valued object from diffraction intensities measured while scanning the slit in the Fresnel-zone plane. The phase retrieval is based on the properties of entire functions, which are effective under the approximation of the inverse Fourier transform (i.e., the sinc function) of the slit aperture to a Gaussian

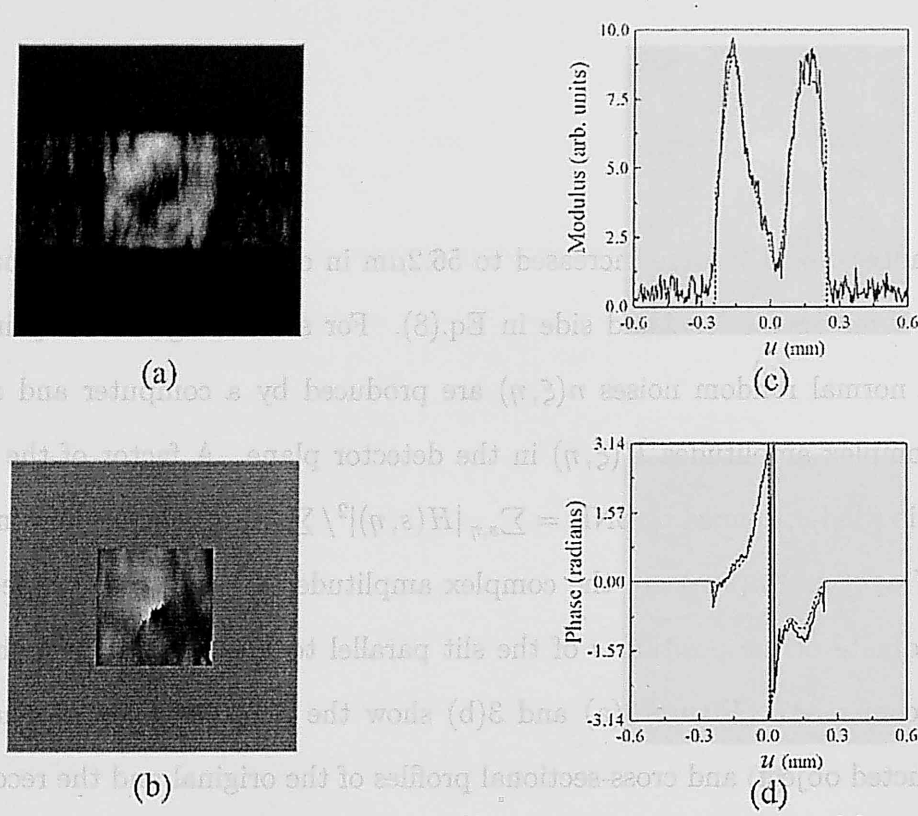

Fig.3. Reconstruction of the object shown in Fig.2 from noisy intensities: (a) modulus and (b) phase of the reconstructed object, and (c) and (d) are cross-sectional profiles of the figures in (a) and (b), respectively, taken along horizontal lines passing through the center of each figure. The dotted and solid curves represent the original and the reconstructed objects, respectively.

function. It was found from computer simulations that this approximation can be used for phase retrieval provided that the main part of the spatial frequency spectrum of an object is smaller than $\sim \dfrac{1}{2}$ of the first zero's extent of the sinc function. On this condition, the phase in the Fresnel-zone plane can be retrieved from two series of intensities recorded at two coordinates in a diffraction plane for the wave passed through the slit aperture. In particular, a new phase calculation technique has been presented here, in which the phase can simply be retrieved by using Fourier transforms in contrast to the solution of the simultaneous equations for unknown coefficients of the phase in the previous papers.[15,16] It has been shown that the resolution of the object reconstruction using the present method is limited by two times the width of the slit in the case that the measurement points are depended to only the position of the slit, but this limitation can be changed into the reciprocal of the width of the scanning area with the slit on the Fresnel-zone plane by the measurement of the intensities at the coordinates depending to the position of the slit and the two distances between the object and Fresnel-zone planes and between the Fresnel-zone and detector planes. The computer-simulated example of this method in the former case of the measurement has presented here the good reconstruction of a complex object with phase vortices. The performance of this method in the latter case of the measurement will be shown in a forthcoming paper.

Since the phase retrieval method used here is noniterative and analytic, there are not convergence problems such as the stagnation and the slow speed convergence. In addition, unlike the other noniterative methods,[11–14] this method can cope with the existence of vortices in the phase distribution. Consequently, these facts exhibit that the present method provides potentially very useful means for lensless coherent imaging in a wide class of wave fields such as x-rays and electron waves.

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