The Killing Vectors on PSU(2|2)/{SU(2)×U(1)} and D(2,1; γ)/{SU(2)×SU(2)×U(1)}

The Killing Vectors on PSU(2|2)/{SU(2)⊗U(1)} and D(2, 1; γ)/{SU(2)⊗SU(2)⊗U(1)}

Yuco Honda*

Department of Physics
Shizuoka University
Ohya 836, Shizuoka
Japan

April 9, 2015

Abstract

We calculate the Killing vectors on specific supercoset spaces, $\mathrm{PSU}(2|2)/\{\mathrm{SU}(2)\otimes\mathrm{U}(1)\}$ and $\mathrm{D}(2,1;\gamma)/\{\mathrm{SU}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)\}$. They represent the symmetry of $\mathrm{D}(2,1;\gamma)$ non-linearly.

*e-mail: yufrau@hotmail.co.jp

Contents

1 Introduction

It is common knowledge that we can construct a coset space G/H on which a group symmetry G is realized as coordinate transformations. It is true even though G is a supergroup. Such transformations are called Killing vectors.

In this letter we calculate the Killing vectors for two supercoset spaces. In senction **2** we warm up with PSU(2|2)/{SU(2)⊗U(1)} as an exercise for the calculation. In section **3** we focus on D$(2,1;\gamma)$/{SU(2)⊗SU(2)⊗U(1)} and work out the Killing vectors on the coset space.

If H is Y⊗U$(1)^n$ where Y is a certain simple or semi-simple group with $n = 1, 2, \cdots$, G/H is Kählerian. For this case generators of G can be decomposed as

$$\left\{T^{\Xi}\right\} = \left\{X^i, \bar{X}^{\bar{i}}, H^I\right\}$$

according to the charge with respect to the centralizer U$(1)^n$. That is, $\{H^I\}$ are Hermetean having charge zero, while $\{X^i\}$ and $\{\bar{X}^{\bar{i}}\}$ are Hermete conjugates with each other having negative and positive charges respectively. Hence G/H gets complex structure. The coset space PSU(2|2)/{SU(2)⊗U(1)} and D$(2,1;\gamma)$/{SU(2)⊗SU(2)⊗U(1)} which we study in this paper are this type.

2 PSU(2|2)/{SU(2)⊗U(1)}

2.1 Setting the algebrae

We consider the supergroup PSU(2|2) for G. PSU(2|2) is generated by two triplets of the SU(2) generators and two quartets of fermionic ones. The generators $\{T^{\Xi}\}$ are decomposed as

$$\left\{T^{\Xi}\right\}=\left\{L^{\alpha}{}_{\beta}, R^{a}{}_{b}, Q^{a}{}_{\alpha}, S^{\alpha}{}_{a}\right\} \tag{2.1}$$

with $\alpha, \beta, a, b = 1$ or 2. Here $L^{\alpha}{}_{\beta}$ are bosonic generators of one of SU(2)s and $R^{a}{}_{b}$ are those of the other. On the other hand $Q^{a}{}_{\alpha}$ and $S^{\alpha}{}_{a}$ are fermionic ones. We study the algebrae of PSU(2|2) in the following 3 steps.

At the first step, we represent $\{T^{\Xi}\}$ by a $(2|2) \times (2|2)$ supermatrix which acts on a supervector v such as

$$T^{A}{}_{B} = \left(\begin{array}{c|c} T^{\alpha}{}_{\beta} & T^{\alpha}{}_{b} \\ \hline T^{a}{}_{\beta} & T^{a}{}_{b} \end{array}\right), \quad v = \left(\begin{array}{c} B^{\beta} \\ \hline F^{b} \end{array}\right).$$

Here B^{β} and F^{b} are bosonic and fermionic 2-component vectors of the respective SU(2) subgroup. Correspondingly to this matrix representation we arrange the generators of (2.1) in a table such as

$$\left\{T^{\Xi}\right\}=\left\{\begin{array}{c|c} L^{\alpha}{}_{\beta} & S^{\alpha}{}_{b} \\ \hline Q^{a}{}_{\beta} & R^{a}{}_{b} \end{array}\right\} \quad \approx \quad \left(\begin{array}{c|c} \mathrm{SU}(2) & e^{S} \\ \hline e^{Q} & \mathrm{SU}(2) \end{array}\right). \tag{2.2}$$

The last table indicates symbolically that the diagonal blocks generate the subgroup SU(2)⊗SU(2). The commutation relations are given by

$$\begin{aligned}
&[L^{\alpha}{}_{\beta}, L^{\gamma}{}_{\delta}] = -\delta^{\gamma}{}_{\beta} L^{\alpha}{}_{\delta} + \delta^{\alpha}{}_{\delta} L^{\gamma}{}_{\beta}, && [R^{a}{}_{b}, R^{c}{}_{d}] = -\delta^{c}{}_{b} R^{a}{}_{d} + \delta^{a}{}_{d} R^{c}{}_{b},\\
&[Q^{a}{}_{\alpha}, L^{\beta}{}_{\gamma}] = -\delta^{\beta}{}_{\alpha} Q^{a}{}_{\gamma} + \frac{1}{2}\delta^{\beta}{}_{\gamma} Q^{a}{}_{\alpha}, && [S^{\alpha}{}_{a}, L^{\beta}{}_{\gamma}] = \delta^{\alpha}{}_{\gamma} S^{\beta}{}_{a} - \frac{1}{2}\beta^{\beta}{}_{\gamma} S^{\alpha}{}_{a},\\
&[Q^{a}{}_{\alpha}, R^{b}{}_{c}] = \delta^{a}{}_{c} Q^{b}{}_{\alpha} - \frac{1}{2}\delta^{b}{}_{c} Q^{a}{}_{\alpha}, && [S^{\alpha}{}_{a}, R^{b}{}_{c}] = -\delta^{b}{}_{a} S^{\alpha}{}_{c} + \frac{1}{2}\delta^{b}{}_{c} S^{\alpha}{}_{a},\\
&\left\{Q^{a}{}_{\alpha}, Q^{b}{}_{\beta}\right\} = 0, && \left\{S^{\alpha}{}_{a}, S^{\beta}{}_{b}\right\} = 0,
\end{aligned}$$

$$\left\{Q^{a}{}_{\alpha}, S^{\beta}{}_{b}\right\}= \delta^{a}{}_{b} L^{\beta}{}_{\alpha} + \delta^{\beta}{}_{\alpha} R^{a}{}_{b}. \tag{2.3}$$

Of course, $L^{\alpha}{}_{\beta}$ commute with $R^{a}{}_{b}$.

In order to construct a Kählerian coset space G/H, we choose SU(2)⊗U(1) for H decomposing the generators $L^{\alpha}{}_{\beta}$ into U(1) and the remaining parts as

$$L^{\alpha}{}_{\beta} = \left\{\begin{array}{c|c} L^{1}{}_{1} & L^{1}{}_{2} \\ \hline L^{2}{}_{1} & L^{2}{}_{2} \end{array}\right\} \tag{2.4}$$

with $L^{2}{}_{2} = -L^{1}{}_{1}$. Here $L^{1}{}_{1}$ is the U(1) centralizer and $L^{1}{}_{2}$ is hermitian conjugate of $L^{2}{}_{1}$.

Putting (2.4) in (2.1) gives

$$\Big\{T^{\Xi}\Big\}=\Big\{\underbrace{L^2{}_1, Q^a{}_1, S^2{}_a}_{X^i}, \underbrace{L^1{}_2, Q^a{}_2, S^1{}_a}_{\overline{X}^{\bar{i}}}, \underbrace{L^1{}_1, R^a{}_b}_{H^I}\Big\}. \tag{2.5}$$

Here each set of the generators $\{X^i\}, \{\overline{X}^{\bar{i}}\}$ and $\{H^I\}$ has definite charges with respect to the centralizer. It will be discussed later (see (2.9)). According to (2.5) we rearrange the table (2.2) as

$$\Big\{T^{\Xi}\Big\}=\left\{\begin{array}{c|c|c} L^1{}_1 & S^1{}_b & L^1{}_2 \\ \hline Q^a{}_1 & R^a{}_b & Q^a{}_2 \\ \hline L^2{}_1 & S^2{}_b & L^2{}_2 \end{array}\right\} \quad \approx \quad \left(\begin{array}{c|c|c} \mathrm{U}(1) & & e^{\overline{X}} \\ \hline & \mathrm{SU}(2) & \\ \hline e^{X} & & \mathrm{U}(1) \end{array}\right). \tag{2.6}$$

That is, $\{X^i\}$ are found in the upper triangular part of the table, $\{\overline{X}^{\bar{i}}\}$ in the lower one and $\{H^I\}$ fall into the diagonal part.

At the second step, we redefine the fermionic generators by using a unified notation $F^{\alpha a\dot{\alpha}}$ as

$$\left\{\frac{\varepsilon^{ab}S^{\alpha}{}_b}{\varepsilon^{\alpha\beta}Q^a{}_\beta}\right\}^{\dot{\alpha}} = \Big\{F^{\alpha a}\Big\}^{\dot{\alpha}} \tag{2.7}$$

with $\dot{\alpha}=1$ or 2. Here ε^{ab} and $\varepsilon^{\alpha\beta}$ are Levi-Civita symbols. The index $\dot{\alpha}$ refers an extra SU(2). This SU(2) is important to generalize PSU(2|2) to a larger supergroup D$(2,1;\gamma)$, as will be shown in the section **3**. Then (2.7) reads

$$\left\{\frac{\varepsilon^{ab}S^2{}_b}{-Q^a{}_1}\right\}^{\dot{\alpha}} = \Big\{F^{2a}\Big\}^{\dot{\alpha}}, \qquad \left\{\frac{\varepsilon^{ab}S^1{}_b}{Q^a{}_2}\right\}^{\dot{\alpha}} = \Big\{F^{1a}\Big\}^{\dot{\alpha}}.$$

With this redifinition, (2.5) becomes

$$\Big\{T^{\Xi}\Big\}=\Big\{\underbrace{L^2{}_1, F^{2a\dot{\alpha}}}_{X^i}, \underbrace{L^1{}_2, F^{1a\dot{\alpha}}}_{\overline{X}^{\bar{i}}}, \underbrace{L^1{}_1, R^a{}_b}_{H^I}\Big\}. \tag{2.8}$$

The algebrae (2.3) are rewritten as

$$[L^2{}_1, L^1{}_2] = 2L^1{}_1, \qquad [R^a{}_b, R^c{}_d] = -\delta^c{}_b R^a{}_d + \delta^a{}_d R^c{}_b,$$

$$[R^a{}_b, L^1{}_1] = 0,$$

$$\begin{aligned}
&[L^2{}_1, L^1{}_1] = -L^2{}_1, && [L^1{}_2, L^1{}_1] = L^1{}_2,\\
&[F^{2a\dot{\alpha}}, L^1{}_1] = -\frac{1}{2}F^{2a\dot{\alpha}}, && [F^{1a\dot{\alpha}}, L^1{}_1] = \frac{1}{2}F^{1a\dot{\alpha}},\\
&[F^{2a\dot{\alpha}}, L^2{}_1] = 0, && [F^{1a\dot{\alpha}}, L^2{}_1] = F^{2a\dot{\alpha}},\\
&[F^{2a\dot{\alpha}}, L^1{}_2] = F^{1a\dot{\alpha}}, && [F^{1a\dot{\alpha}}, L^1{}_2] = 0,\\
&[F^{2a\dot{\alpha}}, R^b{}_c] = \delta^a{}_c F^{2b\dot{\alpha}} - \frac{1}{2}\delta^b{}_c F^{2a\dot{\alpha}}, && [F^{1a\dot{\alpha}}, R^b{}_c,] = \delta^a{}_c F^{1b\dot{\alpha}} - \frac{1}{2}\delta^b{}_c F^{1a\dot{\alpha}},
\end{aligned}$$

$$\left\{F^{2a\dot{\alpha}},F^{2b\dot{\beta}}\right\} = -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L^2{}_1, \qquad \left\{F^{1a\dot{\alpha}},F^{1b\dot{\beta}}\right\} = \varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L^1{}_2,$$

$$\left\{F^{2a\dot{\alpha}},F^{1b\dot{\beta}}\right\} = -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L^1{}_1 + \varepsilon^{ac}\varepsilon^{\dot{\alpha}\dot{\beta}}R^b{}_c. \tag{2.9}$$

The commutator of the 2nd line implies that $\{H^I\}$ has charge zero, while those of the 3rd and 4th lines imply $\{X^i\}$ or $\{\overline{X}^{\bar{i}}\}$ has negative or positive charges respectively.

At the third step, we furthermore redifine some of the generators as

$$L^2{}_1 = L, \qquad L^1{}_2 = \overline{L}, \qquad L^1{}_1 = L^0, \qquad F^{2a\dot{\alpha}} = F^{a\dot{\alpha}}, \qquad F^{1a\dot{\alpha}} = \overline{F}^{a\dot{\alpha}}$$

and write (2.8) down as

$$\left\{T^{\Xi}\right\} = \Big\{\underbrace{L, F^{a\dot{\alpha}}}_{X^i}, \underbrace{\overline{L}, \overline{F}^{a\dot{\alpha}}}_{\overline{X}^{\bar{i}}}, \underbrace{L^0, R^a{}_b}_{H^I}\Big\}. \tag{2.10}$$

Finally the algebrae (2.9) become

$$[L,\overline{L}] = 2L^0, \qquad [R^a{}_b, R^c{}_d] = -\delta^c{}_b R^a{}_d + \delta^a{}_d R^c{}_b,$$

$$[R^a{}_b, L^0] = 0,$$

$$\begin{aligned}
[L, L^0] &= -L, & [\overline{L}, L^0] &= \overline{L},\\
[F^{a\dot{\alpha}}, L^0] &= -\frac{1}{2}F^{a\dot{\alpha}}, & [\overline{F}^{a\dot{\alpha}}, L^0] &= \frac{1}{2}\overline{F}^{a\dot{\alpha}},\\
[F^{a\dot{\alpha}}, L] &= 0, & [\overline{F}^{a\dot{\alpha}}, L] &= F^{a\dot{\alpha}},\\
[F^{a\dot{\alpha}}, \overline{L}] &= \overline{F}^{a\dot{\alpha}}, & [\overline{F}^{a\dot{\alpha}}, \overline{L}] &= 0,\\
[F^{a\dot{\alpha}}, R^b{}_c] &= \delta^a{}_c F^{b\dot{\alpha}} - \frac{1}{2}\delta^b{}_c F^{a\dot{\alpha}}, & [\overline{F}^{a\dot{\alpha}}, R^b{}_c,] &= \delta^a{}_c \overline{F}^{b\dot{\alpha}} - \frac{1}{2}\delta^b{}_c \overline{F}^{a\dot{\alpha}},\\
\left\{F^{a\dot{\alpha}}, F^{b\dot{\beta}}\right\} &= -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L, & \left\{\overline{F}^{a\dot{\alpha}}, \overline{F}^{b\dot{\beta}}\right\} &= \varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{L},
\end{aligned}$$

$$\left\{F^{a\dot{\alpha}}, \overline{F}^{b\dot{\beta}}\right\} = -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L^0 + \varepsilon^{ac}\varepsilon^{\dot{\alpha}\dot{\beta}}R^b{}_c. \tag{2.11}$$

In the following argument we use (2.10) and (2.11).

2.2 Coset space construction

In this section we discuss the construction on PSU(2|2)/{SU(2)⊗U(1)}. Start with a known theorem.

Theorem 1 (Coset decomposition)**.** Let G/H be $\{a\mathrm{H}\}$. For any element $g \in \mathrm{G}$, there exit $a\mathrm{H} \in \mathrm{G/H}$ and $h \in \mathrm{H}$ such that $g = ah$.

In general G/H can be constructed in the real basis. When G/H is Kählerian, the construction can be done in the complex basis as well. That is, adding $\{\overline{X}^{\bar{i}}\}$ to $\{H^I\}$, we enlarge H to $\hat{\mathrm{H}}$ such as

$$\left\{\hat{H}^{\hat{I}}\right\}=\left\{\overline{X}^{\bar{i}},H^{I}\right\}=\left\{\overline{L},\overline{F}^{a\dot{\alpha}},L^{0},R^{a}{}_{b}\right\}. \tag{2.12}$$

We now consider $\mathrm{G^C}/\hat{\mathrm{H}}$. Here $\mathrm{G^C}$ is a complexification of G[1]. This coset space is isomorophic to G/H [1][2]:

$$\mathrm{G^C}/\hat{\mathrm{H}}\simeq \mathrm{G/H}.$$

The coset element is given by

$$e^{\phi\cdot X}=e^{zL+\theta_{a\dot{\alpha}}F^{a\dot{\alpha}}} \tag{2.13}$$

with $\phi=(z,\theta_{a\dot{\alpha}})$ coordinates of $\mathrm{G^C}/\hat{\mathrm{H}}$. Here z is a bosonic number while $\theta_{a\dot{\alpha}}$ are fermionic ones. Let $e^{i\epsilon\cdot T}\in\mathrm{G}$ and $e^{i\lambda\cdot\hat{H}}\in\hat{\mathrm{H}}$ be

$$e^{\epsilon\cdot T}=e^{\epsilon_{L}L+\epsilon_{Fa\dot{\alpha}}F^{a\dot{\alpha}}+\epsilon_{\overline{L}}\overline{L}+\epsilon_{\overline{F}a\dot{\alpha}}\overline{F}^{a\dot{\alpha}}+\epsilon_{L^0}L^0+\epsilon_{R}{}^{b}{}_{a}R^{a}{}_{b}}, \tag{2.14}$$

$$e^{\lambda\cdot\hat{H}}=e^{\lambda_{\overline{L}}\overline{L}+\lambda_{\overline{F}a\dot{\alpha}}\overline{F}^{a\dot{\alpha}}+\lambda_{L^0}L^0+\lambda_{R}{}^{b}{}_{a}R^{a}{}_{b}}. \tag{2.15}$$

Here ϵ and λ are parameters which correspond to $\{T^{\Xi}\}$ of (2.10) and $\{\hat{H}^{\hat{I}}\}$ of (2.12) respectively. We note that $\epsilon_{R}{}^{b}{}_{a}$ and $\lambda_{R}{}^{b}{}_{a}$ are constrained by

$$\mathrm{tr}\epsilon_R=\mathrm{tr}\lambda_R=0 \tag{2.16}$$

because they are parameters of SU(2).

By **Theorem 1.**, we have

$$e^{i\epsilon\cdot T}e^{\phi\cdot X}=e^{\phi'(\phi,\epsilon)\cdot X}e^{i\lambda(\phi,\epsilon)\cdot\hat{H}}$$

with appropriate functions $\phi'(\phi,\epsilon)$ and $\lambda(\phi,\epsilon)$, which depend on ϕ and ϵ. This defines a holomorophic transformation of $\phi\to\phi'(\phi,\epsilon)$ as

$$e^{\phi'(\phi,\epsilon)\cdot X}=e^{i\epsilon\cdot T}e^{\phi\cdot X}e^{-i\lambda(\phi,\epsilon)\cdot\hat{H}}. \tag{2.17}$$

For $\epsilon\ll 1$, the l.h.s. of (2.17) becomes

$$e^{\phi'(\phi,\epsilon)\cdot X}=e^{\phi\cdot X+\delta\phi\cdot X+O(\epsilon^2)}. \tag{2.18}$$

Let us parameterize $\delta\phi$ as $\delta\phi=\epsilon_A R^A$. We then call R^A Killing vectors. By (2.17) and (2.18), we obtain

$$e^{\phi\cdot X+\epsilon_A R^A\cdot X+O(\epsilon^2)}=e^{i\epsilon\cdot T}e^{\phi\cdot X}e^{-i\lambda\cdot\hat{H}}. \tag{2.19}$$

To write the r.h.s in the same form as the l.h.s, we use the following theorem.

Theorem 2. For any matrices $\mathcal{E}$ and X

$$\exp\mathcal{E}\,\exp X=\exp\left(X+\sum_{n=0}^{\infty}\alpha_n(\mathrm{ad}\;X)^n\mathcal{E}+O(\mathcal{E}^2)\right),$$

[1] $\mathrm{G^C}\ni e^{i\epsilon\cdot T}$ with complex parameters ϵ.

$$\exp X \, \exp \mathcal{E} = \exp \left(X + \sum_{n=0}^{\infty} (-1)^n \alpha_n (\mathrm{ad}\ X)^n \mathcal{E} + O(\mathcal{E}^2) \right)$$

where $\mathcal{E} \ll 1$. Here $(\mathrm{ad}\ X)^n \mathcal{E}$ is a n-ple commutator

$$(\mathrm{ad}\ X)^n \mathcal{E} = \underbrace{[X, [X, \cdots [X}_{n}, \mathcal{E}] \cdots]].$$

The constants α_n are given by

$$\alpha_n + \frac{\alpha_{n-1}}{2!} + \cdots + \frac{\alpha_0}{(n+1)!} = 0.$$

For a small n they are computed as

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{1}{2}, \quad \alpha_2 = \frac{1}{12}, \quad \alpha_3 = 0, \quad \alpha_4 = -\frac{1}{720}, \quad \alpha_5 = 0, \quad \cdots . \tag{2.20}$$

The theorem can be proved from Hausdorff formula. For detailed proof refer to [3].

Now we assume that ϵ and λ commute with the generators of $\{T^{\Xi}\}$ irrespectively of their gradings. Using **Theorem 2.**, (2.19) becomes

$$e^{\phi \cdot X + \epsilon_A R^A \cdot X + O(\epsilon^2)} =$$

$$\exp\Big(\phi \cdot X + i \sum_{n=0}^{\infty} \alpha_n (\mathrm{ad}\ \phi \cdot X)^n \epsilon \cdot T - i \sum_{n=0}^{\infty} (-1)^n \alpha_n (\mathrm{ad}\ \phi \cdot X)^n \lambda \cdot \hat{H} + O(\epsilon^2) \Big). \tag{2.21}$$

We expand $R(\phi)^A$ and $\lambda(\phi)$ in series of ϕ:

$$R(\phi)^A = {R_{(0)}}^A(\phi) + {R_{(1)}}^A(\phi) + {R_{(2)}}^A(\phi) + \cdots + {R_{(n)}}^A(\phi) + \cdots , \tag{2.22}$$

$$\lambda(\phi) = \lambda_{(0)}(\phi) + \lambda_{(1)}(\phi) + \lambda_{(2)}(\phi) + \cdots + \lambda_{(n)}(\phi) + \cdots , \tag{2.23}$$

and plug (2.22) and (2.23) in (2.21). Comparing exponents on the both sides, we get

$$\begin{aligned} &\epsilon_A ({R_{(0)}}^A + {R_{(1)}}^A + \cdots + {R_{(n)}}^A + \cdots) \cdot X + O(\epsilon^2) \\ &= i \sum_{n=0}^{\infty} \alpha_n (\mathrm{ad}\ \phi \cdot X)^n \epsilon \cdot T \\ &- i \sum_{n=0}^{\infty} (-1)^n \alpha_n (\mathrm{ad}\ \phi \cdot X)^n (\lambda_{(0)} + \lambda_{(1)} + \cdots + \lambda_{(n)} + \cdots) \cdot \hat{H} + O(\epsilon^2). \end{aligned} \tag{2.24}$$

2.3 Calculation of the Killing vectors

By evaluating both sides of (2.24) order by order of ϕ, we can explicitly find the Killing vectors $R(\phi)^A$ as well as $\lambda(\phi)$. We show the way of the calculation with the help of the appendix **A**.

0th order of ϕ Extracting the 0th-order terms from (2.24) we have

$$\epsilon_A R_{(0)}{}^A L + \epsilon_A R_{(0)}{}^A{}_{a\dot{\alpha}} F^{a\dot{\alpha}} = i\epsilon \cdot T - i\lambda_{(0)} \cdot \hat{H}. \tag{2.25}$$

Let us assume

$$\epsilon_A R_{(0)}{}^A = i\epsilon_L, \quad \epsilon_A R_{(0)}{}^A{}_{a\dot{\alpha}} = i\epsilon_{Fa\dot{\alpha}} \tag{2.26}$$

as the initial condition. Using (2.26) in (2.25) we find $\lambda_{(0)}$:

$$\begin{aligned} \lambda_{(0)\overline{L}} &= \epsilon_{\overline{L}}, \\ \lambda_{(0)\overline{F}a\dot{\alpha}} &= \epsilon_{\overline{F}a\dot{\alpha}}, \\ \lambda_{(0)L^0} &= \epsilon_{L^0}, \\ \lambda_{(0)R^b{}_a} &= \epsilon_{R^b{}_a}. \end{aligned} \tag{2.27}$$

1st order of ϕ Extracting the 1st-order terms from (2.24) we have

$$\epsilon_A R_{(1)}{}^A L + \epsilon_A R_{(1)}{}^A{}_{a\dot{\alpha}} F^{a\dot{\alpha}} = -i\frac{1}{2}[\phi \cdot X, \epsilon \cdot T] - i\lambda_{(1)} \cdot \hat{H} - i\frac{1}{2}[\phi \cdot X, \lambda_{(0)} \cdot \hat{H}]. \tag{2.28}$$

In the r.h.s. replace $\lambda_{(0)}$ by (2.27). We add up the first and the last term, and calculate the summation by using (A.1). Compare the coefficients both sides of (2.28). We find $R_{(1)}{}^A$:

$$\begin{aligned} \epsilon_A R_{(1)}{}^A &= i\Big(z\epsilon_{L^0} + \frac{1}{2}\theta_{a\dot{\alpha}}\epsilon_{Fb\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}\Big), \\ \epsilon_A R_{(1)}{}^A{}_{a\dot{\alpha}} &= i\Big(z\epsilon_{\overline{F}a\dot{\alpha}} + \frac{1}{2}\theta_{a\dot{\alpha}}\epsilon_{L^0} - \theta_{b\dot{\alpha}}\epsilon_{R^b{}_a}\Big), \end{aligned} \tag{2.29}$$

and $\lambda_{(1)}$:

$$\begin{aligned} \lambda_{(1)\overline{L}} &= 0, \\ \lambda_{(1)\overline{F}a\dot{\alpha}} &= -\theta_{a\dot{\alpha}}\epsilon_{\overline{L}}, \\ \lambda_{(1)L^0} &= -2z\epsilon_{\overline{L}} + \theta_{a\dot{\alpha}}\epsilon_{\overline{F}b\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}, \\ \lambda_{(1)R^b{}_a} &= -\theta_{c\dot{\alpha}}\epsilon_{\overline{F}a\dot{\beta}}\varepsilon^{cb}\varepsilon^{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\delta^b{}_a\theta_{c\dot{\alpha}}\epsilon_{\overline{F}d\dot{\beta}}\varepsilon^{cd}\varepsilon^{\dot{\alpha}\dot{\beta}}. \end{aligned} \tag{2.30}$$

The last term of $\lambda_{(1)R^b{}_a}$ was put additionally so as to satisfy the constraint (2.16).

2nd order of ϕ Extracting the 2nd-order terms from (2.24) we have

$$\begin{aligned} \epsilon_A R_{(2)}{}^A L + \epsilon_A R_{(2)}{}^A{}_{a\dot{\alpha}} F^{a\dot{\alpha}} =& \frac{i}{12}[\phi \cdot X, [\phi \cdot X, \epsilon \cdot T]] - i\lambda_{(2)} \cdot \hat{H} \\ &- \frac{i}{2}[\phi \cdot X, \lambda_{(1)} \cdot \hat{H}] - \frac{i}{12}[\phi \cdot X, [\phi \cdot X, \lambda_{(0)} \cdot \hat{H}]]. \end{aligned} \tag{2.31}$$

In the r.h.s. replace $\lambda_{(0)}$ by (2.27). When we subtract the last term from the first one we obtain the l.h.s. of (A.7), which is vanishing. Calculate the 3rd term by using (A.2) and then replace $\lambda_{(1)}$ by (2.30). We find ${R_{(2)}}^A$:

$$\epsilon_A {R_{(2)}}^A = -\frac{i}{2}\Big(2z^2\epsilon_{\overline{L}} - z\theta_{a\dot{\alpha}}\epsilon_{\overline{F}b\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}\Big),$$
$$\epsilon_A {R_{(2)}}^A{}_{a\dot{\alpha}} = -\frac{i}{2}\left(2z\theta_{a\dot{\alpha}}\epsilon_{\overline{L}} - \theta_{b\dot{\alpha}}\theta_{c\dot{\gamma}}\epsilon_{\overline{F}a\dot{\beta}}\varepsilon^{bc}\varepsilon^{\dot{\beta}\dot{\gamma}}\right), \tag{2.32}$$

and $\lambda_{(2)}$:

$$\lambda_{(2)\overline{L}} = 0,$$
$$\lambda_{(2)\overline{F}a\dot{\alpha}} = 0,$$
$$\lambda_{(2)L^0} = -\frac{1}{2}\theta_{a\dot{\alpha}}\theta_{b\dot{\beta}}\epsilon_{\overline{L}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}} = 0,$$
$$\lambda_{(2)R^b{}_a} = \frac{1}{2}\theta_{c\dot{\alpha}}\theta_{a\dot{\beta}}\epsilon_{\overline{L}}\varepsilon^{cb}\varepsilon^{\dot{\alpha}\dot{\beta}}. \tag{2.33}$$

For the result of $\lambda_{(2)L^0}$ we have used $\theta_{a\dot{\alpha}}\theta_{b\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}} = 0$ due to anti-commutativity of fermionic numbers.

3rd order of ϕ Taking out the 3rd-order terms from (2.24) we have

$$\epsilon_A {R_{(3)}}^A L + \epsilon_A {R_{(3)}}^A{}_{a\dot{\alpha}} F^{a\dot{\alpha}}$$
$$= i\alpha_3[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\epsilon\cdot T]]] - i\lambda_{(3)}\cdot\hat{H} - \frac{i}{2}[\phi\cdot X,\lambda_{(2)}\cdot\hat{H}] - \frac{i}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(1)}\cdot\hat{H}]] \tag{2.34}$$

with $\alpha_3 = 0$ by (2.20). In the r.h.s. calculate the 3rd and 4th terms by using (A.2) and (A.3) respectively. Then replace $\lambda_{(2)}$ by (2.33) and $\lambda_{(1)}$ by (2.30) in the result. We find ${R_{(3)}}^A$:

$$\epsilon_A {R_{(3)}}^A = -\frac{i}{12}\theta_{a\dot{\alpha}}\theta_{b\dot{\beta}}\theta_{c\dot{\gamma}}\epsilon_{\overline{F}d\dot{\delta}}\varepsilon^{cb}\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon^{ad}\varepsilon^{\dot{\alpha}\dot{\beta}},$$
$$\epsilon_A {R_{(3)}}^A{}_{a\dot{\alpha}} = -\frac{i}{6}\theta_{b\dot{\alpha}}\theta_{c\dot{\gamma}}\theta_{a\dot{\beta}}\epsilon_{\overline{L}}\varepsilon^{bc}\varepsilon^{\dot{\beta}\dot{\gamma}}, \tag{2.35}$$

and $\lambda_{(3)}$:

$$\lambda_{(3)} = 0. \tag{2.36}$$

4th order of ϕ Taking out the 4th-order terms from (2.24) we have

$$\epsilon_A {R_{(4)}}^A L + \epsilon_A {R_{(4)}}^A{}_{a\dot{\alpha}} F^{a\dot{\alpha}}$$
$$= -\frac{i}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\epsilon\cdot T]]]] - i\lambda_{(4)}\cdot\hat{H} - \frac{i}{2}[\phi\cdot X,\lambda_{(3)}\cdot\hat{H}]$$
$$-\frac{i}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(2)}\cdot\hat{H}]] + i\alpha_3[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(1)}\cdot\hat{H}]]]$$

$$+\frac{i}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(0)}\cdot\hat{H}]]]]. \tag{2.37}$$

We use $\lambda_{(3)}=0$ by (2.36) and $\alpha_3=0$ in the r.h.s. and replace $\lambda_{(0)}$ by (2.27). When we subtract the last term from the first we obtain the l.h.s. of (A.8), which gets vanishing. Calculate the 4th term using (A.3) and replace $\lambda_{(2)}$ by (2.33). We find ${R_{(4)}}^A$:

$$\begin{aligned}\epsilon_A {R_{(4)}}^A &= \frac{i}{24}\theta_{a\dot\alpha}\theta_{b\dot\beta}\theta_{c\dot\gamma}\theta_{d\dot\delta}\epsilon_{\overline{L}}\varepsilon^{ac}\varepsilon^{\dot\gamma\dot\delta}\varepsilon^{db}\varepsilon^{\dot\alpha\dot\beta},\\ \epsilon_A {R_{(4)}}^A{}_{a\dot\alpha} &= 0,\end{aligned} \tag{2.38}$$

and $\lambda_{(4)}$:

$$\lambda_{(4)}=0. \tag{2.39}$$

5th order of ϕ Extracting the 5th-order terms from (2.24) we have

$$\begin{aligned}&\epsilon_A {R_{(5)}}^A L+\epsilon_A {R_{(5)}}^A{}_{a\dot\alpha}F^{a\dot\alpha}\\ &\quad= i\alpha_5[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\epsilon\cdot T]]]]]-i\lambda_{(5)}\cdot\hat{H}-\frac{i}{2}[\phi\cdot X,\lambda_{(4)}\cdot\hat{H}]\\ &\qquad-\frac{i}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(3)}\cdot\hat{H}]]+i\alpha_3[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(2)}\cdot\hat{H}]]]\\ &\qquad+\frac{i}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(1)}\cdot\hat{H}]]]]\\ &\qquad+i\alpha_5[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(0)}\cdot\hat{H}]]]]]\end{aligned} \tag{2.40}$$

with $\alpha_5=\alpha_3=0$. When we replace $\lambda_{(4)}=0$ by (2.39) and $\lambda_{(3)}=0$ by (2.36), the 2nd and 6th terms remain in the r.h.s.. Calculate the 6th term using (A.5) and then replace $\lambda_{(1)}$ by (2.30). We find ${R_{(5)}}^A$:

$$\begin{aligned}\epsilon_A {R_{(5)}}^A &= 0,\\ \epsilon_A {R_{(5)}}^A{}_{a\dot\alpha} &= 0,\end{aligned} \tag{2.41}$$

and $\lambda_{(5)}$:

$$\lambda_{(5)}=0. \tag{2.42}$$

More than 5th order of ϕ For higher orders of ϕ, $\lambda_{(n)}$ with $n>3$ and n-ple commutators with $n>5$ are vanishing. Hence the Killing vectors ${R_{(n)}}^A$ with $n>5$ also vanish.

Finally we sum up the results of all order of ϕ as form (2.22). We get the Killing vectors R^A on PSU(2|2)/{SU(2)⊗U(1)}:

$$\begin{aligned}-i\epsilon_A R^A =&\epsilon_L+z\epsilon_{L^0}+\frac{1}{2}\theta_{a\dot\alpha}\epsilon_{Fb\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}-\frac{1}{2}\Big(2z^2\epsilon_{\overline{L}}-z\theta_{a\dot\alpha}\epsilon_{\overline{F}b\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}\Big)\\ &-\frac{1}{12}\theta_{a\dot\alpha}\theta_{b\dot\beta}\theta_{c\dot\gamma}\epsilon_{\overline{F}d\dot\delta}\varepsilon^{cb}\varepsilon^{\dot\gamma\dot\delta}\varepsilon^{ad}\varepsilon^{\dot\alpha\dot\beta}+\frac{1}{24}\theta_{a\dot\alpha}\theta_{b\dot\beta}\theta_{c\dot\gamma}\theta_{d\dot\delta}\epsilon_{\overline{L}}\varepsilon^{ac}\varepsilon^{\dot\gamma\dot\delta}\varepsilon^{db}\varepsilon^{\dot\alpha\dot\beta},\end{aligned} \tag{2.43}$$

$$
\begin{aligned}
-i\epsilon_A R^A{}_{a\dot\alpha} =& \epsilon_{Fa\dot\alpha} + z\epsilon_{\overline{F}a\dot\alpha} + \frac{1}{2}\theta_{a\dot\alpha}\epsilon_{L^0} - \theta_{b\dot\alpha}\epsilon_{R}{}^{b}{}_{a} \\
& - \frac{1}{2}\left(2z\theta_{a\dot\alpha}\epsilon_{\overline{L}} - \theta_{b\dot\alpha}\theta_{c\dot\gamma}\epsilon_{\overline{F}a\dot\beta}\varepsilon^{bc}\varepsilon^{\dot\beta\dot\gamma}\right) - \frac{1}{6}\theta_{b\dot\alpha}\theta_{c\dot\gamma}\theta_{a\dot\beta}\epsilon_{\overline{L}}\varepsilon^{bc}\varepsilon^{\dot\beta\dot\gamma},
\end{aligned}
\tag{2.44}
$$

and λ:

$$
\lambda_{\overline{L}} = \epsilon_{\overline{L}}, \tag{2.45}
$$

$$
\lambda_{\overline{F}a\dot\alpha} = \epsilon_{\overline{F}a\dot\alpha} - \theta_{a\dot\alpha}\epsilon_{\overline{L}}, \tag{2.46}
$$

$$
\lambda_{L^0} = \epsilon_{L^0} - 2z\epsilon_{\overline{L}} + \theta_{a\dot\alpha}\epsilon_{\overline{F}b\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}, \tag{2.47}
$$

$$
\lambda_{R}{}^{b}{}_{a} = \epsilon_{R}{}^{b}{}_{a} - \theta_{c\dot\alpha}\epsilon_{\overline{F}a\dot\beta}\varepsilon^{cb}\varepsilon^{\dot\alpha\dot\beta} + \frac{1}{2}\delta^{b}{}_{a}\theta_{c\dot\alpha}\epsilon_{\overline{F}d\dot\beta}\varepsilon^{cd}\varepsilon^{\dot\alpha\dot\beta} + \frac{1}{2}\theta_{c\dot\alpha}\theta_{a\dot\beta}\epsilon_{\overline{L}}\varepsilon^{cb}\varepsilon^{\dot\alpha\dot\beta}. \tag{2.48}
$$

3 $\mathbf{D}(2,1;\gamma)/\{\mathbf{SU}(2)\otimes\mathbf{SU}(2)\otimes\mathbf{U}(1)\}$

3.1 Relationship between $\mathbf{D}(2,1;\gamma)$ and $\mathbf{PSU}(2|2)$

As the main topic, we consider the exceptional supergroup $\mathrm{D}(2,1;\gamma)$. It includes the subgroup PSU(2|2) and SU(2) so that

$$
\mathrm{D}(2,1;\gamma) \supset \mathrm{PSU}(2|2)\otimes \mathrm{SU}(2),
$$

$$
\mathrm{PSU}(2|2) \supset \mathrm{SU}(2)\otimes\mathrm{SU}(2).
$$

Consequently $\mathrm{D}(2,1;\gamma)$ contains three SU(2)s as the subgroup. The Lie-algebra of $\mathrm{D}(2,1;\gamma)$ consists of three triplets of the SU(2) generators and an octet of fermionic generators.

For G/H we may choose any homogeneous subgroups for H such as

$$
\mathrm{H} = \begin{cases} \mathrm{SU}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{SU}(2), \\ \mathrm{SU}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1), \\ \mathrm{SU}(2)\otimes\mathrm{U}(1)\otimes\mathrm{U}(1), \\ \mathrm{U}(1)\otimes\mathrm{U}(1)\otimes\mathrm{U}(1). \end{cases}
$$

The larger H we choose, the simpler G/H we get, since the number of the coset generators gets reduced. Therefore we want G/H to be a maximal Kähler coset space, so that we choose H to be SU(2)⊗SU(2)⊗U(1).

3.2 Setting the algebrae

Let decompose 17 generators of $\mathrm{D}(2,1;\gamma)$ as

$$
\left\{T^{\Xi}\right\} = \left\{L^{\alpha}{}_{\beta}, R^{a}{}_{b}, \dot{L}^{\dot\alpha}{}_{\dot\beta}, F^{\alpha a\dot\alpha}\right\}. \tag{3.1}
$$

Here $\dot{L}^{\dot\alpha}{}_{\dot\beta}$ were added to the PSU(2|2) generators of (2.8). The commutation relations are given by

$$
\begin{aligned}
&[L^\alpha{}_\beta, L^\gamma{}_\delta] = -\delta^\gamma{}_\beta L^\alpha{}_\delta + \delta^\alpha{}_\delta L^\gamma{}_\beta, \qquad && [R^a{}_b, R^c{}_d] = -\delta^c{}_b R^a{}_d + \delta^a{}_d R^c{}_b,\\
&[\dot{L}^{\dot\alpha}{}_{\dot\beta}, \dot{L}^{\dot\gamma}{}_{\dot\delta}] = -\delta^{\dot\gamma}{}_{\dot\beta}\dot{L}^{\dot\alpha}{}_{\dot\delta} + \delta^{\dot\alpha}{}_{\dot\delta}\dot{L}^{\dot\gamma}{}_{\dot\beta}, &&\\
&[F^{\alpha a\dot\alpha}, L^\beta{}_\gamma] = \delta^\alpha{}_\gamma F^{\beta a\dot\alpha} - \frac{1}{2}\delta^\beta{}_\gamma F^{\alpha a\dot\alpha}, && [F^{\alpha a\dot\alpha}, R^b{}_c] = \delta^a{}_c F^{\alpha b\dot\alpha} - \frac{1}{2}\delta^b{}_c F^{\alpha a\dot\alpha},\\
&[F^{\alpha a\dot\alpha}, \dot{L}^{\dot\beta}{}_{\dot\gamma}] = \delta^{\dot\alpha}{}_{\dot\gamma} F^{\alpha a\dot\beta} - \frac{1}{2}\delta^{\dot\beta}{}_{\dot\gamma} F^{\alpha a\dot\alpha}, &&
\end{aligned}
$$

$$
\left\{F^{\alpha a\dot\alpha}, F^{\beta b\dot\beta}\right\} = \alpha\varepsilon^{\alpha\gamma}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}L^\beta{}_\gamma + \beta\varepsilon^{\alpha\beta}\varepsilon^{ac}\varepsilon^{\dot\alpha\dot\beta}R^b{}_c + \gamma\varepsilon^{\alpha\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\gamma}\dot{L}^{\dot\beta}{}_{\dot\gamma} \tag{3.2}
$$

with the indices $\alpha, a, \dot\alpha, \beta, \cdots = 1$ or 2. The SU(2) bosonic generators $L^\alpha{}_\beta, R^a{}_b, \dot{L}^{\dot\alpha}{}_{\dot\beta}$ commute canonically. They form algebrae with the fermionic generators $F^{\alpha a\dot\alpha}$ so that $F^{\alpha a\dot\alpha}$ transform in the fundamental representation under each SU(2) generator. The anti-commutator of $F^{\alpha a\dot\alpha}$ is non-trivial. Its form is justified by the Jacobi identity. It turns out that the coefficients should be constrained by $\alpha + \beta + \gamma = 0$. Since the overall rescaling does not change the algebraic structure, the parameter γ/α essentially characterizes the algebrae (3.2) [4][5][2].

Next we decompose $\dot{L}^{\dot\alpha}{}_{\dot\beta}$ and $F^{\alpha a\dot\alpha}$ as

$$
\dot{L}^{\dot\alpha}{}_{\dot\beta} = \left\{\begin{array}{c|c} C & K \\ \hline -P & -C \end{array}\right\}^{\dot\alpha}_{\dot\beta}, \quad \left\{F^{\alpha a}\right\}^{\dot\alpha} = \left\{\begin{array}{c} \varepsilon^{ab}S^\alpha{}_b \\ \hline \varepsilon^{\alpha\beta}Q^a{}_\beta \end{array}\right\}^{\dot\alpha}.
$$

The decomposition of (3.1) reads

$$
\left\{T^\Xi\right\} = \left\{\underbrace{P, Q^a{}_\alpha}_{X^i}, \underbrace{K, S^\alpha{}_a}_{\bar{X}^{\bar{i}}}, \underbrace{L^\alpha{}_\beta, R^a{}_b, C}_{H^I}\right\} \tag{3.3}
$$

and the algebrae (3.2) are rewritten as

$$
\begin{aligned}
&[L^\alpha{}_\beta, L^\gamma{}_\delta] = -\delta^\gamma{}_\beta L^\alpha{}_\delta + \delta^\alpha{}_\delta L^\gamma{}_\beta, \qquad && [R^a{}_b, R^c{}_d] = -\delta^c{}_b R^a{}_d + \delta^a{}_d R^c{}_b,\\
&[Q^a{}_\alpha, L^\beta{}_\gamma] = -\delta^\beta{}_\alpha Q^a{}_\gamma + \frac{1}{2}\delta^\beta{}_\gamma Q^a{}_\alpha, && [S^\alpha{}_a, L^\beta{}_\gamma] = \delta^\alpha{}_\gamma S^\beta{}_a - \frac{1}{2}\beta^\beta{}_\gamma S^\alpha{}_a,\\
&[Q^a{}_\alpha, R^b{}_c] = \delta^a{}_c Q^b{}_\alpha - \frac{1}{2}\delta^b{}_c Q^a{}_\alpha, && [S^\alpha{}_a, R^b{}_c] = -\delta^b{}_a S^\alpha{}_c + \frac{1}{2}\delta^b{}_c S^\alpha{}_a,\\
&[L^\alpha{}_\beta, C] = 0, && [R^a{}_b, C] = 0,\\
&[P, C] = -P, && [K, C] = K,
\end{aligned}
$$

$$
[P, K] = -2C,
$$

$$
[Q^a{}_\alpha, C] = -\frac{1}{2}Q^a{}_\alpha, \qquad [S^\alpha{}_a, C] = \frac{1}{2}S^\alpha{}_a,
$$

[2]In (2.9) we choose $\alpha = 1$ and $\gamma \to 0$

$$
\begin{aligned}
[Q^a{}_\alpha, P] &= 0, & [S^\alpha{}_a, P] &= \varepsilon^{\alpha\beta}\varepsilon_{ab}Q^b{}_\beta, \\
[Q^a{}_\alpha, K] &= -\varepsilon_{\alpha\beta}\varepsilon^{ab}S^\beta{}_b, & [S^\alpha{}_a, K] &= 0, \\
\left\{Q^a{}_\alpha, Q^b{}_\beta\right\} &= \gamma\varepsilon_{\alpha\beta}\varepsilon^{ab}P, & \left\{S^\alpha{}_a, S^\beta{}_b\right\} &= \gamma\varepsilon^{\alpha\beta}\varepsilon_{ab}K,
\end{aligned}
$$

$$
\left\{Q^a{}_\alpha, S^\beta{}_b\right\} = \alpha\delta^a{}_b L^\beta{}_\alpha - \beta\delta^\beta{}_\alpha R^a{}_b + \gamma\delta^\beta{}_\alpha\delta^a{}_b C. \tag{3.4}
$$

Here $C \in \mathrm{U}(1)$ serves as a centralizer. The generators with zero charges belong to $\{H^I\}$. On the other hand $\{X^i\}$ and $\{\overline{X}^{\bar{i}}\}$ are split according to negative or positive charges with respect to the centralizer as explained in the section **2.1**. In the following arguments we use the notation of (3.3) and (3.4).

3.3 Coset space construction

We construct the Kähler coset space $\mathrm{D}(2,1;\gamma)/\{\mathrm{SU}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)\}$. The same method can be used as has been discussed in the section **2.2**. For complexification of the coset space, we define a enlarged subgroup $\hat{\mathrm{H}}$ as

$$
\left\{\hat{H}^{\hat{I}}\right\} = \left\{\overline{X}^{\bar{i}}, H^I\right\} = \left\{K, S^\alpha{}_a, L^\alpha{}_\beta, R^a{}_b, C\right\} \tag{3.5}
$$

and consider $\mathrm{G^C}/\hat{\mathrm{H}}$. The coset element is given by

$$
e^{\phi\cdot X} = e^{xP + \theta^\alpha{}_a Q^a{}_\alpha} \tag{3.6}
$$

with $\phi = (x, \theta^\alpha{}_a)$ the coordinates of $\mathrm{G^C}/\hat{\mathrm{H}}$. Here x is bosonic while $\theta^\alpha{}_a$ are fermionic. For $e^{i\epsilon\cdot T} \in \mathrm{G}$ and $e^{i\lambda\cdot\hat{H}} \in \hat{\mathrm{H}}$, we can write

$$
e^{\epsilon\cdot T} = e^{\epsilon_P P + \epsilon_Q{}^\alpha{}_a Q^a{}_\alpha + \epsilon_K K + \epsilon_S{}^a{}_\alpha S^\alpha{}_a + \epsilon_L{}^\beta{}_\alpha L^\alpha{}_\beta + \epsilon_R{}^b{}_a R^a{}_b + \epsilon_C C}, \tag{3.7}
$$

$$
e^{\lambda\cdot\hat{H}} = e^{\lambda_K K + \lambda_S{}^a{}_\alpha S^\alpha{}_a + \lambda_L{}^\beta{}_\alpha L^\alpha{}_\beta + \lambda_R{}^b{}_a R^a{}_b + \lambda_C C} \tag{3.8}
$$

by using (3.3) and (3.5). Here $\epsilon_L{}^\beta{}_\alpha, \epsilon_R{}^b{}_a, \lambda_L{}^\beta{}_\alpha, \lambda_R{}^b{}_a$ are traceless as were explained in (2.16). ϵ and λ are assumed to commute with all generators. Similaly to (2.17) we reconsider the transformation of the coordinates $\phi = (x, \theta^\alpha{}_a)$ as

$$
e^{\phi'(\phi,\epsilon)\cdot X} = e^{i\epsilon\cdot T} e^{\phi\cdot X} e^{-i\lambda(\phi,\epsilon)\cdot\hat{H}}. \tag{3.9}
$$

For $\epsilon \ll 1$, we can define the Killing vectors R^A as infinitesimal transformations as

$$
\delta\phi = \epsilon_A R^A \quad \Leftrightarrow \quad (\delta^A x, \delta^A \theta^\alpha{}_a) = (R^A, R^{A\alpha}{}_a).
$$

In the same argument from (2.19) to (2.24), we have

$$
\begin{aligned}
&\epsilon_A(R_{(0)}{}^A + R_{(1)}{}^A + \cdots + R_{(n)}{}^A + \cdots)\cdot\overline{X} + O(\epsilon^2) \\
&= i\sum_{n=0}^{\infty}\alpha_n(\mathrm{ad}\ \phi\cdot X)^n\epsilon\cdot T
\end{aligned}
$$

$$-i\sum_{n=0}^{\infty}(-1)^n\alpha_n(\mathrm{ad}\ \phi\cdot X)^n(\lambda_{(0)}+\lambda_{(1)}+\cdots+\lambda_{(n)}+\cdots)\cdot\hat{H}+O(\epsilon^2) \tag{3.10}$$

for ϕ,ϵ and λ in (3.9).

3.4 Calculation of the Killing vectors

We can calculate the Killing vectors R^A and λ similarly to the section **2.3**. We present the result following the same processes. Appendix **B** is helpful for the calculation.

0th order of ϕ Extracting the 0th-order terms from (3.10) we have

$$\epsilon_A R_{(0)}{}^A P+\epsilon_A R_{(0)}{}^{A\alpha}{}_a Q^a{}_\alpha = i\epsilon\cdot T - i\lambda_{(0)}\cdot\hat{H}. \tag{3.11}$$

Let us assume

$$\begin{aligned}\epsilon_A R_{(0)}{}^A &= i\epsilon_P,\\ \epsilon_A R_{(0)}{}^{A\alpha}{}_a &= i\epsilon_Q{}^\alpha{}_a\end{aligned} \tag{3.12}$$

as the initial condition. We find $\lambda_{(0)}$:

$$\begin{aligned}\lambda_{(0)K} &= \epsilon_K,\\ \lambda_{(0)S}{}^a{}_\alpha &= \epsilon_S{}^a{}_\alpha,\\ \lambda_{(0)L}{}^\beta{}_\alpha &= \epsilon_L{}^\beta{}_\alpha,\\ \lambda_{(0)R}{}^b{}_a &= \epsilon_R{}^b{}_a,\\ \lambda_{(0)C} &= \epsilon_C.\end{aligned} \tag{3.13}$$

1st order of ϕ Taking out the 1st-order terms from (3.10) we have

$$-i\Big(\varepsilon_A R_{(1)}{}^A P+\varepsilon_A R_{(1)}{}^{A\alpha}{}_a Q^a{}_\alpha\Big)= -\frac{1}{2}[\phi\cdot X,\varepsilon\cdot T]-\lambda_{(1)}\cdot\hat{H}-\frac{1}{2}[\phi\cdot X,\lambda_{(0)}\cdot\hat{H}].$$

We replace $\lambda_{(0)}$ by (3.13) and calculate the commutator by using (B.1). The r.h.s. becomes

$$\begin{aligned}&-\lambda_{(1)}\cdot\hat{H}\\ &-\Big[xP+\theta^\alpha{}_a Q^a{}_\alpha,\frac{1}{2}\varepsilon_P P+\frac{1}{2}\varepsilon_Q{}^\beta{}_b Q^b{}_\beta+\varepsilon_K K+\varepsilon_S{}^b{}_\beta S^\beta{}_b+\varepsilon_L{}^\gamma{}_\beta L^\beta{}_\gamma+\varepsilon_R{}^c{}_b R^b{}_c+\varepsilon_C C\Big]\\ =&-\Big(\lambda_{(1)K}K+\lambda_{(1)S}{}^a{}_\alpha S^\alpha{}_a+\lambda_{(1)L}{}^\beta{}_\alpha L^\alpha{}_\beta+\lambda_{(1)R}{}^b{}_a R^a{}_b+\lambda_{(1)C}C\Big)\\ &-\Big\{\Big(-x\epsilon_C+\frac{\gamma}{2}\theta^\alpha{}_a\epsilon_Q{}^\beta{}_b\varepsilon_{\alpha\beta}\varepsilon^{ab}\Big)P+\Big(-x\epsilon_S{}^b{}_\beta\varepsilon_{ab}\varepsilon^{\alpha\beta}-\theta^\beta{}_a\epsilon_L{}^\alpha{}_\beta+\theta^\alpha{}_b\epsilon_R{}^b{}_a-\frac{1}{2}\theta^\alpha{}_a\epsilon_C\Big)Q^a{}_\alpha\\ &-\theta^\alpha{}_a\epsilon_K\varepsilon_{\alpha\beta}\varepsilon^{ab}S^\beta{}_b+\alpha\theta^\alpha{}_a\epsilon_S{}^a{}_\beta L^\beta{}_\alpha-\beta\theta^\alpha{}_a\epsilon_S{}^b{}_\alpha R^a{}_b+(\gamma\theta^\alpha{}_a\epsilon_S{}^a{}_\alpha-2x\epsilon_K)C\Big\}.\end{aligned}$$

We find ${R_{(1)}}^A$:

$$\epsilon_A {R_{(1)}}^A = i\left(x\epsilon_C - \frac{\gamma}{2}\theta^\alpha{}_a \epsilon_Q{}^\beta{}_b \varepsilon_{\alpha\beta}\varepsilon^{ab}\right),$$

$$\begin{aligned}\epsilon_A {R_{(1)}}^{A\alpha}{}_a &= i\left(x\epsilon_S{}^b{}_\beta \varepsilon_{ab}\varepsilon^{\alpha\beta} + \theta^\beta{}_a \epsilon_L{}^\alpha{}_\beta - \theta^\alpha{}_b \epsilon_R{}^b{}_a + \frac{1}{2}\theta^\alpha{}_a \epsilon_C\right)\\ &\equiv i\left\{x\epsilon_S{}^b{}_\beta \varepsilon_{ab}\varepsilon^{\alpha\beta} + (\theta\epsilon_L)^\alpha{}_a - (\theta\epsilon_R)^\alpha{}_a + \frac{1}{2}\theta^\alpha{}_a \epsilon_C\right\}\end{aligned} \tag{3.14}$$

and $\lambda_{(1)}$:

$$\begin{aligned}\lambda_{(1)K} &= 0,\\ \lambda_{(1)S}{}^a{}_\alpha &= \theta^\beta{}_b \epsilon_K \varepsilon_{\alpha\beta}\varepsilon^{ab},\\ \lambda_{(1)L}{}^\beta{}_\alpha &= -\alpha\left(\theta^\alpha{}_a \epsilon_S{}^a{}_\beta - \frac{1}{2}\delta^\beta{}_\alpha \theta^\gamma{}_c \epsilon_S{}^c{}_\gamma\right) \equiv -\alpha\left\{(\theta\epsilon_S)^\beta{}_\alpha - \frac{1}{2}\delta^\beta{}_\alpha(\theta\epsilon_S)\right\},\\ \lambda_{(1)R}{}^b{}_a &= \beta\left(\theta^\alpha{}_a \epsilon_S{}^b{}_\alpha - \frac{1}{2}\delta^b{}_a \theta^\gamma{}_c \epsilon_S{}^c{}_\gamma\right) \equiv \beta\left\{(\theta\epsilon_S)^b_a - \frac{1}{2}\delta^b{}_a(\theta\epsilon_S)\right\},\\ \lambda_{(1)C} &= -\gamma\theta^\alpha{}_a \epsilon_S{}^a{}_\alpha + 2x\epsilon_K \equiv -\gamma(\theta\epsilon_S) + 2x\epsilon_K.\end{aligned} \tag{3.15}$$

Hereafter we use abbreviation such that

$$\theta^\beta{}_a \epsilon_L{}^\alpha{}_\beta \equiv (\theta\epsilon_L)^\alpha{}_a, \quad \theta^\alpha{}_b \epsilon_R{}^b{}_a \equiv (\theta\epsilon_R)^\alpha{}_a, \quad \theta^\alpha{}_a \epsilon_S{}^a{}_\alpha = (\theta\epsilon_S), \quad \cdots \quad \text{etc.}$$

As has been done for (2.30), the last terms of $\lambda_{(1)L}{}^\beta{}_\alpha$ and $\lambda_{(1)R}{}^b{}_a$ were added by considering the traceless conditions.

2nd order of ϕ Extracting the 2nd-order terms from (3.10) we have

$$\begin{aligned}&-i\Big(\epsilon_A {R_{(2)}}^A P + \epsilon_A {R_{(2)}}^{A\alpha}{}_a Q^a{}_\alpha\Big)\\ &\quad=\frac{1}{12}[\phi\cdot X,[\phi\cdot X,\epsilon\cdot T]] - \lambda_{(2)}\cdot\hat{H} - \frac{1}{2}[\phi\cdot X, \lambda_{(1)}\cdot\hat{H}] - \frac{1}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(0)}\cdot\hat{H}]].\end{aligned}$$

We replaced $\lambda_{(0)}$ by (3.13) and calculate the commutators by using (B.2) and (B.7). The r.h.s. becomes

$$\begin{aligned}&-\lambda_{(2)}\cdot\hat{H} + \frac{1}{12}[xP + \theta^\alpha{}_a Q^a{}_\alpha,[xP + \theta^\beta{}_b Q^b{}_\beta, \epsilon_P P + \epsilon_Q{}^\gamma{}_c Q^c{}_\gamma]]\\ &-\frac{1}{2}[xP + \theta^\alpha{}_a Q^a{}_\alpha, \lambda_{(1)K}K + \lambda_{(1)S}{}^b{}_\beta S^\beta{}_b + \lambda_{(1)L}{}^\gamma{}_\beta L^\beta{}_\gamma + \lambda_{(1)R}{}^c{}_b R^b{}_c + \lambda_{(1)C}C]\\ =&-\lambda_{(2)}\cdot\hat{H}\\ &-\frac{1}{2}\Big\{-x\lambda_{(1)C}P + \left(-x\lambda_{(1)S}{}^b{}_\beta \varepsilon_{ab}\varepsilon^{\alpha\beta} - \theta^\beta{}_a \lambda_{(1)L}{}^\alpha{}_\beta + \theta^\alpha{}_b \lambda_{(1)R}{}^b{}_a - \frac{1}{2}\theta^\alpha{}_a \lambda_{(1)C}\right) Q^a{}_\alpha\\ &-\theta^\alpha{}_a \lambda_{(1)K}\varepsilon_{\alpha\beta}\varepsilon^{ab} S^\beta{}_b + \alpha\theta^\beta{}_a \lambda_{(1)S}{}^a{}_\alpha L^\alpha{}_\beta - \beta\theta^\alpha{}_a \lambda_{(1)S}{}^b{}_\alpha R^a{}_b + \Big(\gamma\theta^\alpha{}_a \lambda_{(1)S}{}^a{}_\alpha - 2x\lambda_{(1)K}\Big) C\Big\}.\end{aligned}$$

We find ${R_{(2)}}^A$:

$$\begin{aligned}
\epsilon_A {R_{(2)}}^A =& \frac{i}{2} x \lambda_{(1)C} = \frac{i}{2}\left(2x\epsilon_K - \gamma\theta\epsilon_S\right)x, \\
\epsilon_A {R_{(2)}}^{A\alpha}{}_a =& -\frac{i}{2}\left(-x{\lambda_{(1)S}}^b{}_\beta \varepsilon_{ab}\varepsilon^{\alpha\beta} - \theta^\beta{}_a {\lambda_{(1)L}}^\alpha{}_\beta + \theta^\alpha{}_b {\lambda_{(1)R}}^b{}_a - \frac{1}{2}\theta^\alpha{}_a \lambda_{(1)C}\right) \\
=& -\frac{i}{2}\left\{-2x\theta^\alpha{}_a\epsilon_K + (\alpha-\beta)\theta^\beta{}_a\theta^\alpha{}_b {\epsilon_S}^b{}_\beta - \frac{1}{2}(\alpha+\beta-\gamma)\theta^\alpha{}_a(\theta\epsilon_S)\right\}
\end{aligned} \tag{3.16}$$

and $\lambda_{(2)}$:

$$\begin{aligned}
\lambda_{(2)K} &= 0, \\
{\lambda_{(2)S}}^a{}_\alpha &= \frac{1}{2}\theta^\beta{}_b \lambda_{(1)K}\varepsilon_{\alpha\beta}\varepsilon^{ab} = 0, \\
{\lambda_{(2)L}}^\beta{}_\alpha &= -\frac{\alpha}{2}\theta^\beta{}_a {\lambda_{(1)S}}^a{}_\alpha = -\frac{\alpha}{2}\theta^\beta{}_a\theta^\gamma{}_c\epsilon_K\varepsilon_{\alpha\gamma}\varepsilon^{ac}, \\
{\lambda_{(2)R}}^b{}_a &= \frac{\beta}{2}\theta^\alpha{}_a {\lambda_{(1)S}}^b{}_\alpha = \frac{\beta}{2}\theta^\alpha{}_a\theta^\gamma{}_c\epsilon_K\varepsilon_{\alpha\gamma}\varepsilon^{bc}, \\
\lambda_{(2)C} &= -\frac{1}{2}\left(\gamma\theta^\alpha{}_a {\lambda_{(1)S}}^a{}_\alpha - 2x\lambda_{(1)K}\right) = -\frac{\gamma}{2}\theta^\alpha{}_a\theta^\beta{}_b\epsilon_K\varepsilon_{\alpha\beta}\varepsilon^{ab} = 0.
\end{aligned} \tag{3.17}$$

We used anti-commutativity of $\theta^\alpha{}_a$ for the last equation.

3rd order of ϕ Taking out the 3rd-order terms from (3.10) we have

$$\begin{aligned}
&-i\Big(\epsilon_A {R_{(3)}}^A P + \epsilon_A {R_{(3)}}^{A\alpha}{}_a Q^a{}_\alpha\Big) \\
&\qquad = -\lambda_{(3)}\cdot\hat{H} - \frac{1}{2}[\phi\cdot X, \lambda_{(2)}\cdot\hat{H}] - \frac{1}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(1)}\cdot\hat{H}]].
\end{aligned}$$

Calculate the commutators by using (B.2) and (B.3). We then replace $\lambda_{(2)}$ by (3.17) and $\lambda_{(1)}$ by (3.15). Noting $(\theta\lambda_{(1)S}) = 0$ in (B.3), the r.h.s. becomes

$$\begin{aligned}
&-\lambda_{(3)}\cdot\hat{H} - \frac{1}{2}\left(-\theta^\beta{}_a {\lambda_{(2)L}}^\alpha{}_\beta + \theta^\alpha{}_b {\lambda_{(2)R}}^b{}_a\right)Q^a{}_\alpha \\
&-\frac{1}{12}\Bigg\{\gamma\theta^\alpha{}_a\left(-x{\lambda_{(1)S}}^c{}_\gamma\varepsilon_{cb}\varepsilon^{\gamma\beta} - \theta^\gamma{}_b {\lambda_{(1)L}}^\beta{}_\gamma + \theta^\beta{}_c {\lambda_{(1)R}}^c{}_b - \frac{1}{2}\theta^\beta{}_b\lambda_{(1)C}\right)\varepsilon_{\alpha\beta}\varepsilon^{ab}P \\
&-\left(\alpha\theta^\beta{}_a\theta^\alpha{}_b {\lambda_{(1)S}}^b{}_\beta + \beta\theta^\alpha{}_b\theta^\beta{}_a {\lambda_{(1)S}}^b{}_\beta\right)Q^a{}_\alpha\Bigg\} \\
=& -\lambda_{(3)}\cdot\hat{H} - \frac{\gamma}{12}\theta^\alpha{}_a\left\{-2x\theta^\beta{}_b\epsilon_K + (\alpha-\beta)\theta^\gamma{}_b\theta^\beta{}_c {\epsilon_S}^c{}_\gamma - \frac{1}{2}(\alpha+\beta-\gamma)\theta^\beta{}_b(\theta\epsilon_S)\right\}\varepsilon_{\alpha\beta}\varepsilon^{ab}P \\
&+\frac{1}{6}(\alpha-\beta)\theta^\alpha{}_b\theta^\beta{}_a {\lambda_{(1)S}}^b{}_\beta Q^a{}_\alpha.
\end{aligned}$$

Using the notation $\theta^\alpha{}_a\theta^\beta{}_b\varepsilon_{\alpha\beta}\varepsilon^{ab}=0$ we find ${R_{(3)}}^A$:

$$\begin{aligned}\epsilon_A{R_{(3)}}^A =& -\frac{i}{12}\gamma(\alpha-\beta)\theta^\alpha{}_a\theta^\gamma{}_b\theta^\beta{}_c\epsilon_S{}^c{}_\gamma\varepsilon_{\alpha\beta}\varepsilon^{ab},\\ \epsilon_A{R_{(3)}}^{A\alpha}{}_a =& \frac{i}{6}(\alpha-\beta)\theta^\alpha{}_b\theta^\beta{}_a\theta^\gamma{}_c\epsilon_K\varepsilon_{\beta\gamma}\varepsilon^{bc}\end{aligned}\tag{3.18}$$

and $\lambda_{(3)}$:

$$\lambda_{(3)}=0.\tag{3.19}$$

4th order of $\boldsymbol{\phi}$ Extracting the 4th-order terms from (3.10) we have

$$\begin{aligned}-i\Big(&\epsilon_A{R_{(4)}}^AP+\epsilon_A{R_{(4)}}^{A\alpha}{}_aQ^a{}_\alpha\Big)\\ &=-\frac{1}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\epsilon\cdot T]]]]-\lambda_{(4)}\cdot\hat{H}-\frac{1}{2}[\phi\cdot X,\lambda_{(3)}\cdot\hat{H}]\\ &\quad-\frac{1}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(2)}\cdot\hat{H}]]+\frac{1}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(0)}\cdot\hat{H}]]]].\end{aligned}$$

Replace $\lambda_{(3)}$ by (3.19) and $\lambda_{(0)}$ by (3.13). We then calculate the commutators by using (B.3) and (B.8). The r.h.s. becomes

$$\begin{aligned}&-\lambda_{(4)}\cdot\hat{H}-\frac{1}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(2)}\cdot\hat{H}]]-\frac{1}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\epsilon\cdot X]]]]\\ =&-\lambda_{(4)}\cdot\hat{H}-\frac{\gamma}{12}\theta^\alpha{}_a(-\theta^\gamma{}_b\lambda_{(2)L}{}^\beta{}_\gamma+\theta^\beta{}_c\lambda_{(2)R}{}^c{}_b)\varepsilon_{\alpha\beta}\varepsilon^{ab}P.\end{aligned}$$

We find ${R_{(4)}}^A$:

$$\begin{aligned}\epsilon_A{R_{(4)}}^A =& -\frac{i}{24}\gamma(\alpha-\beta)\theta^\alpha{}_a\theta^\gamma{}_b\theta^\beta{}_c\theta^\delta{}_d\epsilon_K\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\varepsilon^{ab}\varepsilon^{cd},\\ \epsilon_A{R_{(4)}}^{A\alpha}{}_a =& 0\end{aligned}\tag{3.20}$$

and $\lambda_{(4)}$:

$$\lambda_{(4)}=0.\tag{3.21}$$

5th order of $\boldsymbol{\phi}$ Extracting the 5th-order terms from (3.10) we have

$$\begin{aligned}-i\Big(\epsilon_A{R_{(5)}}^AP+\epsilon_A{R_{(5)}}^{A\alpha}{}_aQ^a{}_\alpha\Big)=&-\lambda_{(5)}\cdot\hat{H}-\frac{1}{2}[\phi\cdot X,\lambda_{(4)}\cdot\hat{H}]-\frac{1}{12}[\phi\cdot X,[\phi\cdot X,\lambda_{(3)}\cdot\hat{H}]]\\ &+\frac{1}{720}[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda_{(1)}\cdot\hat{H}]]]]=0.\end{aligned}$$

Replace each $\lambda_{(n)}$ by (3.21), (3.19), (3.15) and calculate the commutators by using (B.5). We find $R^A_{(5)}$:

$$\begin{aligned}\epsilon_A{R_{(5)}}^A =& 0,\\ \epsilon_A{R_{(5)}}^{A\alpha}{}_a =& 0\end{aligned}\tag{3.22}$$

and $\lambda_{(5)}$:

$$\lambda_{(5)} = 0. \tag{3.23}$$

More than 5th order of ϕ The Killing vectors ${R_{(n)}}^A$ for $n \geq 5$ are vanishing because

$$\begin{aligned} \lambda_{(n)} = 0 \qquad &\text{for} \quad n \geq 3, \\ \underbrace{[X, [X, \cdots [X}_{n}, T] \cdots]] = 0 \qquad &\text{for} \quad n \geq 5. \end{aligned} \tag{3.24}$$

To summarize, we obtain the Killing vectors R^A on D$(2, 1; \gamma)/\{$SU(2)$\otimes$SU(2)$\otimes$U(1)$\}$:

$$\begin{aligned} -i\epsilon_A R^A =& \epsilon_P + \left(x\epsilon_C - \frac{\gamma}{2}\theta^\alpha{}_a {\epsilon_Q}^\beta{}_b \varepsilon_{\alpha\beta}\varepsilon^{ab} \right) + \frac{1}{2}\Big(2\epsilon_K x - \gamma\theta\epsilon_S \Big) x \\ & - \frac{1}{12}\gamma(\alpha - \beta)\theta^\alpha{}_a \theta^\gamma{}_b \theta^\beta{}_c {\epsilon_S}^c{}_\gamma \varepsilon_{\alpha\beta}\varepsilon^{ab} \\ & - \frac{1}{24}\gamma(\alpha - \beta)\theta^\alpha{}_a \theta^\gamma{}_b \theta^\beta{}_c \theta^\delta{}_d \epsilon_K \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\varepsilon^{ab}\varepsilon^{cd}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} -i\epsilon_A R^{A\alpha}{}_a =& {\epsilon_Q}^\alpha{}_a + \left\{ x{\epsilon_S}^b{}_\beta \varepsilon_{ab}\varepsilon^{\alpha\beta} + (\theta\epsilon_L)^\alpha{}_a - (\theta\epsilon_R)^\alpha{}_a + \frac{1}{2}\theta^\alpha{}_a \epsilon_C \right\} \\ & - \frac{1}{2}\left\{ -2x\theta^\alpha{}_a \epsilon_K + (\alpha - \beta)\theta^\beta{}_a \theta^\alpha{}_b {\epsilon_S}^b{}_\beta - \frac{1}{2}(\alpha + \beta - \gamma)\theta^\alpha{}_a (\theta\epsilon_S) \right\} \\ & + \frac{1}{6}(\alpha - \beta)\theta^\alpha{}_b \theta^\beta{}_a \theta^\gamma{}_c \epsilon_K \varepsilon_{\beta\gamma}\varepsilon^{bc} \end{aligned} \tag{3.26}$$

and λ:

$$\lambda_K = \epsilon_K, \tag{3.27}$$

$$\lambda_S{}^a{}_\alpha = \epsilon_S{}^a{}_\alpha + \theta^\beta{}_b \epsilon_K \varepsilon_{\alpha\beta}\varepsilon^{ab}, \tag{3.28}$$

$$\lambda_L{}^\beta{}_\alpha = \epsilon_L{}^\beta{}_\alpha - \alpha\left\{ (\theta\epsilon_S)^\beta{}_\alpha - \frac{1}{2}\delta^\beta{}_\alpha(\theta\epsilon_S) + \frac{1}{2}\theta^\beta{}_a \theta^\gamma{}_c \epsilon_K \varepsilon_{\alpha\gamma}\varepsilon^{ac} \right\}, \tag{3.29}$$

$$\lambda_R{}^b{}_a = \epsilon_R{}^b{}_a + \beta\left\{ (\theta\epsilon_S)^b_a - \frac{1}{2}\delta^b{}_a(\theta\epsilon_S) + \frac{1}{2}\theta^\alpha{}_a \theta^\gamma{}_c \epsilon_K \varepsilon_{\alpha\gamma}\varepsilon^{bc} \right\}, \tag{3.30}$$

$$\lambda_C = \epsilon_C - \gamma(\theta\epsilon_S) + 2x\epsilon_K. \tag{3.31}$$

3.5 The contraction to PSU$(2|2)\otimes$U$(1)^3$

As discussed in the section **3.1**, D$(2, 1; \gamma)$ contains PSU$(2|2)$ as a maximal subgroup. By rescaling (3.4) as $(C, P, K) \to 1/\gamma(C, P, K)$ and then calculating the limit as γ approaches

0, we can obtain the centrally extended algebrae of PSU(2|2)⊗U(1)^3 such as

$$[L^\alpha{}_\beta, L^\gamma{}_\delta] = -\delta^\gamma{}_\beta L^\alpha{}_\delta + \delta^\alpha{}_\delta L^\gamma{}_\beta, \qquad [R^a{}_b, R^c{}_d] = -\delta^c{}_b R^a{}_d + \delta^a{}_d R^c{}_b,$$

$$[Q^a{}_\alpha, L^\beta{}_\gamma] = -\delta^\beta{}_\alpha Q^a{}_\gamma + \frac{1}{2}\delta^\beta{}_\gamma Q^a{}_\alpha, \qquad [S^\alpha{}_a, L^\beta{}_\gamma] = \delta^\alpha{}_\gamma S^\beta{}_a - \frac{1}{2}\beta^\beta{}_\gamma S^\alpha{}_a,$$

$$[Q^a{}_\alpha, R^b{}_c] = \delta^a{}_c Q^b{}_\alpha - \frac{1}{2}\delta^b{}_c Q^a{}_\alpha, \qquad [S^\alpha{}_a, R^b{}_c] = -\delta^b{}_a S^\alpha{}_c + \frac{1}{2}\delta^b{}_c S^\alpha{}_a,$$

$$[L^\alpha{}_\beta, C] = 0, \qquad [R^a{}_b, C] = 0,$$

$$[P, C] = 0, \qquad [K, C] = 0,$$

$$[P, K] = 0,$$

$$[Q^a{}_\alpha, C] = 0, \qquad [S^\alpha{}_a, C] = 0,$$

$$[Q^a{}_\alpha, P] = 0, \qquad [S^\alpha{}_a, P] = 0,$$

$$[Q^a{}_\alpha, K] = 0, \qquad [S^\alpha{}_a, K] = 0,$$

$$\left\{Q^a{}_\alpha, Q^b{}_\beta\right\} = \varepsilon_{\alpha\beta}\varepsilon^{ab}P, \qquad \left\{S^\alpha{}_a, S^\beta{}_b\right\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}K,$$

$$\left\{Q^a{}_\alpha, S^\beta{}_b\right\} = \alpha\delta^a{}_b L^\beta{}_\alpha - \beta\delta^\beta{}_\alpha R^a{}_b + \delta^\beta{}_\alpha\delta^a{}_b C$$

with $\alpha + \beta = 0$. Here C, P and K are central charges of three U(1)s. By this contraction, SU(2) generated by $\dot{L}^{\dot\alpha}{}_{\dot\beta}$ is broken into U(1)^3 [4][5].

4 Conclusion

In this letter we have calculated completely the Killing vectors on PSU(2|2)/{SU(2)⊗U(1)} and $\mathrm{D}(2,1;\gamma)$/{SU(2)⊗SU(2)⊗U(1)} which were Kählerian. The Killing vectors were found by purely algebraic use of (2.11) or (3.4) with the initial condition of (2.26) or (3.12). It is worth remarking that they were given as polynomials of the holomorphic coordinates ϕs due to the nilpotency (3.24).

It is expected that a non-linear σ-model on G/H is equivalent to a spin-chain with the symmetry of G. The work by Beisert [4] showed that the spin-chain with the symmetry of PSU(2|2)⊗U(1)^3 is equivalent to the $\mathcal{N} = 4$ SUSY Yang-Mills theory. It is then natural to think that the non-linear σ-model on PSU(2|2)⊗U(1)^3/H is equivalent to that spin-chain, and consequently to the $\mathcal{N} = 4$ SUSY Yang-Mills theory. However the coset space G/H is not well-defined for a non-simple G such as PSU(2|2)⊗U(1)^3. To overcome the difficulty and claim the equivalence of the two theories, in [2] PSU(2|2)⊗U(1)^3/H was discussed in a limit of symmetry contraction of an enlarged coset space $\mathrm{D}(2,1;\gamma)$/H. The study on $\mathrm{D}(2,1;\gamma)$/H with H=SU(2)⊗SU(2)⊗U(1) in this paper took a crucially important part in the work [2].

Acknowledgement

The author would like to thank S.Aoyama for discussions. The author was excited about a journey into the physics with him.

A Commutation relations of PSU$(2|2)$

We show the result of commutators by (2.11).

$$
\begin{aligned}
[\phi\cdot X,\epsilon\cdot T] &= \Big[zL+\theta_{a\dot\alpha}F^{a\dot\alpha},\epsilon L+\epsilon_{Fb\dot\beta}F^{b\dot\beta}+\epsilon_{\overline{L}}\overline{L}+\epsilon_{\overline{F}b\dot\beta}\overline{F}^{b\dot\beta}+\epsilon_{L^0}L^0+\epsilon_{R}{}^{c}{}_{b}R^{b}{}_{c}\Big]\\
&=[zL,\epsilon_{\overline{L}}\overline{L}]+[zL,\epsilon_{\overline{F}b\dot\beta}\overline{F}^{b\dot\beta}]+[zL,\epsilon_{L^0}L^0]+\Big\{\theta_{a\dot\alpha}F^{a\dot\alpha},\epsilon_{Fb\dot\beta}F^{b\dot\beta}\Big\}+[\theta_{a\dot\alpha}F^{a\dot\alpha},\epsilon_{\overline{L}}\overline{L}]\\
&\quad+\Big\{\theta_{a\dot\alpha}F^{a\dot\alpha},\epsilon_{\overline{F}b\dot\beta}\overline{F}^{b\dot\beta}\Big\}+[\theta_{a\dot\alpha}F^{a\dot\alpha},\epsilon_{L^0}L^0]+[\theta_{a\dot\alpha}F^{a\dot\alpha},\epsilon_{R}{}^{b}{}_{c}R^{c}{}_{b}]\\
&=\Big(-z\epsilon_{L^0}-\theta_{a\dot\alpha}\epsilon_{Fb\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}\Big)L+\Big(-z\epsilon_{\overline{F}a\dot\alpha}-\tfrac{1}{2}\theta_{a\dot\alpha}\epsilon_{L^0}+\theta_{b\dot\alpha}\epsilon_{R}{}^{b}{}_{a}\Big)F^{a\dot\alpha}\\
&\quad+\theta_{a\dot\alpha}\epsilon_{\overline{L}}\overline{F}^{a\dot\alpha}+\Big(2z\epsilon_{\overline{L}}-\theta_{a\dot\alpha}\epsilon_{\overline{F}b\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}\Big)L^0\\
&\quad+\Big(\theta_{c\dot\alpha}\epsilon_{\overline{F}a\dot\beta}\varepsilon^{cb}\varepsilon^{\dot\alpha\dot\beta}-\tfrac{1}{2}\delta^{b}{}_{a}\theta_{c\dot\alpha}\epsilon_{\overline{F}d\dot\beta}\varepsilon^{cd}\varepsilon^{\dot\alpha\dot\beta}\Big)R^{a}{}_{b},
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
[\phi\cdot X,\lambda\cdot\hat H] &= -z\lambda_{L^0}L+\Big(-z\lambda_{\overline{F}a\dot\alpha}-\tfrac{1}{2}\theta_{a\dot\alpha}\lambda_{L^0}+\theta_{b\dot\alpha}\lambda_{R}{}^{b}{}_{a}\Big)F^{a\dot\alpha}+\theta_{a\dot\alpha}\lambda_{\overline{L}}\overline{F}^{a\dot\alpha}\\
&\quad+\Big(2z\lambda_{\overline{L}}-\theta_{a\dot\alpha}\lambda_{\overline{F}b\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}\Big)L^0+\Big(\theta_{c\dot\alpha}\lambda_{\overline{F}a\dot\beta}\varepsilon^{cb}\varepsilon^{\dot\alpha\dot\beta}-\tfrac{1}{2}\delta^{b}{}_{a}\theta_{c\dot\alpha}\lambda_{\overline{F}d\dot\beta}\varepsilon^{cd}\varepsilon^{\dot\alpha\dot\beta}\Big)R^{a}{}_{b}\\
&\equiv\lambda'_{L}L+\lambda'_{Fa\dot\alpha}F^{a\dot\alpha}+\lambda'_{\overline{F}a\dot\alpha}\overline{F}^{a\dot\alpha}+\lambda'_{L^0}L^0+\lambda'_{R}{}^{b}{}_{a}R^{a}{}_{b},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
[\phi\cdot X,[\phi\cdot X,\lambda\cdot\hat H]] &= -\Big(z\lambda'_{L^0}+\theta_{a\dot\alpha}\lambda'_{Fb\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}\Big)L-\Big(z\lambda'_{Fa\dot\alpha}+\theta_{a\dot\alpha}\lambda'_{Fb\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}-\theta_{b\dot\alpha}\lambda'_{R}{}^{b}{}_{a}\Big)F^{a\dot\alpha}\\
&\quad-\theta_{a\dot\alpha}\lambda'_{\overline{F}b\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}L^0+\theta_{a\dot\alpha}\lambda'_{\overline{F}b\dot\beta}\varepsilon^{ac}\varepsilon^{\dot\alpha\dot\beta}R^{b}{}_{c}\\
&=\Big(-2z^2\lambda_{\overline{L}}+2z\theta_{a\dot\alpha}\lambda_{\overline{F}b\dot\beta}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}-\theta_{a\dot\alpha}\theta_{c\dot\beta}\lambda_{R}{}^{c}{}_{b}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}\Big)L\\
&\quad+\Big(-2z\theta_{a\dot\alpha}\lambda_{\overline{L}}+\theta_{b\dot\alpha}\theta_{c\dot\beta}\lambda_{Fa\dot\gamma}\varepsilon^{cb}\varepsilon^{\dot\beta\dot\gamma}\Big)F^{a\dot\alpha}-\theta_{a\dot\alpha}\theta_{b\dot\beta}\lambda_{\overline{L}}\varepsilon^{ab}\varepsilon^{\dot\alpha\dot\beta}L^0\\
&\quad-\theta_{c\dot\alpha}\theta_{a\dot\beta}\lambda_{\overline{L}}\varepsilon^{cb}\varepsilon^{\dot\alpha\dot\beta}R^{a}{}_{b}\\
&\equiv\lambda''_{L}L+\lambda''_{Fa\dot\alpha}F^{a\dot\alpha}+\lambda''_{R}{}^{b}{}_{a}R^{a}{}_{b},
\end{aligned}
\tag{A.3}
$$

$$[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda\cdot\hat{H}]]] = -\theta_{a\dot{\alpha}}\lambda''_{Fb\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L + \theta_{b\dot{\alpha}}\lambda''_{R}{}^{b}{}_{a}F^{a\dot{\alpha}}$$
$$= \theta_{a\dot{\alpha}}\theta_{d\dot{\beta}}\theta_{c\dot{\delta}}\lambda_{Fb\dot{\gamma}}\varepsilon^{cd}\varepsilon^{\dot{\delta}\dot{\gamma}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L - \theta_{b\dot{\alpha}}\theta_{c\dot{\gamma}}\theta_{a\dot{\beta}}\lambda_{\overline{L}}\varepsilon^{bc}\varepsilon^{\dot{\beta}\dot{\gamma}}F^{a\dot{\alpha}}$$
$$\equiv \lambda'''_{L}L + \lambda'''_{Fa\dot{\alpha}}F^{a\dot{\alpha}}, \tag{A.4}$$

$$[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda\cdot\hat{H}]]]] = -\theta_{a\dot{\alpha}}\lambda'''_{Fb\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L, \tag{A.5}$$

$$[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\lambda\cdot\hat{H}]]]]] = 0, \tag{A.6}$$

$$[\phi\cdot X,[\phi\cdot X,\epsilon\cdot X]] = [zL + \theta_{a\dot{\alpha}}F^{a\dot{\alpha}},[zL + \theta_{b\dot{\beta}}F^{b\dot{\beta}},\epsilon_L L + \epsilon_{Fc\dot{\gamma}}F^{c\dot{\gamma}}]] = 0, \tag{A.7}$$

$$[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,[\phi\cdot X,\epsilon\cdot X]]]] = 0. \tag{A.8}$$

Note that appropriate terms were added in (A.1) and (A.2) because of the constraints $\mathrm{tr}\epsilon_R = \mathrm{tr}\lambda_R = 0$. To obtain the last line in (A.3), we used $\theta_{a\dot{\alpha}}\theta_{b\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}} = 0$.

B Commutation relations of $\mathbf{D}(2,1;\gamma)$

We show the result of commutators by (3.4). These contractions mean $(\theta\epsilon_L)^{\alpha}{}_{a} = \theta^{\beta}{}_{a}\epsilon_L{}^{\alpha}{}_{\beta}$, $(\theta\epsilon_R)^{\alpha}{}_{a} = \theta^{\alpha}{}_{b}\epsilon_R{}^{b}{}_{a}$, $(\theta\epsilon_S) = \theta^{\alpha}{}_{a}\epsilon_S{}^{a}{}_{\alpha}$ for examples.

$$[\phi\cdot X,\epsilon\cdot T]$$
$$= \Big[xP + \theta^{\alpha}{}_{a}Q^{a}{}_{\alpha}, \epsilon_P P + \epsilon_Q{}^{\beta}{}_{b}Q^{b}{}_{\beta} + \epsilon_K K + \epsilon_S{}^{b}{}_{\beta}S^{\beta}{}_{b} + \epsilon_L{}^{\beta}{}_{\alpha}L^{\alpha}{}_{\beta} + \epsilon_R{}^{b}{}_{c}R^{c}{}_{b} + \epsilon_C C\Big]$$
$$= \Big(\gamma\theta^{\alpha}{}_{a}\epsilon_Q{}^{\beta}{}_{b}\varepsilon_{\alpha\beta}\varepsilon^{ab} - x\epsilon_C\Big)P + \Big\{-x\epsilon_S{}^{b}{}_{\beta}\varepsilon_{ba}\varepsilon^{\beta\alpha} - (\theta\epsilon_L)^{\alpha}{}_{a} + (\theta\epsilon_R)^{\alpha}{}_{a} - \tfrac{1}{2}\epsilon_C\theta^{\alpha}{}_{a}\Big\}Q^{a}{}_{\alpha}$$
$$-\epsilon_K\theta^{\beta}{}_{b}\varepsilon_{\alpha\beta}\varepsilon^{ab}S^{\alpha}{}_{a} + \alpha\Big\{(\theta\epsilon_S)^{\beta}{}_{\alpha} - \tfrac{1}{2}\delta^{\beta}{}_{\alpha}(\theta\epsilon_S)\Big\}L^{\alpha}{}_{\beta} - \beta\Big\{(\theta\epsilon_S)^{b}{}_{a} - \tfrac{1}{2}\delta^{b}{}_{a}(\theta\epsilon_S)\Big\}R^{a}{}_{b}$$
$$+\Big\{\gamma(\theta\epsilon_S) - 2x\epsilon_K\Big\}C, \tag{B.1}$$

$$[\phi\cdot X,\lambda\cdot\hat{H}]$$
$$= -x\lambda_C P + \Big\{-x\lambda_S{}^{b}{}_{\beta}\varepsilon_{ba}\varepsilon^{\beta\alpha} - (\theta\lambda_L)^{\alpha}{}_{a} + (\theta\lambda_R)^{\alpha}{}_{a} - \tfrac{1}{2}\lambda_C\theta^{\alpha}{}_{a}\Big\}Q^{a}{}_{\alpha}$$
$$-\theta^{\beta}{}_{b}\lambda_K\varepsilon_{\alpha\beta}\varepsilon^{ab}S^{\alpha}{}_{a} + \alpha\Big\{(\theta\lambda_S)^{\beta}{}_{\alpha} - \tfrac{1}{2}\delta^{\beta}{}_{\alpha}(\theta\lambda_S)\Big\}L^{\alpha}{}_{\beta} - \beta\Big\{(\theta\lambda_S)^{b}{}_{a} - \tfrac{1}{2}\delta^{b}{}_{a}(\theta\lambda_S)\Big\}R^{a}{}_{b}$$
$$+\Big\{\gamma(\theta\lambda_S) - 2x\lambda_K\Big\}C$$

$$\equiv \lambda'_P P + \lambda'_Q{}^{\alpha}{}_a Q^a{}_\alpha + \lambda'_S{}^a{}_\alpha S^\alpha{}_a + \lambda'_L{}^\beta{}_\alpha L^\alpha{}_\beta + \lambda'_R{}^b{}_a R^a{}_b + \lambda'_C C, \tag{B.2}$$

$$[\phi \cdot X, [\phi \cdot X, \lambda \cdot \hat{H}]]$$
$$= \left(\gamma\theta^\alpha{}_a \lambda'_Q{}^\beta{}_b \varepsilon_{\alpha\beta}\varepsilon^{ab} - x\lambda'_C\right) P + \left(-x\lambda'_S{}^b{}_\beta \varepsilon_{ba}\varepsilon^{\beta\alpha} - \theta^\gamma{}_a \lambda'_L{}^\alpha{}_\gamma + \theta^\alpha{}_b \lambda'_R{}^b{}_a - \tfrac{1}{2}\theta^\alpha{}_a \lambda'_C\right) Q^a{}_\alpha$$
$$+ \theta^\alpha{}_a \lambda'_S{}^b{}_\beta \left(\alpha\delta^a{}_b L^\beta{}_\alpha - \beta\delta^\beta{}_\alpha R^a{}_b + \gamma\delta^\beta{}_\alpha \delta^a{}_b C\right)$$
$$= \left(\gamma\theta^\alpha{}_a \left\{-x\lambda_S{}^c{}_\gamma \varepsilon_{cb}\varepsilon^{\gamma\beta} - (\theta\lambda_L)^\beta{}_b + (\theta\lambda_R)^\beta{}_b - \tfrac{1}{2}\lambda_C \theta^\beta{}_b\right\} \varepsilon_{\alpha\beta}\varepsilon^{ab} - x\left\{\gamma(\theta\lambda_S) - 2x\lambda_K\right\}\right) P$$
$$+ \left\{x\lambda_K \theta^\gamma{}_c \varepsilon_{\gamma\beta}\varepsilon_{ba}\varepsilon^{\beta\alpha}\varepsilon^{bc} + \theta^\alpha{}_a x\lambda_K - \alpha\theta^\beta{}_a (\theta\lambda_S)^\alpha{}_\beta - \beta\theta^\alpha{}_b (\theta\lambda_S)^b{}_a + \tfrac{1}{2}(\alpha+\beta-\gamma)\theta^\alpha{}_a (\theta\lambda_S)\right\} Q^a{}_\alpha$$
$$- \alpha\lambda_K \theta^\beta{}_a \theta^\gamma{}_c \varepsilon_{\gamma\alpha}\varepsilon^{ac} L^\alpha{}_\beta + \beta\lambda_K \theta^\alpha{}_a \theta^\gamma{}_c \varepsilon_{\gamma\alpha}\varepsilon^{bc} R^a{}_b - \gamma\lambda_K \theta^\alpha{}_a \theta^\beta{}_b \varepsilon_{\beta\alpha}\varepsilon^{ab} C$$
$$\equiv \lambda''_P P + \lambda''_Q{}^\alpha{}_a Q^a{}_\alpha + \lambda''_L{}^\beta{}_\alpha L^\alpha{}_\beta + \lambda''_R{}^b{}_a R^a{}_b, \tag{B.3}$$

$$[\phi \cdot X, [\phi \cdot X, [\phi \cdot X, \lambda \cdot \hat{H}]]]$$
$$= \left(\gamma\theta^\alpha{}_a \lambda''_Q{}^\beta{}_b \varepsilon_{\alpha\beta}\varepsilon^{ab}\right) P + \left(-\theta^\gamma{}_a \lambda''_L{}^\alpha{}_\gamma + \theta^\alpha{}_b \lambda''_R{}^b{}_a\right) Q^a{}_\alpha \equiv \lambda'''_P P + \lambda'''_Q{}^\alpha{}_a Q^a{}_\alpha, \tag{B.4}$$

$$[\phi \cdot X, [\phi \cdot X, [\phi \cdot X, [\phi \cdot X, \lambda \cdot \hat{H}]]]] = \gamma\theta^\alpha{}_a \lambda'''_Q{}^\beta{}_b \varepsilon_{\alpha\beta}\varepsilon^{ab} P, \tag{B.5}$$

$$[\phi \cdot X, [\phi \cdot X, [\phi \cdot X, [\phi \cdot X, [\phi \cdot X, \lambda \cdot \hat{H}]]]]] = 0, \tag{B.6}$$

$$[\phi \cdot X, [\phi \cdot X, \epsilon \cdot X]] = [xP + \theta^\alpha{}_a Q^a{}_\alpha, [xP + \theta^\beta{}_b Q^b{}_\beta, \epsilon_P P + \epsilon_Q{}^\gamma{}_c Q^c{}_\gamma]] = 0, \tag{B.7}$$

$$[\phi \cdot X, [\phi \cdot X, [\phi \cdot X, [\phi \cdot X, \epsilon \cdot X]]]] = 0. \tag{B.8}$$

Similarly to the calculations of **A**, the coefficients of $L^\alpha{}_\beta$ and $R^a{}_b$ were constrained by traceless conditions so that appropriate terms were put additionally in (B.1) and (B.2). To obtain the last line in (B.3), we used $\theta^\alpha{}_a \theta^\beta{}_b \varepsilon_{\alpha\beta}\varepsilon^{ab} = 0$.

References

[1] K. Itoh, T.Kugo, H.Kunitomo, "Supersymmetric nonlinear realization for arbitrary Kählerian coset space G/H", Nucl. Phys. B **263** (1986) 295.

[2] S. Aoyama, Y. Honda, "Spin-chain with PSU(2|2)⊗U(1)^3 and non-linear σ-model with D$(2,1;\gamma)$", Phys. Lett. B **743** (2015) 531, arXiv:1502.03684[hep-th].

[3] S. Aoyama, "PSU(2, 2|4) exchange algebra of $\mathcal{N} = 4$ superconformal multiplets", arXiv:1412.7808[hep-th].

[4] N. Beisert, “The $\mathfrak{su}(2|2)$ dynamic S-matrix”, Adv. Theor. Math. Phys. **12** (2008) 945, arXiv:hep-th/0511082.

[5] T. Matsumoto, S, Moriyama, “An exceptional algebraic origin of the AdS/CFT Yangian symmetry”, JHEP **0804** (2008) 022, arXiv:0803.1212[hep-th].