Elementary Submodel Arguments in Balogh's Dowker Spaces

ELEMENTARY SUBMODEL ARGUMENTS IN BALOGH'S DOWKER SPACES

HARUTO OHTA AND TERUYUKI YORIOKA

Abstract. We prove a combinatorial lemma which enables us to prove that Balogh's natural Dowker space is not countably paracompact without using elementary submodels.

1. Introduction

A space is called a *Dowker space* if it is normal but not countably paracompact. Z. Balogh [1, 2, 3, 4, 5] constructed a variety of Dowker spaces of size $2^{\aleph_0}$ in ZFC. They have similar constructions except for the first one in [1], and, as he stated in [4], it is relatively easy to read the construction (from which the normality directly follows). The hard part is to show that the space is not countably paracompact. He carries out it by some diagonal arguments and a reflection lemma through countable elementary submodels. The purpose of this note is to consider how we can prove it without going through elementary submodels. In particular, we prove Lemma 1.1 below, which enables us to prove that a Dowker space described in [5] is not countably paracompact without using elementary submodels. The space in [5] is the most basic and plain (i.e., without any additional properties) one among Balogh's Dowker spaces and is entitled a *natural Dowker space*. We believe that mild modifications of Lemma 1.1 are available for other Dowker spaces in [2, 3, 4].

Lemma 1.1. *Let κ be an infinite cardinal with $\mathrm{cf}(\kappa) > \omega$, $\{c(x) : x \in \kappa \times \omega\}$ a set of finite functions into $2 = \{0, 1\}$, and H a countable set. Then, there exist $\delta \in \kappa$ and a function φ such that*

(1) $\mathrm{dom}(\varphi)$ *is a countable subset of* $\delta \times \omega$,
(2) *for each* $x \in \mathrm{dom}(\varphi)$, $\varphi(x) \subseteq \mathrm{dom}(c(x))$, *and*

2010 *Mathematics Subject Classification.* 54D15, 54D20, 03E05.
Key words and phrases. Dowker space, Δ-system, elementary submodel.
The first author was partially supported by KAKENHI 2011 (No. 22540123).
The second author was partially supported by KAKENHI 2011 (No. 22540124).

(3) *for each* $\beta \in \kappa \setminus \delta$ *and each* $n \in \omega$*, if we put* $y = \langle \beta, n\rangle$*, then there exists* $x = \langle \alpha, n\rangle \in \mathrm{dom}(\varphi)$ *such that* $\mathrm{dom}(c(x)) \cap \mathrm{dom}(c(y)) = \varphi(x)$ *and* $c(x)|\varphi(x) = c(y)|\varphi(x)$*.*

Moreover, if we put $M = (\bigcup \mathrm{ran}(\varphi)) \cup H$*, then*

(4) *for each* $x \in \mathrm{dom}(\varphi)$*,* $\mathrm{dom}(c(x)) \cap M = \varphi(x)$*, and*
(5) $\{\mathrm{dom}(c(x)) \setminus M : x \in \mathrm{dom}(\varphi)\}$ *is pairwise disjoint.*

Lemma 1 will be proved in Section 2. In Section 3, we sketch the definition of a natural Dowker space in [5] and, in Section 4, we apply Lemma 1.1 to prove that the space is not countably paracompact.

As usual, a cardinal is identified with the initial ordinal, and an ordinal is the set of smaller ordinals. For a set A and a cardinal κ, let $[A]^\kappa = \{B : B \subseteq A, |B| = \kappa\}$ and $[A]^{<\kappa} = \{B : B \subseteq A, |B| < \kappa\}$. For a function f, $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ denote the domain and the range of f, respectively, and $f|A$ stands for the restriction of f to a set A. By a *finite function* we mean a function defined on a finite set. Other terms and notation will be used as in [6].

2. Proof of Lemma 1.1

Recall that a family $\{a_\lambda : \lambda \in \Lambda\}$ of sets is a Δ*-system* if there is a fixed set r, called the *root* of the Δ-system, such that $a_\lambda \cap a_\mu = r$ whenever $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$. It is known ([6, Ch. 2, Theorem 1.5]) that if each a_λ is finite and Λ is uncountable, then there is an uncountable $\Lambda' \subseteq \Lambda$ such that $\{a_\lambda : \lambda \in \Lambda'\}$ forms a Δ-system.

Proof of Lemma 1.1. For a while, we fix $n \in \omega$. By induction on $\nu \in \omega_1 + 1$, we define subsets D^n_ν, E^n_ν of κ, a finite set R^n_ν and a function $c^n_\nu : R^n_\nu \to 2$ as follows. If the set $F^n_\nu = \kappa \setminus \bigcup_{\mu<\nu} E^n_\mu$ is uncountable, then there exist an uncountable $D^n_\nu \subseteq F^n_\nu$, a finite set R^n_ν and a function $c^n_\nu : R^n_\nu \to 2$ such that $\{\mathrm{dom}(c(\alpha, n)) : \alpha \in D^n_\nu\}$ forms a Δ-system with root R^n_ν,

$$(\forall \alpha \in D^n_\nu)(c(\alpha, n)|R^n_\nu = c^n_\nu), \tag{2.1}$$

and for any set R, if $|R| < |R^n_\nu|$, then there is no uncountable $D \subseteq F^n_\nu$ such that $\{\mathrm{dom}(c(\alpha, n)) : \alpha \in D\}$ forms a Δ-system with root R. Then, we define

$$E^n_\nu = \{\alpha \in F^n_\nu : \mathrm{dom}(c(\alpha, n)) \supseteq R^n_\nu \text{ and } c(\alpha, n)|R^n_\nu = c^n_\nu\}. \tag{2.2}$$

We note that $D^n_\nu \subseteq E^n_\nu \subseteq F^n_\nu$ and

$$(\forall \mu < \nu)(\langle R^n_\nu, c^n_\nu\rangle \neq \langle R^n_\mu, c^n_\mu\rangle \text{ and } |R^n_\mu| \leq |R^n_\nu|). \tag{2.3}$$

Because, if for some $\mu < \nu$, either $\langle R^n_\nu, c^n_\nu\rangle = \langle R^n_\mu, c^n_\mu\rangle$ or $|R^n_\mu| > |R^n_\nu|$, then the former equality implies $D^n_\nu \subseteq E^n_\mu$, which contradicts the fact that $D^n_\nu \cap E^n_\mu = \emptyset$, and the latter inequality contradicts the minimality of $|R^n_\mu|$ since $D^n_\nu \subseteq F^n_\mu$. On the other hand, if either F^n_ν is countable or $\nu = \omega_1$, then we stop the induction and define $\xi(n) = \nu$.

Now, we show that $\xi(n) < \omega_1$. Suppose on the contrary that $\xi(n) = \omega_1$. Then by induction on $\nu \in \omega_1$, we can choose $\alpha_\nu \in D^n_\nu$ such that

$$\{\mathrm{dom}(c(\alpha_\nu, n)) \setminus R^n_\nu : \nu \in \omega_1\} \text{ is pairwise disjoint.} \tag{2.4}$$

This can be done because for each $\nu \in \omega_1$, $\{\mathrm{dom}(c(\alpha, n)) \setminus R^n_\nu : \alpha \in D^n_\nu\}$ is uncountable and consists of pairwise disjoint finite sets. Note that the sequence $\langle |R^n_\nu| : \nu \in \omega_1\rangle$ is constant on $\omega_1 \setminus \mu$ for some $\mu < \omega_1$. Hence, there is an uncountable set $\Gamma \subseteq \omega_1$ and a finite set R such that $\{\mathrm{dom}(c(\alpha_\nu, n)) : \nu \in \Gamma\}$ forms a Δ-system with root R and

$$(\forall \mu, \nu \in \Gamma)(c(\alpha_\mu, n)|R = c(\alpha_\nu, n)|R \text{ and } |R^n_\mu| = |R^n_\nu|). \tag{2.5}$$

Fix $\lambda = \min \Gamma$. Then, by (2.4), we can find $\mu_0, \mu_1 \in \Gamma \setminus \{\lambda\}$ such that $\mu_0 \neq \mu_1$ and for each $i < 2$,

$$\mathrm{dom}(c(\alpha_\lambda, n)) \cap (\mathrm{dom}(c(\alpha_{\mu_i}, n)) \setminus R^n_{\mu_i}) = \emptyset,$$

which implies that $R \subseteq R^n_{\mu_i}$, since $R = \mathrm{dom}(c(\alpha_\lambda, n)) \cap \mathrm{dom}(c(\alpha_{\mu_i}, n))$. By the second equality of (2.5) and the minimality of $|R^n_\lambda|$, $|R^n_{\mu_i}| = |R^n_\lambda| \leq |R|$ for each $i < 2$. Hence, $R^n_{\mu_0} = R = R^n_{\mu_1}$. By the first equality of (2.5),

$$\begin{aligned} c^n_{\mu_0} = c(\alpha_{\mu_0}, n)|R^n_{\mu_0} &= c(\alpha_{\mu_0}, n)|R \\ &= c(\alpha_{\mu_1}, n)|R = c(\alpha_{\mu_1}, n)|R^n_{\mu_1} = c^n_{\mu_1}. \end{aligned}$$

Hence, $\langle R^n_{\mu_0}, c^n_{\mu_0}\rangle = \langle R^n_{\mu_1}, c^n_{\mu_1}\rangle$, which contradicts (2.3).

Put $M = \bigcup\{R^n_\nu : \nu \in \xi(n), n \in \omega\} \cup H$. Since M is countable, by induction, we can take $\alpha^n_{\nu,k} \in D^n_\nu$ for each $n \in \omega$, $\nu \in \xi(n)$ and $k \in \omega$ such that $\alpha^n_{\nu,k} \neq \alpha^{n'}_{\nu',k'}$ whenever $\langle n, \nu, k\rangle \neq \langle n', \nu', k'\rangle$,

$$\{\mathrm{dom}(c(\alpha^n_{\nu,k}, n)) \setminus R^n_\nu : n \in \omega, \nu \in \xi(n), k \in \omega\} \text{ is pairwise disjoint,} \tag{2.6}$$

and

$$(\forall n \in \omega, \nu \in \xi(n), k \in \omega)(\mathrm{dom}(c(\alpha^n_{\nu,k}, n)) \cap M = R^n_\nu). \tag{2.7}$$

Define a function φ by $\mathrm{dom}(\varphi) = \{\langle \alpha^n_{\nu,k}, n\rangle : n \in \omega, \nu \in \xi(n), k \in \omega\}$ and

$$\varphi(\alpha^n_{\nu,k}, n) = R^n_\nu$$

for n, ν and k. Note that $\mathrm{dom}(\varphi)$ and $F^n_{\xi(n)}$ are countable for each $n < \omega$. Since $\mathrm{cf}(\kappa) > \omega$, we can take $\delta \in \kappa$ such that $\sup(\bigcup_{n\in\omega} F^n_{\xi(n)}) < \delta$ and $\mathrm{dom}(\varphi) \subseteq \delta \times \omega$. We will check that these φ and δ work. Let $\beta \in \kappa \setminus \delta$ and $n \in \omega$. Then, since $\beta \notin F^n_{\xi(n)}$, there is unique $\nu \in \xi(n)$ such that $\beta \in E^n_\nu$. By (2.6), there is $k \in \omega$ such that

$$(\mathrm{dom}(c(\alpha^n_{\nu,k}, n)) \setminus R^n_\nu) \cap (\mathrm{dom}(c(\beta, n)) \setminus R^n_\nu) = \emptyset. \tag{2.8}$$

Now, we put $\alpha = \alpha^n_{\nu,k}$. Since $\alpha \in D^n_\nu$ and $\beta \in E^n_\nu$,

$$\varphi(\alpha, n) = R^n_\nu \subseteq \mathrm{dom}(c(\alpha, n)) \cap \mathrm{dom}(c(\beta, n)).$$

This combined with (2.8) implies that

$$\varphi(\alpha, n) = \mathrm{dom}(c(\alpha, n)) \cap \mathrm{dom}(c(\beta, n)),$$

and (2.1) and (2.2) imply that

$$c(\alpha, n)|\varphi(\alpha, n) = c(\alpha, n)|R^n_\nu = c^n_\nu = c(\beta, n)|R^n_\nu = c(\beta, n)|\varphi(\alpha, n).$$

Hence, we have (3) in Lemma 1.1. Finally, by (2.6), $\{\mathrm{dom}(c(x)) \setminus \varphi(x) : x \in \mathrm{dom}(\varphi)\}$ is pairwise disjoint and, by (2.7), $\mathrm{dom}(c(x)) \cap M = \varphi(x)$ for each $x \in \mathrm{dom}(\varphi)$. Hence, we have (4) and (5) in Lemma 1.1. □

Remark 2.1. There is a short proof of Lemma 1.1 using an elementary submodel. Let M be a countable elementary submodel of the set $H((2^\kappa)^+)$ such that M contains κ, H and the set $\{c(\alpha, n) : \alpha \in \kappa, n \in \omega\}$ as members. Then we note that for every $\beta \in \kappa \setminus M$ and $n \in \omega$, there is $I \in [\kappa]^\kappa \cap M$ such that the set $\{\mathrm{dom}(c(\alpha, n)) : \alpha \in I\}$ forms a Δ-system with root $\mathrm{dom}(c(\beta, n)) \cap M$ and for every $\alpha \in I$, $c(\alpha, n)|\big(\mathrm{dom}(c(\beta, n)) \cap M\big) = c(\beta, n)|\big(\mathrm{dom}(c(\beta, n)) \cap M\big)$. We enumerate all tuples $\langle n, I, R, c\rangle$ in M such that $I \in \mathcal{P}(\kappa) \cap M$ and the set $\{\mathrm{dom}(c(\alpha, n)) : \alpha \in I\}$ forms an infinite Δ-system with root R and for every $\alpha \in I$, $c(\alpha, n)|R = c$ by $\langle\langle n_i, I_i, R_i, c_i\rangle : i \in \omega\rangle$. By induction on $i \in \omega$, we take $\alpha_i \in I_i \cap M$ such that $\alpha_i \notin \{\alpha_j : j < i\}$ and the set $\{\mathrm{dom}(c(\alpha_i, n_i)) \setminus R_i : i \in \omega\}$ is pairwise disjoint. This can be done because for each $i \in \omega$, the set $\{\mathrm{dom}(c(\alpha, n_i)) \setminus R_i : \alpha \in I_i\}$ is a pairwise disjoint family of finite sets and belongs to M. Then we define $\delta := \sup(\kappa \cap M)$ and $\varphi := \{\langle\langle \alpha_i, n_i\rangle, R_i\rangle : i \in \omega\}$. Then $\delta < \kappa$ because M is countable and $\mathrm{cf}(\kappa) > \omega$. These are as desired.

3. Balogh's natural Dowker space

For reader's convenience, we sketch the construction of a natural Dowker space in [5]. Let $X = \mathfrak{c} \times \omega$, where $\mathfrak{c} = 2^{\aleph_0}$, and for all $n \in \omega$, $W_n = \mathfrak{c} \times n$. The collection $\mathcal{B}_0 = \{W_n : n \in \omega\} \cup \{X \setminus \{x\} : x \in X\}$ is a subbase for an initial topology. Balogh's idea is to add more open sets in $2^\mathfrak{c}$ steps to $\mathcal{B}_0$ to make X a Dowker space. For $U^0, U^1 \subseteq X$, the pair $\langle U^0, U^1\rangle$ is called a *covering pair* if $U^0 \cup U^1 = X$. Fix a list $\langle\langle U^0_\xi, U^1_\xi\rangle\rangle_{\xi<2^\mathfrak{c}}$ of all covering pairs mentioning each $2^\mathfrak{c}$ many times. Inductively, we will define $H \subseteq 2^\mathfrak{c}$ and, for all $\xi \in H$, a covering pair $\langle B^0_\xi, B^1_\xi\rangle$ such that $B^i_\xi \subseteq U^i_\xi$ for each $i < 2$ and $B^0_\xi \cap B^1_\xi = \emptyset$. Then, we will set

$$\mathcal{B}_\xi = \mathcal{B}_0 \cup \{B^0_\eta, B^1_\eta : \eta \in H, \eta < \xi\}.$$

A subset of X is called *ξ-open* if it is open in the topology generated by $\mathcal{B}_\xi$ as a subbase. For $A \subseteq X$, let

$$\mathcal{S}_A = \{\langle U^0 \cap A, U^1 \cap A\rangle : \langle U^0, U^1\rangle \text{ is a covering pair}\}.$$

Definition 3.1. A pair $\langle A, u\rangle$ is a *control pair* if $A \in [X]^{\aleph_0}$ and u is a function with $\mathrm{dom}(u) \subseteq A$ such that

(C_1) for each $x \in \mathrm{dom}(u)$, $u(x) \in [\mathcal{S}_A]^{<\aleph_0}$, and
(C_2) for each $x, x' \in \mathrm{dom}(u)$, if $x \neq x'$, then $u(x) \cap u(x') = \emptyset$.

Remark 3.2. Balogh [5] used the notion of a *control triple*, however, the above definition is enough for our purpose. Note that the size of the set of all control pairs is $\mathfrak{c}$ since $|\mathcal{S}_A| = \mathfrak{c}$ for $A \in [X]^{\aleph_0}$.

Let $\langle\langle A_\beta, u_\beta\rangle\rangle_{\beta<\mathfrak{c}}$ be a list of all control pairs mentioning each $\mathfrak{c}$ many times and where $A_\beta \subseteq \beta \times \omega$ for each $\beta \in \mathfrak{c}$. Suppose $\xi < 2^{\mathfrak{c}}$ and for every $\eta < \xi$ we have already decided whether $\eta \in H$ and, if so, what B^0_η, B^1_η are. We now decide if $\xi \in H$ and, if so, show how to define B^0_ξ, B^1_ξ.

Case 1. Suppose that U^0_ξ, U^1_ξ are ξ-open and there is no $\eta < \xi$ such that $\langle U^0_\eta, U^1_\eta\rangle = \langle U^0_\xi, U^1_\xi\rangle$ and U^0_η, U^1_η are η-open. Then $\xi \in H$ and we need to define B^0_ξ, B^1_ξ.

Suppose $\beta < \mathfrak{c}$ and for every $\alpha < \beta$ and every $k \in \omega$, we already decided $\langle \alpha, k\rangle(\xi)$ equal to the unique $i < 2$ with $\langle\alpha, k\rangle \in B^i_\xi$.

Subcase 1.1. If $\langle U^0_\xi \cap A_\beta, U^1_\xi \cap A_\beta\rangle \notin \bigcup \mathrm{ran}(u_\beta)$, then take the biggest $m \leq \omega$ such that there exists $i < 2$ with $\{\beta\} \times m \subseteq U^i_\xi$. Fix an i with $\{\beta\} \times m \subseteq U^i_\xi$ and make sure $\{\beta\} \times m \subseteq B^i_\xi$. If this $m < \omega$, then, for each k with $m \leq k < \omega$, pick any $i < 2$ such that $\langle \beta, k\rangle \in U^i_\xi$ and set $\langle\beta, k\rangle \in B^i_\xi$.

Subcase 1.2. If $\langle U^0_\xi \cap A_\beta, U^1_\xi \cap A_\beta\rangle \in u_\beta(x)$ for some $x \in \mathrm{dom}(u_\beta)$, then there is only one such x by (C_2). Since $x \in A_\beta \subseteq \beta \times \omega$, $x(\xi)$ has been defined. For every $j \in \omega$, set $\langle\beta, j\rangle(\xi) = x(\xi)$ if $\langle \beta, j\rangle \in U^{x(\xi)}_\xi$, and $\langle\beta, j\rangle(\xi) = 1 - x(\xi)$ otherwise.

Case 2. Not Case 1. Then $\xi \notin H$ and B^0_ξ, B^1_ξ need not be defined.

The space $X = \mathfrak{c} \times \omega$ with

$$\mathcal{B} = \bigcup_{\xi<2^{\mathfrak{c}}} \mathcal{B}_\xi = \mathcal{B}_0 \cup \{B^0_\xi, B^1_\xi : \xi \in H\}$$

as a subbase for the topology is normal by construction and is named a *natural Dowker space* in [5]. The following claims immediately follow from the construction.

Claim 3.3. *For every $\xi, \xi' \in H$, if $\xi \neq \xi'$, then $\langle U^0_\xi, U^1_\xi\rangle \neq \langle U^0_{\xi'}, U^1_{\xi'}\rangle$.*

Claim 3.4. *Let $\xi \in H$ and $\beta \in \kappa$. Then,*

(1) *in Subcase 1.1 above, if $\{\beta\} \times (k+1) \subseteq U^j_\xi$ for some $k < \omega$ and $j < 2$, then $\{\beta\} \times (k+1) \subseteq B^{\langle \beta,k\rangle(\xi)}_\xi$, and*
(2) *in Subcase 1.2 above, $(\{\beta\} \times \omega) \cap U^{x(\xi)}_\xi \subseteq B^{x(\xi)}_\xi$.*

Claim 3.5. *If F^0 and F^1 are disjoint closed sets in X, then there exists $\xi \in H$ such that $F^i \subseteq B^i_\xi$ for each $i < 2$.*

Each basic neighborhood for $x = \langle \alpha, k\rangle \in X$ is of the form

$$V_{t,K}(x) = \bigcap_{\xi \in t} B^{x(\xi)}_\xi \cap (W_{k+1} \setminus K)$$

for some $t \in [H]^{<\aleph_0}$ and $K \in [X \setminus \{x\}]^{<\aleph_0}$. A neighborhood $V_{t,K}(x)$ is called *complete* if for every $\xi \in t$, $V_{t\cap\xi,K}(x) \subseteq U^{x(\xi)}_\xi$.

Lemma 3.6 ([5, Lemma 2.1])**.** *For every neighborhood $V_{t,K}(x)$, there exist $t^* \supseteq t$ and $K^* \supseteq K$ such that $V_{t^*,K^*}(x)$ is a complete neighborhood.*

4. The space X is not countably paracompact

For $\beta \in \mathfrak{c}$ and $\xi \in H$, we say that β is *ξ-homogeneous* if there exists $i < 2$ such that $\{\beta\} \times \omega \subseteq B^i_\xi$. Balogh [5] proved the following lemma.

Lemma 4.1 ([5, Lemma 3.1])**.** *For every countable subset H' of H, there exists $\beta \in \mathfrak{c}$ such that β is ξ-homogeneous for every $\xi \in H'$.*

Remark 4.2. We note that Lemma 4.1 implies that X is not countably paracompact. Suppose that X is countably paracompact. Then there is a countable closed cover $\{F_n : n \in \omega\}$ of X such that $F_n \subseteq W_n$ for each $n \in \omega$. By Claim 3.5, there is a function $\xi : \omega \to H$ such that $F_n \subseteq B^0_{\xi(n)}$ and $X \setminus W_n \subseteq B^1_{\xi(n)}$ for each $n \in \omega$. Then for every $\beta \in \mathfrak{c}$, there is $n \in \omega$ such that β is not $\xi(n)$-homogeneous, since $\bigcup_{n\in\omega} F_n = X$. This contradicts Lemma 4.1.

Now, we give an alternative proof of Lemma 4.1 using Lemma 1.1.

Proof of Lemma 4.1. Let $H' = \{\xi_n : n \in \omega\} \subseteq H$. By Lemma 3.6, there exist functions $t : X \to [H]^{<\aleph_0}$ and $K : X \to [X]^{<\aleph_0}$ such that $V_{t(x),K(x)}(x)$ is a complete neighborhood and

$$\text{if } x = \langle \alpha, n\rangle \in X, \text{then } \{\xi_j : j \le n\} \subseteq t(x). \tag{4.1}$$

For each $x \in X$, define a function $c(x) : t(x) \to 2$ by $c(x)(\xi) = x(\xi)$ for $\xi \in t(x)$. Then by Lemma 1.1, there exist $\delta \in \mathfrak{c}$ and a function φ such that

(1) $\mathrm{dom}(\varphi) \in [\delta \times \omega]^{\aleph_0}$,

(2) for each $x \in \mathrm{dom}(\varphi)$, $\varphi(x) \subseteq t(x)$, and
(3) for each $\beta \in \mathfrak{c} \setminus \delta$ and each $n \in \omega$, there is $\langle \alpha, n \rangle \in \mathrm{dom}(\varphi)$ such that $t(\alpha, n) \cap t(\beta, n) = \varphi(\alpha, n)$ and $c(\alpha, n)|\varphi(\alpha, n) = c(\beta, n)|\varphi(\alpha, n)$.

Moreover, if we put $M = (\bigcup \mathrm{ran}(\varphi)) \cup H'$, then

(4) for each $x \in \mathrm{dom}(\varphi)$, $t(x) \cap M = \varphi(x)$, and
(5) $\{t(x) \setminus M : x \in \mathrm{dom}(\varphi)\}$ is pairwise disjoint.

Put $D = \bigcup\{t(x) : x \in \mathrm{dom}(\varphi)\} \cup H'$. Since D is countable, by Claim 3.3 there exists $A \in [X]^{\aleph_0}$ such that $\mathrm{dom}(\varphi) \subseteq A$ and

$$(4.2)\quad (\forall \xi, \xi' \in D)(\text{if } \xi \neq \xi', \text{then } \langle C^0_\xi \cap A, C^1_\xi \cap A\rangle \neq \langle C^0_{\xi'} \cap A, C^1_{\xi'} \cap A\rangle).$$

Define a function u by $\mathrm{dom}(u) = \mathrm{dom}(\varphi)$ and

$$u(x) = \{\langle C^0_\xi \cap A, C^1_\xi \cap A\rangle : \xi \in t(x) \setminus \varphi(x)\}$$

for $x \in \mathrm{dom}(u)$. Then by (4), (5) and (4.2), $u(x) \cap u(x') = \emptyset$ whenever $x \neq x'$. Hence, $\langle A, u\rangle$ is a control pair. Since this appears cofinally many times in our list of control pairs, we can choose $\beta \in \mathfrak{c}$ such that $\beta > \delta$, $\bigcup_{x \in \delta \times \omega} K(x) \subseteq \beta \times \omega$ and $\langle A, u\rangle = \langle A_\beta, u_\beta\rangle$. Note that, since $M \subseteq D$ and $(t(x) \setminus \varphi(x)) \cap M = \emptyset$ for each $x \in \mathrm{dom}(\varphi)$, it follows from (4.2) that

$$(4.3)\qquad (\forall \xi \in M)\left(\langle C^0_\xi \cap A_\beta, C^1_\xi \cap A_\beta\rangle \notin \bigcup \mathrm{ran}(u_\beta)\right).$$

We show that β is the desired ordinal. Our proof of the remaining part is almost same as in [5]. Suppose that β is not μ-homogeneous for some $\mu \in H'$. Then by (4.1), there is $n \in \omega$ such that $\mu \in t(\beta, n)$ and

$$(4.4)\qquad (\{\beta\} \times (n+1)) \cap B^i_\mu \neq \emptyset \text{ for each } i < 2.$$

Put $y = \langle \beta, n\rangle$. Then by (3), there exists $x = \langle \alpha, n\rangle \in \mathrm{dom}(u_\beta)$ such that

$$(4.5)\qquad t(x) \cap t(y) = \varphi(x) \quad\text{and}\quad c(x)|\varphi(x) = c(y)|\varphi(x).$$

Claim 1. $\{\beta\} \times (n+1) \subseteq V_{t(x)\cap\mu, K(x)}(x)$.

Proof of Claim 1. Remember that

$$V_{t(x)\cap\mu, K(x)}(x) = \bigcap_{\xi \in t(x)\cap\mu} B^{x(\xi)}_\xi \cap (W_{n+1} \setminus K(x)).$$

Since $K(x) \subseteq \beta \times \omega$, it suffices to show that for each $\xi \in t(x) \cap \mu$,

$$(4.6)\qquad \{\beta\} \times (n+1) \subseteq B^{x(\xi)}_\xi.$$

We show this by induction on $\xi \in t(x) \cap \mu$. Suppose that for each $\eta \in t(x)\cap\xi$, $\{\beta\} \times (n+1) \subseteq B^{x(\eta)}_\eta$. Then by the completeness of $V_{t(x),K(x)}(x)$,

$$(4.7)\qquad \{\beta\} \times (n+1) \subseteq V_{t(x)\cap\xi, K(x)}(x) \subseteq U^{x(\xi)}_\xi.$$

We distinguish two cases.

Case A. $\xi \in t(x) \cap M \ (= \varphi(x))$.

In this case, $\langle C^0_\xi \cap A_\beta, C^1_\xi \cap A_\beta \rangle \notin \bigcup \mathrm{ran}(u_\beta)$ by (4.3). Hence, by (4.7) and Claim 3.4 (1), $\{\beta\} \times (n+1) \subseteq B^{y(\xi)}_\xi$. Since $y(\xi) = c(y)(\xi) = c(x)(\xi) = x(\xi)$ by (4.5), we have (4.6).

Case B. $\xi \in t(x) \setminus M$.

In this case, $\langle C^0_\xi \cap A_\beta, C^1_\xi \cap A_\beta \rangle \in u_\beta(x)$ by the definition of $u = u_\beta$. Hence, by (4.7) and Claim 3.4 (2), $\{\beta\} \times (n+1) \subseteq (\{\beta\} \times \omega) \cap U^{x(\xi)}_\xi \subseteq B^{x(\xi)}_\xi$. This completes the proof of Claim 1.

Finally, by Claim 1 and the completeness of $V_{t(x),K(x)}(x)$, $\{\beta\} \times (n+1) \subseteq U^{x(\mu)}_\mu$. Since $\mu \in H' \subseteq M$, $\langle C^0_\mu \cap A_\beta, C^1_\mu \cap A_\beta \rangle \notin \bigcup \mathrm{ran}(u_\beta)$ by (4.3). Hence, by Claim 3.4 (1), $\{\beta\} \times (n+1) \subseteq B^{y(\mu)}_\mu$, which contradicts (4.4). This completes the proof of Lemma 4.1. □

References

[1] Z. Balogh, *A small Dowker space in ZFC*, Proc. Amer. Math. Soc. **124** (1996), 2555–2560.

[2] Z. Balogh, *A normal screenable nonparacompact space in ZFC*, Proc. Amer. Math. Soc. **126** (1998), 1835–1844.

[3] Z. Balogh, *Nonshrinking open covers and K. Morita's third conjecture*, Topology and its Appl. **84** (1998), 185–198.

[4] Z. Balogh, *Dowker spaces and paracompactness questions*, Topology and its Appl. **114** (2001), 49–60.

[5] Z. Balogh, *A natural Dowker space*, Topology Proc. **27** (2003), 1–7.

[6] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, North-Holland, 1980.

(Ohta) Faculty of Education, Shizuoka University, Ohya, Shizuoka, 422-8529 Japan

E-mail address: echohta@ipc.shizuoka.ac.jp

(Yorioka) Deaprtment of Mathematics, Faculty of Science, Shizuoka University, Ohya, Shizuoka, 422-8529 Japan

E-mail address: styorio@ipc.shizuoka.ac.jp