# A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees

Teruyuki Yorioka

*Department of Mathematics, Shizuoka University, Ohya 836, Shizuoka, 422-8529, JAPAN*

**Abstract**

We introduce a property of forcing notions, called the anti-$\mathsf{R}_{1,\aleph_1}$, which comes from Aronszajn trees. This property canonically defines a new chain condition stronger than the countable chain condition, which is called the property $\mathsf{R}_{1,\aleph_1}$.

In this paper, we investigate the property $\mathsf{R}_{1,\aleph_1}$. For example, we show that a forcing notion with the property $\mathsf{R}_{1,\aleph_1}$ does not add random reals. We prove that it is consistent that every forcing notion with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$ and $\mathsf{MA}_{\aleph_1}$ for forcing notions with the property $\mathsf{R}_{1,\aleph_1}$ fails. This negatively answers a part of one of classical problems about implications between fragments of $\mathsf{MA}_{\aleph_1}$.

*Key words:* Martin's Axiom and its fragments, Aronszajn trees, $(\omega_1, \omega_1)$-gaps, unbounded families, entangled sets of reals, adding no random reals, non-special Aronszajn trees.
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## 1. Introduction

Martin's Axiom was introduced by Martin and Solovay in [21]. In 1980's, Todorčević investigated Martin's Axiom from the view point of Ramsey theory, and introduced the countable chain condition, abbreviated as the ccc, for partitions on the set of finite sets of countable ordinals, $[\omega_1]^{<\aleph_0}$. In [35], Todorčević and Veličković proved that $\mathsf{MA}_{\aleph_1}$, which is Martin's Axiom for $\aleph_1$ many dense sets, is equivalent to the statement $\mathcal{H}$ that every ccc partition $K_0 \cup K_1$ on $[\omega_1]^{<\aleph_0}$ has an uncountable $K_0$-homogeneous set. Todorčević also introduced many fragments of $\mathsf{MA}_{\aleph_1}$ in his many papers e.g. [32, 35]. Some of them are as follows: For each $n \in \omega$, $\mathcal{K}_n$ is the statement that every uncountable subset of a ccc forcing notion has an uncountable $n$-linked subset, $\mathcal{K}'_n$ is the statement

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*Email addresses:* `styorio@ipc.shizuoka.ac.jp` (Teruyuki Yorioka)

that every ccc partition $K_0 \cup K_1 = [\omega_1]^n$ has an uncountable $K_0$-homogeneous set, and $\mathcal{C}^2$ is the statement that every product of ccc forcing notions has the countable chain condition ([1]). We note that they have many applications. For example, $\mathcal{C}^2$ implies Suslin's Hypothesis, every $(\omega_1, \omega_1)$-gap is indestructible, and $\mathfrak{b} > \aleph_1$, and $\mathcal{K}_2'$ implies that every Aronszajn tree is special. We also note the following diagram of implications between them:

$$
\begin{tikzcd}
\mathsf{MA}_{\aleph_1} \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \mathcal{K}_{n+1} \arrow[r] \arrow[d] & \mathcal{K}_n \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[d] & \mathcal{C}^2 \\
\mathcal{H} \arrow[r] & \cdots \arrow[r] & \mathcal{K}'_{n+1} \arrow[r] & \mathcal{K}'_n \arrow[r] & \cdots \arrow[r] & \mathcal{K}'_2 &
\end{tikzcd}
$$

But it is unknown whether any other implications hold in $\mathsf{ZFC}$. For example, the following two types of problems have not been settled yet:

**Problem 1** Let $n \in \omega$. Does $\mathcal{K}'_n$ imply $\mathcal{K}_n$? ([2])

**Problem 2** Let $n \in \omega$. Does $\mathcal{K}_n$ imply $\mathsf{MA}_{\aleph_1}$? Or does $\mathcal{C}^2$ imply $\mathsf{MA}_{\aleph_1}$?

In [40], the author introduced a new chain condition, called the anti-rectangle refining property, and some fragments of Martin's Axiom related to this condition. A typical example of a forcing notion with the anti-rectangle refining property is an Aronszajn tree. In [20], Larson and Todorčević introduced the statement $\mathcal{K}_2(\text{rec})$ that every partition on $K_0 \cup K_1 = [\omega_1]^2$ with the rectangle refining property has an uncountable $K_0$-homogeneous set, to completely answer Katětov's problem. In [40], an affirmative answer of Problem 1 on $\mathcal{K}_2(\text{rec})$ has been presented by using a class of forcing notions with the anti-rectangle refining property.

In this paper, we introduced other chain conditions, called the anti-$\mathsf{R}_{1,\aleph_1}$ and the property $\mathsf{R}_{1,\aleph_1}$, and their fragments of Martin's Axiom. A typical example of forcing notions with the anti-$\mathsf{R}_{1,\aleph_1}$ is also an Aronszajn tree, and the property $\mathsf{R}_{1,\aleph_1}$ canonically comes from the anti-$\mathsf{R}_{1,\aleph_1}$ and is a stronger property than the countable chain condition. For each $n \in \omega$, $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ is denoted as $\mathcal{K}_n$ for forcing notions with the property $\mathsf{R}_{1,\aleph_1}$, $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ is denoted as $\mathsf{MA}_{\aleph_1}$ for the class of forcing notions with the property $\mathsf{R}_{1,\aleph_1}$. It follows from results in [35] that $\mathsf{MA}_{\aleph_1}$ is equivalent that every ccc forcing notion has precaliber $\aleph_1$, that is, if $\mathbb{P}$ is a ccc forcing notion, then every uncountable subset of $\mathbb{P}$ has an uncountable subset any of whose finite subsets can be extended in $\mathbb{P}$. (See section 6, also in [8, Chapter 3].) We note that if $\mathbb{P}$ has precaliber $\aleph_1$, then $\mathcal{K}_n$ for $\mathbb{P}$ holds for every $n \in \omega$. So if every forcing notion with the property $\mathsf{R}_{1,\aleph_1}$

[1] They are defined by Todorčević in several papers. In [19, Definition 4.9] and [35, §2], $\mathcal{K}_n$'s are defined as statements for ccc forcing notions, however in [20, §4] and [32, §7], $\mathcal{K}_n$'s are defined as statements for ccc partitions. To separate them, we use notation as above.

[2] Moore announced to the author that if $\mathcal{K}'_2$ implies $\mathcal{K}_2$, then $\mathcal{K}'_n$ implies $\mathcal{K}_n$ for every $n \in \omega$.

has precaliber $\aleph_1$, then $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ holds for every $n \in \omega$. We will show that it is consistent that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ fails and every forcing notion with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$, hence $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ holds for every $n \in \omega$. This gives a negative answer of Problem 2 for forcing notions with the property $\mathsf{R}_{1,\aleph_1}$.

In section 2, we introduce the anti-$\mathsf{R}_{1,\aleph_1}$ and the property $\mathsf{R}_{1,\aleph_1}$, and prove some basic results about them.

In section 3, we give two examples of the anti-$\mathsf{R}_{1,\aleph_1}$. One is an $(\omega_1, \omega_1)$-gap, and the other is an unbounded sequence of functions in $\omega^\omega$. These observations are used in applications of $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$.

In section 4, we consider fragments of forcing notions with the property $\mathsf{R}_{1,\aleph_1}$. In particular, we show that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ implies that every Aronszajn tree is special, and we also show that it is consistent that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ holds and $\mathsf{MA}_{\aleph_1}$ fails. This says that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ is a weak fragment of $\mathsf{MA}_{\aleph_1}$.

In section 5, we give two preservation theorems of forcing notions with the property $\mathsf{R}_{1,\aleph_1}$. One is the preservation of the additivity of the measure zero ideal, and the other is on not adding random reals. We show that forcing notions with the property $\mathsf{R}_{1,\aleph_1}$ don't add random reals. This presents a new type of ccc forcing notions not adding random reals.

In section 6, we show the main consistency result, which is an answer of Problem 2 above for the class of forcing notions with the property $\mathsf{R}_{1,\aleph_1}$. To show this, we use Shelah's technique for proving the consistency of the statement that Suslin's Hypothesis holds and there exists a non-special Aronszajn tree.

## 2. Two properties which come from Aronszajn trees

In this article, a forcing notion means a non-atomic partially ordered structure $(\mathbb{P}, \leq_{\mathbb{P}})$. Members of forcing notions are called conditions. For a forcing notion $\mathbb{P}$ and conditions $p$ and $q$ of $\mathbb{P}$, $p \leq_{\mathbb{P}} q$ means that $p$ is stronger than (or equal to) $q$, $p$ extends $q$, or $p$ is an extension of $q$. we say that $p$ and $q$ are compatible in $\mathbb{P}$ if there exists a condition $r$ of $\mathbb{P}$ such that $r \leq_{\mathbb{P}} p$ and $r \leq_{\mathbb{P}} q$, that is, $r$ is a common extension of $p$ and $q$ in $\mathbb{P}$. When $p$ and $q$ are not compatible in $\mathbb{P}$ (that is, incompatible in $\mathbb{P}$), we write $p \perp_{\mathbb{P}} q$. So the formula $p \not\perp_{\mathbb{P}} q$ means that $p$ is compatible with $q$ in $\mathbb{P}$. In this paper, we focus on forcing notions of size $\aleph_1$ because we will study fragments of $\mathsf{MA}_{\aleph_1}$, Martin's Axiom for $\aleph_1$ many dense sets. Moreover we introduce the following class of forcing notions.

**Definition 2.1.** $\mathsf{FSCO}$ *is the collection of forcing notions $\mathbb{Q}$ such that*

- *a condition of $\mathbb{Q}$ is a finite set of countable ordinals,*
- *$\mathbb{Q}$ is uncountable, and*
- *$\leq_{\mathbb{Q}}$ is equal to the superset $\supseteq$, that is, for any $\sigma$ and $\tau$ in $\mathbb{Q}$, $\sigma \leq_{\mathbb{Q}} \tau$ iff $\sigma \supseteq \tau$,*
- *$\mathbb{Q}$ is closed under subsets, that is, if $\sigma \in \mathbb{Q}$ and $\tau \subseteq \sigma$, then $\tau \in \mathbb{Q}$.*

We note that if $\mathbb{Q}$ is a forcing notion in $\mathsf{FSCO}$ and condition $\sigma$ and $\tau$ are compatible in $\mathbb{Q}$, then $\sigma \cup \tau$ is a common extension of $\sigma$ and $\tau$. From a view point of applications of $\mathsf{MA}_{\aleph_1}$, many forcing notions can be considered in $\mathsf{FSCO}$. The most important example in this paper is the specialization of Aronszajn trees [7]. An Aronszajn tree is an $\omega_1$-tree which does not have an uncountable chain. An Aronszajn tree is called special if it is a union of countably many antichains. For an $\omega_1$-tree $T$, the specialization $\mathcal{S}_T$ of $T$ by finite approximations is a forcing notion which consists of finite partial functions $f$ from $T$ into $\omega$ such that for every $n \in \operatorname{ran}(f)$, the set $f^{-1}[\{n\}]$ is an antichain in $T$, inversely ordered by extension. It is proved in [7] that if $T$ is an Aronszajn tree, then $\mathcal{S}_T$ has the countable chain condition. (A more stronger result is proved in [40, Proposition 3.1.], and we prove an another stronger result, Proposition 4.2.) We can consider $T$ as an order structure on $\omega_1$ and then each condition of the specialization of $T$ is a finite subset of the set $\omega_1 \times \omega$, and its order relation is equal to the superset, so it can be considered as a member of $\mathsf{FSCO}$.

In this section, we introduce two properties of forcing notions which come from Aronszajn trees. One has been defined in [40] as follows.

**Definition 2.2** ([40, Definition 2.1.])**.** *A forcing notion $\mathbb{P}$ has the anti-rectangle refining property if $\mathbb{P}$ is uncountable and for any pair of uncountable subsets $I$ and $J$ of $\mathbb{P}$, there are uncountable subsets $I'$ and $J'$ of $I$ and $J$ respectively such that any member of $I'$ is incompatible with any member of $J'$ in $\mathbb{P}$.*

A typical example of this property is an Aronszajn tree (as a forcing notion). The anti-rectangle refining property can be considered as a variation of the rectangle refining property for partitions on $[\omega_1]^2$, which was introduced to solve Katětov's problem, due to Larson–Todorčević [20, Definition 4.1.]. In fact, we gave a variation of $\mathcal{K}_2$, using the anti-rectangle refining property, equivalent to $\mathcal{K}_2'$ for all partitions with the rectangle refining property [40, Proposition 4.5.]. This gives a positive answer of Problem 1 in section 1 for partitions with the rectangle refining property.

For a forcing notion $\mathbb{P}$, let $a(\mathbb{P})$ be a forcing notion which consists of finite antichains in $\mathbb{P}$, ordered by supersets, that is, for finite antichains $\sigma$ and $\tau$ in $\mathbb{P}$,

$$\sigma \leq_{a(\mathbb{P})} \tau :\iff \sigma \supseteq \tau.$$

When $\mathbb{P}$ is of size $\aleph_1$, then $a(\mathbb{P})$ can be considered as a forcing notion in $\mathsf{FSCO}$. If $a(\mathbb{P})$ has the countable chain condition, it can generically add an uncountable antichain in $\mathbb{P}$. A forcing notion of this form is essentially used in [7] to force that every Aronszajn tree is special for the first time. In [7], it is essentially proved that if $T$ is an Aronszajn tree, then $a(T)$ has the countable chain condition.

We say that a forcing notion $\mathbb{P}$ has the property K if every uncountable subset of $\mathbb{P}$ has an uncountable pairwise compatible subset in $\mathbb{P}$ [15]. So the property K implies the countable chain condition. We notice that, since a forcing notion with the property K preserves any ccc forcing notion in the ground model (see e.g. [18]), for any forcing notion $\mathbb{P}$, if $\mathbb{P}$ has the countable chain condition, then $a(\mathbb{P})$ doesn't have the property K. Moreover, for an uncountable forcing notion $\mathbb{P}$, the set $\{\langle p, \{p\}\rangle\,; p \in \mathbb{P}\}$ is an uncountable antichain in the product $\mathbb{P} \times a(\mathbb{P})$. So if there exists a ccc forcing notion $\mathbb{P}$ such that $a(\mathbb{P})$ also has the countable chain condition, then $\mathcal{C}^2$, which is defined in section 1, fails. A typical example of this is a Suslin tree, so the principle $\mathcal{C}^2$ can be considered as a generalization of Suslin's Hypothesis.

In [40, Proposition 2.2.], we note that if $\mathbb{P}$ has the anti-rectangle refining property, then for any pair of uncountable subsets $I$ and $J$ of the forcing notion $a(\mathbb{P})$, whenever $I \cup J$ forms a $\Delta$-system, then there are uncountable subsets $I'$ and $J'$ of $I$ and $J$ respectively such that any member of $I'$ is compatible with any member of $J'$ in $a(\mathbb{P})$. So we introduce the following property.

**Definition 2.3.** *A forcing notion $\mathbb{Q}$ in* $\mathsf{FSCO}$ *has the rectangle refining property if for any pair of uncountable subsets $I$ and $J$ of the forcing notion $\mathbb{Q}$, whenever $I \cup J$ forms a $\Delta$-system, then there are uncountable subsets $I'$ and $J'$ of $I$ and $J$ respectively such that any member of $I'$ is compatible with any member of $J'$ in $\mathbb{Q}$.*

We note that this property is stronger than the countable chain condition, in fact, this is closed under finite support products. This property used in [42] to investigate Rudin's construction of a Dowker space by a Suslin tree. It seems that this property is similar to the property $\mathsf{R}_{1,\aleph_1}$, which is defined below. But we don't know they are same or not.

We introduce the other property of forcing notion as follows. In this paper, we focus on this property.

**Definition 2.4.** *A forcing notion $\mathbb{P}$ has the anti-$\mathsf{R}_{1,\aleph_1}$ if $\mathbb{P}$ is uncountable and for any regular cardinal $\kappa$ larger than $\aleph_1$, countable elementary submodel $N$ of $H(\kappa)$ which has the set $\{\mathbb{P}\}$, $I \in [\mathbb{P}]^{\aleph_1} \cap N$ and $p \in \mathbb{P} \setminus N$, there exists $I' \in [I]^{\aleph_1} \cap N$ such that every member of $I'$ is incompatible with $p$ in $\mathbb{P}$.*

In the definition of the properness of a forcing notion $\mathbb{P}$, it suffices to consider countable elementary submodels of $H(\kappa)$ for large enough regular cardinals $\kappa$, and a *large enough* regular cardinal means a regular cardinal larger than the size $2^{|\mathbb{P}|}$ [27, Chapter III §§1–2]. It should be same for the anti-$\mathsf{R}_{1,\aleph_1}$. But as seen below in the proofs of the propositions in section 3, all of our examples have the above property for every countable elementary submodel of $H(\kappa)$ for a regular cardinal larger than $\aleph_1$. Because $H(\aleph_2)$ has the *real* first uncountable cardinal, so the uncountability in a countable elementary submodel of $H(\aleph_2)$ is same to the real uncountability. So the author defined as above. A typical example of the anti-$\mathsf{R}_{1,\aleph_1}$ is also an Aronszajn tree.

**Proposition 2.5.** *An $\omega_1$-tree $T$ has the anti-$\mathsf{R}_{1,\aleph_1}$ iff $T$ is Aronszajn, that is, there is no uncountable chain.*

*Proof.* Let $T$ be an $\omega_1$-tree, and we denote an order of $T$ by $<_T$. When we consider $T$ as a forcing notion, a stronger condition means a higher node, that is, for conditions $s$ and $t$ in $T$, $s$ is stronger than $t$ iff $t <_T s$. We note that $T$ is Aronszajn iff for every uncountable subset $I$ of $T$, there are two nodes $s_0$ and $s_1$ such that $s_0 \perp_T s_1$ and the sets $\{u \in I; s_0 <_T u\}$ and $\{u \in I; s_1 <_T u\}$ are both uncountable.

Suppose that $T$ is not Aronszajn and let $I$ be an uncountable chain through $T$. We take a countable elementary submodel $N$ of $H(\aleph_2)$ which has the set $\{T, I\}$, and take $t \in I \setminus N$. Then since every member of $I$ is compatible with $t$ in $T$, the tuple $\langle T, N, I, t\rangle$ is a witness that $T$ does not have the anti-$\mathsf{R}_{1,\aleph_1}$.

Suppose that $T$ is Aronszajn, and let $N$ be a countable elementary submodel $N$ of $H(\aleph_2)$ which has the set $\{T\}$, $I$ an uncountable subset of $T$ in $N$, and $t \in T \setminus N$. Then since $T$ is Aronszajn, there are incompatible condition $s_0$ and $s_1$ of $T$ in $N$ such that the sets $\{u \in I; s_0 <_T u\}$ and $\{u \in I; s_1 <_T u\}$ are both uncountable. Since $t \notin N$ and both $s_0$ and $s_1$ are in $N$, the height of $t$ is larger than ones of $s_0$ and $s_1$. So for some $i \in \{0, 1\}$, $s_i \perp_T t$ holds. Let $I' := \{u \in I; s_i <_T u\}$. Since $s_i \in N$, so is $I'$. Then we note that $t$ is incompatible with all members of $I'$ in $T$. Therefore $T$ has the anti-$\mathsf{R}_{1,\aleph_1}$. $\square$

As seen in the proof above, if a forcing notion has an uncountable pairwise compatible subset, then it doesn't have the anti-$\mathsf{R}_{1,\aleph_1}$. Like the rectangle refining property for forcing notions in $\mathsf{FSCO}$, we can introduce the property for the compatibility-variation of the anti-$\mathsf{R}_{1,\aleph_1}$ as follows.

**Definition 2.6.** *A forcing notion $\mathbb{Q}$ in $\mathsf{FSCO}$ has the property $\mathsf{R}_{1,\aleph_1}$ if for any regular cardinal $\kappa$ larger than $\aleph_1$, countable elementary submodel $N$ of $H(\kappa)$ which has the set $\{\mathbb{Q}\}$, $I \in [\mathbb{Q}]^{\aleph_1} \cap N$ and $\sigma \in \mathbb{Q} \setminus N$, if $I$ forms a $\Delta$-system with root $\nu$ and $\sigma \cap N \subseteq \nu$, then there exists $I' \in [I]^{\aleph_1} \cap N$ such that every member of $I'$ is compatible with $\sigma$ in $\mathbb{Q}$.*

As seen on the rectangle refining property, we have the following propositions.

**Proposition 2.7.** *If $\mathbb{P}$ has the anti-$\mathsf{R}_{1,\aleph_1}$, then $a(\mathbb{P})$ has the property $\mathsf{R}_{1,\aleph_1}$.*

*Proof.* Let $\kappa$ be a regular cardinal larger than $\aleph_1$, $N$ a countable elementary submodel of $H(\kappa)$ which has the set $\{\mathbb{P}\}$, $I$ an uncountable subset of $a(\mathbb{P})$ in $N$, $\sigma \in a(\mathbb{P}) \setminus N$, and assume that $I$ forms a $\Delta$-system with root $\nu$ such that $\sigma \cap N \subseteq \nu$.

By shrinking $I$ in $N$ if necessary, we may assume that every member of $I$ is of size $n$ for some fixed $n \in \omega$. Let $m := |\sigma \setminus N|$, which is larger than 0, because $\sigma \notin N$. By applying the anti-$\mathsf{R}_{1,\aleph_1}$ of $\mathbb{P}$ for $n \cdot m$ many times, we can find $I' \in [I]^{\aleph_1} \cap N$ such that for each $i \in n$ and $j \in m$ and $\tau \in I'$, the $i$-th element of $\tau$ is incompatible with the $j$-th element of $\sigma \setminus N$. Then every member of $I'$ is compatible with $\sigma$ in $a(\mathbb{P})$, which finishes the proof. $\square$

**Proposition 2.8.** *The property $\mathsf{R}_{1,\aleph_1}$ is closed under finite support products in the following sense.*

*If $\{\mathbb{Q}_\xi; \xi \in \Sigma\}$ is a set of forcing notions in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$, $\kappa$ is a large enough regular cardinal, $N$ is a countable elementary submodel of $H(\kappa)$ which has the set $\{\{\mathbb{Q}_\xi; \xi \in \Sigma\}\}$, $I$ is an uncountable subset of the finite support product $\prod_{\xi\in\Sigma} \mathbb{Q}_\xi$ in $N$, $\vec{\sigma} \in \prod_{\xi\in\Sigma} \mathbb{Q}_\xi \setminus N$, $I$ forms a $\Delta$-system with root $\vec{\nu}$ such that $\vec{\sigma} \cap N \subseteq \vec{\nu}$, that is,*

- *the set $\{\mathrm{supp}(\vec{\tau}); \tau \in I\}$ forms a $\Delta$-system with root $\Gamma$ (or $\{\mathrm{supp}(\vec{\tau}); \tau \in I\}$ is a singleton), where $\mathrm{supp}(\vec{\tau}) := \{\xi \in \Sigma; \vec{\tau}(\xi) \neq \emptyset\}$,*
- $\mathrm{supp}(\vec{\sigma}) \cap N \subseteq \Gamma = \mathrm{supp}(\vec{\nu})$*,*
- *for each $\xi \in \mathrm{supp}(\vec{\sigma}) \cap N$, the set $\{\vec{\tau}(\xi); \tau \in I\}$ forms a $\Delta$-system with root $\vec{\nu}(\xi)$ such that $\vec{\sigma}(\xi) \cap N \subseteq \vec{\nu}(\xi)$,*

*then there exists $I' \in [I]^{\aleph_1} \cap N$ such that every element of $I'$ is compatible with $\vec{\sigma}$ in $\prod_{\xi\in\Sigma} \mathbb{Q}_\xi$.*

*Proof.* Under the assumption, we apply the property $\mathsf{R}_{1,\aleph_1}$ of $\mathbb{Q}_\xi$'s for finitely many times to find $I' \in [I]^{\aleph_1} \cap N$ such that for each $\vec{\tau} \in I'$ and $\xi \in \mathrm{supp}(\vec{\nu}) \cap N$, $\vec{\tau}(\xi)$ is compatible with $\vec{\sigma}(\xi)$, which is as desired. □

We give other these examples as follows.

**Proposition 2.9.** 1. *$Coll(\omega, \omega_1)$, the collapse of $\omega_1$ to $\omega$, has both the anti-rectangle refining property and the anti-$\mathsf{R}_{1,\aleph_1}$.*

2. *$\mathbb{C}_{\omega_1}$, Cohen forcing on $\omega_1$, has both the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$.*

*Proof.* We only prove for the anti-$\mathsf{R}_{1,\aleph_1}$ and the property $\mathsf{R}_{1,\aleph_1}$. Results for the anti-rectangle refining property and the rectangle refining property can be proved similarly.

(1) $Coll(\omega, \omega_1)$ consists of functions such that the domain is a natural number and the range is a subset of $\omega_1$, ordered by supersets. Let $N$ be a countable elementary submodel of $H(\aleph_2)$ with $\omega_1 \in N$, $I$ an uncountable subset of $Coll(\omega, \omega_1)$, and $f \in Coll(\omega, \omega_1) \setminus N$. Let $k \in \omega$ be the least number such that the set $\{g(k); g \in I\}$ is uncountable.

If the length of the function $f$ is not longer than $k$, then all members of $I$ have to be incompatible with $f$. (Since $f \notin N$, there exists $l \in \mathrm{dom}(f)$ such that $f(l) \notin N$, that is, $f(l) \geq \omega_1 \cap N$. By the minimality of $k$ and $l < k$, for every $g \in I$, $g(l)$ have to be in $N$, hence $g(l) \neq f(l)$.)

When the length of the function $f$ is longer than $k$, we take uncountable subsets $I_0$ and $I_1$ of $I$ in $N$ such that

$$\{g(k); g \in I_0\} \cap \{g(k); g \in I_1\} = \emptyset.$$

Then we can find $i \in \{0,1\}$ such that $f(k)$ is not in the set $\{g(k); g \in I_i\}$. Then $I_i$ is our desired uncountable subset of $I$.

(2) $\mathbb{C}_{\omega_1}$ consists of binary functions such that the domain is a finite subset of $\omega_1$, ordered by supersets. ($\mathbb{C}_{\omega_1}$ can also be considered a forcing notion in $\mathsf{FSCO}$.) Let $N$ be a countable elementary submodel of $H(\aleph_2)$ with $\omega_1 \in N$, $I$ an uncountable subset of $\mathbb{C}_{\omega_1}$, and $f \in \mathbb{C}_{\omega_1} \setminus N$, and suppose that $I$ forms a $\Delta$-system with root $h$ and $f \cap N \subseteq h$, that is, for any $i \in \mathrm{dom}(f) \cap N$, $i$ is in $\mathrm{dom}(h)$ and $h(i) = f(i)$.

By shrinking $I$ in $N$ if necessary, we may assume that there exists $k \in \omega$ such that for every member $g$ of $I$, $g \setminus h$ is of size $k$, and let $\langle \{\alpha_i^g; i \in k\}; g \in I\rangle$ be an enumeration of the domains of functions $g \setminus h$ for all $g \in I$, in $N$. Let $\{\beta_j; j \in l\}$ be an enumeration of the domain of the function $f \setminus N$. By taking subsets in $N$ for $k \cdot l$ many times as in the proof for $Coll(\omega, \omega_1)$ above, we can find an uncountable subset $I'$ of $I$ in $N$ such that for each $i \in k$ and $j \in l$, $\beta_j$ is not in the set $\{\alpha_i^g; g \in I'\}$. (We should notice that for each $j \in l$, $\beta_j$ is not in $N$, however the sequence $\langle \{\alpha_i^g; i \in k\}; g \in I\rangle$ belongs to $N$. So we can take such an $I'$ in $N$.) Then $I'$ is our desired subset of $I$. $\square$

At last in this section, we present a key combinatorial proposition of two properties, the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$. It follows from natural applications of these properties, and these properties will be used often in these forms.

**Proposition 2.10.**

1. *Let $\mathbb{Q}$ be a forcing notion in $\mathsf{FSCO}$ with the rectangle refining property. Suppose that $\langle I_i; i \in n\rangle$ is a finite sequence of uncountable subsets of $\mathbb{Q}$ such that the union $\bigcup_{i \in n} I_i$ forms a $\Delta$-system.*
   *Then there exists uncountably many members $\langle \sigma_i; i \in n\rangle \in \prod_{i \in n} I_i$ such that there exists a common extension of the $\sigma_i$ in $\mathbb{Q}$.*
2. *Let $\mathbb{Q}$ be a forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$. Suppose that $\kappa$ is a regular cardinal larger than $\aleph_1$, $N$ is a countable elementary submodel of $H(\kappa)$ which has the set $\{\mathbb{Q}\}$, $\langle I_i; i \in n\rangle$ is a finite sequence of members of the set $[\mathbb{Q}]^{\aleph_1} \cap N$, and $\sigma \in \mathbb{Q} \setminus N$ such that the union $\bigcup_{i \in n} I_i$ forms a $\Delta$-system with root $\nu$ such that $\sigma \cap N \subseteq \nu$.*
   *Then there exists $\langle \tau_i; i \in n\rangle \in \prod_{i \in n} I_i$ such that there exists a common extension of $\sigma$ and the $\tau_i$ in $\mathbb{Q}$.*

*Proof.* (1) At first, we enumerate each $I_i$ by $\left\langle \sigma_\xi^i; \xi \in \omega_1 \right\rangle$. By applying the rectangle refining property of $\mathbb{Q}$ recursively for $n-1$ many times, we can find a sequence $\left\langle K_i^j; j \in n-1 \;\&\; i \leq j+1 \right\rangle$ of uncountable subsets of $\omega_1$ as follows.

- By applying the rectangle refining property for sets $I_0$ and $I_1$, we find $K_0^0$ and $K_1^0$ such that for every $\xi \in K_0^0$ and $\eta \in K_1^0$, $\sigma_\xi^0$ and $\sigma_\eta^1$ are compatible in $\mathbb{Q}$.

- For each $j \in n-2$, letting $\xi^j_{i,\alpha}$ be the $\alpha$-th member of $K^j_i$ for each $i \le j+1$ and $\alpha \in \omega$, by applying the rectangle refining property for sets $\left\{ \bigcup_{i \le j+1} \sigma^i_{\xi^j_{i,\alpha}} ; \alpha \in \omega_1 \right\}$ and $I_{j+1}$, we find $K^{j+1}_0, \cdots, K^{j+1}_{j+2}$ such that for every $\alpha \in \omega_1$ and $\eta \in K^{j+1}_{j+2}$, $\bigcup_{i \le j+1} \sigma^i_{\xi^{j+1}_{i,\alpha}}$ and $\sigma^{j+1}_\eta$ are compatible in $\mathbb{Q}$.

Then for each $\alpha \in \omega_1$, $\bigcup_{i \in n} \sigma^{n-1}_{\xi^{n-1}_{i,\alpha}}$ is a condition in $\mathbb{Q}$.

(2) By applying the property $\mathsf{R}_{1,\aleph_1}$ of $\mathbb{Q}$ recursively for $n$ many times, we can take sequences $\langle I'_i; i \in n \rangle$ and $\langle \tau_i; i \in n \rangle$ such that for each $i \in n$,

- $I'_i$ is an uncountable subset of $I_i$ in $N$ and $\tau_i \in I'_i \setminus N$,
- $\min(\tau_i \setminus N) \ge \omega_1 \cap N$, and
- every member of $I'_i$ is compatible with the condition $\sigma \cup \bigcup_{j<i} \tau_j$ in $\mathbb{Q}$.

This can be done because the union $\bigcup_{i \in n} I_i$ forms a $\Delta$-system with root $\nu$ such that $\sigma \cap N \subseteq \nu$ and $N$ is countable, and this finishes the proof. $\square$

## 3. Examples – $(\omega_1, \omega_1)$-gaps and unbounded families in $\omega^\omega$

*3.1. $(\omega_1, \omega_1)$-gaps*

In this article, an $(\omega_1, \omega_1)$-pregap is defined ([3]) as a sequence $\langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ of infinite sets of natural numbers such that

- for every $\alpha < \beta$ in $\omega_1$, $a_\alpha$ and $b_\alpha$ are almost contained in $a_\beta$ and $b_\beta$ respectively, that is, both $a_\alpha \setminus a_\beta$ and $b_\alpha \setminus b_\beta$ are finite (and then we denote $a_\alpha \subseteq^* a_\beta$ and $b_\alpha \subseteq^* b_\beta$),
- for every $\alpha \ne \beta$ in $\omega_1$, $a_\alpha$ and $b_\beta$ are almost disjoint, that is, the set $a_\alpha \cap b_\beta$ is finite,
- for every $\alpha \in \omega_1$, $a_\alpha \cap b_\alpha = \emptyset$,
- it is closed under finite modifications, that is, for every $\alpha \in \omega_1$ and a pair $\langle c, d \rangle$ of infinite sets of natural numbers such that if $c$ almost coincides with $a_\alpha$ (that is, $c \setminus n = a_\alpha \setminus n$ for some $n \in \omega$) and $d$ almost coincides with $b_\alpha$, then $\langle c, d \rangle = \langle a_\beta, b_\beta \rangle$ for some $\beta \in \omega$.

An $(\omega_1, \omega_1)$-pregap $\langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ is called a gap if there is no infinite set $c$ of natural numbers such that for all $\alpha \in \omega_1$, $a_\alpha \subseteq^* c$ and $b_\alpha \cap c$ is finite. If such a $c$ exists, it is called an interpolation of $(\mathcal{A}, \mathcal{B})$. So we note that $(\mathcal{A}, \mathcal{B})$ is a gap iff there is no interpolation of $(\mathcal{A}, \mathcal{B})$. For an $(\omega_1, \omega_1)$-pregap $(\mathcal{A}, \mathcal{B}) =$

[3] In four conditions, the last two conditions are additional ones to the usual definition of $(\omega_1, \omega_1)$-pregaps. But it doesn't matter.

$\langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$, let $\mathcal{S}(\mathcal{A}, \mathcal{B})$ be the forcing notion which consists of countable ordinals such that for each $\alpha$ and $\beta$ in $\omega_1$,

$$\alpha \leq_{\mathcal{S}(\mathcal{A},\mathcal{B})} \beta :\iff a_\beta \subseteq a_\alpha \;\&\; b_\beta \subseteq b_\alpha.$$

We notice that when $G$ is an $\mathcal{S}(\mathcal{A}, \mathcal{B})$-generic filter, then the set $\bigcup_{\alpha \in G} a_\alpha$ is an interpolation of $(\mathcal{A}, \mathcal{B})$.

The first discovery of an $(\omega_1, \omega_1)$-gap is due to Hausdorff (see e.g. [12, Theorem 29.7]). Kunen and Todorčević have studied $(\omega_1, \omega_1)$-pregaps independently and they pointed out that Hausdorff's $(\omega_1, \omega_1)$-gap $(\mathcal{A}, \mathcal{B})$ is absolute, that is, it is always a gap in any forcing extension with the same $\omega_1$ as the ground model, in fact, then $\mathcal{S}(\mathcal{A}, \mathcal{B})$ collapses $\aleph_1$. Such an $(\omega_1, \omega_1)$-pregap is called indestructible. Todorčević proved that adding a Cohen real adds an $(\omega_1, \omega_1)$-gap $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{S}(\mathcal{A}, \mathcal{B})$ has the countable chain condition [34, Theorem 9.3], and so such an $(\omega_1, \omega_1)$-gap exists if $\diamondsuit$ holds [11, Theorem 1.2. and Lemma 2.5.] (see also [10, Proposition 2.5.]). This is an analogous result of the existence of a Suslin tree due to Shelah (see [12, Theorem 28.12] and [34, Theorem 3.1]) and Jensen [13] (see also [17, Chapter II 7.8. Theorem]). For an $(\omega_1, \omega_1)$-pregap $(\mathcal{A}, \mathcal{B})$, when $\mathcal{S}(\mathcal{A}, \mathcal{B})$ has the countable chain condition, $(\mathcal{A}, \mathcal{B})$ is called destructible. We have the following Ramsey-theoretic characterizations of the destructibility and being a gap. From the following characterizations, Abraham–Todorčević pointed out that $(\omega_1, \omega_1)$-destructible gaps can be considered an analogue of Suslin trees in [2].

**Theorem 3.1** (E.g. [9, 16, 25, 32]). *Let $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ be an $(\omega_1, \omega_1)$-pregap.*

1. *$(\mathcal{A}, \mathcal{B})$ is destructible iff for every uncountable subset $I$ of $\omega_1$, there exists two ordinals $\alpha$ and $\beta$ in $I$ such that*

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset.$$

2. *$(\mathcal{A}, \mathcal{B})$ is a gap, iff for every uncountable subset $I$ of $\omega_1$, there exists two ordinals $\alpha$ and $\beta$ in $I$ such that*

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset.$$

Therefore if $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ is an $(\omega_1, \omega_1)$-pregap and there is an uncountable subset $I$ of $\omega_1$ such that for every $\alpha$ and $\beta$ in $I$ with $\alpha \neq \beta$,

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset,$$

then it has to be a gap in any forcing extension not collapsing $\aleph_1$, that is, it is indestructible.

Let $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ be an $(\omega_1, \omega_1)$-pregap. For countable ordinals $\alpha$ and $\beta$ in $\omega_1$, since the gap is closed under finite modifications, $\alpha$ and $\beta$ are compatible in $\mathcal{S}(\mathcal{A}, \mathcal{B})$ iff

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset.$$

Therefore $(\mathcal{A},\mathcal{B})$ is indestructible iff $\mathcal{S}(\mathcal{A},\mathcal{B})$ does not have the countable chain condition, and so the forcing notion $a(\mathcal{S}(\mathcal{A},\mathcal{B}))$ adds a witness of the indestructibility if it does not collapse $\aleph_1$. In fact, $a(\mathcal{S}(\mathcal{A},\mathcal{B}))$ is equal to the standard forcing notion forcing $(\mathcal{A},\mathcal{B})$ to be indestructible, which is denoted by $\mathcal{F}(\mathcal{A},\mathcal{B})$ in [39].

In [40, §3.2], it is proved that for an $(\omega_1,\omega_1)$-gap $(\mathcal{A},\mathcal{B})$, $\mathcal{S}(\mathcal{A},\mathcal{B})$ has the anti-rectangle refining property, hence $a(\mathcal{S}(\mathcal{A},\mathcal{B}))$ has the rectangle refining property. It follows from the following proposition that for an $(\omega_1,\omega_1)$-gap $(\mathcal{A},\mathcal{B})$, $a(\mathcal{S}(\mathcal{A},\mathcal{B}))$ has the property $\mathsf{R}_{1,\aleph_1}$, hence doesn't collapse $\aleph_1$.

**Proposition 3.2.** *For an $(\omega_1,\omega_1)$-pregap $(\mathcal{A},\mathcal{B})$, $(\mathcal{A},\mathcal{B})$ is a gap iff $\mathcal{S}(\mathcal{A},\mathcal{B})$ has the anti-$\mathsf{R}_{1,\aleph_1}$.*

*Proof.* Suppose that $(\mathcal{A},\mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha\in\omega_1\rangle$ has an interpolation $c$, that is, for every $\alpha\in\omega_1$, $a_\alpha\subseteq^* c$ and $c\cap b_\alpha$ is finite. Then there are $n\in\omega$ and an uncountable subset $I$ of $\omega_1$ such that

- for every $\alpha\in I$, $a_\alpha\setminus n\subseteq c$,
- for every $\alpha\in I$, $c\cap b_\alpha\subseteq n$, and
- for any $\alpha$ and $\beta$ in $I$, $a_\alpha\cap n = a_\beta\cap n$ and $b_\alpha\cap n = b_\beta\cap n$.

Then we note that if $\alpha$ and $\beta$ are in $I$, then

$$(a_\alpha\cap b_\beta)\cup(a_\beta\cap b_\alpha)=\emptyset,$$

so are compatible in $\mathcal{S}(\mathcal{A},\mathcal{B})$. Therefore $\mathcal{S}(\mathcal{A},\mathcal{B})$ does not have the anti-$\mathsf{R}_{1,\aleph_1}$.

Suppose that $(\mathcal{A},\mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha\in\omega_1\rangle$ is a gap. Let $N$ be a countable elementary submodel of $H(\aleph_2)$ which has the set $\{(\mathcal{A},\mathcal{B})\}$, $I$ an uncountable subset of $\mathcal{S}(\mathcal{A},\mathcal{B})$ in $N$, $\delta\in\omega_1\setminus N$. By shrinking $I$ in $N$ if necessary, we may assume that

- for any $n\in\omega$, if the set $\{\alpha\in I; n\in a_\alpha\}$ is countable, then $n\notin a_\alpha$ for all $\alpha\in\omega_1$, and
- for any $n\in\omega$, if the set $\{\beta\in I; n\in b_\beta\}$ is countable, then $n\notin b_\beta$ for all $\beta\in\omega_1$.

We will show that there exists $n\in\omega$ such that " $n\in a_\delta$ and the set $\{\beta\in I; n\in b_\beta\}$ is uncountable " or " $n\in b_\delta$ and the set $\{\alpha\in I; n\in a_\alpha\}$ is uncountable "

Suppose not, that is, for every $n\in a_\delta$, the set $\{\beta\in I; n\in b_\beta\}$ is countable, and for every $n\in b_\delta$, the set $\{\alpha\in I; n\in a_\alpha\}$ is countable. Then by our assumption, we notice that for every $n\in a_\delta$, $n\notin b_\beta$ holds for all $\beta\in I$, and for every $n\in b_\delta$, $n\notin a_\alpha$ holds for all $\alpha\in I$. Let

$$c := \{n\in\omega;\ \text{the set } \{\alpha\in I; n\in a_\alpha\} \text{ is uncountable}\}$$

and

$$d := \{n\in\omega;\ \text{the set } \{\beta\in I; n\in b_\beta\} \text{ is uncountable}\}.$$

Since both $(\mathcal{A}, \mathcal{B})$ and $I$ are members of $N$, we note that both $c$ and $d$ are also in $N$. Moreover by our assumption, we note that

$$c \cup d = \bigcup_{\gamma \in I} a_\gamma \cup b_\gamma, \qquad c \cap b_\delta = \emptyset, \qquad a_\delta \cap d = \emptyset.$$

Since for any $\gamma \in I \cap N$, $a_\gamma \subseteq^* a_\delta$ and $b_\gamma \subseteq^* b_\delta$, it follows from the above observations that

$$N \models \text{“ for every } \gamma \in I, \text{ both } c \cap b_\gamma \text{ and } a_\gamma \cap d \text{ are finite ”}.$$

So by the elementarity of $N$, for every $\gamma \in I$, both $c \cap b_\gamma$ and $a_\gamma \cap d$ are finite. Therefore for every $\gamma \in I$, $a_\gamma \subseteq^* c \setminus d$ and $(c \setminus d) \cap b_\gamma$ is finite, and since $I$ is uncountable, it follows that $(\mathcal{A}, \mathcal{B})$ is not a gap, which is a contradiction.

Without loss of generality, we assume now that there exists $n \in \omega$ such that $n \in a_\delta$ and the set $I' := \{\beta \in I; n \in b_\beta\}$ is uncountable. Since all of members of the set $\{(\mathcal{A}, \mathcal{B}), I, n\}$ belong to $N$, so does $I'$. Then we note that every member $\beta$ of $I'$ is incompatible with $\delta$, because $n \in a_\delta \cap b_\beta$, which finishes the proof. $\square$

*3.2. Unbounded families in $\omega^\omega$*

$\omega^\omega$ denotes the set of functions from $\omega$ into $\omega$. For functions $f$ and $g$ in $\omega^\omega$, $f <^* g$ means that there exists $n \in \omega$ such that for every natural number $m \geq n$, $f(m) < g(m)$, and we say $g$ bounds $f$, or $g$ dominates $f$. An unbounded family is an unbounded subset of the ordered structure $(\omega^\omega, <^*)$, and the (un)bounding number $\mathfrak{b}$ is the smallest size of unbounded families in $\omega^\omega$.

We note that there exists an unbounded $<^*$-increasing sequence $\langle f_\alpha; \alpha \in \mathfrak{b} \rangle$ of strictly increasing functions in $\omega^\omega$, that is, for any $\alpha$ and $\beta$ in $\mathfrak{b}$ with $\alpha \leq \beta$, $f_\alpha <^* f_\beta$, and the set $\{f_\alpha; \alpha \in \mathfrak{b}\}$ is unbounded. So the equality $\mathfrak{b} = \aleph_1$ is equivalent to the statement that there exists an unbounded $<^*$-increasing sequence of $\omega_1$-many strictly increasing functions in $\omega^\omega$. Suppose that $F = \langle f_\alpha; \alpha \in \omega_1 \rangle$ is a $<^*$-increasing sequence of strictly increasing functions in $\omega^\omega$. In [41], the forcing notion $\mathbb{P}(F)$ is defined as follows. $\mathbb{P}(F)$ consists of finite subsets $\sigma$ of $\omega_1$ such that for every $\xi \in \sigma$ and $n \in \omega$,

**either** $\max\{f_\zeta(n); \zeta \in \sigma \cap \xi\} < f_\xi(n)$

**or** $f_\xi(n) \in \{f_\zeta(n); \zeta \in \sigma \cap \xi\}$,

and $\mathbb{P}(F)$ is ordered by supersets. In [41], it is shown that if $F$ is unbounded, then $\mathbb{P}(F)$ has the anti-rectangle refining property [41, Lemma 3.2.]. We will use $\mathbb{P}(F)$ in section 4.2.

**Lemma 3.3** ([41, Lemma 3.3.])**.** *If $F$ is an unbounded $<^*$-increasing sequence of $\omega_1$-many strictly increasing functions in $\omega^\omega$, then $\mathbb{P}(F)$ has the countable chain condition.* $\square$

**Proposition 3.4.** *If $F = \langle f_\alpha; \alpha \in \omega_1 \rangle$ is an unbounded $<^*$-increasing sequence of strictly increasing functions in $\omega^\omega$, then $\mathbb{P}(F)$ has the anti-$\mathsf{R}_{1,\aleph_1}$.*

*Proof.* Suppose that $N$ is a countable elementary submodel of $H(\aleph_2)$ with the set $\{F\}$, $I$ is an uncountable subset of $\mathbb{P}(F)$ in $N$, $\sigma \in \mathbb{P}(F) \setminus N$, $I$ forms a $\Delta$-system with root $\mu$ and $\sigma \cap N \subseteq \mu$. By shrinking $I$ in $N$ if necessary, we may assume that there exists $l \in \omega$ such that for any $\tau \in I \cup \{\sigma \cup \mu\}$, $\alpha$ and $\beta$ in $\tau$ with $\alpha < \beta$, and $n \geq l$, $f_\alpha(n) < f_\beta(n)$. (We notice that $\sigma \cap N$ belongs to $N$.) Then we note that $\mu \in N$, and for any $\tau \in I$ and $n \geq l$,

$$\min\{f_\alpha(n); \alpha \in \tau \setminus \mu\} = f_{\min(\tau\setminus\mu)}(n).$$

We will show that there exists $e \in \omega \setminus l$ such that for every $n \in \omega \setminus e$ and $k \in \omega$, the set $\left\{\tau \in I; f_{\min(\tau\setminus\mu)}(n) \geq k\right\}$ is uncountable. (A similar statement is proved in [41, Proposition 3.1.].) Assume not, that is, there exists an infinite set $Z$ of natural numbers such that for every $n \in Z$, there exists $k_n \in \omega$ such that the set $\left\{\tau \in I; f_{\min(\tau\setminus\mu)}(n) \geq k_n\right\}$ is countable. Let $\delta \in \omega_1$ be such that for all $n \in Z$, the set $\left\{\tau \in I; f_{\min(\tau\setminus\mu)}(n) \geq k_n\right\}$ is a subset of the set $[\delta]^{<\aleph_0}$. Let $\{n_i; i \in \omega\}$ be an increasing enumeration of $Z$, and we define a function $g$ on $\omega$ by

$$g(m) := \max\Big(\{f_\delta(m)\} \cup \{k_{n_i}; i \in m+1\} \cup \{g(i)+1; i \in m\}\Big)$$

for each $m \in \omega$. We notice that for each $\alpha \in \delta$, $f_\alpha <^* g$. Moreover for each $\tau \in I \setminus [\delta]^{<\aleph_0}$ and $m \in \omega \setminus l$, since $m \leq n_m$,

$$f_{\min(\tau\setminus\mu)}(m) \leq f_{\min(\tau\setminus\mu)}(n_m) < k_{n_m} \leq g(m).$$

So the set $\left\{f_{\min(\tau\setminus\mu)}; \tau \in I\right\}$ is bounded by $g$. But since $\mu$ is the root of the uncountable $\Delta$-system $I$, the set $\left\{f_{\min(\tau\setminus\mu)}; \tau \in I\right\}$ is unbounded, so we have a contradiction.

We let $\delta := \min(\sigma \setminus N)$ (which is not smaller than $\omega_1 \cap N$), and let

$$I' := \left\{\tau \in I; f_{\min(\tau\setminus\mu)}(e) \geq f_\delta(e) + 1\right\}.$$

Now $f_\delta$ doesn't belong to the model $N$, however $f_\delta(e)$ is in $N$, hence we note that $I'$ is also in $N$. Moreover, we notice that for every $\tau \in I'$, if $(\tau \setminus \mu) \cap \delta$ is not empty, that is, $\min(\tau \setminus \mu) < \delta$, then

$$\begin{array}{rcll} \max\{f_\alpha(e); \alpha \in \mu\} & < & f_\delta(e) & \text{(by the property of } l\text{)} \\ & < & \min\{f_\alpha(e); \alpha \in (\tau \setminus \mu) \cap \delta\} & \text{(because } \tau \in I'\text{)} \\ & \leq & \min\{f_\alpha(e); \alpha \in ((\tau \cup \sigma) \setminus \mu) \cap \delta\} & \text{(because } \sigma \cap N \subseteq \mu\text{)} \\ & \leq & \max\{f_\alpha(e); \alpha \in ((\tau \cup \sigma) \setminus \mu) \cap \delta\} & \end{array}$$

and so, because of the property of $l$,

$$f_\delta(e) \notin \{f_\alpha(e); \alpha \in (\tau \cup \sigma) \cap \delta\}.$$

Therefore if $\tau \in I'$ and $(\tau \setminus \mu) \cap \delta$ is not empty, then $\tau$ and $\sigma$ are incompatible in $\mathbb{P}(F)$.

We will show that there exists $d \in \omega \setminus e$ such that the set

$$\{\tau \in I'; \max\{f_\alpha(d); \alpha \in \tau\} < f_\delta(d)\}$$

is uncountable. Suppose not, that is, for every $n \in \omega \setminus e$, the set

$$\{\tau \in I'; \max\{f_\alpha(n); \alpha \in \tau\} < f_\delta(n)\}$$

is countable. Let $g$ be a function in $\omega^\omega$ such that for each $n \in \omega \setminus e$,

$$g(n) := \min\{k \in \omega; \{\tau \in I'; \max\{f_\alpha(n); \alpha \in \tau\} < k\} \text{ is uncountable}\}.$$

This is well defined because of the property of our $e$. We note that $g$ belongs to the model $N$ and for every $n \in \omega \setminus e$, $f_\delta(n) < g(n)$ by our assumption. Since $F$ is unbounded in $N$, there is $\beta \in \omega_1 \cap N$ such that $f_\beta \not<^* g$ holds. Then $f_\beta \not<^* f_\delta$ also holds, however now $\beta$ has to be less than $\delta$, so we have a contradiction.

Let

$$I'' := \{\tau \in I'; \max\{f_\alpha(d); \alpha \in \tau\} < f_\delta(d)\}.$$

Then we note that $I'' \in N$ and by taking a subset of $I''$ in $N$ if necessary as in the proof of Proposition 2.9 (1), we may assume that $\min(\tau \setminus \mu) \neq \delta$ for every $\tau \in I''$. Then for every $\tau \in I''$, if $(\tau \setminus \mu) \cap \delta$ is empty, then $\min(\tau \setminus \mu) > \delta$, so

$$\begin{aligned} f_{\min(\tau\setminus\mu)}(d) &< f_\delta(d) && \text{(because } \tau \in I''\text{)} \\ &\leq \max\{f_\alpha(d); \alpha \in (\tau \cup \sigma) \cap \min(\tau \setminus \mu)\} \end{aligned}$$

and so, since now $(\tau \cup \sigma) \cap \min(\tau \setminus \mu) = \mu \cup (\sigma \cap \min(\tau \setminus \mu))$ and $d \geq l$,

$$f_{\min(\tau\setminus\mu)}(d) \notin \{f_\alpha(d); \alpha \in (\tau \cup \sigma) \cap \min(\tau \setminus \mu)\}.$$

Thus if $\tau \in I''$ and $(\tau \setminus \mu) \cap \delta$ is empty, then $\tau$ and $\sigma$ are incompatible in $\mathbb{P}(F)$. Therefore every member of $I''$ is incompatible with $\sigma$ in $\mathbb{P}(F)$, which finishes the proof. □

## 4. Fragments of Martin's Axiom

*4.1. Fragments of $\mathsf{MA}_{\aleph_1}$ for forcing notions with the property $\mathsf{R}_{1,\aleph_1}$*

Martin's Axiom for $\aleph_1$ many dense sets, denoted by $\mathsf{MA}_{\aleph_1}$, is the statement that for every ccc forcing notion $\mathbb{P}$ and every set $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{P}$, there exists a filter of $\mathbb{P}$ which meets all of members of $\mathcal{D}$. In this section, we consider two fragments of $\mathsf{MA}_{\aleph_1}$ as follows.

**Definition 4.1.** 1. $\mathsf{MA}_{\aleph_1}(\text{rec})$ *is the statement that for any forcing notion $\mathbb{Q}$ in $\mathsf{FSCO}$ with the rectangle refining property and any set $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{Q}$, there exists a filter $G$ of $\mathbb{Q}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.*

2. $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ *is the statement that for any forcing notion* $\mathbb{Q}$ *in* $\mathsf{FSCO}$ *with the property* $\mathsf{R}_{1,\aleph_1}$ *and any set* $\mathcal{D}$ *of* $\aleph_1$ *many dense subsets of* $\mathbb{Q}$*, there exists a filter* $G$ *of* $\mathbb{Q}$ *such that* $G \cap D \neq \emptyset$ *for every* $D \in \mathcal{D}$*.*

Since both the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$ are stronger than the countable chain condition, both $\mathsf{MA}_{\aleph_1}(\text{rec})$ and $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ are fragments of $\mathsf{MA}_{\aleph_1}$, that is, they follow from $\mathsf{MA}_{\aleph_1}$. In [7], it is proved that $\mathsf{MA}_{\aleph_1}$ implies that every Aronszajn tree is special. The following is an important application of $\mathsf{MA}_{\aleph_1}(\text{rec})$ and $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$.

**Proposition 4.2.** $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ *implies that every Aronszajn tree is special. The same holds for* $\mathsf{MA}_{\aleph_1}(\text{rec})$*.*

*Proof.* Let $T$ be an Aronszajn tree. It suffices to find a forcing notion which specializes $T$ and has the property $\mathsf{R}_{1,\aleph_1}$ (and the rectangle refining property). We will show that $\mathcal{S}_T$, which defined above in section 2, has the property $\mathsf{R}_{1,\aleph_1}$. (We can show $\mathcal{S}_T$ also has the rectangle refining property, but we omit the proof.)

Let $N$ be a countable elementary submodel of $H(\aleph_2)$ which has the set $\{T\}$, $I$ an uncountable subset of $\mathcal{S}_T$ in $N$, and $f \in \mathcal{S}_T \setminus N$, and suppose that $I$ forms a $\Delta$-system with root $h$ and $f \cap N \subseteq h$. By shrinking $I$ in $N$ if necessary, we may assume that

- the set $\{\text{dom}(g); g \in I\}$ also form a $\Delta$-system, and let $h'$ be the root of the $\Delta$-system $I$,
- there exists $k \in \omega$ such that for every $g \in I$, the set $g \setminus h'$ is of size $k$, and so let $\{t_i^g; i \in k\}$ enumerate the set $\text{dom}(g \setminus h')$,
- for any $g$ and $g'$ in $I$ and $i \in k$, $g(t_i^g) = g'(t_i^{g'})$.

Let $\nu := \{g(t_i^g); i \in k\}$ for some (any) $g \in I$. If the set $\text{ran}(f) \cap \nu$ is empty, we have nothing to do, because then every condition in $I$ is compatible with $f$. For each $m \in \text{ran}(f) \cap \nu$, we enumerate the set $f^{-1}[\{m\}]$ by $\left\{s_j^m; j \in l_m\right\}$.

Since $T$ is Aronszajn and the set $\text{ran}(f) \cap \nu$ is in $N$, by applying the anti-$\mathsf{R}_{1,\aleph_1}$ of $T$ in $N$ for finitely many times, we can find an uncountable subset $I'$ of $I$ in $N$ such that for every $m \in \text{ran}(f) \cap \nu$, $i \in k$ and $j \in l_m$, if $g(t_i^g) = m$, then $t_i^g$ is incompatible with $s_j^m$. Therefore for every $g \in I'$, $g \cup f$ is also a condition in $\mathcal{S}_T$, which finishes the proof. $\square$

In [40, Theorem 4.6.], it is proved that $\mathsf{MA}_{\aleph_1}(a(\text{arec}))$, which is $\mathsf{MA}_{\aleph_1}$ for every forcing notion which forms $a(\mathbb{P})$ for some $\mathbb{P}$ with the anti-rectangle refining property, is weaker than $\mathsf{MA}_{\aleph_1}$, that is, it is consistent that $\mathsf{MA}_{\aleph_1}(a(\text{arec}))$ holds but $\mathsf{MA}_{\aleph_1}$ fails. By a similar argument, we can show the same holds for $\mathsf{MA}_{\aleph_1}(\text{rec})$. In this section, we will show that the same holds for $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ later, Theorem 4.9. Before that, we will see other fragments of $\mathsf{MA}_{\aleph_1}$ which will argue in this paper.

For a forcing notion $\mathbb{P}$ and $n \in \omega$, $I$ is called an $n$-linked subset of $\mathbb{P}$ if $I$ is a subset of $\mathbb{P}$ such that for every $\sigma \in [I]^n$, there exists a condition in $\mathbb{P}$ which extends all of members of $\sigma$. For each $n \in \omega$, $\mathcal{K}_n$ is defined as the statement that for every ccc forcing notion $\mathbb{P}$, any uncountable subset of $\mathbb{P}$ has an uncountable $n$-linked subset. (See e.g. [32, §7].) We notice that if natural numbers $m$ and $n$ are such that $m < n$, then $\mathcal{K}_n$ implies $\mathcal{K}_m$, and it is proved that $\mathsf{MA}_{\aleph_1}$ implies all of the $\mathcal{K}_n$. An important point is that many applications of $\mathsf{MA}_{\aleph_1}$ are applications of some $\mathcal{K}_n$. For example, it follows from $\mathcal{K}_2$ that the inequality $\mathfrak{b} > \aleph_1$ holds [32, 7.8. Theorem], (see also [40, §3.3]) and every Aronszajn tree is special [33, Theorem 1], the inequality $\mathsf{add}(\mathcal{N}) > \aleph_1$ follows from $\mathcal{K}_3$ [23, Theorem 6.1.]. (For other examples, see e.g. [19].) In this paper, we also consider two fragments of the $\mathcal{K}_n$ as follows.

**Definition 4.3.** *For each $n \in \omega$,*

1. $\mathcal{K}_n(\mathrm{rec})$ *is the statement that for every forcing notion $\mathbb{Q}$ in $\mathsf{FSCO}$ with the rectangle refining property, any uncountable subset of $\mathbb{Q}$ has an uncountable $n$-linked subset, and*
2. $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ *is the statement that for every forcing notion $\mathbb{Q}$ in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$, any uncountable subset of $\mathbb{P}$ has an uncountable $n$-linked subset.*

We note that for each $n \in \omega$, $\mathcal{K}_n$ implies both $\mathcal{K}_n(\mathrm{rec})$ and $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$. In section 6 below, we will show that it is consistent that for every $n \in \omega$, $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ holds and $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ fails, which is a negative answer of Problem 2 in section 1 for forcing notions in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$.

*4.2. Two generalizations of Suslin's Hypothesis*

Suslin's Hypothesis is the statement that there doesn't exist a Suslin line (which is introduced by Suslin [28]), and equivalent to the statement that there doesn't exist a Suslin tree (which is due to Kurepa, see e.g. [17, 5.13. Theorem]). A Suslin tree is an Aronszajn tree which does not have an uncountable antichain, that is, which has the countable chain condition as a forcing notion. Therefore Suslin's Hypothesis says that there are no ccc Aronszajn tree.

**Definition 4.4.**
1. $\neg\mathcal{C}(\mathrm{arec})$ *is the statement that there are no ccc forcing notions with the anti-rectangle refining property.*
2. $\neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1})$ *is the statement that there are no ccc forcing notions with the anti-*$\mathsf{R}_{1,\aleph_1}$.

Since an Aronszajn tree has both the anti-rectangle refining property and the anti-$\mathsf{R}_{1,\aleph_1}$, $\neg\mathcal{C}(\mathrm{arec})$ implies Suslin's Hypothesis and $\neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1})$ also implies Suslin's Hypothesis. So both $\neg\mathcal{C}(\mathrm{arec})$ and $\neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1})$ can be considered as generalizations of Suslin's Hypothesis.

Let $\mathfrak{A}$ be a set of partially ordered structures on $\omega_1$, like $\omega_1$-trees or $(\omega_1, \omega_1)$-pregaps, and suppose that every structure in $\mathfrak{A}$ has the anti-rectangle refining property. Then the statement that every structure in $\mathfrak{A}$ has an uncountable antichain can be considered as an analogue of Suslin's Hypothesis. (See [2], also [40].) The same things can be considered for the anti-$\mathsf{R}_{1,\aleph_1}$.

**Proposition 4.5.** *The following statements follow from $\neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1})$.*

1. *Suslin's Hypothesis.*
2. *Every $(\omega_1, \omega_1)$-gap is indestructible.*
3. *The inequality $\mathfrak{b} > \aleph_1$ holds.*

*Proof.* The first two statements are straightforward.

For the third one, suppose that $\mathfrak{b} = \aleph_1$. Then there exists an unbounded $<^*$-increasing sequence $F$ of $\omega_1$-many strictly increasing functions in $\omega^\omega$. By Lemma 3.3 and Proposition 3.4, $\mathbb{P}(F)$ has the countable chain condition and the anti-$\mathsf{R}_{1,\aleph_1}$. Therefore $\neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1})$ fails. $\square$

$\mathcal{C}^2$ is the statement that if $\mathbb{P}$ and $\mathbb{Q}$ are ccc forcing notions, then $\mathbb{P} \times \mathbb{Q}$ is also ccc. We saw that if $\mathbb{P}$ is a ccc forcing notion with the anti-rectangle refining property or the anti-$\mathsf{R}_{1,\aleph_1}$, then this $\mathbb{P}$ is a witness of the failure of $\mathcal{C}^2$ in section 2. So $\mathcal{C}^2$ implies both $\neg\mathcal{C}(\mathrm{arec})$ and $\neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1})$. Therefore we have the following diagram.

$$\begin{tikzcd}
\mathsf{MA}_{\aleph_1} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{K}_2 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C}^2 \arrow[dr] \arrow[dd] & \\
& \mathsf{MA}_{\aleph_1}(\mathrm{rec}) \arrow[rr] & & \mathcal{K}_2(\mathrm{rec}) \arrow[rr] & & \neg\mathcal{C}(\mathrm{arec}) \\
\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1}) \arrow[rr] & & \mathcal{K}_2(\mathsf{R}_{1,\aleph_1}) \arrow[rr] & & \neg\mathcal{C}(\mathrm{a}\mathsf{R}_{1,\aleph_1}) &
\end{tikzcd}$$

*4.3. A consistency result*

In this subsection, we show that it is consistent that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ holds and $\mathcal{C}^2$ fails. So $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ is a weak fragment of $\mathsf{MA}_{\aleph_1}$. This is an analogical consistency result that $\mathsf{MA}_{\aleph_1}(\mathrm{rec})$ holds and $\mathcal{C}^2$ fails in [40, §4.2]. To show this, as in [40, §4.2], we use the entangled set of reals and the theorem below.

**Definition 4.6** (Abraham–Rubin–Shelah [1, §8.])**.** *A set $E$ of reals is entangled if $E$ is uncountable and for every $n \in \omega$ and every $s \in 2^n$ and every uncountable family $F$ of increasing (with respect to the usual ordering of the reals) pairwise disjoint $n$-tuples of elements in $E$, there exist $x$ and $y$ in $F$ such that for every $i \in n$, $x(i) < y(i)$ iff $s(i) = 0$.*

**Theorem 4.7** (Todorčević [30, Theorem 6])**.** *If $\mathcal{C}^2$ holds, then there are no entangled set of reals.* $\square$

To show the theorem below, we use the next proposition. For this proof, see also [40, §4.2].

**Proposition 4.8** ([3, Fact 3.13.]). *If $E$ is an entangled set of reals, then for every $n \in \omega$, every $s \in 2^n$ and every uncountable family $F$ of increasing pairwise disjoint $n$-tuples of elements in $E$, there are $F_0$ and $F_1$ in $[F]^{\aleph_1}$ such that for every $x \in F_0$ and every $y \in F_1$ and every $i \in n$, $x(i) < y(i)$ iff $s(i) = 0$.* $\square$

**Theorem 4.9.** *It is consistent that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ holds and there exists an entangled set of reals.*

*Proof.* We note that the preservation of the entangledness of sets of reals by ccc forcing is preserved by finite support iterations (e.g. [3, Lemma 3.14]). So it suffices to show that if $E$ is an entangled set of reals and a forcing notion $\mathbb{Q}$ in $\mathsf{FSCO}$ has the property $\mathsf{R}_{1,\aleph_1}$, then $E$ is still entangled in the forcing extension with $\mathbb{Q}$.

Suppose that $E$ is an entangled set of reals and $\mathbb{Q}$ is a forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$. Let $n \in \omega$, $s \in 2^n$, $\dot{F}$ a $\mathbb{Q}$-name for an uncountable family of increasing pairwise disjoint $n$-tuples of elements of $E$, and $\sigma \in \mathbb{Q}$. We will find an extension $\mu$ of $\sigma$ in $\mathbb{Q}$, and increasing $n$-tuples $x$ and $y$ of $E$ such that for all $i \in n$, $x(i) < y(i)$ iff $s(i) = 0$, and

$$\mu \Vdash_{\mathbb{Q}} \text{“ } x \in \dot{F} \ \& \ y \in \dot{F} \text{ ”}.$$

We can find a sequence $\langle \langle \tau_\alpha, x_\alpha \rangle \, ; \alpha \in \omega_1 \rangle$ of pairs such that

- for each $\alpha \in \omega_1$, $\tau_\alpha \in \mathbb{Q}$ and $\tau_\alpha \leq_{\mathbb{Q}} \sigma$, and $x_\alpha$ is an $n$-tuples of elements in $E$,
- for each $\alpha \in \omega_1$, $\tau_\alpha \Vdash_{\mathbb{Q}}$ “ $x_\alpha \in \dot{F}$ ”,
- the set $\{\tau_\alpha; \alpha \in \omega_1\}$ forms a $\Delta$-system with root $\nu$, and
- the family $\{x_\alpha; \alpha \in \omega_1\}$ is pairwise disjoint.

By Proposition 4.8, there are uncountable subsets $I_0$ and $I_1$ of $\omega_1$ such that for all $\alpha \in I_0$, $\beta \in I_1$ and $i \in n$, $x_\alpha(i) < x_\beta(i)$ iff $s(i) = 0$. Here we take a countable elementary submodel $N$ which has the set $\{E, \mathbb{Q}, I_0\}$ and take $\delta \in I_1 \setminus N$ such that $\tau_\delta \cap N = \nu$. Since the set $\{\tau_\alpha; \alpha \in I_0\}$ also forms a $\Delta$-system with root $\nu$, by the property $\mathsf{R}_{1,\aleph_1}$ of the forcing notion $\mathbb{Q}$, there exists an uncountable subset $I_0'$ of $I_0$ in $N$ such that for every $\alpha \in I_0'$, $\tau_\alpha$ and $\tau_\delta$ are compatible in $\mathbb{Q}$. Therefore for each $\alpha \in I_0'$, there exists a common extension $\mu$ of $\tau_\alpha$ and $\tau_\delta$ in $\mathbb{Q}$, and then

$$\mu \Vdash_{\mathbb{Q}} \text{“ } x_\alpha \in \dot{F} \ \& \ x_\delta \in \dot{F} \text{ ”},$$

which finishes the proof. $\square$

## 5. Forcing notions with the property $\mathsf{R}_{1,\aleph_1}$

In this section, we consider two preservation theorems of forcing notions in $\mathsf{FSCO}$ with the rectangle refining property or the property $\mathsf{R}_{1,\aleph_1}$. One is the additivity $\mathsf{add}(\mathcal{N})$ of the null ideal, which is the smallest size of sets $\mathcal{X}$ of measure zero sets such that the union of $\mathcal{X}$ is not measure zero. The other is the covering number $\mathsf{cov}(\mathcal{N})$ of the null ideal, which is the smallest size of sets $\mathcal{X}$ of measure zero sets such that the union of $\mathcal{X}$ is the set of the reals. Two theorems are ones for both the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$. We give only proofs for the property $\mathsf{R}_{1,\aleph_1}$, but proofs for the rectangle refining property are almost same. The only difference is the way to find compatible conditions in uncountable sets of conditions, that is, a use of (1) or (2) in Proposition 2.10.

### *5.1. The friendly-ness*

In [43, §2], Zapletal proved that it is consistent that every Aronszajn tree is special and $\mathsf{add}(\mathcal{N}) = \aleph_1$ holds by showing the preservation theorem on Bartoszyński's characterization of the additivity of the null ideal for the specialization of an Aronszajn tree by finitely approximation.

**Definition and Theorem 5.1** (Bartoszyński [5])**.** *A slalom is a function in the set*

$$\prod_{n\in\omega}\left([\omega]^{\leq n+1}\setminus\{\emptyset\}\right).$$

*For a function $f$ in $\omega^\omega$ and a slalom $\varphi$, $\varphi$ captures $f$ if for all but finitely many $n\in\omega$, $f(n)\in\varphi(n)$.*

*The additivity of the null ideal is equal to the smallest size of sets $\mathcal{F}$ of functions in $\omega^\omega$ such that for every slalom $\varphi$, there exists a member of $\mathcal{F}$ which is not captured by $\varphi$.*

**Definition 5.2** (Judah–Shelah [14, 3.3. definition])**.** *A forcing notion $\mathbb{P}$ is friendly* ([4]) *if for any large enough regular cardinal $\theta$, countable elementary submodel $N$ of $H(\theta)$ which has the set $\{\mathbb{P}\}$, $p\in\mathbb{P}\cap N$, and $f\in\omega^\omega$, if $f$ is not captured by any slalom in $N$, then there exists $q\leq_{\mathbb{P}} p$ which is $(N,\mathbb{P})$-generic such that*

$$q\Vdash_{\mathbb{P}} \text{“ } f \text{ is not captured by any slalom in } N \text{ ”}.$$

We note that a friendly forcing notion preserves the additivity number of the measure zero ideal, and the friendly-ness is closed under countable support iterations [14, 3.3. and 3.4. Lemmas] (see also [43, Lemma 13.]). So it follows from the next theorem that it is consistent that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ (and-or $\mathsf{MA}_{\aleph_1}(\text{rec})$) holds and $\mathsf{add}(\mathcal{N}) = \aleph_1$. In [43, §2], Zapletal proved that if $T$ is an Aronszajn tree, then $\mathcal{S}_T$, which is the specialization of $T$ by finite approximations defined in section 2, is friendly. So the next theorem is an expansion of Zapletal's theorem.

[4]In [14], this property is called good, and in [43], this is called friendly.

**Theorem 5.3.** *A forcing notion in* $\mathsf{FSCO}$ *with the property* $\mathsf{R}_{1,\aleph_1}$ *(or the rectangle refining property) is friendly.*

*Proof.* Suppose that $\mathbb{Q} \in \mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$, $\theta$ is a large enough regular cardinal, $N$ is a countable elementary submodel of $H(\theta)$ which has the set $\{\mathbb{Q}\}$, $f \in \omega^\omega$, $\sigma \in \mathbb{Q} \cap N$, and $\dot\varphi$ is a $\mathbb{Q}$-name for a slalom such that $\dot\varphi \in N$ and

$$\sigma \Vdash_{\mathbb{Q}} \text{“ } \dot\varphi \text{ captures } f \text{ ”}.$$

We will show that there exists a slalom $\psi$ in $N$ which captures $f$. It suffices to show the theorem because by the property $\mathsf{R}_{1,\aleph_1}$, $\mathbb{Q}$ is ccc, hence every condition of $\mathbb{Q}$ is generic.

By our assumption, there are $\tau \leq_{\mathbb{Q}} \sigma$ and $n \in \omega$ such that

$$\tau \Vdash_{\mathbb{Q}} \text{“ } \forall m \geq n \, (f(m) \in \dot\varphi(m)) \text{ ”}.$$

By strengthening $\tau$ if necessary, we may assume that $\tau$ is not in $N$, that is, $\tau \setminus N \neq \emptyset$. We note that both $\tau \cap N$ and $n$ are in $N$. And then for every $m \geq n$ and $\alpha \in \omega_1$, there exists $\mu \in \mathbb{Q}$ such that

- $\mu \supseteq \tau \cap N$,
- $\min(\mu \setminus (\tau \cap N)) > \alpha$, and
- $\mu \Vdash_{\mathbb{Q}}$ “ $f(m) \in \dot\varphi(m)$ ”.

To show this, let $m \geq n$, and assume that there exists $\alpha \in \omega_1$ such that for every $\mu \in \mathbb{Q}$, if $\mu \supseteq \tau \cap N$ and $\mu \Vdash_{\mathbb{Q}}$ “ $f(m) \in \dot\varphi(m)$ ”, then $\min(\mu \setminus (\tau \cap N)) \leq \alpha$. We notice that $f$ may not be in $N$, but $f(m)$ is in $N$. So by the the elementarity of $N$, we can find such an $\alpha$ in $\omega_1 \cap N$. However for our $\tau$, $\tau \supseteq \tau \cap N$, $\min(\tau \setminus (\tau \cap N)) > \omega_1 \cap N > \alpha$ and $\tau \Vdash_{\mathbb{Q}}$ “ $f(m) \in \dot\varphi(m)$ ”, which is a contradiction.

Let $\psi$ be a slalom such that for each $m \geq n$,

$$\begin{aligned}\psi(m) := \big\{ & k \in \omega; \text{ for every } \alpha \in \omega_1, \text{ there is } \mu \in \mathbb{Q} \text{ such that} \\ & \mu \supseteq \tau \cap N, \ \min(\mu \setminus (\tau \cap N)) > \alpha \text{ and } \mu \Vdash_{\mathbb{Q}} \text{“ } k \in \dot\varphi(m) \text{ ”} \big\}\end{aligned}$$

By the elementarity of $N$, we note that $\psi \in N$, and by the previous observation, for every $m \geq n$, $f(m) \in \psi(m)$. So the rest of the proof is that $\psi$ is a slalom, that is for every $m \geq n$, the size of the set $\psi(m)$ is not bigger than $m + 1$.

Assume that there exists a subset $\{k_i; i \in m+2\} \in [\omega]^{m+2}$ of $\psi(m)$ for some $m \geq n$. Then for each $i \in m+2$, there is an uncountable subset $I_i$ of $\mathbb{Q}$ in $N$ such that

- for every $i \in m+2$ and $\mu \in I_i$, $\mu \supseteq \tau \cap N$,

- for every $i \in m+2$ and $\mu \in I_i$, $\mu \Vdash_{\mathbb{Q}}$ " $k_i \in \dot{\varphi}(m)$ ",
- the set $\bigcup_{i \in m+2} I_i$ forms a $\Delta$-system with root $\tau \cap N$.

(We notice that the sequence $\langle I_i; i \in m+2\rangle$ belongs to $N$.) By Proposition 2.10, there exists a sequence $\langle \mu_i; i \in m+2\rangle \in \prod_{i \in m+2} I_i$ such that there exists a common extension $\mu$ of members of the set $\{\mu_i; i \in m+2\}$ in $\mathbb{Q}$, and then

$$\mu \Vdash_{\mathbb{Q}} \text{" } \{k_i; i \in m+2\} \subseteq \dot{\varphi}(m) \text{ ",}$$

which is a contradiction. □

*5.2. Not adding random reals*

We note that a $\sigma$-centered forcing notion is a ccc forcing notion which doesn't add random reals [6, §6.5.D]. This is a typical example of ccc forcing notions not adding random reals. There are many non-ccc forcing notions which doesn't add random reals [24], but not so many for ccc forcing notions. For example, it is consistent that there exists a ccc forcing notion with the Sacks property (see [37]). This does not add random reals. In [29], Talagrand gives a counterexample of the Control Measure Problem, which is a Maharam algebra which is not a measure algebra (see [38, §2] and [4, §4]). It seems an example not adding random reals, but we don't know that it doesn't add random reals. Anyway, we should notice that both of two examples are $\omega^\omega$-bounding.

The next theorem gives a new type of ccc forcing notions not adding random reals. Because both $\mathsf{MA}_{\aleph_1}(\text{rec})$ and $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$, in fact both $\neg\mathcal{C}(\text{arec})$ and $\neg\mathcal{C}(\text{a}\mathsf{R}_{1,\aleph_1})$, imply $\mathfrak{b} > \aleph_1$ ([41] and Proposition 4.5), so some of forcing notions in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$ don't have to be $\omega^\omega$-bounding. (Because since the $\omega^\omega$-bounding property is preserved by countable support iterations (which is due to Shelah, see e.g. [6, 6.3.A]), if all of them are $\omega^\omega$-bounding, we can force that $\mathsf{MA}_{\aleph_1}(\text{rec})$ (and-or $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$) holds and $\mathfrak{b} = \aleph_1$ by countable support iteration.)

**Theorem 5.4.** *A forcing notion in* $\mathsf{FSCO}$ *with the property* $\mathsf{R}_{1,\aleph_1}$ *(or the rectangle refining property) doesn't add random reals.*

*Proof.* Suppose that $\mathbb{Q}$ is a forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$, $\dot{r}$ is a $\mathbb{Q}$-name for a function in $2^\omega$, and $\sigma \in \mathbb{Q}$. Let $\theta$ be a large enough regular cardinal and $N$ a countable elementary submodel of $H(\theta)$ which has the set $\{\mathbb{Q}, \dot{r}, \sigma\}$, and we take a sequence $\langle U_n; n \in \omega\rangle$ of open subsets of $2^\omega$ such that for each $n \in \omega$, the Lebesgue measure of $U_n$ is less than $2^{-n}$ and

$$2^\omega \cap N \subseteq \bigcap_{n \in \omega} \bigcup_{m \geq n} U_m.$$

We note that the right-hand set in the above formula is of Lebesgue measure zero (in fact, $G_\delta$ null). We will show that

$$\sigma \nVdash_{\mathbb{Q}} \text{" } \dot{r} \notin \bigcap_{n \in \omega} \bigcup_{m \geq n} U_m \text{ ",}$$

which finishes the proof.

Suppose that

$$\sigma \Vdash_{\mathbb{Q}} \text{“ } \dot{r} \notin \bigcap_{n\in\omega} \bigcup_{m\geq n} U_m \text{ ”},$$

and take $\tau \leq_{\mathbb{Q}} \sigma$ and $n \in \omega$ such that

$$\tau \Vdash_{\mathbb{Q}} \text{“ } \forall m \geq n\, (\dot{r} \notin U_m) \text{ ”}.$$

We note that $n \in N$, and by extending $\tau$ if necessary, we may assume that $\tau \notin N$, that is, $\tau \setminus N \neq \emptyset$. Even then, we note that $\tau \cap N \in N$.

For each $k \in \omega$, let

$$\begin{aligned} S_k := \big\{ s \in 2^k;\ & \text{there is } \alpha \in \omega_1 \text{ such that for every } \mu \in \mathbb{Q} \text{ with } \mu \supseteq \tau \cap N, \\ & \text{if } \mu \Vdash_{\mathbb{Q}} \text{“ } \dot{r}{\restriction}k \neq s \text{ ”, then } \min(\mu \setminus (\tau \cap N)) \leq \alpha \big\}. \end{aligned}$$

We note that the sequence $\langle S_k; k \in \omega \rangle$ belongs to the model $N$. Let $k \in \omega$, and assume that $S_k = \emptyset$. Then for all $s \in 2^k$ and $\alpha \in \omega_1$, there exists $\mu \in \mathbb{Q}$ such that $\mu \supseteq \tau \cap N$, $\mu \Vdash_{\mathbb{Q}}$ “ $\dot{r}{\restriction}k \neq s$ ” and $\min(\mu \setminus (\tau \cap N)) > \alpha$. So we can find a sequence $\langle I_s; s \in 2^k \rangle$ of uncountable subsets of $\mathbb{Q}$ in $N$ such that

- the set $\bigcup_{s\in 2^k} I_s$ forms a $\Delta$-system with root $\tau \cap N$, and
- for any $s \in 2^k$ and $\mu \in I_s$, $\mu \Vdash_{\mathbb{Q}}$ “ $\dot{r}{\restriction}k \neq s$ ”.

By Proposition 2.10, we can find a sequence $\langle \mu_s; s \in 2^k \rangle \in \prod_{s\in 2^k} I_s$ such that there exists a common extension $\mu$ of members of the set $\{\mu_s; s \in 2^k\}$ in $\mathbb{Q}$, and then

$$\mu \Vdash_{\mathbb{Q}} \text{“ } \dot{r}{\restriction}k \notin 2^k \text{ ”},$$

which is a contradiction. Therefore for every $k \in \omega$, $S_k$ is not empty.

We note that for any $s$ and $t$ in $2^{<\omega}$, if $s \subseteq t$ and $t \in S_{|t|}$, then $s$ is also in $S_{|s|}$, hence the set $\bigcup_{k\in\omega} S_k$ forms an infinite subtree of $2^{<\omega}$. Since it is in the model $N$, there exists $u \in 2^\omega \cap N$ such that for every $k \in \omega$, $u{\restriction}k \in S_k$. Since $u \in N$, there exists $m \geq n$ such that $u \in U_m$, and since $U_m$ is an open set, there exists $k \geq m$ such that

$$[u{\restriction}k] := \{v \in 2^\omega; u{\restriction}k \subseteq v\} \subseteq U_m.$$

Since $u{\restriction}k \in S_k$ in $N$, there is $\alpha \in \omega_1 \cap N$ such that for any $\mu \in \mathbb{Q}$ with $\mu \supseteq \tau \cap N$, if

$$\mu \Vdash_{\mathbb{Q}} \text{“ } \dot{r}{\restriction}k \neq u{\restriction}k \text{ ”},$$

then $\min(\mu \setminus (\tau \cap N)) \leq \alpha$. Since $\min(\tau \setminus (\tau \cap N)) \geq \omega_1 \cap N > \alpha$, it follows that

$$\tau \nVdash_{\mathbb{Q}} \text{“ } \dot{r}{\restriction}k \neq u{\restriction}k \text{ ”},$$

so there is $\nu \leq_{\mathbb{Q}} \tau$ such that

$$\nu \Vdash_{\mathbb{Q}} \text{“ } \dot{r}\restriction k = u\restriction k \text{ ”}.$$

Then since

$$\nu \Vdash_{\mathbb{Q}} \text{“ } [\dot{r}\restriction k] = [u\restriction k] \subseteq U_k \text{ ”},$$

it follows that

$$\nu \Vdash_{\mathbb{Q}} \text{“ } \dot{r} \in U_k \text{ ”}$$

which is a contradiction. □

We note that the property that “no random reals are added” is preserved by finite support iterations [6, §6.5.D], and $\mathsf{cov}(\mathcal{N})$ is still $\aleph_1$ in any forcing extension which doesn't add random reals over a ground model satisfying $\mathsf{CH}$. Therefore we have the following corollary.

**Corollary 5.5.** *It is consistent that* $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ *holds and* $\mathsf{cov}(\mathcal{N}) = \aleph_1$.

We note that if $T$ is an Aronszajn tree such that every node in $T$ has infinitely many successors, then $a(T)$ adds a Cohen real ([5]). So it follows from this fact and Proposition 4.5 that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ implies that the additivity $\mathsf{add}(\mathcal{M})$ of the meager ideal is larger than $\aleph_1$, because of the equation $\mathsf{add}(\mathcal{M}) = \min\{\mathsf{cov}(\mathcal{M}), \mathfrak{b}\}$, which is due to Miller and Truss independently [22, 36].

## 6. Problem 2 for forcing notions with the property $\mathsf{R}_{1,\aleph_1}$

In this section, we show that it is consistent that for every $n \in \omega$, $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ holds and $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ fails. This is a negative answer of Problem 2 for the class of forcing notions in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$.

We will prove more stronger result thinking of the following property.

**Definition 6.1** (E.g. [31, §3])**.** *Let $\mathbb{P}$ be a forcing notion.*

1. *An uncountable subset $I$ of $\mathbb{P}$ has the finite compatibility property if every finite subset $s$ of $I$ has a common extension $p$ in $\mathbb{P}$, that is, $p \leq_{\mathbb{P}} q$ for all $q \in s$.*
2. *$\mathbb{P}$ has precaliber $\aleph_1$ if every uncountable subset of $\mathbb{P}$ has an uncountable subset which has the finite compatibility property.*

[5] For each $n \in \omega$, let $\langle t_i^n; i \in \omega\rangle$ be an enumeration of nodes of $T$ of level $n$. Then in the extension by $a(T)$, letting $G$ be $a(T)$-generic,

$$c := \left\{ n \in \omega; \min\left\{ i \in \omega; t_i^{n+1} \in \bigcup G \right\} \text{ is odd} \right\}$$

is a Cohen real.

In [35], Todorčević and Veličiković proved that $\mathsf{MA}_{\aleph_1}$ is equivalent to the statement that every uncountable ccc forcing notion has precaliber $\aleph_1$ [35]. If a forcing notion $\mathbb{P}$ has precaliber $\aleph_1$, then all $\mathcal{K}_n$ hold for this $\mathbb{P}$. So if every forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$, then $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ holds for every $n \in \omega$. We have proved that $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ implies that every Aronszajn tree is special (Proposition 4.2).

**Theorem 6.2.** *It is consistent that every forcing notion in* $\mathsf{FSCO}$ *with the property* $\mathsf{R}_{1,\aleph_1}$ *has precaliber* $\aleph_1$ *and there exists a non-special Aronszajn tree.*

*Therefore it is consistent that* $\mathcal{K}_n(\mathsf{R}_{1,\aleph_1})$ *holds for all* $n \in \omega$ *and* $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ *fails.*

To show this, we use the technique due to Shelah showing the consistency that Suslin's Hypothesis holds and there exists a non-special Aronszajn tree [27, Chapter IX].

The rest of this section is devoted to the proof of Theorem 6.2. In this proof, we use two kinds of notion of levels. One is levels of nodes of an Aronszajn tree $T$, called heights in this paper, and denote the height of a node $t$ of $T$ by $\mathsf{ht}_T(t)$. The other is defined as below. This is necessary to define a forcing notion $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$ below.

**Definition 6.3.** *Let* $\mathbb{P}$ *be a forcing notion of size* $\aleph_1$, *and* $\vec{M} = \langle M_\alpha; \alpha \in \omega_1 \rangle$ *a sequence of countable elementary submodels of* $H(\aleph_2)$ *such that* $\mathbb{P} \in M_0$, *and for every* $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha \rangle \in M_\alpha$. *Then for each* $p \in \mathbb{P}$, *define*

$$\mathsf{lv}_{\mathbb{P},\vec{M}}(p) = \mathsf{lv}(p) := \min \{\alpha \in \omega_1; p \in M_\alpha\}.$$

**Definition 6.4.** *Let* $\mathbb{Q}$ *be a forcing notion in* $\mathsf{FSCO}$, $I$ *an uncountable subset of* $\mathbb{Q}$, *and* $\vec{M} = \langle M_\alpha; \alpha \in \omega_1 \rangle$ *a sequence of countable elementary submodels of* $H(\aleph_2)$ *such that* $\{\mathbb{Q}, I\} \in M_0$, *and for every* $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha \rangle \in M_\alpha$. *(Then we notice that for each* $\alpha \in \omega_1$, *the set*

$$I \cap \left( M_\alpha \setminus \bigcup_{\beta \in \alpha} M_\beta \right)$$

*is not empty, in fact, it is uncountable in* $M_\alpha$.*)*

*Define* $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$, *whose conditions* $\langle h, f \rangle$ *satisfies that*

- $h$ *is a finite partial function from* $\omega_1$ *into* $\omega_1$,
- *for any* $\alpha, \beta \in \mathrm{dom}(h)$, $\alpha \le h(\alpha)$, *and if* $\alpha < \beta$, *then* $h(\alpha) < \beta$,
- $f$ *is a finite partial function from the set* $\bigcup_{\alpha \in \mathrm{dom}(h)} \{\sigma \in I; \mathsf{lv}(\sigma) = h(\alpha)\}$ *into* $\omega_1$,
- *for any* $\sigma \in \mathrm{dom}(f)$, $f(\sigma) < \mathsf{lv}(\sigma)$,

- *for any $\sigma \in \mathrm{dom}(f)$, there exists a common extension of all members of the set $f^{-1}[f(\sigma)]$ in $\mathbb{Q}$,*

*inversely ordered by extension, that is, for any $\langle h, f\rangle$ and $\langle h', f'\rangle$ in $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$,*

$$\langle h, f\rangle \leq_{\mathcal{Q}(\mathbb{Q},I,\vec{M})} \langle h', f'\rangle :\iff h \supseteq h \ \&\ f \supseteq f'.$$

*Moreover for a subset $S$ of the set $\omega_1 \setminus \{0\}$, let*

$$\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S) := \left\{\langle h, f\rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}); \forall\alpha \in \mathrm{dom}(h) \cap S\,(h(\alpha) = \alpha)\right\},$$

*whose order is the inherited one.*

At first, we prove two lemmas, from which it follows that it is consistent that every forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$. To prove the following two lemmas, we don't need the full strength of the property $\mathsf{R}_{1,\aleph_1}$.

**Definition 6.5.** *A forcing notion $\mathbb{P}$ is called powerfully ccc if the finite support products of any number of copies of $\mathbb{P}$ have the countable chain condition.*

Powerfully ccc forcing notions play an important role to introduced a Ramsey-theoretic characterization of Martin's Axiom in [35] (see also [8, Chapter 3]). In [33, p 837], Todorčević presented a problem whether Martin's Axiom for powerfully ccc forcing notions is actually the same as the full Martin's Axiom.

We note that forcing notions with the rectangle refining property or the property $\mathsf{R}_{1,\aleph_1}$ are powerfully ccc (Proposition 2.8). The following two lemmas hold for powerfully ccc forcing notions in $\mathsf{FSCO}$, so the two lemmas are also available for forcing notions in $\mathsf{FSCO}$ with the rectangle refining property.

**Lemma 6.6.** *Suppose that $\mathbb{Q}$ is a powerfully ccc forcing notion in $\mathsf{FSCO}$, $I$ is an uncountable subset of $\mathbb{Q}$, $\vec{M} = \langle M_\alpha; \alpha \in \omega_1\rangle$ is a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{\mathbb{Q}, I\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha\rangle \in M_\alpha$, and $S \subseteq \omega_1 \setminus \{0\}$.*

*Then both $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$ and $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ are proper. In fact, for a large enough regular cardinal $\theta$, a countable elementary submodel $N$ of $H(\theta)$ which has the set $\left\{\mathbb{Q}, \vec{M}, S\right\}$, a condition $\langle h, f\rangle$ of $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$ (or $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$), and a countable ordinal $\delta$ larger than the ordinal $\omega_1 \cap N$, $\langle h \cup \{\langle\omega_1 \cap N, \delta\rangle\}, f\rangle$ is $(N, \mathcal{Q}(\mathbb{Q}, I, \vec{M}))$-generic (or $(N, \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$-generic if it is a condition of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$).*

*Proof.* We will prove only for $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$. A proof for $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is completely the same. Let $\mathbb{Q}$, $N$, $\langle h, f\rangle$, $\delta$ be as in the assumption of the statement of the lemma, and let $\langle h', f'\rangle \leq_{\mathcal{Q}(\mathbb{Q},I,\vec{M})} \langle h \cup \{\langle\omega_1 \cap N, \delta\rangle\}, f\rangle$ and $D$ a dense open subset of $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$. We will find a condition in $D \cap N$ which is compatible with $\langle h', f'\rangle$ in $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$.

Without loss of generality, we may assume that $\langle h', f'\rangle \in D$. We note that $\langle h' \restriction N, f' \restriction N\rangle$ is in $\mathcal{Q}(\mathbb{Q}, I, \vec{M}) \cap N$ because $\omega_1 \cap N \in \mathrm{dom}(h')$. Let

$$L := \{f'(\sigma); \sigma \in \mathrm{dom}(f') \setminus N \ \&\ f'(\sigma) \in \omega_1 \cap N\},$$

which is a finite subset of $N$, and hence is in $N$. Let

$$D' := \left\{ \langle k, g \rangle \in D; \langle k, g \rangle \leq_{\mathcal{Q}(\mathbb{Q}, I, \vec{M})} \langle h' \restriction N, f' \restriction N \rangle \ \& \ g^{-1}[\{\alpha\}] \neq \emptyset \text{ for all } \alpha \in L \right\}.$$

We note that $D'$ is in $N$, and is dense in $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$ below $\langle h' \restriction N, f' \restriction N \rangle$. So by the definition of $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$, there exists a subset $J$ of the product ${}^{L}\mathbb{Q}$ of $L$ many copies of $\mathbb{Q}$ in $N$ such that for each $\langle k, g \rangle \in D'$, there is $\langle \mu_\alpha; \alpha \in L \rangle$ in $J$ such that for every $\alpha \in L$, $\mu_\alpha$ is an extension of all members of $g^{-1}[\{\alpha\}]$ in $\mathbb{Q}$. Since $\langle h', f' \rangle \in D'$, there exists $\langle \tau_\alpha; \alpha \in L \rangle$ in $J$ such that for each $\alpha \in L$, $\tau_\alpha$ is an extension of all members of $(f')^{-1}[\{\alpha\}]$ in $\mathbb{Q}$. Let $J' \in N$ be a maximal antichain in $J$, that is,

- for every $\vec{\mu}$ and $\vec{\nu}$ in $J'$, if $\vec{\mu} \neq \vec{\nu}$, then $\vec{\mu}$ and $\vec{\nu}$ are incompatible in ${}^{L}\mathbb{Q}$,
- for every $\vec{\mu}$, there exists $\vec{\nu} \in J'$ which is compatible with $\vec{\mu}$ in ${}^{L}\mathbb{Q}$.

Since ${}^{L}\mathbb{Q}$ is ccc (which follows from the powerful cccness of $\mathbb{Q}$), $J'$ is countable, hence $J' \subseteq N$. Since $\langle \tau_\alpha; \alpha \in L \rangle$ is not in $N$, we notice that $\langle \tau_\alpha; \alpha \in L \rangle \notin J'$. Let $\langle \mu_\alpha; \alpha \in L \rangle \in J'$ be compatible with $\langle \tau_\alpha; \alpha \in L \rangle$ in ${}^{L}\mathbb{Q}$, and let $\langle k, g \rangle \in D' \cap N$ be such that for each $\alpha \in L$, $\mu_\alpha$ is an extension of all members of $g^{-1}[\{\alpha\}]$ in $\mathbb{Q}$. Then $\langle h' \cup k, f' \cup g \rangle$ is a common extension of $\langle h', f' \rangle$ and $\langle k, g \rangle$ in $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$. $\square$

**Lemma 6.7.** *Suppose that $\mathbb{Q}$ is a powerfully ccc forcing notion in* $\mathsf{FSCO}$*, $I$ is an uncountable subset of $\mathbb{Q}$, $\vec{M} = \langle M_\alpha; \alpha \in \omega_1 \rangle$ is a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{\mathbb{Q}, I\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha \rangle \in M_\alpha$, and $S \subseteq \omega_1 \setminus \{0\}$ is stationary.*

*Then $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ adds an uncountable subset $I'$ of $I$ such that every finite subset of $I'$ has a common extension in $\mathbb{Q}$.*

*Proof.* Let $\mathbb{Q}$, $I$, $\vec{M}$, $S$ be as in the assumption of the statement of the lemma, and let $G$ be a $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$-generic filter. For each $\alpha \in \omega_1$, the set

$$\left\{ \langle h, f \rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S); \mathrm{dom}(h) \not\subseteq \alpha \ \& \ \exists \sigma \in \mathrm{dom}(f)\, (\mathsf{lv}(\sigma) \geq \alpha) \right\}$$

is dense in $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$. So in the extension $\mathbf{V}[G]$, the sets

$$h_G := \bigcup_{\langle h, f \rangle \in G} h \qquad \text{and} \qquad f_G := \bigcup_{\langle h, f \rangle \in G} f$$

are both uncountable.

Next, we show that $\mathrm{dom}(h_G) \cap S$ is stationary in $\mathbf{V}[G]$. To show this, we work in the ground model $\mathbf{V}$. Let $\dot{C}$ be a $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$-name for a club on $\omega_1$, and $\langle h, f \rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$. We take a countable elementary submodel $N$ of $H(\theta)$ which has the set $\left\{ \mathbb{Q}, I, \vec{M}, S, \dot{C}, \langle h, f \rangle \right\}$ such that $\omega_1 \cap N \in S$. Then by the previous lemma, $\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}, f \rangle$ is $(N, \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S))$-generic and so

$$\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}, f \rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{“ } \omega_1 \cap N \in \dot{C} \text{ ”}.$$

Thus

$$\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}, f \rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{“ } \omega_1 \cap N \in \operatorname{dom}(h_{\dot{G}}) \cap S \cap \dot{C} \text{ ”}.$$

In $\mathbf{V}[G]$, by genericity, for every $\alpha \in \operatorname{dom}(h_G)$, we can pick $\sigma_\alpha \in \operatorname{dom}(f_G)$ such that $\mathsf{lv}(\sigma_\alpha) = h_G(\alpha)$. Then for every $\alpha \in \operatorname{dom}(h_G) \cap S$,

$$f_G(\sigma_\alpha) < \mathsf{lv}(\sigma_\alpha) = h_G(\alpha) = \alpha,$$

so by Fodor's lemma, there exists a stationary subset $S'$ of $\operatorname{dom}(h_G) \cap S$ and $\gamma \in \omega_1$ such that for every $\alpha \in S'$, $f_G(\sigma_\alpha) = \gamma$. Then the set $\{\sigma_\alpha ; \alpha \in S'\}$ is an uncountable subset of $I$ and has the finite compatibility property in $\mathbb{Q}$. $\square$

Therefore by a bookkeeping argument, we can force by the countable support iteration that every forcing notion in $\mathsf{FSCO}$ with the rectangle refining property or the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$.

To force that a fixed non-special Aronszajn tree (in fact, we start with a fixed Suslin tree) is still not special, we use the following preservation condition.

**Definition 6.8** (Shelah [27, Chapter IX, 4.5 Definition])**.** *Let $T$ be an Aronszajn tree and $S$ a subset of $\omega_1$.*

*A forcing notion $\mathbb{P}$ is $(T, S)$-preserving if for a large enough regular cardinal $\theta$, a countable elementary submodel $N$ of $H(\theta)$ which has the set $\{\mathbb{P}, T, S\}$ and $p \in \mathbb{P} \cap N$, there exists $q \leq_{\mathbb{P}} p$ which is $(N, \mathbb{P})$-generic such that if $\omega_1 \cap N \notin S$, then*

*for any $x \in T$ of height $\omega_1 \cap N$,*
*if $\forall A \in \mathcal{P}(T) \cap N \, (x \in A \to \exists y \in A \, (y <_T x))$,*
*then for every $\mathbb{P}$-name $\dot{A}$, which is in $N$, for a subset of $T$,*

$$q \Vdash_{\mathbb{P}} \text{“ } x \in \dot{A} \to \exists y \in \dot{A} \, (y <_T x) \text{ ”}.$$

We note that if the above $T$ is Suslin, then it follows ([6]) that

$$\forall A \in \mathcal{P}(T) \cap N \, (x \in A \to \exists y \in A \, (y <_T x)).$$

To show the next lemma, we use the full strength of the property $\mathsf{R}_{1,\aleph_1}$. We don't know whether or not the same holds for forcing notions in $\mathsf{FSCO}$ with the rectangle refining property.

[6]Let $A \in \mathcal{P}(T) \cap N$ and $x \in A \cap T_{\omega_1 \cap N}$, and let $A' \in N$ be a maximal pairwise compatible subset of $A$. Then since $T$ is Suslin, $A'$ have to be countable, so $A' \subseteq N$. Since $x \in A$, there exists $y \in A'$ which is comparable with $x$. Since $\mathsf{ht}_T(y) < \omega_1 \cap N = \mathsf{ht}_T(x)$, $y <_T x$.

**Lemma 6.9.** *Suppose that* $\mathbb{Q}$ *is a forcing notion in* $\mathsf{FSCO}$ *with the property* $\mathsf{R}_{1,\aleph_1}$, $I$ *is an uncountable subset of* $\mathbb{Q}$, $\vec{M} = \langle M_\alpha; \alpha \in \omega_1 \rangle$ *is a sequence of countable elementary submodels of* $H(\aleph_2)$ *such that* $\{\mathbb{Q}, I\} \in M_0$, *and for every* $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha \rangle \in M_\alpha$, *and* $S \subseteq \omega_1 \setminus \{0\}$ *is stationary.*

*Then* $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ *is* $(T,S)$*-preserving.*

*Proof.* Let $\mathbb{Q}$, $I$, $\vec{M}$, $S$ be as in the assumption of the statement of the lemma, and $T$, $\theta$, $N$ as in the statement of the definition of the $(T,S)$-preservation, (moreover we suppose $\vec{M} \in N$, to calculate levels of conditions in $\mathbb{Q}$) and $\langle h, f \rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S) \cap N$. Suppose that $\omega_1 \cap N \notin S$, because if $\omega_1 \cap N \in S$, then the condition $\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}, f \rangle$ is as desired.

Let

$$\delta := \sup \left\{ F(\omega_1 \cap N) + 1; F \in ({}^{\omega_1}\omega_1) \cap N \right\}.$$

Since $N$ is countable, $\delta$ is a countable ordinal. We will show that the condition $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$ of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is our desired one.

By Lemma 6.6, $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$ is $(N, \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S))$-generic. Suppose that $x \in T$ of height $\omega_1 \cap N$ such that for any subset $A \in N$ of $T$, if $x \in A$, then there is $y \in A$ such that $y <_T x$. Let $\dot{A} \in N$ be a $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$-name for a subset of $T$. We will show that

$$\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle \Vdash_{\mathbb{Q}} \text{“ } x \notin \dot{A} \text{ or } \exists y \in \dot{A}\, (y <_T x) \text{ ”}.$$

Let $\langle h', f' \rangle \leq_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$, and assume that

$$\langle h', f' \rangle \nVdash_{\mathbb{Q}} \text{“ } x \notin \dot{A} \text{ ”}.$$

By strengthening $\langle h', f' \rangle$ if necessary, we may assume that

$$\langle h', f' \rangle \Vdash_{\mathbb{Q}} \text{“ } x \in \dot{A} \text{ ”}.$$

We note that $\langle h' \restriction N, f' \restriction N \rangle$ is in $N$ (because $\omega_1 \cap N \in \mathrm{dom}(h')$) and for every $\sigma \in \mathrm{dom}(f') \setminus N$, $\mathsf{lv}(\sigma) > \delta$ by the definition of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$. Let

$$L := \left\{ f'(\sigma); \sigma \in \mathrm{dom}(f') \ \&\ f'(\sigma) \in \omega_1 \cap N \right\},$$

which is a finite subset of $N$, hence is in $N$. Let $\langle \tau_\alpha; \alpha \in L \rangle$ be in the product ${}^L\mathbb{Q}$ such that for each $\alpha \in L$, $\tau_\alpha$ is an extension of all members of $(f')^{-1}[\{\alpha\}]$ in $\mathbb{Q}$. The sequence $\langle \tau_\alpha; \alpha \in L \rangle$ does not belong to $N$, however we notice that the sequence $\langle \tau_\alpha \cap N; \alpha \in L \rangle$ belongs to $N$. We define a function $F$ with the domain

$$\{ t \in T; \mathsf{ht}_T(t) > \max(\mathrm{dom}(h' \restriction N)) \}$$

such that for each $t \in T$ of height larger than $\max(\mathrm{dom}(h' \restriction N))$,

$$F(t) := \sup \Big\{ \beta \in \omega_1; \quad \text{there exists } \langle k, g \rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S) \text{ such that} \\ \bullet \min(\mathrm{dom}(k)) = \mathsf{ht}(t), \\ \bullet\ k(\mathsf{ht}_T(t)) = \beta, \\ \bullet \langle (h' \restriction N) \cup k, (f' \restriction N) \cup g \rangle \text{ is a condition of } \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S), \\ \bullet \langle (h' \restriction N) \cup k, (f' \restriction N) \cup g \rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{“ } t \in \dot{A} \text{ ”, and} \\ \bullet \text{ there is } \langle \mu_\alpha; \alpha \in L \rangle \text{ such that for all } \alpha \in L, \\ \mu_\alpha \supseteq \tau_\alpha \cap N \text{ and } \mu_\alpha \leq_{\mathbb{Q}} \sigma \\ \text{for all } \sigma \in (f' \restriction N)^{-1}[\{\alpha\}] \cup g^{-1}[\{\alpha\}] \Big\}.$$

Then $F$ belongs to $N$. Let

$$B := \{ t \in T; \mathsf{ht}_T(t) > \max(\mathrm{dom}(h' \restriction N)) \ \&\ F(t) = \omega_1 \},$$

which is also in $N$. We define a function $F'$ with the domain

$$[\max(\mathrm{dom}(h' \restriction N)) + 1, \omega_1)$$

such that for a countable ordinal $\beta$ larger than $\max(\mathrm{dom}(h' \restriction N))$,

$$F'(\beta) := \sup \left\{ F(t) + 1; t \in T \setminus B \ \&\ \mathsf{ht}_T(t) \in \big( \max(\mathrm{dom}(h' \restriction N)), \beta \big] \right\}.$$

This $F'$ is a function from $\omega_1$ into $\omega_1$ and also in $N$. Hence $F'(\omega_1 \cap N) < \delta$ by the definition of $\delta$. Since $\langle h', f' \rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)}$ “ $x \in \dot{A}$ ”, $\tau_\alpha \supseteq \tau_\alpha \cap N$ and $\tau_\alpha$ is a common extension of conditions in $(f')^{-1}[\{\alpha\}]$ for all $\alpha \in L$, and $h'(\mathsf{ht}_T(x)) = h'(\omega_1 \cap N) = \delta$, $F(x) \geq \delta$ holds. Therefore $x$ have to belong to $B$. Thus by our assumption, there exists $y \in B$ such that $y <_T x$. Let

$$E := \Big\{ \langle k, g \rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S); \quad \bullet \min(\mathrm{dom}(k)) = \mathsf{ht}_T(y), \\ \bullet \langle (h' \restriction N) \cup k, (f' \restriction N) \cup g \rangle \\ \text{is a condition of } \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S), \\ \bullet \langle (h' \restriction N) \cup k, (f' \restriction N) \cup g \rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{“ } y \in \dot{A} \text{ ”,} \\ \text{and} \\ \bullet \text{ there is } \langle \mu_\alpha; \alpha \in L \rangle \text{ such that for all } \alpha \in L, \\ \mu_\alpha \supseteq \tau_\alpha \cap N \text{ and } \mu_\alpha \leq_{\mathbb{Q}} \sigma \\ \text{for all } \sigma \in (f' \restriction N)^{-1}[\{\alpha\}] \cup g^{-1}[\{\alpha\}] \Big\}.$$

We note that $E$ is in $N$, and uncountable because $F(y) = \omega_1$.

Then there exists a subset $J$ of the product ${}^L\mathbb{Q}$ in $N$ such that for each $\langle k, g \rangle \in E$, there is $\langle \mu_\alpha; \alpha \in L \rangle$ in $J$ such that for every $\alpha \in L$, $\mu_\alpha \supseteq \tau_\alpha \cap N$ and $\mu_\alpha$ is an extension of all members of $(f' \restriction N)^{-1}[\{\alpha\}] \cup g^{-1}[\{\alpha\}]$ in $\mathbb{Q}$. Then there exists $J' \in [J]^{\aleph_1} \cap N$ such that for every $\alpha \in L$, the set $\{\mu_\alpha; \langle \mu_\gamma; \gamma \in L \rangle \in J'\}$ forms a $\Delta$-system with root $\nu_\alpha$. We note that the sequence $\langle \nu_\alpha; \alpha \in L \rangle$ is in $N$

and for each $\alpha \in L$, $\nu_\alpha \supseteq \tau_\alpha \cap N$. So by the property $\mathsf{R}_{1,\aleph_1}$ of ${}^L\mathbb{Q}$, there exists $J'' \in [J']^{\aleph_1} \cap N$ such that every member of $J''$ is compatible with $\langle \tau_\alpha; \alpha \in L\rangle$ in ${}^L\mathbb{Q}$. Therefore when we take any $\langle \mu_\alpha; \alpha \in L\rangle \in J'' \cap N$ and $\langle k, g\rangle \in E \cap N$, for every $\alpha \in L$, a common extension of $\mu_\alpha$ and $\tau_\alpha$ is an extension of all members of $(f' \restriction N)^{-1}[\{\alpha\}] \cup g^{-1}[\{\alpha\}]$ in $\mathbb{Q}$, so $\langle h' \cup k, f' \cup g\rangle$ is a common extension of $\langle h', f'\rangle$ and $\langle k, g\rangle$ in $\mathbb{Q}$. Moreover it follows that

$$\langle h' \cup k, f' \cup g\rangle \Vdash_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \text{“ } y \in \dot{A} \text{ ”}.$$

□

The rest of the proof of Theorem 6.2 is the same to Shelah's original one [27, Chapter IX, 4.8 Conclusion] ([7]). We start in the ground model where $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and there exists a Suslin tree $T$. Let $S$ be a stationary costationary subset of the set $\omega_1 \setminus \{0\}$. We define an $\aleph_1$-free iteration $\langle P_\xi, Q_\eta; \xi \le \omega_2 \;\&\; \eta < \omega_2\rangle$ such that

- $Q_0 = Q(T, S)$, which is not a forcing notion defined above, but one defined by Shelah [27, Chapter IX, 4.2 and 4.3 Definitions],
- each $Q_\eta$ satisfies one of the following:
  1. $Q_\eta$ is proper and $(T, S)$-preserving of size $\aleph_1$,
  2. for some $P_\xi$-name of an antichain $\dot{A}$ of $T$, $\mathsf{ht}_T[\dot{A}] \cap S = \emptyset$ and $Q_\eta = Q_{\text{club}}(\omega_1 \setminus \mathsf{ht}_T[\dot{A}])$, which shoots a club through the set $\omega_1 \setminus \mathsf{ht}_T[\dot{A}]$ by countable approximations,
- for every $\xi \in \omega_2$ and $P_\xi$-names $\dot{\mathbb{Q}}$ and $\dot{I}$ for a forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$ and an uncountable subset of $\dot{\mathbb{Q}}$ respectively, there exists $\eta \in \omega_2$ such that

  $\Vdash_{P_\eta}$ “ $Q_\eta = \dot{\mathcal{Q}}(\dot{\mathbb{Q}}', \dot{I}, \underset{\sim}{\vec{M}}, S)$, where $\dot{\mathbb{Q}}'$ is a forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$ and $\vec{M} = \langle M_\alpha; \alpha \in \omega_1\rangle$ is a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{\dot{\mathbb{Q}}, \dot{I}\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha\rangle \in M_\alpha$ , and if $\dot{\mathbb{Q}}$ has the property $\mathsf{R}_{1,\aleph_1}$, then $\dot{\mathbb{Q}} = \dot{\mathbb{Q}}'$ ”,
- for every $\xi \in \omega_2$ and a $P_\xi$-name $\dot{A}$ of an antichain of $T$, there exists $\eta \in \omega_2$ such that

  $\Vdash_{P_\eta}$ “ $\mathsf{ht}_T[\dot{A}] \cap S = \emptyset$ and $Q_\eta = \dot{Q}_{\text{club}}(\omega_1 \setminus \mathsf{ht}_T[\dot{A}'])$ for some $\dot{A}'$ such that $\dot{A} = \dot{A}'$ ”.

[7] Shelah's proof uses an $\aleph_1$-free iteration. This is different from a countable support iteration. But Schlindwein proved in [26] that the same proof works for a countable support iterations. So our theorem can be shown by a countable support iteration.

The reason why this works well is also same to [27, Chapter IX, 4.8 Conclusion]. (Since this iteration is proper, $S$ is a stationary costationary subset of $\omega_1$ in the extension. The third condition guarantees that every forcing notion in $\mathsf{FSCO}$ with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$ in the extension, and the other conditions guarantee that $T$ is $S$-$st$-special (so $T$ is a non-Suslin Aronszajn tree) and for every antichain $A$ of $T$, the set of levels of $A$ minus the set $S$ is nonstationary (so $T$ is not special) in the extension.)

## 7. Questions

From the result of the previous section, we conclude that $\mathcal{K}_2(\mathsf{R}_{1,\aleph_1})$ does not imply that every Aronszajn tree is special. This is quite different from Todorčević's result in [33, Theorem 1] that $\mathcal{K}_2$ implies that every Aronszajn tree is special. We can modify his result in a straightforward manner by considering the following property. This looks very similar to the definition in [42].

**Definition 7.1.** *A partition $K_0 \cup K_1 = [\omega_1]^2$ has the weak rectangle refining property if there exists an absolute property $P$ with the parameter $K_0$ such that*

- *any uncountable subset of $\omega_1$ has an uncountable subset which satisfies the property $P$, and*
- *for any two uncountable subsets $I$ and $J$ of $\omega_1$, if $I \cup J$ satisfies the property $P$, then there are uncountable subsets $I'$ and $J'$ of $I$ and $J$ respectively such that for every $\alpha \in I'$ and $\beta \in J'$ with $\alpha < \beta$, $\{\alpha, \beta\} \in K_0$.*

Let $\mathcal{K}_2'(\text{wrec})$ be the $\mathcal{K}_2'$ for every partition on $[\omega_1]^2$ with the weak rectangle refining property. Since the rectangle refining property is stronger than the weak rectangle refining property, $\mathcal{K}_2'(\text{wrec})$ implies $\mathcal{K}_2'(\text{rec})$. In [33, Theorem 1], Todorčević actually proved that $\mathcal{K}_2'(\text{wrec})$ implies that every Aronszajn tree is special.

**Question 7.2.** 1. *Does $\mathcal{K}_2'(\text{rec})$ imply $\mathcal{K}_2'(\text{wrec})$? That is, is $\mathcal{K}_2'(\text{rec})$ equivalent to $\mathcal{K}_2(\text{wrec})$?*

2. *Does $\mathcal{K}_2'(\text{rec})$ imply that every Aronszajn tree is special?*

As seen through this paper, we see common results about the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$. The only difference we found in this paper is Theorem 6.9. We showed that if $\mathbb{Q}$ has the property $\mathsf{R}_{1,\aleph_1}$, then $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is $(T,S)$-preserving. But we don't know that whenever $\mathbb{Q}$ has the rectangle refining property, whether $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is $(T,S)$-preserving or not.

**Question 7.3.** *What is a difference between the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$? Is the $(T,S)$-preservation one of differences between the rectangle refining property and the property $\mathsf{R}_{1,\aleph_1}$?*

**Question 7.4.** *Is it consistent that $\neg\mathcal{C}(\text{arec})$ holds and there exists a non-special Aronszajn tree?*

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## References

[1] U. Abraham, M. Rubin and S. Shelah. *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types.* Ann. Pure Appl. Logic 29 (1985), no. 2, 123–206.

[2] U. Abraham and S. Todorčević. *Partition properties of $\omega_1$ compatible with CH.* Fund. Math. 152 (1997), 165–180.

[3] J. Bagaria. *Fragments of Martin's axiom and $\Delta^1_3$ sets of reals.* Ann. Pure Appl. Logic 69 (1994), no. 1, 1–25.

[4] B. Balcar and T. Jech. *Weak distributivity, a problem of von Neumann and the mystery of measurability.* Bull. Symbolic Logic 12 (2006), no. 2, 241–266.

[5] T. Bartoszyński. *Additivity of measure implies additivity of category.* Trans. Amer. Math. Soc. 281 (1984), no. 1, 209–213.

[6] T. Bartoszyński and H. Judah. *Set theory. On the structure of the real line.* A K Peters, Ltd., Wellesley, MA, 1995.

[7] J. Baumgartner, J. Malitz and W. Reinhardt. *Embedding trees in the rationals.* Proc. Nat. Acad. Sci. U.S.A. 67 (1970), 1748–1753.

[8] M. Bekkali. *Topics in set theory. Lebesgue measurability, large cardinals, forcing axioms, rho-functions. Notes on lectures by Stevo Todorčević.* Lecture Notes in Mathematics, 1476. Springer-Verlag, Berlin, 1991.

[9] H. Dales and H. Woodin. *An introduction to independence for analysts.* London Mathematical Society Lecture Note Series, 115.

[10] A. Dow. *More set-theory for topologists.* Topology Appl. 64 (1995), no. 3, 243–300.

[11] I. Farah, R. Ketchersid, P. Larson and M. Magidor. *Absoluteness for universally Baire sets and the uncountable II.* Computational Prospects of Infinity Part II, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, Vol. 15, 163–191.

[12] T. Jech. *Set theory. The third millennium edition, revised and expanded.* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[13] R. Jensen. *The fine structure of the constructible hierarchy. With a section by Jack Silver.* Ann. Math. Logic 4 (1972), 229–308.

[14] H. Judah and S. Shelah. *The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing).* J. Symbolic Logic 55 (1990), no. 3, 909–927.

[15] B. Knaster. *Sur une propriété caractéristique de l'ensemble des nombres réela.* Mat. Sb. 16 (1945), 281–288.

[16] K. Kunen. *$(\kappa, \lambda^*)$-gaps under MA.* handwritten note, 1976.

[17] K. Kunen. *Set theory. An introduction to independence proofs.* Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1980.

[18] K. Kunen and F. Tall. *Between Martin's axiom and Souslin's hypothesis.* Fund. Math. 102 (1979), no. 3, 173–181.

[19] P. Larson and S. Todorčević. *Chain conditions in maximal models.* Fund. Math. 168 (2001), no. 1, 77–104.

[20] P. Larson and S. Todorčević. *Katětov's problem.* Trans. Amer. Math. Soc. 354 (2002), no. 5, 1783–1791.

[21] D. Martin and R. Solovay. *Internal Cohen Extensions.* Ann. Math. Logic 2 (1970), 143–178.

[22] A. Miller. *Some properties of measure and category.* Trans. Amer. Math. Soc. 266 (1981), no. 1, 93–114.

[23] J. Moore. *Continuous colorings associated with certain characteristics of the continuum.* Discrete Math. 214 (2000), no. 1-3, 263–273.

[24] A. Rosłanowski and S. Shelah. *Norms on possibilities. I. Forcing with trees and creatures.* Mem. Amer. Math. Soc. 141 (1999), no. 671.

[25] M. Scheepers. *Gaps in $\omega^\omega$*. In *Set Theory of the Reals*. volume 6 of *Israel Mathematical Conference Proceedings*. 439–561, 1993.

[26] C. Schlindwein. *Consistency of Suslin's hypothesis, a nonspecial Aronszajn tree, and GCH.* J. Symbolic Logic 59 (1994), no. 1, 1–29.

[27] S Shelah. *Proper and improper forcing. Second edition.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.

[28] M. Suslin. *Problème 3.* Fund. Math. 1 (1920), 223.

[29] M. Talagrand. *Maharam's problem.* C. R. Math. Acad. Sci. Paris 342 (2006), no. 7, 501–503.

[30] S. Todorčević. *Remarks on chain conditions in products.* Compositio Math. 55 (1985), no. 3, 295–302.

[31] S. Todorčević. *Remarks on cellularity in products.* Compositio Math. 57 (1986), no. 3, 357–372.

[32] S. Todorčević. *Partition Problems in Topology.* volume 84 of *Contemporary mathematics.* American Mathematical Society, Providence, Rhode Island, 1989.

[33] S. Todorčević. *Remarks on Martin's axiom and the continuum hypothesis.* Canad. J. Math. 43 (1991), no. 4, 832–851.

[34] S. Todorčević and I. Farah. *Some applications of the method of forcing.* Yenisei Series in Pure and Applied Mathematics. Yenisei, Moscow; Lycée, Troitsk, 1995.

[35] S. Todorčević and B. Veličiković. *Martin's axiom and partitions.* Compositio Math. 63 (1987), no. 3, 391–408.

[36] J. Truss. *Sets having calibre $\aleph_1$.* Logic Colloquium 76 (Oxford, 1976), pp. 595–612. Studies in Logic and Found. Math., Vol. 87, North-Holland, Amsterdam, 1977.

[37] B. Veličković. *CCC posets of perfect trees.* Compositio Math. 79 (1991), no. 3, 279–294.

[38] B. Veličković. *The basis problem for CCC posets.* Set theory (Piscataway, NJ, 1999), 149–160, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 58, Amer. Math. Soc., Providence, RI, 2002.

[39] T. Yorioka. *Independent families of destructible gaps.* Tsukuba J. Math. 31 (2007), no. 1, 129–141.

[40] T. Yorioka. *Some weak fragments of Martin's axiom related to the rectangle refining property.* Arch. Math. Logic 47 (2008), no. 1, 79–90.

[41] T. Yorioka. *The inequality $\mathfrak{b} > \aleph_1$ can be considered as an analogue of Suslin's Hypothesis.* Axiomatic Set Theory and Set-theoretic Topology (Kyoto 2007), Sūrikaisekikenkyūsho Kōkyūroku No. 1595 (2008), 84–88.

[42] T. Yorioka. *Rudin's Dowker space in the extension with a Suslin tree.* Fund. Math. 201 (2008), no. 1, 53–89.

[43] J. Zapletal. *Keeping additivity of the null ideal small.* Proc. Amer. Math. Soc. 125 (1997), no. 8, 2443–2451.