Schmitt-Vogel type lemma for reductions

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Abstract. The lemma given by Schmitt and Vogel is an important tool in the study of the arithmetical rank of squarefree monomial ideals. In this paper, we give a Schmitt–Vogel type lemma for reductions as an analogous result.

Mathematics Subject Classification (2010). 13A30.

Keywords. analytic spread, arithmetical rank, reduction.

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring with non-zero identity. Let I be an ideal of R. Then the *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min\{r \ : \ \text{there exist } a_1, \ldots, a_r \in R \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\}.$$

If $\sqrt{(a_1, \ldots, a_r)} = \sqrt{I}$ holds, then we say that $a_1, \ldots, a_r$ generate I *up to radical.*

Assume that R is a polynomial ring over a field K and I is generated by squarefree monomials. Then we have the following inequalities:

$$\operatorname{height} I \le \operatorname{pd}_R R/I = \operatorname{cd}(I) \le \operatorname{ara} I \le \mu(I),$$

where $\operatorname{height} I$ (resp. $\operatorname{pd}_R R/I$, $\operatorname{cd}(I)$, $\mu(I)$) denotes the height of I (resp. the projective dimension of R/I over R, the cohomological dimension of I, the minimal number of generators of I); see e.g. [7]. Many researchers, e.g. Barile [1, 2, 3, 4, 5], Schmitt and Vogel [12] and the authors [7, 8] have proved that, in many cases, $\operatorname{ara} I = \operatorname{pd}_R R/I$ using the following lemma given by Schmitt and Vogel [12] or its generalizations.

Fact (Schmitt and Vogel [12, Lemma, p. 249]**).** *Let $\mathcal{P}$ be a finite subset of R, and let I be the ideal generated by $\mathcal{P}$. Let $r \ge 0$ be an integer. Assume that there exist subsets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ of $\mathcal{P}$ such that the following conditions are satisfied*:

(i) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r$.
(ii) $\sharp\mathcal{P}_0 = 1$.
(iii) *For each* ℓ $(0 < \ell \leq r)$ *and for every* $a, a'' \in \mathcal{P}_\ell$ *with* $a \neq a''$*, there exist an integer* ℓ' $(0 \leq \ell' < \ell)$*, and an element* $a' \in \mathcal{P}_{\ell'}$*, such that* $aa'' \in (a')$.

If we set

$$g_\ell = \sum_{a \in \mathcal{P}_\ell} a, \quad \ell = 0, 1, \ldots, r,$$

then $\sqrt{I} = \sqrt{(g_0, g_1, \ldots, g_r)}$.

An ideal $J \subset I$ is said to be a *reduction* of I if there exists some integer $s \geq 1$ such that $I^{s+1} = JI^s$ holds. When this is the case, $\sqrt{J} = \sqrt{I}$ holds. If J is minimal among all reductions of I with respect to inclusion, then it is said to be a *minimal reduction* of I. Let R be a polynomial ring over a field K and I a homogeneous ideal of R, or let R be a local ring with unique maximal ideal $\mathfrak{m}$ and $K = R/\mathfrak{m}$ and I an ideal of R. If K is infinite, then any ideal I has a minimal reduction J and the minimal number of generators of J is independent of the choice of J; see [10]. The number of generators of J is called the *analytic spread* of I (denoted by $\ell(I)$) and it gives an upper bound for $\operatorname{ara} I$. In the commutative ring theory, minimal reductions play an important role because they admit the same integral closure as the original ideal. Moreover, the analytic spread is equal to the Krull dimension of the fiber cone $F(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ of I in a local ring $(R, \mathfrak{m})$, and hence it is an important invariant.

The main purpose of this note is to give results on reductions that are analogous to the Schmitt-Vogel Lemma and some of its generalizations. Let us consider the following monomial ideal in a suitable polynomial ring R:

$$I = (x_{11}, \ldots, x_{1h_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qh_q}), \tag{1.1}$$

where the indeterminates are pairwise distinct. In order to give an upper bound for $\operatorname{cd}(I)$, Schenzel and Vogel [11] computed $\operatorname{depth} R/I^\ell$ for all $\ell \geq 1$, and proved that

$$\operatorname{cd}(I) \leq \ell(I) \leq \operatorname{depth} R - \inf_\ell \operatorname{depth} R/I^\ell = \sum_{i=1}^{q} h_i - q + 1 \ \big(= \operatorname{pd}_R R/I\big),$$

where the second inequality is known as Burch's inequality. On the other hand, Schmitt and Vogel [12] constructed $\operatorname{pd}_R R/I$ generators up to radical using their lemma. By using Theorem 2.1 instead of their lemma, we can provide a minimal reduction with $\operatorname{pd}_R R/I$ generators; see Proposition 2.3.

The proof of our main theorem is given in Section 3. It is based on refinements of the results presented by Barile in [1] and [3].

2. Schmitt–Vogel type lemma for reductions and its application

The following theorem is the main result in this paper, and is analogous to the Schmitt-Vogel Lemma. It is an immediate consequence of Theorem 3.1, which will be proved in Section 3.

Theorem 2.1 (Schmitt–Vogel type lemma for reductions). *Let $\mathcal{P}$ be a finite subset of R, and let I be the ideal generated by $\mathcal{P}$. Let $r \geq 0$ be an integer. Assume that there exist subsets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ of $\mathcal{P}$ such that the following conditions are satisfied:*

(SV1) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r$.
(SV2) $\sharp\mathcal{P}_0 = 1$.
(SV3) *For each ℓ $(0 < \ell \leq r)$ and for every $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$, there exist an integer ℓ' $(0 \leq \ell' < \ell)$, and elements $a' \in \mathcal{P}_{\ell'}$, $b \in I$ such that $aa'' = a'b$.*

If we set

$$g_\ell = \sum_{a \in \mathcal{P}_\ell} a, \quad \ell = 0, 1, \ldots, r,$$

then $J = (g_0, g_1, \ldots, g_r)$ is a reduction of I.

We now restrict our attention to the following case: R is a polynomial ring over a field K and I is a squarefree monomial ideal of R. In this case, as an application of the above theorem, we have the following result.

Corollary 2.2. *Let R be a polynomial ring and I a squarefree monomial ideal of R. Assume that there exist finite subsets $\mathcal{P}_0, \ldots, \mathcal{P}_r$ of I satisfying the assumptions in Theorem 2.1 for $r = \operatorname{pd}_R R/I - 1$. Then $(g_0, g_1, \ldots, g_r)$ is a minimal reduction of I, and $\ell(I) = \operatorname{ara} I = \operatorname{pd}_R R/I = r + 1$.*

Proof. Since I is a squarefree monomial ideal, we have

$$r + 1 = \operatorname{pd}_R R/I = \operatorname{cd}(I) \leq \operatorname{ara} I \leq \ell(I),$$

where the second equality follows from [9]. On the other hand, Theorem 2.1 implies $\ell(I) \leq r + 1$. This proves the claim. $\square$

We can apply our results to Alexander dual of complete intersection monomial ideals; see below.

Proposition 2.3 (Alexander dual of complete intersection monomial ideals). *Let $I \subseteq R$ be a squarefree monomial ideal of the following shape:*

$$(x_{11}, \ldots, x_{1h_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qh_q}), \tag{2.1}$$

where $R = K[x_{11}, \ldots, x_{1h_1}, \ldots, x_{q1}, \ldots, x_{qh_q}]$ is a polynomial ring over a field K. Note that I can be regarded as the Alexander dual of the complete intersection monomial ideal $(x_{11} \cdots x_{1h_1}, \ldots, x_{q1} \cdots x_{qh_q})$ if $h_1, \ldots, h_q \geq 2$.

Set $r = h_1 + \cdots + h_q - q$ and

$$g_\ell = \sum_{\ell_1 + \cdots + \ell_q = \ell} x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q}, \quad \ell = 0, 1, \ldots, r.$$

Then $(g_0, g_1, \dots, g_r)$ is a minimal reduction of I. In particular,

$$\ell(I) = \operatorname{ara} I = \operatorname{pd}_R R/I = \sum_{i=1}^{q} h_i - q + 1.$$

Proof. It is known that

$$r + 1 = \operatorname{pd}_R R/I = \operatorname{ara} I \le \ell(I);$$

see e.g. [12, Theorem] or [7, Section 5].

For each $\ell = 0, 1, \dots, r$, we set

$$\mathcal{P}_\ell = \left\{ x_{1\ell_1} \cdots x_{q\ell_q} \;:\; 1 \le \ell_j \le h_j,\ \ell_1 + \cdots + \ell_q = \ell + q \right\}.$$

Then I is generated by all monomials in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_r$, and $\mathcal{P}_0$ consists of only one element, namely $x_{11} \cdots x_{q1}$. Thus it suffices to show that condition (SV3) of Theorem 2.1 is fulfilled. Let a, $a'' \in \mathcal{P}_\ell$, say

$$a = x_{1i_1} x_{2i_2} \cdots x_{qi_q}, \qquad a'' = x_{1j_1} x_{2j_2} \cdots x_{qj_q},$$

where $i_1 + \cdots + i_q = j_1 + \cdots + j_q = \ell + q$. As $a \ne a''$, there exists an integer k $(1 \le k \le q)$ such that $i_k > j_k$. We may assume without loss of generality, that $k = 1$. If we set

$$a' = a \cdot \frac{x_{1j_1}}{x_{1i_1}} = x_{1j_1} x_{2i_2} \cdots x_{qi_q}, \qquad b = a'' \cdot \frac{x_{1i_1}}{x_{1j_1}} = x_{1i_1} x_{2j_2} \cdots x_{qj_q} \in I,$$

we then have $aa'' = a'b$ and $a' \in \mathcal{P}_{\ell'}$, where

$$\ell' = j_1 + i_2 + \cdots + i_q - q < i_1 + i_2 + \cdots + i_q - q = \ell.$$

Hence we can apply Corollary 2.2. $\square$

Remark 2.4. Let $I \subseteq R$ be the ideal that appears in Proposition 2.3. Then Schmitt and Vogel [12] proved that $\operatorname{ara} I = \operatorname{pd}_R R/I$ by showing that $g_0, g_1, \dots, g_r$ generate I up to radical. Thus the above proposition gives an improvement of their result.

We can generalize Proposition 2.3 as follows.

Proposition 2.5. *For each $i = 1, 2, \dots, s$, let I_i be a squarefree monomial ideal of the shape* (2.1)*:*

$$I_i = (x_{11}^{(i)}, \dots, x_{1h_1^{(i)}}^{(i)}) \cap \cdots \cap (x_{q^{(i)}1}^{(i)}, \dots, x_{q^{(i)}h_{q^{(i)}}^{(i)}}^{(i)}).$$

Let $G(I_i)$ be the minimal set of monomial generators of I_i. Suppose that there are no variables which appear both in $G(I_i)$ and in $G(I_j)$ for each i, j with $i \ne j$. For all i, define $g_\ell^{(i)}$ as in Proposition 2.3*. Then*

$$(g_\ell^{(i)} \;:\; i = 1, \dots, s, \ell = 0, 1, \dots, h_1^{(i)} + \cdots + h_{q^{(i)}}^{(i)} - q^{(i)})$$

is a minimal reduction of $I_1 + \cdots + I_s$. In particular, $\ell(I_1 + \cdots + I_s) = \ell(I_1) + \cdots + \ell(I_s)$.

In order to prove Proposition 2.5, it is enough to show the following lemma.

Lemma 2.6. *Let R, S be polynomial rings over a field K with no common variables, and put $T = R \otimes_K S$. Let $I \subseteq R$ (resp. $J \subseteq S$) be a squarefree monomial ideal. Then:*

(1) $\mathrm{pd}_T\, T/(IT + JT) = \mathrm{pd}_R\, R/I + \mathrm{pd}_S\, S/J$.
(2) *Assume that $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r \subseteq R$ (resp. $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_s \subseteq S$) satisfy $(SV1)$, $(SV2)$ and $(SV3)$ in Theorem 2.1. Then $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r, \mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_s$ also satisfy the same conditions as finite subsets of T.*

Proof. (1) Let $F_\bullet$ (resp. $G_\bullet$) be a minimal free resolution of R/I over R (resp. S/J over S). Then $F_\bullet \otimes_K G_\bullet$ is a minimal free resolution of $T/(IT + JT)$. Thus we have $\mathrm{pd}_T\, T/(IT + JT) = \mathrm{pd}_R\, R/I + \mathrm{pd}_S\, S/J$.

(2) It is clear by definition. □

Remark 2.7. Under the same notation as in Lemma 2.6, it is easy to see that $\mathrm{ara}(IT + JT) \le \mathrm{ara}\, I + \mathrm{ara}\, J$ holds. If both $\mathrm{ara}\, I = \mathrm{pd}_R\, R/I$ and $\mathrm{ara}\, J = \mathrm{pd}_S\, S/J$ hold, then equality holds. But we do *not* know whether it is always true. Moreover, it seems that a similar result holds for analytic spreads, but we do *not* have any proof in general.

3. Proof of the theorem

In this section, we prove Theorem 2.1, which is the analogue of the Schmitt-Vogel Lemma in the framework of reductions. Some generalizations of this lemma have been given by Barile [1, 3, 5]. Theorems 3.1 and 3.3 are analogous results for reductions.

The following theorem gives an analogous result for Barile [3, Lemma 2.1], which is a generalization of Theorem 2.1.

Theorem 3.1. *Let $\mathcal{P} \subset R$ be a finite subset, and put $I = (\mathcal{P})$. Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be subsets of $\mathcal{P}$. Assume that the following conditions:*

(B1) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r$.
(B2) $\sharp \mathcal{P}_0 = 1$.
(B3) *For each ℓ $(0 < \ell \le r)$ and for every $a, a'' \in \mathcal{P}_\ell$ with $a \ne a''$, there exists an integer $m \ge 1$ such that $(aa'')^m \in (\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{\ell-1}) I^{2m-1}$.*

Set

$$g_\ell = \sum_{a \in \mathcal{P}_\ell} a, \quad \ell = 0, 1, \ldots, r.$$

Then $J = (g_0, g_1, \ldots, g_r)$ is a reduction of I.

Remark 3.2. The difference between Theorem 3.1 and the original result of Barile [3] lies in condition (B3). The condition of the original result corresponding to (B3) is

(B3)' For each ℓ $(0 < \ell \le r)$ and for every $a, a'' \in \mathcal{P}_\ell$ with $a \ne a''$, there exists an integer $m \ge 1$ such that $(aa'')^m \in (\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{\ell-1})$.

Proof of Theorem 3.1. Since $J \subseteq I$, it suffices to show that $I^{s+1} \subset JI^s$ for some s. To this end, we set $\sharp\mathcal{P}_\ell = c_\ell$ and $I_\ell = (\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_\ell)$ for each $\ell = 0, 1, \ldots, r$. Moreover, for each ℓ, we choose an integer $m_\ell \geq 1$ such that

$$(aa'')^{m_\ell} \in I_{\ell-1} I^{2m_\ell - 1} \tag{3.1}$$

for all $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$. Set $n_0 = 1$ and $n_j = c_1 \cdots c_j m_1 \cdots m_j$ for each $j = 1, \ldots, r$. Then it is enough to prove that

$$I_j^{n_j} \subset I_{j-1}^{n_{j-1}} I^{n_j - n_{j-1}} + JI^{n_j - 1} \tag{3.2}$$

for each $j = 0, 1, \ldots, r$. Indeed, $I_0 = (\mathcal{P}_0) = (g_0) \subset J$. Then (3.2) implies that

$$\begin{aligned} I^{n_r} = I_r^{n_r} &\subset I_{r-1}^{n_{r-1}} I^{n_r - n_{r-1}} + JI^{n_r - 1} \\ &\subset (I_{r-2}^{n_{r-2}} I^{n_{r-1} - n_{r-2}} + JI^{n_{r-1} - 1}) I^{n_r - n_{r-1}} + JI^{n_r - 1} \\ &= I_{r-2}^{n_{r-2}} I^{n_r - n_{r-2}} + JI^{n_r - 1} \\ &\subset \cdots \subset I_0^{n_0} I^{n_r - n_0} + JI^{n_r - 1} = JI^{n_r - 1}. \end{aligned}$$

Now suppose $j = \ell \geq 1$ and assume that (3.2) holds for every $j \leq \ell - 1$. To prove (3.2) for $j = \ell$, it is enough to show that for arbitrary n_ℓ elements (it is allowed to take the same element more than once) in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_\ell$, the product of all elements is contained in the right hand side of (3.2). We divide these elements into $n_{\ell-1}$ sequences of $c_\ell m_\ell$ elements, and show that the product of the elements in each sequence is in $I_{\ell-1} I^{c_\ell m_\ell - 1} + JI^{c_\ell m_\ell - 1}$.

In what follows, we consider one of such sequences. If this sequence contains an element of $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{\ell-1}$, then it is clear that the product is in $I_{\ell-1} I^{c_\ell m_\ell - 1}$. Therefore, we may assume that all elements in the sequence are in $\mathcal{P}_\ell$. If we can find a pair (a, a'') with $a \neq a''$ which appears at least m_ℓ times in this sequence, then from (3.1) we deduce that the product of all elements in the sequence belongs to $I_{\ell-1} I^{c_\ell m_\ell - 1}$. Otherwise, we pick an element a_1 in $\mathcal{P}_\ell$ such that the number of times (say, d) it appears in the sequence is maximal. Note that $d > m_\ell$. Let $\mathcal{P}_\ell = \{a_1, a_2, \ldots, a_{c_\ell}\}$. Then the product of all elements in the sequence is

$$\begin{aligned} a_1^d a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} &= a_1^{m_\ell} a_1^{d - m_\ell} a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} \\ &= a_1^{m_\ell} \left(g_\ell - \sum_{i=2}^{c_\ell} a_i \right)^{d - m_\ell} a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} \\ &= g_\ell \cdot (\text{an element of } I^{c_\ell m_\ell - 1}) + a_1^{m_\ell} \left(- \sum_{i=2}^{c_\ell} a_i \right)^{d - m_\ell} a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} \\ &= g_\ell \cdot (\text{an element of } I^{c_\ell m_\ell - 1}) \pm \sum a_1^{m_\ell} a_2^{k'_2} \cdots a_{c_\ell}^{k'_{c_\ell}}, \end{aligned}$$

where $k_2 + \ldots + k_{c_\ell} = c_\ell m_\ell - d$ and $k'_2 + \ldots + k'_{c_\ell} = (c_\ell - 1) m_\ell$. Then there exists an integer j with $2 \leq j \leq c_\ell$ such that $k'_j \geq m_\ell$. Now consider the last term. Since the pair (a_1, a_j) appears at least m_ℓ times in the sequence, we deduce that the product of all elements in the sequence belongs to $I_{\ell-1} I^{c_\ell m_\ell - 1}$ by assumption. Hence the right-hand side is contained in $JI^{c_\ell m_\ell - 1} + I_{\ell-1}I^{c_\ell m_\ell - 1}$. Hence we have finished the proof. □

Proof of Theorem 2.1. Assume that I satisfies (SV1),(SV2), and (SV3). Then it also satisfies (B1), (B2) and (B3). Hence the assertion immediately follows from Theorem 3.1. □

In the proof of the following two examples, we need Theorem 3.1 instead of Theorem 2.1.

Example. Let K be a field, and let $m \geq 2$ be an integer. Consider the hypersurface $R = K[[x, y, z]]/(x^m y^m - z^{2m})$. Let $I = (x, y, z)R$ and set

$$\mathcal{P}_0 = \{z\}, \quad \mathcal{P}_1 = \{x, y\}.$$

Then since $(xy)^m = z \cdot z^{2m-1} \in (\mathcal{P}_0)I^{2m-1}$, by virtue of Theorem 3.1, we can conclude that $(x + y, z)$ is a (minimal) reduction of I. But we cannot apply Theorem 2.1 because $xy \notin (z)$.

Example. Let $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$ be a polynomial ring over a field K. Consider the ideal

$$I = (x_1x_2{+}x_1x_3,\ x_1x_4,\ x_1x_5,\ x_1x_6, x_2x_5,\ x_2x_6,\ x_3x_4,\ x_3x_6,\ x_4x_5,\ x_4x_6,\ x_5x_6),$$

and set

$$\begin{array}{lcllcl} \mathcal{P}_0 & = & \{x_1x_6\}, & \mathcal{P}_1 & = & \{x_1x_5,\ x_2x_6\}, \\ \mathcal{P}_2 & = & \{x_1x_4,\ x_3x_6\}, & \mathcal{P}_3 & = & \{x_2x_5,\ x_4x_6\}, \\ \mathcal{P}_4 & = & \{x_3x_4,\ x_5x_6\}, & \mathcal{P}_5 & = & \{x_1x_2 + x_1x_3,\ x_4x_5\}. \end{array}$$

Then, by Theorem 3.1, we can conclude that

$$J = (x_1x_6,\ x_1x_5{+}x_2x_6,\ x_1x_4{+}x_3x_6,\ x_2x_5{+}x_4x_6,\ x_3x_4{+}x_5x_6,\ x_1x_2{+}x_1x_3{+}x_4x_5)$$

is a (minimal) reduction of I because the product of any two elements in $\mathcal{P}_\ell$ belongs to $(\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{\ell-1})I$ for every $\ell = 1, 2, \ldots, 5$. But we cannot apply Theorem 2.1 because the product of $(x_1x_2 + x_1x_3) \in \mathcal{P}_5$ and $x_4x_5 \in \mathcal{P}_5$ is not contained in the ideal (a') for any element $a' \in \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_4$.

Next, we refine the result by Barile [1, Proposition 1.1].

Theorem 3.3. *Assume that R is a local ring. Let $\mathcal{P} \subset R$ be a finite subset, and let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be subsets of $\mathcal{P}$. We set $\sharp\mathcal{P}_\ell = c_\ell$ for all ℓ and $I = (\mathcal{P})$. Assume that the following conditions are satisfied*:

(Ba1) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r$.

(Ba2) $\sharp\mathcal{P}_0 = 1$.

(Ba3) *For each ℓ $(0 < \ell \leq r)$ with $c_\ell \geq 2$, there exists an integer n_ℓ with $2 \leq n_\ell \leq c_\ell$ such that for arbitrary n_ℓ distinct elements $p_1, p_2, \ldots, p_{n_\ell} \in \mathcal{P}_\ell$, there exist an integer ℓ' with $0 \leq \ell' < \ell$, elements $p' \in \mathcal{P}_{\ell'}$ and $b \in I^{n_\ell - 1}$ such that $p_1p_2 \cdots p_{n_\ell} = p'b$.*

For $0 \le \ell \le r$ with $c_\ell = 1$, we set $n_\ell = 2$. For each $\ell = 0, 1, \ldots, r$, let $A^{(\ell)} = (a_{ij}^{(\ell)})$ be an $(n_\ell - 1) \times c_\ell$ matrix with $a_{ij}^{(\ell)} \in R$. Assume that all maximal minors of $A^{(\ell)}$ are units in R. Set

$$\mathcal{P}_\ell = \{p_1^{(\ell)}, p_2^{(\ell)}, \ldots, p_{c_\ell}^{(\ell)}\}, \quad 0 \le \ell \le r,$$

$$g_i^{(\ell)} = \sum_{j=1}^{c_\ell} a_{ij}^{(\ell)} p_j^{(\ell)}, \quad 1 \le i \le n_\ell - 1, \quad 0 \le \ell \le r,$$

$$J = (g_i^{(\ell)} \,:\, 0 \le \ell \le r,\ 1 \le i \le n_\ell - 1).$$

Then J is a reduction of I.

Remark 3.4. The difference between Theorem 3.3 and the original result of Barile [1] lies in condition (Ba3). The condition of the original result corresponding to (Ba3) is

(Ba3)' For each ℓ $(0 < \ell \le r)$ with $c_\ell \ge 2$, there exists some integer n_ℓ, $2 \le n_\ell \le c_\ell$ such that for arbitrary n_ℓ distinct elements $p_1, p_2, \ldots, p_{n_\ell} \in \mathcal{P}_\ell$, there exist ℓ' with $0 \le \ell' < \ell$ and $p' \in \mathcal{P}_{\ell'}$, such that $p_1 p_2 \cdots p_{n_\ell} \in (p')$.

Proof of Theorem 3.3. It is enough to show that $I^{s+1} \subset JI^s$ for some $s \ge 0$.

For each $\ell = 0, 1, \ldots, r$, we set $I_\ell = (\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_\ell)$. Then it is enough to prove that

$$I_j^{n_0 n_1 \cdots n_j} \subset I_{j-1}^{n_0 n_1 \cdots n_{j-1}} I^{(n_0 n_1 \cdots n_{j-1})(n_j - 1)} + JI^{n_0 n_1 \cdots n_j - 1} \tag{3.3}$$

for each $j = 0, 1, \ldots, r$.

The case of $j = 0$ is clear because $p_0 = g_0 \in J$ by assumption (Ba2).

Now suppose $j = \ell \ge 1$ and assume that (3.3) holds for every $j \le \ell - 1$. In order to prove (3.3) for $j = \ell$, it is enough to show that for arbitrary $n_0 n_1 \cdots n_\ell$ elements (it is allowed to take the same element more than once) in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_\ell$, the product of these elements is contained in the right hand side of (3.3). We divide these elements into $n_0 n_1 \cdots n_{\ell-1}$ sequences of n_ℓ elements, and show that the product of all elements in each sequence is contained in $I_{\ell-1} I^{n_\ell - 1} + JI^{n_\ell - 1}$.

Fix one of these sequences. If this sequence contains an element of $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{\ell-1}$ in the sequence, then it is clear that the product is contained in $I_{\ell-1} I^{n_\ell - 1}$. Therefore, we may assume that all elements in the sequence belong to $\mathcal{P}_\ell$.

In the following, we omit the symbol ℓ for simplicity. Consider the product

$$\mu = p_1^{k_1} p_2^{k_2} \cdots p_c^{k_c}, \quad k_1 + k_2 + \cdots + k_c = n, \quad k_i \ge 0$$

and set

$$t := t(\mu) := \sharp\{i \,:\, k_i = 1\}.$$

We prove that $\mu \in I_{\ell-1} I^{n-1} + JI^{n-1}$ by descending induction on t $(0 \le t \le n)$.

If $t = n$, then μ is a product of n distinct elements in $\mathcal{P}_\ell$. It follows that $\mu \in I_{\ell-1} I^{n-1}$ by assumption (Ba3).

Now we consider the case where $0 \le t \le n-1$. Then we can assume without loss of generality that $k_1 = k_2 = \cdots = k_t = 1$ and $k_{t+1} \ge 2$. Notice that $t \le n-2$. Let A' be the $(n-1)\times(n-1)$ submatrix of A consisting of the first $n-1$ columns of A. By assumption, A' is invertible. Since R is local, it is possible to transform, by elementary row operations, the matrix A into the matrix $B = (b_{ij})$ having the same size as A with $b_{ij} = \delta_{ij}$ for $1 \le i \le n-1$, $1 \le j \le n-1$. Then we put

$$g'_{t+1} = p_{t+1} + \sum_{j=t+2}^{c} b_{t+1,j}p_j \in J.$$

Since $k_{t+1} \ge 2$, we have

$$\begin{aligned}
\mu &= p_1p_2\cdots p_tp_{t+1}\left(g'_{t+1} - \sum_{j=t+2}^{c} b_{t+1,j}p_j\right)^{k_{t+1}-1} p_{t+2}^{k_{t+2}}\cdots p_c^{k_c}\\
&= g'_{t+1}(\text{an element of } I^{n-1})\\
&+ p_1p_2\cdots p_tp_{t+1}\left(-\sum_{j=t+2}^{c} b_{t+1,j}p_j\right)^{k_{t+1}-1} p_{t+2}^{k_{t+2}}\cdots p_c^{k_c}\\
&= (\text{an element of } JI^{n-1})\\
&+ \sum(\text{an element of } R)\cdot p_1p_2\cdots p_tp_{t+1}\, p_{t+2}^{k'_{t+2}}\cdots p_c^{k'_c},
\end{aligned}$$

where

$$t+1+k'_{t+2}+\cdots+k'_c = t+k_{t+1}+k_{t+2}+\cdots+k_c = n.$$

Then the induction hypothesis implies that the second term in the last equation is contained in $I_{\ell-1}I^{n-1} + JI^{n-1}$. This completes the proof. $\square$

In the next example, the analytic spread of I is known, but we can provide a concrete minimal reduction using Theorem 3.3.

Example. Let $r \ge 2$ be an integer. Set $I = (x_1x_2, x_2x_3, \ldots, x_{2r-1}x_{2r}, x_{2r}x_1)$, the edge ideal of the $2r$-cycle ($r \ge 2$). Put

$$\begin{aligned}
\mathcal{P}_\ell &= \{x_{2\ell+1}x_{2\ell+2}\}, \quad \ell = 0,1,\ldots,r-1,\\
\mathcal{P}_r &= \{x_2x_3, x_4x_5, \ldots, x_{2r-2}x_{2r-1}, x_{2r}x_1\}.
\end{aligned}$$

Then the assumptions of Theorem 3.3 are satisfied with $n_\ell = 2$ for $\ell = 0,1,\ldots,r-1$ and $n_r = r$. Moreover, since all maximal minors of the matrix

$$A^{(r)} = \begin{pmatrix} 1 & & & & 1\\ & 1 & & & 1\\ & & \ddots & & \vdots\\ & & & 1 & 1 \end{pmatrix}$$

are units in R, from Theorem 3.3, we obtain that

$$x_1x_2,\ x_3x_4, \ldots, x_{2r-1}x_{2r},\ x_2x_3+x_{2r}x_1,\ x_4x_5+x_{2r}x_1,\ \ldots, x_{2r-2}x_{2r-1}+x_{2r}x_1$$

is a reduction of I.

On the other hand, we have $\ell(I) = 2r-1$ due to Vasconcelos [14, Section 1.3.3, p.50] because any $2r$-cycle is a bipartite graph. In particular, the above reduction is a minimal reduction of I.

In the following example, we cannot apply the above theorem, but we can find a minimal reduction by a similar argument as in the proof.

Example. Let $R = K[x_1, x_2, x_3, x_4, x_5]$ be a polynomial ring over an infinite field K, and let $a, b, c, d \in K \setminus \{0\}$ be pairwise distinct elements. Let I be the edge ideal of the complete graph K_5, that is, I is the ideal generated by the following squarefree monomials of degree 2:

$$x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_2x_3,\ x_2x_4,\ x_2x_5,\ x_3x_4,\ x_3x_5,\ x_4x_5.$$

Set

$$\begin{array}{rclrcl} \mathcal{P}_0 &=& \{x_1x_2\}, & \mathcal{P}_1 &=& \{x_2x_3,\ x_4x_5\}, \\ \mathcal{P}_2 &=& \{x_3x_4,\ x_1x_5\}, & \mathcal{P}_3 &=& \{x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5\}, \end{array}$$

and $I_\ell = (\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_\ell)$ for each $\ell = 0, 1, 2$. If we put

$$\begin{array}{rcl} g_0 &=& x_1x_2, \\ g_1 &=& x_2x_3 + x_4x_5, \\ g_2 &=& x_3x_4 + x_1x_5, \\ g_3 &=& x_1x_3 + ax_1x_4 + bx_2x_4 + cx_2x_5 + dx_3x_5, \\ g_4 &=& x_1x_3 + a^2x_1x_4 + b^2x_2x_4 + c^2x_2x_5 + d^2x_3x_5, \end{array}$$

then $J = (g_0, g_1, g_2, g_3, g_4)$ is a (minimal) reduction of I. Indeed, we note that $I_2^3 \subseteq (g_0, g_1, g_2)I^2$. Moreover, one can easily see that $\mathcal{P}_3$ satisfies the condition (Ba3) as $n_3 = 3$. Then, by a similar argument as in the proof of Theorem 3.3, we obtain that $I^3 = I_3^3 \subset I_2I^2 + JI^2$. Therefore

$$I^9 = I_2^3 I^6 + JI^8 \subset (g_0, g_1, g_2)I^2 \cdot I^6 + JI^8 = JI^8.$$

Ackowledgment. The first-named author was supported by JST, CREST. The second-named author was supported by JSPS 20540047. The third-named author was supported by JSPS 19340005.

The authors thank the referee for reading our manuscript carefully.

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