PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 8, Pages 2461–2469
S 0002-9939(02)06343-8
Article electronically published on February 4, 2002

FRÉCHET-URYSOHN SPACES IN FREE TOPOLOGICAL GROUPS

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(Communicated by Alan Dow)

ABSTRACT. Let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group on a Tychonoff space X. For every natural number n we denote by $F_n(X)$ ($A_n(X)$) the subset of $F(X)$ ($A(X)$) consisting of all words of reduced length $\leq n$. It is well known that if a space X is not discrete, then neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn, and hence first countable. On the other hand, it is seen that both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn for a paracompact Fréchet-Urysohn space X. In this paper, we prove first that for a metrizable space X, $F_3(X)$ ($A_3(X)$) is Fréchet-Urysohn if and only if the set of all non-isolated points of X is compact and $F_5(X)$ is Fréchet-Urysohn if and only if X is compact or discrete. As applications, we characterize the metrizable space X such that $A_n(X)$ is Fréchet-Urysohn for each $n \geq 3$ and $F_n(X)$ is Fréchet-Urysohn for each $n \geq 3$ except for $n = 4$. In addition, however, there is a first countable, and hence Fréchet-Urysohn subspace Y of $F(X)$ ($A(X)$) which is not contained in any $F_n(X)$ ($A_n(X)$). We shall show that if such a space Y is first countable, then it has a special form in $F(X)$ ($A(X)$). On the other hand, we give an example showing that if the space Y is Fréchet-Urysohn, then it need not have the form.

1. INTRODUCTION

All spaces are assumed to be Tychonoff and we denote by $\mathbb{N}$ the set of all natural numbers. Let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group on a Tychonoff space X in the sense of Markov [4]. For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose reduced length is less than or equal to n. Then each $F_n(X)$ is closed in $F(X)$. This concept is defined for $A(X)$ in the same fashion. It is well known that if a space X is not discrete, then neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn, and hence first countable (see [1]). On the other hand, $F_n(X)$ and $A_n(X)$, $n \in \mathbb{N}$, have a chance to be first countable for a non-discrete space X. In fact, the author [6] recently obtained the following results:

For a metrizable space X, the following are equivalent: (i) $A_n(X)$ *is metrizable for each* $n \in \mathbb{N}$; (ii) $A_n(X)$ *is first countable for each* $n \in \mathbb{N}$; (iii) $A_2(X)$ *is metrizable*; (iv) $A_2(X)$ *is first countable*; (v) *the set of all non-isolated points of*

Received by the editors June 20, 2000 and, in revised form, March 7, 2001.
1991 *Mathematics Subject Classification.* Primary 54H11, 54A35, 54A25.
Key words and phrases. Free topological group, free Abelian topological group, Fréchet-Urysohn space, first countable space, semidirect product.

X is compact. In the non-Abelian case the following are equivalent: (i) $F_n(X)$ *is metrizable for each* $n \in \mathbb{N}$; (ii) $F_n(X)$ *is first countable for each* $n \in \mathbb{N}$; (iii) $F_4(X)$ *is metrizable*; (iv) $F_4(X)$ *is first countable*; (v) *X is compact or discrete. Furthermore, the following are also equivalent*: (i) $F_3(X)$ *is metrizable*; (ii) $F_3(X)$ *is first countable*; (iii) $F_2(X)$ *is metrizable*; (vi) $F_2(X)$ *is first countable*; (v) *the set of all non-isolated points of X is compact.*

In the proofs of the above results, we proved that for a metrizable space X, if $F_2(X)$ $(A_2(X))$ is first countable, then the set of all non-isolated points of X is compact. It is easy to see that both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn for a metrizable space X (see the next section). In this paper, we shall show that for a metrizable space X if $F_n(X)$ $(A_n(X))$ is Fréchet-Urysohn for some $n \geq 3$, then the set of all non-isolated points of X is compact. Moreover we shall prove that for a metrizable space X, if $F_n(X)$ is Fréchet-Urysohn for some $n \geq 5$, then X is compact or discrete. To prove it, we need some algebraic techniques; that is, we shall construct actions of groups on spaces and a semidirect product with respect to the action.

We call a subspace Y of $F(X)$ $(A(X))$ *bounded* in $F(X)$ $(A(X))$ if Y is contained in $F_n(X)$ $(A_n(X))$ for some $n \in \mathbb{N}$. On the other hand, a subspace Y of $F(X)$ $(A(X))$ is called *unbounded* in $F(X)$ $(A(X))$ if Y is not bounded in $F(X)$ $(A(X))$. As we mentioned above, neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn if a space X is not discrete. On the other hand, from the above results, we can construct concrete spaces X and Y such that Y is first countable and bounded in $F(X)$ $(A(X))$. Then the following natural question can be considered:

Are there spaces X and Y such that Y is first countable or Fréchet-Urysohn, and Y is unbounded in $F(X)$ $(A(X))$?

However it is easy to answer the question positively. We shall show, in §3, that every unbounded subspace Y must have a special form in $F(X)$ $(A(X))$ if Y is first countable. That is, the family $\{Y \cap (F_{n+1}(X) \setminus F_n(X)) : n \in \mathbb{N}\}$ $(\{Y \cap (A_{n+1}(X) \setminus A_n(X)) : n \in \mathbb{N}\})$ has to be discrete in Y. On the other hand, we give an example of a locally compact separable metric space X and a Fréchet-Urysohn subspace Y of $F(X)$ $(A(X))$ such that Y does not have the above form. The example also gives us a first countable subspace of $F(X)$ $(A(X))$ such that the above family is not discrete in $F(X)$ $(A(X))$.

We refer to [3] for elementary properties of topological groups and to [1] for the main properties of free topological groups.

2. Fréchet-Urysohn property of $F_n(X)$ and $A_n(X)$

In this section we study metrizable spaces X for which $F_n(X)$ and $A_n(X)$ are Fréchet-Urysohn. Recently the author showed that if a space X is paracompact, then $F_2(X)$ is a closed image of $(X \oplus \{e\} \oplus X^{-1})^2$ and $A_2(X)$ is a closed image of $(X \oplus \{0\} \oplus -X)^2$ (see [7, Proposition 4.8]). Hence, both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn if X is metrizable. In addition, in the same paper [6], he proved that for a metrizable space, (i) if $F_n(X)$ is first countable for some $n \geq 2$, then the set of all non-isolated points of X is compact and the same is true for $A_n(X)$, and (ii) if $F_n(X)$ is first countable for some $n \geq 4$, then X is compact or discrete. We shall improve the result (i) for $n \geq 3$ and the result (ii) for $n \geq 5$ by showing the hypothesis of $F_n(X)$ and $A_n(X)$ is enough to be Fréchet-Urysohn.

For a space X, let $\mathcal{U}_{\overline{X}}$ and $\mathcal{U}_X$ be the universal uniformities on $\overline{X} = X \oplus \{e\} \oplus X^{-1}$ and on X, respectively. For each $n \in \mathbb{N}$, $U \in \mathcal{U}_{\overline{X}}$ and $U' \in \mathcal{U}_X$, the author defined the subsets $W_n(U)$ of $F_{2n}(X)$ in [7] and $V(U')$ of $A_{2n}(X)$ in [6], as follows:

$W_n(U)$ is a subset of $F_{2n}(X)$ which consists of the identity e and all words g satisfying the following conditions;

(1) g can be represented as the reduced form $g = x_1 x_2 \cdots x_{2k}$, where $x_i \in \overline{X}$ for $i = 1, 2, \ldots, k$ and $1 \leq k \leq n$,
(2) there is a partition $\{1, 2, \ldots, 2k\} = \{i_1, i_2, \ldots, i_k\} \cup \{j_1, j_2, \ldots, j_k\}$,
(3) $i_1 < i_2 < \cdots < i_k$ and $i_s < j_s$ for $s = 1, 2, \ldots, k$,
(4) $(x_{i_s}, x_{j_s}^{-1}) \in U$ for $s = 1, 2, \ldots, k$ and
(5) $i_s < i_t < j_s \Longleftrightarrow i_s < j_t < j_s$ for $s, t = 1, 2, \ldots, k$.

$V_n(U') = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U' \text{ for } i = 1, 2, \ldots, k, k \leq n\}$.

Then the following are proved.

Theorem 2.1. *Let X be a space. Then:*

(1) *([7]) $W_n(U)$ is a neighborhood of e in $F_{2n}(X)$ for each $U \in \mathcal{U}_{\overline{X}}$, and*
(2) *([6]) $V_n(U')$ is a neighborhood of 0 in $A_{2n}(X)$ for each $U' \in \mathcal{U}_X$.*

The above neighborhoods are used to prove the following result.

Theorem 2.2. *Let X be a metrizable space. If $F_n(X)$ for some $n \geq 3$ is Fréchet-Urysohn, then the set of all non-isolated points of X is compact. The same is true for $A_n(X)$.*

Proof. It suffices to prove the theorem for $n = 3$. Suppose that the set of all non-isolated points of X is not compact, and take sequences $\{y_i : i \in \mathbb{N} \cup \{0\}\}$, $\{x_i : i \in \mathbb{N}\}$ and $\{x_{i,j} : j \in \mathbb{N}\}$ $(i \in \mathbb{N})$ in X such that

(1) the set $Y = \{y_i : i \in \mathbb{N} \cup \{0\}\} \cup \{x_i : i \in \mathbb{N}\} \cup \{x_{i,j} : i, j \in \mathbb{N}\}$ consists of distinct points of X,
(2) the sequence $\{y_i : i \in \mathbb{N}\}$ converges to y_0,
(3) the sequence $\{x_{i,j} : j \in \mathbb{N}\}$ converges to x_i for each $i \in \mathbb{N}$ and
(4) $\{\{y_i : i \in \mathbb{N} \cup \{0\}\}\} \cup \{\{x_{i,j} : j \in \mathbb{N}\} : i \in \mathbb{N}\}$ is a discrete family of closed subsets of X.

For every $i \in \mathbb{N}$ put $D_i = \{x_{i,j}\, x_i^{-1}\, y_i : j \in \mathbb{N}\}$ and $D = \bigcup_{i=1}^{\infty} D_i$. Then D is a subset of $F_3(X)$ and (3) implies that the sequences D_i converge to y_i, respectively. Hence, by (2), we have that $y_0 \in \overline{D}$. To prove that $F_3(X)$ is not Fréchet-Urysohn, we need to show that there are no sequences in D which converge to y_0.

Let S be an arbitrary sequence in D. If $S \cap D_i$ is infinite for some i, then S cannot converge to y_0 by (1) and (3). So, we may assume that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $S \cap D_i \subseteq \{x_{i,j}\, x_i^{-1}\, y_i : j \leq f(i)\}$ for each $i \in \mathbb{N}$. Let $V = (\{y_i : i \in \mathbb{N} \cup \{0\}\})^2 \cup \bigcup_{i=1}^{\infty} (\{x_{i,j} : j > f(i)\} \cup \{x_i\})^2$. Then V is an open neighborhood of the diagonal Δ_Y in Y^2. Let W be an open neighborhood of the diagonal Δ_X in X^2 such that $W \cap Y^2 = V$ and put $U = W \cup \{(x^{-1}, y^{-1}) : (x, y) \in W\} \cup \{(e, e)\}$. Then $U \in \mathcal{U}_{\overline{X}}$. By Theorem 2.1(1), $W_2(U)$ is a neighborhood of e in $F_4(X)$. In addition, if we put $B_{y_0} = y_0 W_2(U) \cap F_3(X)$, then B_{y_0} is a neighborhood of y_0 in $F_3(X)$ (see [7, Lemma 3.1]). On the other hand, from the definition of V, it is easy to see that $B_{y_0} \cap S = \emptyset$. This means that the sequence S cannot converge to y_0. Consequently, $F_3(X)$ is not Fréchet-Urysohn.

In the Abelian case, put $D_i = \{x_{i,j} - x_i + y_i : j \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Then, applying Theorem 2.1(2), the above argument also implies that $(y_0 + V_2(W)) \cap A_3(X) \cap S = \emptyset$, and hence $A_3(X)$ is not Fréchet-Urysohn. □

Remark 2.3. Let C be a non-trivial convergent sequence $\{x_i : i \in \mathbb{N}\}$ with its limit x and $D = \{d_i : i \in \mathbb{N} \cup \{0\}\}$ be an infinite discrete space consisting of distinct points. Then $C \times D$ is homeomorphic to the space Y which appears in the above proof. Put $D_i = \{(x_j, d_i) - (x, d_i) + (x, d_0) : j \in \mathbb{N}\} \subseteq A_3(C \times D)$ $(i \in \mathbb{N})$ and $D = \bigcup_{i=1}^{\infty} D_i$. Then the above argument yields that, in $A_3(C \times D)$, no sequences in D converge to (x, d_0), however $(x, d_0) \in \overline{D}$. We apply the fact in the proof of the next theorem.

Theorem 2.4. *Let X be a metrizable space. If $F_n(X)$ is Fréchet-Urysohn for some $n \geq 5$, then X is compact or discrete.*

Proof. Suppose that a metrizable space X is neither compact nor discrete, and choose sequences $\{x_i : i \in \mathbb{N}\}$ and $\{d_i : i \in \mathbb{N}\}$ consisting of distinct points in X and a point x in X such that the sequence $\{x_i : i \in \mathbb{N}\}$ converges to x and $\{\{x_i : i \in \mathbb{N}\} \cup \{x\}\} \cup \{\{d_i\} : i \in \mathbb{N}\}$ is a discrete closed family in X. For each $i \in \mathbb{N}$ put $E_i = \{d_i\, x_j\, x^{-1} d_i^{-1} x_i : j \in \mathbb{N}\}$ and $E = \bigcup_{i=1}^{\infty} E_i$. Then E is a subset of $F_5(X)$. Since each sequence E_i converges to x_i, we have that $x \in \overline{E}$. To prove that $F_5(X)$ is not Fréchet-Urysohn, we need to show that there are no sequences in E which converge to x in $F_5(X)$. Let S be an arbitrary sequence in E. Then we may assume that $S \cap E_i$ is a non-empty finite set for each $i \in \mathbb{N}$, that is, we may assume that for each $i \in \mathbb{N}$ there is a non-empty finite set $p_i \subseteq \mathbb{N}$ such that $S \cap E_i = \{g_{i,j} = d_i\, x_j\, x^{-1} d_i^{-1} x_i : j \in p_i\}$. To prove that S cannot converge to x, we need to construct some mappings and topological groups which are defined by Pestov and the author in [5].

Let $C = \{x_i : i \in \mathbb{N}\} \cup \{x\}$ and $D = \{d_i : i \in \mathbb{N}\}$. Define a mapping $\tau : F(D) \times (C \times F(D)) \to C \times F(D)$ by letting $\tau((g, (x, h))) = (x, gh)$ for each $(g, (x, h)) \in F(D) \times (C \times F(D))$. Since $F(D)$ is a discrete space, τ is a continuous action of the group $F(D)$ on the space $C \times F(D)$. For every $g \in F(D)$, the self-homeomorphism $\tau_g : C \times F(D) \to C \times F(D) : (x, h) \to (x, gh)$ can be extended to an automorphism $\overline{\tau_g} : A(C \times F(D)) \to A(C \times F(D))$. Then, put $\overline{\tau} : F(D) \times A(C \times F(D)) \to A(C \times F(D))$ as $\overline{\tau}((g, h)) = \overline{\tau_g}(h)$ for each $(g, h) \in F(D) \times A(C \times F(D))$. Since $F(D)$ is a discrete space, $\overline{\tau}$ is a continuous action of $F(D)$ on $A(C \times F(D))$.

Let $G = F(D) \ltimes_\tau A(C \times F(D))$ be the semidirect product formed with respect to the action $\overline{\tau}$. In other words, as a topological space, G is the product of $F(D)$ and $A(C \times F(D))$ and the group operation is given by $(g, a) \cdot (h, b) = (gh, a + \overline{\tau_g}(b))$, where $g, h \in F(D)$ and $a, b \in A(C \times F(D))$. Since $A(C \times F(D))$ (identified with $\{e\} \times A(C \times F(D))$, where e is the unit element of $F(D)$) forms an open normal subgroup of G, let $\pi : G \to A(C \times F(D))$ be the quotient mapping.

Define a mapping $\psi : C \oplus D \to G$ by

$$\psi(t) = \begin{cases} (e, (t, e)) \in F(D) \ltimes_\tau A(C \times F(D)), & \text{if } t \in C, \\ (t, 0) \in F(D) \ltimes_\tau A(C \times F(D)), & \text{if } t \in D, \end{cases}$$

where 0 denotes the unit element of $A(C \times F(D))$. Since

$$\lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} (e, (x_n, e)) = (e, (x, e)) = \psi(x),$$

the mapping ψ is continuous and therefore extends to a continuous homomorphism $\overline{\psi} : F(C \oplus D) \to G$. Let $Y = C \times (\{e\} \oplus D) \subseteq C \times F(D)$ and

$$f = (\pi \circ \overline{\psi})|_{(\pi\circ\overline{\psi})^{-1}(A(Y))} : (\pi \circ \overline{\psi})^{-1}(A(Y)) \to A(Y).$$

Then, since $A(Y)$ is a topological subgroup of $A(C \times F(D))$, f is a continuous homomorphism.

We return to prove that the sequence S cannot converge to x in $F_5(X)$. Since $F_5(C \oplus D)$ is a subspace of $F_5(X)$ and $S \cup \{x\} \subseteq F_5(C \oplus D)$, it suffices to show that S does not converge to x in $F_5(C \oplus D)$. Let $i \in \mathbb{N}$ and $j \in p_i$. Before we calculate $\overline{\psi}(g_{i,j})$, let us note that the inverse elements of $(d, 0)$ and $(e, (x, e))$ in G are $(d^{-1}, 0)$ and $(e, -(x, e))$ for each $d \in D$ and $x \in C$, respectively. Hence $g_{i,j}$ is mapped by $\overline{\psi}$, as follows:

$$\begin{aligned}
\overline{\psi}(g_{i,j}) &= \overline{\psi}(d_i x_j x^{-1} d_i^{-1} x_i) = \overline{\psi}(d_i)\overline{\psi}(x_j)\overline{\psi}(x)^{-1}\overline{\psi}(d_i)^{-1}\overline{\psi}(x_i)\\
&= \psi(d_i)\psi(x_j)\psi(x)^{-1}\psi(d_i)^{-1}\psi(x_i)\\
&= (d_i, 0)(e, (x_j, e))(e, (x, e))^{-1}(d_i, 0)^{-1}(e, (x_i, e))\\
&= (d_i, \overline{\tau_{d_i}}((x_j, e)))(e, -(x, e))(d_i^{-1}, 0)(e, (x_i, e))\\
&= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e) + \overline{\tau_e}(0))(e, (x_i, e))\\
&= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e))(e, (x_i, e))\\
&= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e) + \overline{\tau_{d_i^{-1}}}(x_i, e))\\
&= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e) + (x_i, d_i^{-1}))\\
&= (e, (x_j, d_i) + \overline{\tau_{d_i}}(-(x, e) + (x_i, d_i^{-1})))\\
&= (e, (x_j, d_i) - \overline{\tau_{d_i}}(x, e) + \overline{\tau_{d_i}}(x_i, d_i^{-1}))\\
&= (e, (x_j, d_i) - (x, d_i) + (x_i, e)).
\end{aligned}$$

Hence $\pi \circ \overline{\psi}(g_{i,j}) = (x_j, d_i) - (x, d_i) + (x_i, e)$. Since $(x_j, d_i), (x, d_i), (x_i, e) \in Y$, $\pi \circ \overline{\psi}(g_{i,j}) \in A(Y)$. This means that $f(g_{i,j}) = (x_j, d_i) - (x, d_i) + (x_i, e)$. Therefore, we have that $f(S) = \{(x_j, d_i) - (x, d_i) + (x_i, e) : j \in p_i, i \in \mathbb{N}\} \subseteq A_3(Y)$. On the other hand, in the same way, we can show that $f(x) = (x, e) \in A(Y)$. Hence, by Remark 2.3, the sequence $f(S)$ does not converge to $f(x)$ in $A_3(Y)$. This yields that S does not converge to x in $F_5(C \oplus D)$. Therefore, we conclude that $F_5(X)$ is not Fréchet-Urysohn. □

From the above theorems, we obtain the following results which improve Theorem 4.5, 4.9 and 4.12 in [7].

Corollary 2.5. *For a metrizable space X, the following are equivalent*:

(1) $A_n(X)$ *is metrizable for each* $n \in \mathbb{N}$;
(2) $A_n(X)$ *is first countable for each* $n \in \mathbb{N}$;
(3) $A_n(X)$ *is Fréchet-Urysohn for each* $n \in \mathbb{N}$;
(4) $A_2(X)$ *is metrizable*;
(5) $A_2(X)$ *is first countable*;
(6) $A_3(X)$ *is Fréchet-Urysohn*;
(7) $F_3(X)$ *is metrizable*;
(8) $F_3(X)$ *is first countable*;
(9) $F_3(X)$ *is Fréchet-Urysohn*;
(10) $F_2(X)$ *is metrizable*;

(11) $F_2(X)$ *is first countable*;
(12) *the set of all non-isolated points of* X *is compact.*

Proof. The equivalence of the statements (1), (2), (4), (5), (7), (8), (10), (11) and (12) are due to Theorem 4.5 and 4.12 in [7]. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (6) and (8) $\Rightarrow$ (9) are trivial. Theorem 2.2 yields the implications (6) $\Rightarrow$ (12) and (9) $\Rightarrow$ (12). $\square$

Corollary 2.6. *For a metrizable space* X*, the following are equivalent*:
(1) $F_n(X)$ *is metrizable for each* $n \in \mathbb{N}$;
(2) $F_n(X)$ *is first countable for each* $n \in \mathbb{N}$;
(3) $F_n(X)$ *is Fréchet-Urysohn for each* $n \in \mathbb{N}$;
(4) i_n *is a closed mapping for each* $n \in \mathbb{N}$;
(5) $F_4(X)$ *is metrizable* ;
(6) $F_4(X)$ *is first countable* ;
(7) $F_5(X)$ *is Fréchet-Urysohn* ;
(8) i_3 *is a closed mapping* ;
(9) X *is compact or discrete.*

Proof. The equivalence of the statements (1), (2), (4), (5), (6), (8) and (9) is due to Lemma 4.7 and Theorem 4.9 in [7], and the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (7) are trivial. Theorem 2.4 yields the implication (7) $\Rightarrow$ (9). $\square$

As we mentioned at the beginning of this section, we have already shown that the mapping i_2 is closed if and only if every neighborhood of the diagonal in X^2 is an element of the universal uniformity $\mathcal{U}_X$ of X (see [7, Proposition 4.8]). In particular, i_2 is closed for a paracompact space. Therefore both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn for a metrizable space X. So, the reader must note that it is not clarified that the equivalent condition of a metrizable space X for $F_4(X)$ be Fréchet-Urysohn. Unfortunately, the author does not know about it. He just conjectures that $F_4(X)$ is Fréchet-Urysohn if the set of all non-isolated points of a metrizable space X is compact, and hence we could add the statement that $F_4(X)$ is Fréchet-Urysohn on the list of equivalences in Corollary 2.5.

3. Unbounded subspaces of $F(X)$ and $A(X)$

As we mentioned in §1, however, neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn for a non-discrete space X, and there are non-discrete spaces X and Y such that Y is a first countable unbounded subspace of $F(X)$ $(A(X))$. For example, let X be any non-discrete first countable space. Fix an element $x \in X$ and for each $n \in \mathbb{N}$ let $X_n = X \cdot x^n$. Then the subspace X_n of $F(X)$ is homeomorphic to X. Let f be a function from X to the additive group of integers such that $f(x) = 1$ for each $x \in X$. Then the homomorphic extension $F(f)$ of f over $F(X)$ is continuous. Since $F(f)(g) = n + 1$ for each $g \in X_n$ and $n \in \mathbb{N}$, it follows that the subspace $Y = \bigcup_{i=1}^{\infty} X_n$ is the sum of $\{X_n : n \in \mathbb{N}\}$. Hence Y is a required unbounded subspace of $F(X)$.

For each $n \in \mathbb{N}$, let $E_n(X) = F_n(X) \setminus F_{n-1}(X)$. Then, for the above subspace Y, the family $\{X_n = Y \cap E_n : n \in \mathbb{N}\}$ is discrete in Y. Generally, we can show that every first countable subspace Y of $F(X)$ $(A(X))$ has this property.

Proposition 3.1. *Let* X *be a Tychonoff space and* Y *a first countable subspace of* $F(X)$*. Then* $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ *is discrete in* Y*. The same is true for* $A(X)$*.*

Proof. Suppose that there is a first countable subspace Y of $F(X)$ such that $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is not discrete at a word g in Y. Since Y is first countable, g has a countable neighborhood base in Y. So, we can choose sequences $\{k_n : n \in \mathbb{N}\}$ of natural numbers and $\{g_n : n \in \mathbb{N}\}$ in Y such that

(1) $k_1 < k_2 < \cdots$,
(2) $g_n \in Y \cap E_{k_n}$ for each $n \in \mathbb{N}$ and
(3) the sequence $\{g_n\}$ converges to g in Y.

Then the compact set $\{g_n : n \in \mathbb{N}\} \cup \{g\}$ is unbounded in $F(X)$. Since every compact subset of $F(X)$ is bounded in $F(X)$, this is a contradiction. □

Corollary 3.2. *Let X be a Tychonoff space and Y a subspace of $F(X)$ satisfying one of the following properties: locally compactness, Čech-completeness, first countability and point-countable type. Then $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is discrete in Y. The same is true for $A(X)$.*

Proof. Recall that a space Y is of *point-countable type* iff for each point $p \in Y$, there is a compact set K and K has countable character. Then, the argument of the proof of Proposition 3.1 implies that $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is discrete in a subspace Y of $F(X)$ if Y is of point-countable type. Since the point-countable type is the weakest property among the above properties, this completes the proof. □

In general, The family $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ of Corollary 3.2 is not necessary to be discrete in $F(X)$ $((A(X))$. Furthermore, the family $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is not necessary to be discrete in Y if Y is Fréchet-Urysohn. The following example shows these facts.

Example 3.3. There is an unbounded subspace Y of the free topological group $F(X)$ on a metrizable space X such that Y is a locally compact separable metric space and $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is not discrete at e in $F(X)$. In addition, if we put $Z = Y \cup \{e\}$, then Z is Fréchet-Urysohn and $\{Z \cap E_n(X) : n \in \mathbb{N}\}$ is not discrete at e in Z.

Proof. Let $X = \bigoplus_{n=1}^{\infty} C_n$, where each C_n is a non-trivial convergent sequence $\{x_{k,n} : k \in \mathbb{N}\}$ with its limit x_n. For each $n \in \mathbb{N}$ let $p(n) = \frac{n(n-1)}{2}$ and

$$S_n = \{{x_{p(n)+1}}^{-1} x_{k,p(n)+1} {x_{p(n)+2}}^{-1} x_{k,p(n)+2} \cdots {x_{p(n)+n}}^{-1} x_{k,p(n)+n} : k \in \mathbb{N}\}.$$

Since each S_n is a subset of E_{2n}, the set $Y = \bigcup_{n=1}^{\infty} S_n$ is unbounded in $F(X)$. To prove that Y is a required subspace of $F(X)$, it suffices to show that the subspace $Z = Y \cup \{e\}$ of $F(X)$ is homeomorphic to the sequential fan $S(\omega)$. It is clear that each sequence S_n converges to e and $S_i \cap S_j = \emptyset$ if $i \neq j$. Hence it suffices to show that a subset L of Z is closed in Z whenever the intersection of L with $S_n \cup \{e\}$ is closed in $S_n \cup \{e\}$ for each $n \in \mathbb{N}$. Let L be a subset of Z such that $L \cap (S_n \cup \{e\})$ is closed in $S_n \cup \{e\}$ for each $n \in \mathbb{N}$ and K be a compact subset of $F(X)$. Since K is bounded in $F(X)$, there is $n \in \mathbb{N}$ such that $L \cap K \subseteq \bigcup_{i=1}^{n} S_i \cup \{e\}$. It follows that $L \cap K = \bigcup_{i=1}^{n} (L \cap (S_i \cup \{e\})) \cap K$, and hence $L \cap K$ is closed in K. Since X is a locally compact separable metrizable space, $F(X)$ is a k-space (see [2, Theorem 2.11]). Thus the above argument yields that L is closed in $F(X)$, and hence in Z. Consequently we can prove that Z is homeomorphic to $S(\omega)$. □

In particular, Corollary 3.2 yields that the subspace $\bigcup_{i=1}^{\infty} E_{n_i}$ does not satisfy any properties of Corollary 3.2 for each subsequence $\{n_i : i \in \mathbb{N}\}$ of natural numbers, as follows.

Corollary 3.4. *Let X be a non-discrete Tychonoff space. Then, for every sequence $\{n_i : n \in \mathbb{N}\}$ of natural numbers, $\bigcup_{i=1}^{\infty} E_{n_i}(X)$ is not of point-countable type, and hence it does not satisfy any properties of Corollary* 3.2. *The same is also true for $A(X)$.*

Proof. Let $Y = \bigcup_{i=1}^{\infty} E_{n_i}(X)$. By Corollary 3.2, it suffices to show that $\mathcal{E} = \{E_{n_i}(X) : i \in \mathbb{N}\}$ is not discrete in Y. To prove that, choose a subsequence $\{m_i : i \in \mathbb{N}\}$ of $\{n_i : i \in \mathbb{N}\}$ and natural numbers k_i such that $m_{i+1} - m_i = 2k_i$ for each $i \in \mathbb{N}$. Fix $g \in E_{m_1}$, a non-isolated point x in X and an open neighborhood U of g in $F(X)$. Denote the set of all elements of X taking part in the reduced form of the word g by $\operatorname{car} g$. Then we can choose a sequence $\{U_i : i \in \mathbb{N}\}$ of an open neighborhood of e in $F(X)$ and sequences $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$ in X satisfying the following properties:

(1) $U_i = U_i^{-1}$ for each $i \in \mathbb{N}$,
(2) $gU_1 \subseteq U$ and ${U_{i+1}}^{2p_i} \subseteq U_i$ for each $i \in \mathbb{N}$, where $p_i = \sum_{j=1}^{i} k_j$,
(3) the family $\{a_i : i \in \mathbb{N}\} \cup \{b_i : i \in \mathbb{N}\} \cup \operatorname{car} g$ consists of distinct points in X and
(4) $a_i, b_i \in xU_{i+1}$.

For each $i \in \mathbb{N}$, since $a_i^{-1}b_i \in (xU_{i+1})^{-1}xU_{i+1} = U_{i+1}^{-1}U_{i+1} = {U_{i+1}}^2$, we have that $(a_i^{-1}b_i)^{p_i} \in ({U_{i+1}}^2)^{p_i} = {U_{i+1}}^{2p_i} \subseteq U_i$. For each $i \in \mathbb{N}$, put $g_i = g(a_i^{-1}b_i)^{p_i}$. Then each $g_i \in gU_i \subseteq gU_1 \subseteq U$. Furthermore, property (3) implies the length of $g_i = m_1 + 2p_i = m_1 + 2\sum_{j=1}^{i} k_j = m_{i+1}$. It follows that $g_i \in E_{m_{i+1}}$. Thus we have that $U \cap E_{m_{i+1}} \neq \emptyset$ for each $i \in \mathbb{N}$. This means that $\mathcal{E}$ is not discrete at g in Y. □

However Proposition 3.1 is not true for Fréchet-Urysohn spaces as is shown in Example 3.3; Corollary 3.4 is also true for Fréchet-Urysohn spaces.

Proposition 3.5. *Let X be a non-discrete Tychonoff space. Then, for every sequence $\{n_i : i \in \mathbb{N}\}$ of natural numbers, $\bigcup_{i=1}^{\infty} E_{n_i}(X)$ is not a Fréchet-Urysohn space.*

Proof. Let $Y = \bigcup_{i=1}^{\infty} E_{n_i}(X)$ and suppose that Y is Fréchet-Urysohn. Choose a subsequence $\{m_i : i \in \mathbb{N}\}$ of $\{n_i : i \in \mathbb{N}\}$ such that $m_{i+1} - m_i = 2k_i$ $(k_i \in \mathbb{N})$ for each $i \in \mathbb{N}$. Then, applying the proof of Corollary 3.4, we can show that for each $i \in \mathbb{N}$ and $g \in E_{m_i}(X)$, $g \in \overline{E_{m_j}(X)}^Y$ for each $j \geq i$. Choose $g \in E_{m_1}$. Since $g \in \overline{E_{m_2}}^Y \setminus E_{m_2}$ there is a non-trivial sequence $\{g_i : i \in \mathbb{N}\}$ in E_{m_2} that converges to g. For each $i \in \mathbb{N}$ since $g_i \in \overline{E_{m_{i+2}}}^Y \setminus E_{m_{i+2}}$ we can take a non-trivial sequence $\{g_{i,j} : j \in \mathbb{N}\}$ in $E_{m_{i+2}}$ that converges to g_i. We put $A = \{g_{i,j} : i, j \in \mathbb{N}\}$. Since the sequence $\{g_i\}$ converges to g, we have that $g \in \overline{A}$. So there is a sequence S in A that converges to g. Each sequence $\{g_{i,j} : j \in \mathbb{N}\}$ converges to g_i and $g_i \neq g$. It follows that the sequence S is unbounded in $F(X)$. Since $S \cup \{g\}$ is compact, this is a contradiction. Consequently Y is not Fréchet-Urysohn. □

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