On local Lp−Lq well-posedness of incompressible two phase flows with phase transitions

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THESIS

On local $L_p - L_q$ well-posedness of incompressible two phase flows with phase transitions

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THESIS

On local $L_p - L_q$ well-posedness of incompressible two phase flows with phase transitions
相転移を伴う非圧縮性2相流の
局所 $L_p - L_q$ 適切性について

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1. Introduction of The Case of non Equal Densities

In this paper, we consider incompressible two-phase flows with phase transitions in $\mathbb{R}^n$ with initial interface is nearly flat. Let

$$\Omega_\pm(t) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ \pm(x_n - h(t, x')) > 0, \ t \geq 0\},$$

and $\Omega(t) = \Omega_-(t) \cup \Omega_+(t)$. A nearly flat interface represented as a graph over $\mathbb{R}^{n-1}$ is given by

$$\Gamma(t) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ x_n - h(t, x') = 0, \ t \geq 0\}.$$

Let $\rho_\pm > 0$ denote the densities of $\Omega_\pm(t)$. From section 1 to section 12, we consider the case of non-equal densities $\rho_+ \neq \rho_-$. In order to economize our notation, we set

$$\rho = \begin{cases} \rho_+ & \text{in } \Omega_+(t) \\ \rho_- & \text{in } \Omega_-(t). \end{cases}$$

Let u denote the velocity vector field, π the pressure field and θ the absolute temperature field. $T(u, \pi, \theta)$ the stress tensor defined by

$$T(u, \pi, \theta) = 2\mu(\theta)D(u) - \pi I,$$

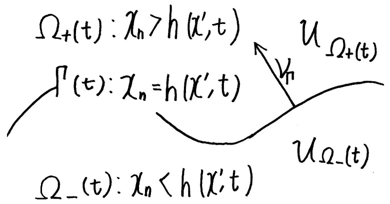
$D(u) = (\nabla u + [\nabla u]^\mathsf{T})/2$ denotes the rate of deformation tensor, $\mu_\pm(\theta) > 0$ the viscosity, I the unit matrix. $\nu_\Gamma = (-\nabla' h, 1)/(|\nabla' h|^2 + 1)^{1/2}$ the outer normal of Ω_+, u_Γ the interface velocity, $V_\Gamma = u_\Gamma \cdot \nu_\Gamma$ the normal velocity of $\Gamma(t)$, $H_\Gamma = H(\Gamma(t)) = \mathrm{div}_\Gamma \nu_\Gamma = \nabla' \cdot \nu_\Gamma$ the curvature of $\Gamma(t)$, $\sigma > 0$ the constant coefficient of surface tension. j , and

$$[\![u]\!] = \big(u|_{\Omega_+(t)} - u|_{\Omega_-(t)}\big)\big|_{\Gamma(t)}$$

denote the jump of a quantity u across $\Gamma(t)$. We define the phase flux j by

$$j = \rho_+(u_+ - u_\Gamma) \cdot \nu_\Gamma = \rho_-(u_- - u_\Gamma) \cdot \nu_\Gamma,$$

because balance of mass across $\Gamma(t)$ requires $[\![\rho(u - u_\Gamma)]\!] \cdot \nu_\Gamma = 0$ (cf. [14, Section 2]).



In the problem without phase transitions, it holds that $V_\Gamma = u \cdot \nu_\Gamma$ i.e. $j = 0$. This means that the normal velocity of $\Gamma(t)$, V_Γ is determined by the only velocity, u. In the problem with phase transitions, V_Γ isn't determined by the only u, hence $j \neq 0$.

Several quantities are derived from the specific *free energy* $\psi_\pm(\theta)$ in phase $\Omega_\pm(t)$ as follows.

- $\epsilon_\pm(\theta) := \psi_\pm(\theta) + \theta\eta_\pm(\theta)$ the internal energy,
- $\eta_\pm(\theta) := -\psi'_\pm(\theta)$ the entropy,
- $\kappa_\pm(\theta) := \epsilon'_\pm(\theta) = -\theta\psi''_\pm(\theta) > 0$ the heat capacity,
- $l(\theta) := \theta[\![\psi'(\theta)]\!] = -\theta[\![\eta(\theta)]\!]$ the latent heat.

Further $d_\pm(\theta) > 0$ denotes the coefficient of heat conduction in Fourier's law. In order to economize our notation, we set

$$d(\theta) = \begin{cases} d_+(\theta) & \text{in } \Omega_+(t) \\ d_-(\theta) & \text{in } \Omega_-(t). \end{cases}$$

We just keep in mind that the coefficients depend on the phases.

We find a family of hypersurfaces $\{\Gamma(t)\}_{t\geq 0}$ and appropriately smooth functions $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, and $\pi, \theta : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ such that

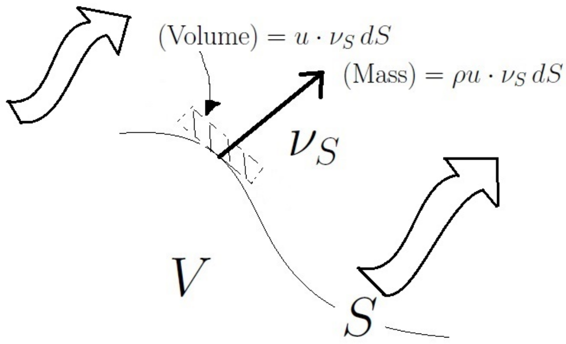
$$\begin{aligned} \rho(\partial_t u + u\cdot\nabla u) - \operatorname{div} T(u,\pi,\theta) &= 0 && \text{in } \ \Omega(t),\ t>0, \\ \operatorname{div} u &= 0 && \text{in } \ \Omega(t),\ t>0, \\ [\![\frac{1}{\rho}]\!] j^2\nu_\Gamma - [\![T(u,\pi,\theta)\nu_\Gamma]\!] - \sigma H_\Gamma \nu_\Gamma &= 0 && \text{on } \ \Gamma(t),\ t>0, \\ [\![u]\!] - [\![\frac{1}{\rho}]\!] j\nu_\Gamma &= 0 && \text{on } \ \Gamma(t),\ t>0, \\ u(0) &= u_0 && \text{in } \ \Omega(t), \end{aligned} \tag{1.1}$$

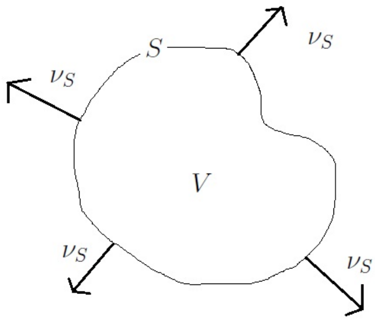
$$\begin{aligned} \rho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) - 2\mu(\theta)|D(u)|_2^2 &= 0 && \text{in } \ \Omega(t),\ t>0, \\ l(\theta)j + [\![d(\theta)\partial_{\nu_\Gamma}\theta]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0, \\ [\![\theta]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0, \\ \theta(0) &= \theta_0 && \text{in } \ \mathbb{R}^n, \end{aligned} \tag{1.2}$$

$$\begin{aligned} [\![\psi(\theta)]\!] + [\![\frac{1}{2\rho^2}]\!] j^2 - [\![\frac{T(u,\pi,\theta)\nu_\Gamma\cdot\nu_\Gamma}{\rho}]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0, \\ V_\Gamma - u\cdot\nu_\Gamma + \frac{1}{\rho}j &= 0 && \text{on } \ \Gamma(t),\ t>0, \\ \Gamma(0) &= \Gamma_0, \end{aligned} \tag{1.3}$$

where $|D|_2^2 = \Sigma_{i,j=1}^n d_{ij}^2$ and $\operatorname{div} D = (\Sigma_{j=1}^n \partial_j d_{1j}, \cdots, \Sigma_{j=1}^n \partial_j d_{nj})^T$ for an $n\times n$ matrix D whose (i,j) element is d_{ij}. The problem is called an incompressible two-phase flow with phase transitions. Here we remark that finding a family of hypersurfaces $\{\Gamma(t)\}_{t\geq 0}$ is equivalent to finding a family of $\{\Omega(t)\}_{t\geq 0}$.

We explain the model concisely. Let V, S and ν_S be any fixed bounded domain in $\Omega_\pm(t)$, the smooth boundary and the outer normal of S, respectively, and suppose $S \cap \Gamma(t) = \emptyset$.





Observing the volume flowing from V in unit time through S, we have:

$$\int\!\!\int_S \rho u \cdot \nu_S \, dS = -\frac{\partial}{\partial t} \int\!\!\int\!\!\int_V \rho \, dV,$$

so it holds that

$$\int\!\!\int_S \rho u \cdot \nu_S \, dS = -\frac{\partial}{\partial t} \int\!\!\int\!\!\int_V \rho \, dV = -\int\!\!\int\!\!\int_V \frac{\partial}{\partial t} \rho \, dV.$$

By Gauss formula, we gain

$$\int\!\!\int_S \rho u \cdot \nu_S \, dS = \int\!\!\int\!\!\int_V \mathrm{div}(\rho u) \, dV,$$

hence we observe

$$\int\!\!\int\!\!\int_V (\partial_t \rho + \mathrm{div}\,(\rho u)) \, dV = 0.$$

From voluntariness of V, the equation:

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \quad \text{in } \Omega(t), \ t > 0$$

denotes balance of mass in $\Omega(t)$. The second equation of (1.1):

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t),\ t > 0$$

stands for the case where ρ is a constant i.e. we consider incompressible flows. Next, we write equations that mean balance of momentum:

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div} T(u,\pi,\theta) &= 0 && \text{in } \Omega(t),\ t > 0, \\ [\![\rho u \otimes (u - u_\Gamma) - T(u,\pi,\theta)]\!]\nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma && \text{on } \Gamma(t),\ t > 0, \end{aligned}$$

where $\otimes$ means tensor product i.e. $a \otimes b = ab^T$ for $a,\ b \in \mathbb{R}^n$. In case ρ is a constant,

$$\begin{aligned} \operatorname{div}(\rho u \otimes u) &= \rho(\Sigma_{j=1}^n \partial_j(u_1 u_j), \cdots, \Sigma_{j=1}^n \partial_j(u_n u_j))^T \\ &= \rho(\Sigma_{j=1}^n (\partial_j u_1)u_j + u_1 \operatorname{div} u, \cdots, \Sigma_{j=1}^n (\partial_j u_n)u_j + u_n \operatorname{div} u)^T \\ &= \rho u \cdot \nabla u, \end{aligned}$$

so

$$\rho(\partial_t u + u \cdot \nabla u) - \operatorname{div} T(u,\pi,\theta) = \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div} T(u,\pi,\theta) = 0$$

in $\Omega(t)$. Utilizing $j = \rho(u - u_\Gamma) \cdot \nu_\Gamma$ i.e. $[\![1/\rho]\!]j = [\![u \cdot \nu_\Gamma]\!]$, we have

$$[\![\rho u \otimes (u - u_\Gamma)]\!]\nu_\Gamma = [\![\rho(u - u_\Gamma) \cdot \nu_\Gamma u]\!] = [\![u]\!]j = [\![\frac{1}{\rho}]\!]j^2 \nu_\Gamma,$$

therefore it holds that

$$[\![\frac{1}{\rho}]\!]j^2 \nu_\Gamma - [\![T(u,\pi,\theta)\nu_\Gamma]\!] - \sigma H_\Gamma \nu_\Gamma = 0$$

on $\Gamma(t)$. This model is explained in more detail in Prüss-Shibata-Shimizu-Simonett [14] (cf. [1], [2], [3], [7], [8], [9], [10], [12], [13]). It is in some sense the simplest sharp interface model for incompressible Newtonian two-phase flows taking into account phase transitions driven by temperature.

Note that in the case of equal densities, the phase flux j does not enter (1.1) because $[\![1/\rho]\!] = 0$, and so in this case we obtain essentially a Stefan problem with surface tension, which is only weakly coupled to the standard two-phase Navier-Stokes problem via temperature dependent viscosities. We call this case *temperature dominated*, and it has been studied in [14] and Section 11. But in the case of different densities, the phase flux j causes a jump in the velocity field on the interface, which leads to so called *Stefan currents* which are convections driven by phase transitions. In this situation it turns out that the heat problem (1.2) is only weakly coupled to (1.1) and (1.3), we call this case *velocity dominated*. The resulting two-phase Navier-Stokes problem is non-standard, therefore it requires a new analysis, and it has been studied in Prüss and Shimizu [15] in L_p-setting in time and space.

The aim of section 1-12 is to prove local L_p-L_q well-posedness of the problem of (1.1) (1.2) (1.3) in the case of non-equal densities and an initial interface which is nearly flat.

We set $\Omega_0 = \Omega(0)$ and $\Gamma_0 = \Gamma(0)$. The main result of this paper is the local-wellposedness of (1.1) (1.2) (1.3) L_p in time L_q in space setting.

Theorem 1.1. *Let $p < \infty$, $n < q < \infty$, $2/p + n/q < 1$ and $\rho_+ \neq \rho_-$, and suppose $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ are such that*

$$\kappa_\pm(s) = -s\psi_\pm''(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0 \quad s \in (0,\infty).$$

Let the initial interface Γ_0 be given by a graph $x' \mapsto (x', h_0(x'))$, $\theta_\infty > 0$ be the constant temperature at infinity. And let

$$(u_0, \theta_0, h_0) \in B^{2-2/p}_{q,p}(\Omega_0)^n \times B^{2-2/p}_{q,p}(\Omega_0) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

be given. Assume that the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = 0 \quad &\text{in } \Omega_0,\\ P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] = 0, \quad P_{\Gamma_0}[\![u_0]\!] = 0 \quad &\text{on } \Gamma_0,\\ [\![\theta_0]\!] = 0, \quad (l(\theta_0)/[\![1/\rho]\!])[\![u_0 \cdot \nu_0]\!] + [\![d(\theta_0)\partial_{\nu_0}\theta_0]\!] = 0 \quad &\text{on } \Gamma_0, \end{aligned}$$

where $P_{\Gamma_0} = I - \nu_{\Gamma_0} \otimes \nu_{\Gamma_0}$ denotes the projection onto the tangent bundle of Γ_0. Then there exists a constant ε_0 depending only on Ω_0, p, q, n such that if h_0 and u_0 satisfy $\|\nabla' h_0\|_{L_\infty(\dot{\mathbb{R}}^n)} + \|u_0\|_{L_\infty(\Omega_0)} \leq \varepsilon_0$, then there exist

$$T = T(\|\theta_0 - \theta_\infty\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}, \varepsilon_0) > 0$$

and a unique L_p-L_q solution (u, π, θ, h) of (1.1)-(1.3) on $[0,T]$ in the class of (2.8) below.

Remark 1.2.

(1) The notion of L_p-L_q-solution is explained in more detail in Section 5.

(2) In Prüss-Shimizu [15], they considered the same problem when $p = q$ and proved local well-posedness in L_p-setting when $n+2 < p < \infty$. Our result may treat the case when $p < \infty$, $n < q < \infty$ and $2/p + n/q < 1$, which covers wider range than the results of [15]. Indeed, if $n + 2 < q < \infty$,

$$\begin{aligned} n + 2 - \frac{2q}{q-n} &= \frac{(n+2)(q-n) - 2q}{q-n}\\ &= \frac{qn - n^2 - 2n}{q-n}\\ &= \frac{(q-n-2)n}{q-n} > 0. \end{aligned}$$

Thus, we know that

$$2q/(q-n) < p \leq n+2$$

is permitted and if $n + 2 < p < \infty$, then $q = n+2$ is permitted.

(3) The restriction of expornents of p, q comes from using the following embedding relations to treat nonlinear terms. When $n < q < \infty$, it holds that

$$W^1_q(\mathbb{R}^n_\pm) \hookrightarrow L_\infty(\mathbb{R}^n_\pm).$$

When $2 < p < \infty$, it holds that

$$B^{2-2/p}_{q,p}(\mathbb{R}^n_\pm) \hookrightarrow W^1_q(\mathbb{R}^n_\pm).$$

Let $J = [0, T]$. When $1 < p,\ q < \infty$, it holds that

$$W^1_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^2_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; B^{2-2/p}_{q,p}(\mathbb{R}^n_\pm)),$$

and when $n < q < \infty$ and $2/p + n/q < 1$, it holds that

$$H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; BUC(\mathbb{R}^n_\pm))$$

(cf. Lemma 5.2, below).

(4) The smallness condition $\|u_0\|_{L_\infty(\Omega_0)} \leq \varepsilon_0$ comes from the nonlinear terms $u_n \nabla' h$ and $u' \cdot \nabla' h$ in the fourth equation of (2.4) and the second equation of (2.6), respectively.

2. Linearized Problem

Let $\mathbb{R}^n_0 = \mathbb{R}^{n-1} \times \{0\}$ and $\dot{\mathbb{R}}^n = \mathbb{R}^n \backslash \mathbb{R}^n_0$. We use contraction mapping principle in order to prove Theorem 1.1.

Theorem 2.1. (contraction mapping principle) Let X be a Banach space and S be a closed subset in X. If a map, $\Phi : S \to S$ is a contraction map i.e. there exists $\rho\,(0 \leq \rho < 1)$ and

$$\|\Phi(u) - \Phi(\upsilon)\|_X \leq \rho \|u - \upsilon\|_X \qquad \text{for u}, \upsilon \in \text{S},$$

there is a unique fixed point of Φ in S.

Here, we mention semi-group theory.

Theorem 2.2. (semi-group and $L_p - L_q$ estimate)

$$\begin{cases} \partial_t u - \Delta u = f & t > 0, \\ u(0) = u_0. \end{cases} \tag{2.1}$$

We could give the unique solution of (2.1)*, u as*

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(s)\, dt$$

$e^{t\Delta} g$ satisfies the following: for any $1 \leq p \leq q < \infty,\ g \in L_q(\mathbb{R}^n)$ and multi-index α,

$$\|\partial_t^k \partial_x^\alpha e^{t\Delta} g\|_{L_p(\mathbb{R}^n)} \leq C t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - k - \frac{|\alpha|}{2}} \|g\|_{L_q(\mathbb{R}^n)} \quad \text{for } t > 0,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n),\ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,\ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \alpha_{x_n}^{\alpha_n}$.

We deal with the next quasi-linear problem:

$$\begin{cases} \partial_t u - \Delta u = P(u, \nabla u, \nabla^2 u) & t > 0, \\ u(0) = u_0. \end{cases} \tag{2.2}$$

The formula:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} P(u, \nabla u, \nabla^2 u)(s)\, ds$$

is called mild solution. Defining $\Phi(u)$ as $\Phi(u) = \int_0^t e^{(t-s)\Delta} P(u, \nabla u, \nabla^2 u)(s)\, ds$, we could write

$$u(t) = e^{t\Delta}u_0 + \Phi(u).$$

We would like estimates of $\Phi(u)$, $\nabla\Phi(u)$ and $\nabla^2\Phi(u)$ to prove contract of $\Phi(s)$. By Theorem 2.2,

$$\|\nabla^2\Phi(u)\|_{L_p(\mathbb{R}^n)} \leq C\int_0^t \frac{1}{t-s}\|P(u, \nabla u, \nabla^2 u)\|_{L_q(\mathbb{R}^n)}\, dt$$

The right hand side of this estimate has singularity at $t = 0$. This fact shows that it is not easy for us to solve (2.2) with contraction mapping principle and properties of semi-group. However, estimates of maximal L_p regularity is useful. Incidentally, we could solve semi-linear problem by properties of semi-group.

Definition 2.3. (Maximal L_p Regularity) Let X be a Banach space and A be a closed operator whose domain, $\mathcal{D}(A)$ is dense in X. We treat the following Cauchy problem:

$$\begin{cases} \partial_t u + Au = f & t > 0, \\ u(0) = 0, \end{cases} \tag{2.3}$$

We say that A has maximal regularity when (2.3) admits a unique solution, u for any $f \in L_p((0,T); X)\,(1 < p < \infty)$, where $0 < t < T \leq \infty$ and u satisfies the estimate:

$$\|\partial_t u\|_{L_p((0,T);X)} + \|Au\|_{L_p((0,T);X)} \leq C\|f\|_{L_p((0,T);X)},$$

where a positive constant, C is independent of f. Moreover, we know that an operator, A is a generator of semi-group, e^{-tA} if A has maximal regularity.

Changing variables of (1.1)-(1.3) with $y_n = x_n - h(x', t)$, we obtain the quasi-linear problem. Indeed, setting $u(x,t) = \upsilon(y,t)$, we obtain

$$\begin{aligned} &\upsilon(y,t) = \upsilon(x', x_n - h(x',t), t) = u(x,t), \quad \partial_k u = -(\partial_k h)\partial_n \upsilon + \partial_n \upsilon \\ &\partial_n u = \partial_n \upsilon, \quad \partial_k^2 u = \partial_k^2 \upsilon + (\partial_k h)^2 \partial_n^2 \upsilon - (\partial_k^2 h)\partial_n \upsilon - 2(\partial_k h)\partial_k \partial_n \upsilon \\ &\partial_n^2 u = \partial_n^2 \upsilon, \quad \partial_t u = -(\partial_t h)\partial_n \upsilon + \partial_t \upsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho\partial_t u - \mu\Delta u = \rho\partial_t \upsilon &- \mu\Delta\upsilon - \rho(\partial_t h)\partial_n \upsilon \\ &- \mu|\nabla' h|^2 \partial_n^2 \upsilon + \mu(\Delta' h)\partial_n \upsilon + 2\mu(\nabla' h)\cdot\nabla'\partial_n \upsilon. \end{aligned}$$

The principal part of the linearized problem in the case of a nearly flat initial interface reads as follows

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u \qquad \text{in } \dot{\mathbb{R}}^n, \quad t > 0, \\ \operatorname{div} u &= f_d \qquad \text{in } \dot{\mathbb{R}}^n, \quad t > 0, \end{aligned}$$

$$
\begin{aligned}
-2[\![\mu D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma\Delta' h\nu &= g_u && \text{on } \mathbb{R}^n_0, \quad t>0,\\
[\![u']\!] &= g && \text{on } \mathbb{R}^n_0, \quad t>0,\\
u(0) &= u_0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
\rho\kappa\partial_t\theta - d\Delta\theta &= f_\theta && \text{in } \dot{\mathbb{R}}^n, \quad t>0,\\
-[\![d\partial_\nu\theta]\!] &= g_\theta && \text{on } \mathbb{R}^n_0, \quad t>0,\\
[\![\theta]\!] &= 0 && \text{on } \mathbb{R}^n_0, \quad t>0,\\
\theta(0) &= \theta_0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
-2[\![\frac{\mu_0 D(u)\nu\cdot\nu}{\rho}]\!] + [\![\frac{\pi}{\rho}]\!] &= g_\pi && \text{on } \mathbb{R}^n_0, \quad t>0,\\
\partial_t h - [\![\rho u\cdot\nu]\!]/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^n_0, \quad t>0,\\
h(0) &= h_0 && \text{on } \mathbb{R}^n_0,
\end{aligned} \tag{2.6}
$$

where $\mu_\pm, \kappa_\pm, d_\pm, \rho_\pm$ are constants, $\nu = e_n = (0,\cdots,0,1)$. We assume as always in this paper $[\![\rho]\!] = \rho_+ - \rho_- \neq 0$. Apparently, (2.5) decouples from the remaining problem. Since it is well-known that this problem has maximal L_p-L_q-regularity (cf. Denk, Hieber and Prüss [4]), we concentrate on the remaining one. It reduces to the problem:

$$
\begin{aligned}
\rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u && \text{in } \dot{\mathbb{R}}^n, \quad t>0,\\
\operatorname{div} u &= f_d && \text{in } \dot{\mathbb{R}}^n, \quad t>0,\\
-2[\![\mu D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma\Delta' h\nu &= g_u && \text{on } \mathbb{R}^n_0, \quad t>0,\\
[\![u']\!] &= g && \text{on } \mathbb{R}^n_0, \quad t>0,\\
-2[\![\mu D(u)\nu\cdot\nu/\rho]\!] + [\![\pi/\rho]\!] &= g_\pi && \text{on } \mathbb{R}^n_0, \quad t>0,\\
\partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^n_0, \quad t>0,\\
u(0) &= 0 && \text{in } \dot{\mathbb{R}}^n,\\
h(0) &= 0 && \text{on } \mathbb{R}^n_0
\end{aligned} \tag{2.7}
$$

with positive constants, ρ, μ, σ and κ.

Remark 2.4. The system (2.7) is the different linear problem from two-phase Stokes problem without phase transitions analyzed by Prüss-Simonett [17, 18], Shibata-Shimizu [23], and Kohne-Prüss-Wilke [11].

We set

$$
\begin{aligned}
\hat{W}^1_q(\mathbb{R}^n) &= \{\theta \in L_{q,loc}(\mathbb{R}^n) \mid \nabla\theta \in L_q(\mathbb{R}^n)^n\},\\
L_{p,0,\gamma_0}(\mathbb{R};X) &= \{f:\mathbb{R}\to X \mid e^{-\gamma_0 t}f(t)\in L_p(\mathbb{R};X),\ f(t)=0 \text{ for } t<0\},\\
W^m_{p,0,\gamma_0}(\mathbb{R};X) &= \{f\in L_{p,0,\gamma_0}(\mathbb{R};X) \mid e^{-\gamma_0 t}D^j_t f(t)\in L_p(\mathbb{R};X),\ j=1,\cdots,m\},
\end{aligned}
$$

for $a \in \mathbb{R}$,

$$
\Lambda^a_\gamma f(t) = \mathcal{L}^{-1}[|s|^a \mathcal{L}[f](s)](t)
$$

$$H^a_{p,0,\gamma_0}(\mathbb{R};X) = \{f : \mathbb{R} \to X \mid e^{-\gamma t}\Lambda^a_\gamma f(t) \in L_p(\mathbb{R};X)$$
$$\text{for any } \gamma \geq \gamma_0,\ f(t) = 0 \text{ for } t < 0\}$$

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ are Laplace transform and its inverse respectively, and set $\hat{W}^{-1}_q(\mathbb{R}^n)$ the dual space of $\hat{W}^1_{q'}(\mathbb{R}^n)$, where $1/q + 1/q' = 1$.

For problem (2.7), we have the following maximal L_p-L_q regularity result.

Theorem 2.5. *Let $1 < p, q < \infty$, and assume that σ, $\rho_\pm$, $\mu_\pm$ are positive constants and $\rho_+ \neq \rho_-$. Suppose the data $(f_u, f_d, g_u, g, g_\pi, g_h)$ satisfy the following regularity conditions:*

$$\begin{aligned}
&f_u \in L_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n))^n,\\
&f_d \in W^1_{p,0,\gamma_0}(\mathbb{R}_+; \hat{W}^{-1}_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^1_q(\dot{\mathbb{R}}^n)),\\
&g_u \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n))^n \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^1_q(\dot{\mathbb{R}}^n))^n,\\
&g \in W^1_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n))^{n-1} \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^2_q(\dot{\mathbb{R}}^n))^{n-1},\\
&g_\pi \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^1_q(\dot{\mathbb{R}}^n)),\\
&g_h \in W^1_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^2_q(\dot{\mathbb{R}}^n))
\end{aligned}$$

and compatibility conditions:

$$f_d(0) = P_{\mathbb{R}^{n-1}} g_u(0) = g(0) = 0 \quad \text{in } \dot{\mathbb{R}}^n,$$

where $P_{\mathbb{R}^{n-1}}$ denotes the projection onto $\mathbb{R}^{n-1}$. Then the asymmetric Stokes problem (2.7) admits a unique solution (u, π, h) with regularity

$$\begin{aligned}
&u \in W^1_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n))^n \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^2_q(\dot{\mathbb{R}}^n))^n,\\
&\pi \in L_{p,0,\gamma_0}(\mathbb{R}_+; \hat{W}^1_q(\dot{\mathbb{R}}^n)),\\
&\pi_\pm \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n_\pm)) \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^1_q(\mathbb{R}^n_\pm)),\\
&h \in W^1_{p,0,\gamma_0}(\mathbb{R}_+; W^2_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^3_q(\dot{\mathbb{R}}^n))\\
&\qquad\qquad \cap H^{3/2}_{p,0,\gamma_0}(\mathbb{R}_+; W^1_q(\dot{\mathbb{R}}^n)) \cap W^2_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n)),
\end{aligned}$$

The solution map $[(f_u, f_d, g_u, g, g_\pi,\ g_h) \mapsto (u, \pi, h)]$ is continuous between the corresponding spaces.

Here, compatibility conditions are necessary conditions that initial values should satisfy. For instance, $\partial_j u(x, 0) = 0\,(j = 1, \cdots, n)$ by $u(0) = u(x, 0) = 0$ for any $x \in \mathbb{R}$, therefore $f_d(0) = f_d(x, 0) = \operatorname{div} u(x, 0) = 0$ for any $x \in \mathbb{R}$.

If $0 < T < \infty$, it holds that

$$\begin{aligned}
\|f\|_{L_p(0,T;L_q(\mathbb{R}^n))} &= \Big(\int_0^T \|f(t)\|^p_{L_q(\mathbb{R}^n)}\,dt\Big)^{1/p}\\
&= \Big(\int_0^T \|e^{\gamma t}e^{-\gamma t} f(t)\|^p_{L_q(\mathbb{R}^n)}\,dt\Big)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq \Big(\int_0^T \Big(\sup_{t\in[0,T]} e^{\gamma t}\Big) \|e^{-\gamma t} f(t)\|^p_{L_q(\mathbb{R}^n)}\, dt\Big)^{1/p} \\
&= e^{\gamma T} \|e^{-\gamma t} f\|_{L_p(\mathbb{R}_+, L_q(\mathbb{R}))}
\end{aligned}$$

for some $\gamma > 0$. Hence, we may view the nonlinear problem in following spaces. Let $J = [0, T]$. We set the function spaces of the solution:

$$\begin{aligned}
\mathbb{E}_u(J) &:= (W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n))^n, \\
\mathbb{E}_\pi(J) &:= L_p(J; \hat{W}^1_q(\dot{\mathbb{R}}^n)), \\
\mathbb{E}_{\pi_\pm}(J) &:= H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm))^n \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n_\pm))^n, \\
\mathbb{E}_\theta(J) &:= W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n)), \\
\mathbb{E}_h(J) &:= W^1_p(J; W^2_q(\dot{\mathbb{R}}^n)) \cap L_p(J; W^3_q(\dot{\mathbb{R}}^n)) \\
&\qquad\qquad \cap H^{3/2}_p(J; W^1_q(\dot{\mathbb{R}}^n)) \cap W^2_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n)), \\
\mathbb{E}(J) &:= \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\pi_\pm}(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_h(J). \qquad (2.8)
\end{aligned}$$

We set the function spaces of right members:

$$\begin{aligned}
\mathbb{F}_u(J) &:= L_p(J; L_q(\mathbb{R}^n))^n, \\
\mathbb{F}_d(J) &:= W^1_p(J; \hat{W}^{-1}_q(\mathbb{R}^n)) \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n)), \\
\mathbb{F}_\theta(J) &:= L_p(J; L_q(\mathbb{R}^n)), \\
\mathbb{G}(J) &:= W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n)), \\
\mathbb{G}_u(J) &:= H^{1/2}_p(J; L_q(\mathbb{R}^n))^n \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n))^n, \\
\mathbb{G}_\theta(J) &:= H^{1/2}_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n)), \\
\mathbb{G}_\pi(J) &:= H^{1/2}_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n)), \\
\mathbb{G}_h(J) &:= W^1_p(J; L_q(\dot{\mathbb{R}}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n)), \\
\mathbb{F}(J) &:= \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{F}_\theta(J) \times \mathbb{G}_u(J) \times \mathbb{G}(J) \times \mathbb{G}_\theta(J) \times \mathbb{G}_\pi(J) \times \mathbb{G}_h(J).
\end{aligned}$$

We know that

$$\begin{aligned}
\mathbb{E}_u(J) &\hookrightarrow BUC(J; B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n))^n, \\
\mathbb{E}_\theta(J) &\hookrightarrow BUC(J; B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)), \\
W^1_p(J; W^2_q(\dot{\mathbb{R}}^n)) \cap L_p(J; W^3_q(\dot{\mathbb{R}}^n)) &\hookrightarrow BUC(J; B^{3-1/p}_{q,p}(\dot{\mathbb{R}}^n),
\end{aligned}$$

so we define the time trace space X_γ of $\mathbb{E}(J)$ as

$$X_\gamma = B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1}).$$

The main result which is maximal L_p-L_q regularity for linearized problem (2.4)-(2.6) is stated as follows.

Theorem 2.6. *Let $1 < p, q < \infty$, and assume that σ, $\rho_\pm$ $\mu_\pm$ are positive constants $\rho_+ \neq \rho_-$, and set $J = [0, T]$. If $(f_u, f_d, f_\theta, g_u, g, g_\theta, g_\pi, g_h) \in \mathbb{F}(J)$, and the initial data*

$$(u_0, \theta_0, h_0) \in X_\gamma = B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

satisfy the compatibility conditions:

$$\begin{aligned}
\operatorname{div} u_0 = f_d(0) \quad & \text{in } \dot{\mathbb{R}}^n, \qquad 2 - 2/p > 1 + 1/q, \\
-[\![\mu P_{\mathbb{R}^{n-1}} D(u_0)]\!] = P_{\mathbb{R}^{n-1}} g_u(0) \quad & \text{on } \mathbb{R}^{n-1}, \quad 2 - 2/p > 1 + 1/q, \\
[\![u_0']\!] = g(0), \quad [\![\theta_0]\!] = 0 \quad & \text{on } \mathbb{R}^{n-1}, \quad 2 - 2/p > 1/q, \\
-[\![d\partial_n \theta_0]\!] = g_\theta(0) \quad & \text{on } \mathbb{R}^{n-1}, \quad 2 - 2/p > 1 + 1/q,
\end{aligned}$$

then the linearized problem (2.4)-(2.6) *admits a unique solution* $(u, \pi, \pi_\pm, \theta, h) \in \mathbb{E}(J)$.

Theorem 2.6 is proved by combining Theorem 2.5 and the results within [11], [23] and [4]. Therefore it is key to prove Theorem 2.5.

The plan for this part is as follows. In Section 4, we prove Theorem 2.5, namely maximal L_p-L_q regularity of (2.7). Section 5 is devoted to prove local L_p-L_q well-posedness of the problem of (1.1) (1.2) (1.3). In Appendix, we calculate the explicit solution formula of (2.7).

3. $\mathcal{R}-$boundedness and Operator Valued Fourier Multiplier Theorem

This section is a quotation from Section 2 in [24]. Let X and Y be two Banach spaces whose norms are $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. $\mathcal{B}$ denote the set of all bounded linear operators from X into Y and $\mathcal{B}(X) = \mathcal{B}(X, X)$.

Definition 3.1. ([24, Definition 2.1]) A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}-$bounded, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for each $m \in \mathbb{N}$, $\mathbb{N}$ being the set of all natural numbers, $T_j \in \mathcal{T}$, $x_j \in X$ $(j = 1, \cdots, N)$ and for all sequence $\{r_j(u)\}_{j=1}^N$ of independent, symmetric, $\{-1, 1\}-$valued random variables on $[0, 1]$, there holds the inequality:

$$\int_0^1 \| \sum_{j=1}^N r_j(u) T_j(x_j) \|_Y^p \, du \le C \int_0^1 \| \sum_{j=1}^N r_j(u) x_j \|_X^p \, du. \tag{3.1}$$

The smallest such C is called $\mathcal{R}-$bound of $\mathcal{T}$, which is denoted by $\mathcal{R}(T)$.

Given $M \in L_{1,loc}(\mathbb{R}; \mathcal{B}(X, Y))$, let us define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to S'(\mathbb{R}, Y)$ by the formula:

$$T_M \phi = \mathcal{F}^{-1}[M \mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathcal{D}, X)) \tag{3.2}$$

We mention operator valued Fourier multiplier theorem by Weis under the definition $\mathcal{R}-$boundedness above.

Theorem 3.2. ([25], [24, Theorem 2.3]) *Let G be a domain in $\mathbb{R}^n$ and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R}\backslash\{0\}, \mathcal{B}(L_q(G)))$ such that*

$$\mathcal{R}(\{M(\rho) \mid \rho \in \mathbb{R}\backslash\{0\}\}) = \kappa_0 < \infty, \quad \mathcal{R}(\{\rho M'(\rho) \mid \rho \in \mathbb{R}\backslash\{0\}\}) = \kappa_1 < \infty.$$

Then, the operator T_M defined in (3.2) is extended to a bounded linear operator from $L_p(\mathbb{R}; L_q(G))$ into $L_p(\mathbb{R}; L_q(G))$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{B}(L_p(\mathbb{R};L_q(G)))} \leq C(\kappa_0 + \kappa_1)$$

for some positive constant C depending on p, q, G.

A sector $\Sigma_{\epsilon,\gamma}$ is defined as

$$\Sigma_{\epsilon,\gamma_0} = \{s \in \mathbb{C}\backslash\{0\} \mid |\arg s| \leq \pi - \epsilon, \ |s| \geq \gamma\}$$

From Theorem 3.2, we obtain the next theorem.

Theorem 3.3. ([24, Theorem 2.8]) *Let $1 < p, q < \infty$, $0 < \epsilon < \pi/2$ and $\gamma_0 \geq 0$. Let G be a domain in $\mathbb{R}^n$ and Φ_s be a function of $\tau \in \mathbb{R}\backslash\{0\}$ when $s = \gamma + i\tau \in \Sigma_{\epsilon,\gamma}$ with its value in $\mathcal{B}(L_q(G))$. Assume that the sets $\{\Phi_s \mid s \in \Sigma_{\epsilon,\gamma_0}\}$ and $\{\tau \frac{d}{d\tau}\Phi_s \mid s = \gamma + i\tau \in \Sigma_{\epsilon,\gamma_0}\}$ are $\mathcal{R}-$bounded families in $\mathcal{B}(L_q(G))$. In addition, we assume that there exists a constant M such that*

$$\mathcal{R}(\{\Phi_s \mid s \in \Sigma_{\epsilon,\gamma_0}\}) \leq M, \quad \mathcal{R}(\{\tau \frac{d}{d\tau}\Phi_s \mid s = \gamma + i\tau \in \Sigma_{\epsilon,\gamma_0}\}) \leq M.$$

Then, we have

$$\|\Phi_s f\|_{L_q(G)} \leq C_q M \|f\|_{L_q(G)} \quad (f \in L_q(G), \ p \in \Sigma\epsilon, \gamma_0)$$

for me constant C_q depending on q.

Moreover, if we define the operator Ψ of a function $f \in L_p(\mathbb{R}; L_q(G))$ by the formula:

$$\Psi f(x,t) = \mathcal{L}_s^{-1}[\Phi_s \mathcal{L}[f](s)](x,t) = e^{\gamma t}\mathcal{F}_\tau^{-1}[\Phi_s \mathcal{F}[e^{-\gamma t}f](\tau)](t) \tag{3.3}$$

where

$$\mathcal{F}[e^{-\gamma t}f](\tau) = \int_{-\infty}^{\infty} e^{-(\gamma+i\tau)t} f(x,t)\,dt,$$

then there exists a constant $C_{p,q}$ depending on p and q such that

$$\|e^{-\gamma t}\Psi f\|_{L_p(\mathbb{R};L_q(G))} \leq C_{p,q} M \|e^{-\gamma t} f\|_{L_p(\mathbb{R};L_q(G))}$$

for any $\gamma \geq \gamma_0$.

Thus, we may investigate whether or not solutions whose forms are (3.3) satisfy the condition of Theorem 3.3. However, it is not easy to understand the definition of $\mathcal{R}-$boundedness and we couldn't directly solve the linearized by Theorem 3.2, hence we use a useful lemma.

Lemma 3.4. ([24, Lemma 5.4]) *Let $0 < \epsilon < \pi/2$, $1 < q < \infty$ and $\gamma_0 \geq 0$. Suppose that m_1 and m_2 satisfy for $l = 0, 1$*

$$|D_{\xi'}^{\alpha'}(\tau D_\tau)^l m_1(s,\xi')| \leq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-|\xi'|},$$
$$|D_{\xi'}^{\alpha'}(\tau D_\tau)^l m_2(s,\xi')| \leq C_{\epsilon,\gamma_0} A^{-|\xi'|},$$

respectively. We define K_1, K_2 and K_3 for $s \in \Sigma_{\epsilon,\gamma_0}$ as

$$[K_1(s)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m_1(s,\xi')|s|^{1/2}e^{-B_\pm x_n}\hat{g}(\xi', y_n)](x')\,dy_n,$$
$$[K_2(s)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m_2(s,\xi')Ae^{-Ax_n}\hat{g}(\xi', y_n)](x')\,dy_n,$$
$$[K_3(s)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m_1(s,\xi')e^{-B_\pm x_n}\hat{g}(\xi', y_n)](x')\,dy_n.$$

Then, for $l = 0, 1$ and $i = 1, 2, 3$, the sets, $\{(\tau D_\tau)^l K_i(s) \mid s \in \Sigma_{\epsilon,\gamma_0}\}$ are $\mathcal{R}-$bounded families in $\mathcal{B}(L_q(\mathbb{R}^n_+))$.

By Lemma 3.4, we may discuss boundedness of functions for $\mathcal{R}$-boundedness of operators.

4. Maximal L_p-L_q Regularity; Proof of Theorem 2.5

In this section, we prove Theorem 2.5 by estimating explicit solution formula of (2.7). (2.7) is written by the following problem in the upper and the lower half spaces:

$$\begin{aligned}
\rho_\pm\partial_t u_\pm - \mu_\pm\Delta u_\pm + \nabla\pi_\pm &= f_u \quad \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
\operatorname{div} u_\pm &= f_d \quad \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
-2[\![\mu D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma(\Delta' h)\nu &= g_u \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
[\![u']\!] &= g \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
u_\pm(0) &= 0 \quad \text{in } \mathbb{R}^n_\pm,\\
-2[\![\mu D(u)\nu\cdot\nu/\rho]\!] + [\![\pi/\rho]\!] &= g_\pi \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
\partial_t h - [\![\rho u\cdot\nu]\!]/[\![\rho]\!] &= g_h \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
h_\pm(0) &= 0 \quad \text{on } \mathbb{R}^n_0,
\end{aligned} \tag{4.1}$$

where $\nu = e_n = (0,\cdots,0,1)^T$, $u_\pm = (u_{\pm 1},\cdots,u_{\pm n})^T$, $u' = (u_1,\cdots,u_{n-1})^T$ and $\rho_\pm$, $\mu_\pm$ and σ are positive constants.

If we set $u = \upsilon + w$ and $\pi = \tau + \kappa$ for a solution (u,π) of (4.1), then (υ,τ) and (w,κ) satisfy the following problems:

$$\begin{aligned}
\rho_\pm\partial_t \upsilon_\pm - \mu_\pm\Delta \upsilon_\pm + \nabla\tau_\pm &= f_u \quad \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
\operatorname{div} \upsilon_\pm &= f_d \quad \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
[\![\mu(\partial_n \upsilon_k + \partial_k \upsilon_n)]\!] &= -g_{u,k} \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
[\![2\mu\partial_n \upsilon_n]\!] - [\![\tau]\!] &= -g_{u,n} \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,
\end{aligned}$$

$$
\begin{aligned}
[\![v_k]\!] = g_k & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
v_\pm(0) = 0 & \quad \text{in } \mathbb{R}^n_\pm,\\
[\![(2\mu/\rho)\partial_n v_n]\!] - [\![\tau/\rho]\!] = -g_\pi & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0.
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\rho_\pm \partial_t w_\pm - \mu_\pm \Delta w_\pm + \nabla \kappa_\pm = 0 & \quad \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
\mathrm{div} w_\pm = 0 & \quad \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
[\![\mu(\partial_n w_k + \partial_k w_n)]\!] = 0 & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
[\![2\mu \partial_n w_n]\!] - [\![\kappa]\!] = -\sigma \Delta' h & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
[\![w_k]\!] = 0 & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
w_\pm(0) = 0 & \quad \text{in } \mathbb{R}^n_\pm,\\
[\![(2\mu/\rho)\partial_n w_n]\!] - [\![\kappa/\rho]\!] = 0 & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
\partial_t h - [\![\rho w_n]\!]/[\![\rho]\!] = g_h + [\![\rho v_n]\!]/[\![\rho]\!] & \quad \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
h_\pm(0) = 0 & \quad \text{on } \mathbb{R}^n_0.
\end{aligned} \tag{4.3}
$$

Let $\mathcal{F}_{x'}$ and $\mathcal{F}^{-1}_{\xi'}$ denote the partial Fourier transform with respect to x' and its inversion transform

$$
\mathcal{F}_{x'}[u(\cdot, x_n)](\xi') = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n)\ dx',
$$

$$
\mathcal{F}^{-1}_{\xi'}[u(\cdot, \xi_n)](x') = (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} u(\xi', \xi_n)\ d\xi',
$$

and let $\mathcal{L}_t$ and $\mathcal{L}^{-1}_s$ denote the Laplace transform and its inversion transform

$$
\mathcal{L}_t[u](s) = \int_{\mathbb{R}} e^{-st} u(t)\ dt, \quad \mathcal{L}^{-1}_s[u](t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{st} u(s)\ d\tau.
$$

We use the symbol: $\hat{u} = \mathcal{F}_{x'}\mathcal{L}_t[u]$. Set

$$
A = |\xi'|\ ,\ B_\pm = \sqrt{\frac{\rho_\pm}{\mu_\pm} s + A^2} \ \text{ with } \ \mathrm{Re}\, B_\pm > 0.
$$

First we solve (4.2). We could deduce the case where $f_u = f_d = 0$ in the problem (4.2) (e.g. Shibata and Shimizu [24, Section 3]). Using the Fourier transform with respect to x' and the Laplace transform with respect to t, we can convert the problem (4.2) into ordinary differential equations of x_n with $f_u = f_d = 0$;

$$
B^2_\pm \hat{v}_{\pm k} - \partial_n^2 \hat{v}_{\pm k} + (i\xi_k/\mu_\pm)\hat{\tau}_\pm = 0 \quad \text{in } \mathbb{R}^n_\pm, \tag{4.4}
$$

$$
B^2_\pm \hat{v}_{\pm n} - \partial_n^2 \hat{v}_{\pm n} + \mu_\pm^{-1}\partial_n \hat{\tau}_\pm = 0 \quad \text{in } \mathbb{R}^n_\pm, \tag{4.5}
$$

$$
\Sigma_{k=1}^{n-1} i\xi_k \hat{v}_{\pm k} + \partial_n \hat{v}_{\pm n} = 0 \quad \text{in } \mathbb{R}^n_\pm, \tag{4.6}
$$

$$
\hat{v}_{+k}|_{x_n=0} - \hat{v}_{-k}|_{x_n=0} = \hat{g}_k \quad \text{on } \mathbb{R}^n_0, \tag{4.7}
$$

$$
[\![\mu(\partial_n \hat{v}_k + i\xi_k \hat{v}_n)_k]\!] = -\hat{g}_{u,k} \quad \text{on } \mathbb{R}^n_0, \tag{4.8}
$$

$$
[\![2\mu \partial_n \hat{v}_n]\!] - [\![\hat{\tau}]\!] = -\hat{g}_{u,n} \quad \text{on } \mathbb{R}^n_0, \tag{4.9}
$$

$$
[\![(2\mu/\rho)\partial_n \hat{v}_n]\!] - [\![\hat{\pi}/\rho]\!] = -\hat{g}_\pi \quad \text{on } \mathbb{R}^n_0, \tag{4.10}
$$

$k = 1, \cdots, n-1$.

From (4.4) and (4.5), it holds that

$$B_\pm^2(\Sigma_{k=1}^{n-1} i\xi_k \hat{v}_{\pm k} + \partial_n \hat{v}_{\pm n}) - \partial_n^2(\Sigma_{k=1}^{n-1} i\xi_k \hat{v}_{\pm k} + \partial_n \hat{v}_{\pm n}) + \mu_\pm^{-1}(-A^2 + \partial_n^2)\hat{\tau} = 0,$$

so by (4.6), we gain

$$(A^2 - \partial_n^2)\mu_\pm^{-1}\hat{\tau}_\pm = 0.$$

Moreover, from this equation, (4.4) and (4.5), it is showed that

$$(A^2 - \partial_n^2)(B_\pm^2 - \partial_n^2)\hat{v}_{\pm m} = 0 \quad \text{for } m = 1, \cdots, n.$$

We look for solutions whose forms are;

$$\hat{\tau}_+(s, \xi', x_n) = \mu_+ R e^{-Ax_n} \quad \text{for } x_n > 0, \tag{4.11}$$

$$\hat{\tau}_-(s, \xi', x_n) = \mu_- R' e^{Ax_n} \quad \text{for } x_n < 0, \tag{4.12}$$

$$\hat{v}_{+m}(s, \xi', x_n) = P_m e^{-Ax_n} + Q_m e^{-B_+ x_n} \quad \text{for } x_n > 0, \tag{4.13}$$

$$\hat{v}_{-m}(s, \xi', x_n) = P'_m e^{Ax_n} + Q'_m e^{B_- x_n} \quad \text{for } x_n < 0. \tag{4.14}$$

Generally, we give the solution of the following equation:

$$(A^2 - \partial_n^2)f = 0$$

as

$$f = C_1 e^{-Ax_n} + C_2 e^{Ax_n}$$

but $e^{Ax_n} \to \infty$ $(x_n \to \infty)$, so we couldn't use Fourier transform for f when $x_n \to \infty$. Oppositely, $e^{-Ax_n} \to \infty (x_n \to -\infty)$ if $x_n < 0$. In the same way, the function:

$$g = C_1 e^{-Ax_n} + C_2 e^{Ax_n} + C_3 e^{-Bx_n} + C_4 e^{Bx_n}$$

is the general solution of

$$(A^2 - \partial_n^2)(B^2 - \partial_n^2)g = 0$$

but we should set $C_2 = C_4 = 0$ in case $x_n > 0$ and $C_1 = C_3 = 0$ if $x_n < 0$ because of $\operatorname{Re} B_\pm > 0$ and $|e^{B_\pm}| = |e^{\operatorname{Re} B_\pm + i\operatorname{Im} B_\pm}| = e^{\operatorname{Re} B_\pm}$. Thus, we look for solutions whose forms are (4.11)-(4.14).

We set

$$\alpha_\pm = -\mu_\pm A^2(3B_\pm - A)/(2B_\pm(B_\pm + A)) \ , \ \beta = (\mu_+ B_+ + \mu_- B_-)/2.$$

R, R', P_m, P'_m, Q_m, Q'_m are determined by

$$\begin{aligned} R = (\alpha_+ + \alpha_- - \beta)^{-1}\{&(-2\alpha_- + \mu_- B_-)\hat{\operatorname{div}_{x'} g} \\ &+ (\mu_-(1 - \rho_+/\rho_-))^{-1}(\alpha_- + \mu_- A^2/(2B_-))\hat{g}_{u,n} \\ &+ (\mu_+(1 - \rho_-/\rho_+))^{-1}(\alpha_- - \beta - \mu_+ A^2/(2B_+))\hat{g}_{u,n} \\ &+ (\mu_-(-1 + \rho_+/\rho_-))^{-1}\rho_+(\alpha_- + \mu_- A^2/(2B_-))\hat{g}_\pi \\ &+ (\mu_+(-1 + \rho_-/\rho_+))^{-1}\rho_-(\alpha_- - \beta - \mu_+ A^2/(2B_+))\hat{g}_\pi \\ &- \Sigma_{k=1}^{n-1} i\xi_k \hat{g}_{u,k}\}, \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
R' &= (\alpha_+ + \alpha_- - \beta)^{-1}\{(2\alpha_+ - \mu_+ B_+)\hat{\mathrm{div}_{x'} g} \\
&\quad + (\mu_+(1 - \rho_-/\rho_+))^{-1}(-\alpha_+ - \mu_+ A^2/(2B_+))\hat{g}_{u,n} \\
&\quad + (\mu_-(1 - \rho_+/\rho_-))^{-1}(-\alpha_+ + \beta + \mu_- A^2/(2B_-))\hat{g}_{u,n} \\
&\quad + (\mu_+(-1 + \rho_-/\rho_+))^{-1}\rho_-(-\alpha_+ - \mu_+ A^2/(2B_+))\hat{g}_\pi \\
&\quad + (\mu_-(-1 + \rho_+/\rho_-))^{-1}\rho_+(-\alpha_+ + \beta + \mu_- A^2/(2B_+))\hat{g}_\pi \\
&\quad - \Sigma_{k=1}^{n-1} i\xi_k \hat{g}_{u,k}\}, \qquad (4.16)\\
P_k &= -i\mu_+\xi_k R/(\rho_+ s), \quad P_n = \mu_+ AR/(\rho_+ s), \qquad (4.17)\\
P'_k &= -i\mu_-\xi_k R'/(\rho_- s), \quad P'_n = -\mu_- AR'/(\rho_- s), \qquad (4.18)\\
Q_k &= (\mu_+ B_+ + \mu_- B_-)^{-1}[-(\mu_+ A + \mu_- B_-)i\mu_+\xi_k R/(\rho_+ s) + \mu_+^2 i\xi_k AR/(\rho_+ s) \\
&\quad + i\xi_k\{(2B_+(1 - \rho_-/\rho_+))^{-1}(-\rho_-\hat{g}_\pi + \hat{g}_{u,n}) \\
&\qquad - (A/B_+)\mu_+^2 AR/(\rho_+ s) - \mu_+ R/(2B_+)\} \\
&\quad - (B_- - A)i\mu_-^2\xi_k R'/(\rho_- s) + \mu_-^2 i\xi_k AR'/(\rho_- s) \\
&\quad - i\xi_k\{(2B_-(1 - \rho_+/\rho_-))^{-1}(-\rho_+\hat{g}_\pi + \hat{g}_{u,n}) + (A/B_-)\mu_-^2 AR'/(\rho_- s) \\
&\quad + \mu_- R'/(2B_-)\} + \mu_- B_-\hat{g}_k + \hat{g}_{u,k}], \qquad (4.19)\\
Q'_k &= (\mu_+ B_+ + \mu_- B_-)^{-1}[-(B_+ - A)i\mu_+^2\xi_k R/(\rho_+ s) + \mu_+^2 i\xi_k AR/(\rho_+ s) \\
&\quad + i\xi_k\{(2B_+(1 - \rho_-/\rho_+))^{-1}(-\rho_-\hat{g}_\pi + \hat{g}_{u,n}) \\
&\qquad - (A/B_+)\mu_+^2 AR/(\rho_+ s) - \mu_+ R/(2B_+)\} \\
&\quad + (\mu_+ B_+ + \mu_- A)i\mu_-\xi_k R'/(\rho_- s) + \mu_-^2 i\xi_k AR'/(\rho_- s) \\
&\quad - i\xi_k\{(2B_-(1 - \rho_+/\rho_-))^{-1}(-\rho_+\hat{g}_\pi + \hat{g}_{u,n}) + (A/B_-)\mu_-^2 AR'/(\rho_- s) \\
&\quad + \mu_- R'/(2B_-)\} - \mu_+ B_+\hat{g}_k - \hat{g}_{u,k}]. \qquad (4.20)
\end{aligned}
$$

Using

$$
\begin{aligned}
f(B_+, B_-, A) &= \mu_+ A^2(3B_+ - A)B_-(B_- + A) + \mu_- A^2(3B_- - A)B_+(B_+ + A) \\
&\quad + (\mu_+ B_+ + \mu_- B_-)B_+ B_-(B_+ + A)(B_- + A), \qquad (4.21)
\end{aligned}
$$

we deform $\hat{\tau}_+$;

$$
\begin{aligned}
\hat{\tau}_+(s, \xi', x_n) &= (f(B_+, B_-, A))^{-1}\times \\
&\quad [-2B_+(B_+ + A)(\mu_- A^2(3B_- - A) + \mu_- B_-^2(B_+ + A))e^{-Ax_n}\hat{\mathrm{div}_{x'} g} \\
&\quad - (2\rho_-\mu_- A^2(B_- - A)B_+(B_+ + A)/(\mu_-[\![\rho]\!]))e^{-Ax_n}\hat{g}_{u,n} \\
&\quad + ((\rho_+(B_+ + A)(\mu_- A^2 B_+(3B_- - A) + \mu_+ A^2 B_-(B_- + A) \\
&\quad + (\mu_+ B_+ + \mu_- B_-)B_+ B_-(B_- + A)))/(\mu_+[\![\rho]\!]))e^{-Ax_n}\hat{g}_{u,n} \\
&\quad + (2\rho_+\rho_-\mu_- A^2(B_- - A)B_+(B_+ + A)/(\mu_-[\![\rho]\!]))e^{-Ax_n}\hat{g}_\pi \\
&\quad - (\rho_+\rho_-(B_+ + A)(\mu_- A^2 B_+(3B_- - A) + \mu_+ A^2 B_-(B_- + A) \\
&\quad + (\mu_+ B_+ + \mu_- B_-)B_+ B_-(B_- + A)))/(\mu_+[\![\rho]\!])e^{-Ax_n}\hat{g}_\pi
\end{aligned}
$$

$$+ \Sigma_{k=1}^{n-1} i\xi_k B_+ B_-(B_+ + A)(B_- + A)e^{-Ax_n}\hat{g}_{u,k}].$$

We consider $\mathcal{R}$-boundedness of solution operators defined in a sector

$$\Sigma_{\epsilon,\gamma_0} = \{s \in \mathbb{C}\backslash\{0\} \mid |\arg s| \leq \pi - \epsilon,\ |s| \geq \gamma_0\}$$

with $0 < \epsilon < \pi/2$ and $\gamma_0 \geq 1$ large enough (see Section 3).

Lemma 4.1. *Let $l = 0, 1$. For every $s \in \Sigma_{\epsilon,\gamma_0}$, we have*

$$|f(B_+, B_-, A)| \geq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^5,$$

$$|D_{\xi'}^{\alpha'}(\tau D_\tau)^l f(B_+, B_-, A)^{-1}| \leq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-5} A^{-|\alpha'|}.$$

Proof. This lemma is proved in the same way as Lemma 5.5 in [20]. □

Refining Lemma 4.6 and Lemma 4.8 in [23], we derive the following estimates for $l = 0, 1$;

$$\begin{aligned}
&|D_{\xi'}^{\alpha'}(\tau D_\tau)^l(2B_+(B_+ + A)(\mu_- A^2(3B_- - A) + \mu_- B_-^2(B_+ + A)))|,\\
&|D_{\xi'}^{\alpha'}(\tau D_\tau)^l(2\rho_-\mu_- A^2(B_- - A)B_+(B_+ + A))|,\\
&|D_{\xi'}^{\alpha'}(\tau D_\tau)^l(\rho_+(B_+ + A)(\mu_- A^2 B_+(3B_- - A) + \mu_+ A^2 B_-(B_- + A)\\
&\qquad\qquad + (\mu_+ B_+ + \mu_- B_-)B_+ B_-(B_- + A)))|,\\
&|D_{\xi'}^{\alpha'}(\tau D_\tau)^l(i\xi_k B_+ B_-(B_+ + A)(B_- + A))| \leq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^5 A^{-|\alpha'|}.
\end{aligned} \tag{4.22}$$

Because of Lemma 4.1 and (4.22), we could make use of the following lemma. By Lemma 3.4 and Volevich trick, we obtain

$$\begin{aligned}
\|e^{-\gamma t}\nabla\tau_\pm\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))} \leq C_{\epsilon,\gamma_0}(&\|e^{-\gamma t}(g, \nabla g, \nabla^2 g)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))}\\
&+ \|e^{-\gamma t}(g_u, \nabla g_u, g_\pi, \nabla g_\pi)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))}).
\end{aligned} \tag{4.23}$$

Now we calculate v_k, $k = 1, \ldots, n-1$. By (4.13), we could deform $\hat{v}_{+k}$ as follows:

$$\begin{aligned}
&\hat{v}_{+k}(s, \xi', x_n)\\
&= -\frac{i\mu_+}{\rho_+}\frac{1}{s}\xi_k R e^{-Ax_n} + m_1(s,\xi')\frac{1}{s}\xi_k R e^{-B_+x_n} + m_2(s,\xi')\frac{1}{s}\xi_k R' e^{-B_+x_n}\\
&\quad - \frac{i\mu_+}{2}\frac{1}{B_+(\mu_+B_+ + \mu_-B_-)}\xi_k R e^{-B_+x_n} - \frac{i\mu_-}{2}\frac{1}{B_-(\mu_+B_+ + \mu_-B_-)}\xi_k R' e^{-B_+x_n}\\
&\quad + \frac{i}{2(1 - \rho_-/\rho_+)}\frac{1}{B_+(\mu_+B_+ + \mu_-B_-)}\xi_k e^{-B_+x_n}(-\rho_-\hat{g}_\pi + \hat{g}_{u,n})\\
&\quad - \frac{i}{2(1 - \rho_+/\rho_-)}\frac{1}{B_-(\mu_+B_+ + \mu_-B_-)}\xi_k e^{-B_+x_n}(-\rho_+\hat{g}_\pi + \hat{g}_{u,n})\\
&\quad + \frac{\mu_-B_-}{\mu_+B_+ + \mu_-B_-}e^{-B_+x_n}\hat{g}_k + \frac{1}{\mu_+B_+ + \mu_-B_-}e^{-B_+x_n}\hat{g}_{u,k},
\end{aligned}$$

where both $m_1(s,\xi')$ and $m_2(s,\xi')$ satisfy

$$|D_{\xi'}^{\alpha'}(\tau D_\tau)^l m_{l'}(s,\xi)| \leq CA^{-|\alpha'|}$$

for $l = 0, 1$ and $l' = 1, 2$. For terms which carry A or ξ_k, we may estimate them like τ_+. For terms which do not carry A or ξ_k;

$$\mu_- B_-(\mu_+ B_+ + \mu_- B_-)^{-1} e^{-B_+ x_n} \hat{g}_k \ , \ (\mu_+ B_+ + \mu - B_-)^{-1} e^{-B_+ x_n} \hat{g}_{u,k},$$

we pay attention that

$$|D^{\alpha'}_{\xi'}(\tau D_\tau)^l \mu_- B_-(\mu_+ B_+ + \mu_- B_-)^{-1}| \le C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-|\alpha'|}$$

and

$$|D^{\alpha'}_{\xi'}(\tau D_\tau)^l (\mu_+ B_+ + \mu_- B_-)^{-1}| \le C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-1-|\alpha'|}$$

hold for $l = 0, 1$ in the view of regularity of g_k and $g_{u,k}$. In order to estimate υ_{+n}, we use the following formula of υ_{+n} in view of (12.3) and (12.4) in Section 12 below,

$$\hat{\upsilon}_{+n}(\xi', x_n, s) = (\mu_+ AR/(\rho_+ s))e^{-Ax_n} + \Sigma_{k=1}^{n-1}(i\xi_k/B_+)Q_k e^{-B_+ x_n}. \tag{4.24}$$

Employing this formula, we could estimate υ_n like υ_k because for $l = 0$, 1 it holds that

$$|D^{\alpha'}_{\xi'}(\tau D_\tau)^l (\xi_k/B_+)| \le C_{\epsilon,\gamma_0} A^{-|\xi'|}.$$

We don't make use of the description of $\hat{\upsilon}_{+n}$:

$$\hat{\upsilon}_{+n}(\xi', x_n, s) = (\mu_+ AR/(\rho_+ s))e^{-Ax_n} + Q_n e^{-B_+ x_n},$$

where

$$Q_n = (2\mu_+ B_+(1 - \rho_-/\rho_+))^{-1}(-\rho_- \hat{g}_\pi + \hat{g}_{u,n}) - (A/B_+)(\mu_+ AR/(\rho_+ s)) - R/(2B_+)$$

(see Section 12 below) because we couldn't apply Lemma 3.4 to the term $(R/(2B_+))e^{-B_+ x_n}$. Lemma 3.4 avails estimate of $\gamma\upsilon_+$ and $\partial_t \upsilon_+$ such that

$$\begin{aligned} \|e^{-\gamma t}(\gamma\upsilon_+, \partial_t\upsilon_+)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} &\le C_{\epsilon,\gamma_0}(\|e^{-\gamma t}(g, \partial_t g, \nabla g, \nabla^2 g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} \\ &+ \|e^{-\gamma t}(g_u, \Lambda^{1/2}_\gamma g_u, \nabla g_u, g_\pi, \Lambda^{1/2}_\gamma g_\pi, \nabla g_\pi)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))}). \end{aligned} \tag{4.25}$$

Moreover, we see by (4.14)

$$\begin{aligned} \|e^{-\gamma t}(\gamma\upsilon_-, \partial_t\upsilon_-)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_-))} &\le C_{\epsilon,\gamma_0}(\|e^{-\gamma t}(g, \partial_t g, \nabla g, \nabla^2 g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_-))} \\ &+ \|e^{-\gamma t}(g_u, \Lambda^{1/2}_\gamma g_u, \nabla g_u, g_\pi, \Lambda^{1/2}_\gamma g_\pi, \nabla g_\pi)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_-))}). \end{aligned} \tag{4.26}$$

By using the identity $(1 - \Delta)\upsilon_\pm = \upsilon_\pm - \mu^{-1}_\pm \rho_\pm \partial_t \upsilon_\pm$, we obtain

$$\begin{aligned} \|e^{-\gamma t}(\upsilon_\pm, \nabla\upsilon_\pm, \nabla^2\upsilon_\pm)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_\pm))} &\le C_{\epsilon,\gamma_0}(\|e^{-\gamma t}(g, \partial_t g, \nabla g, \nabla^2 g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_\pm))} \\ &+ \|e^{-\gamma t}(g_u, \Lambda^{1/2}_\gamma g_u, \nabla g_u, g_\pi, \Lambda^{1/2}_\gamma g_\pi, \nabla g_\pi)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_\pm))}). \end{aligned} \tag{4.27}$$

Now, we investigate the regularity of h. Defining $L(B_+, B_-, A)$ as

$$\begin{aligned} &L(B_+, B_-, A) \\ &= sf(B_+, B_-, A) + [\![\rho]\!]^{-2}\sigma A^2\{2\rho_+\rho_- A^2(B_+ - A)(B_- - A) \\ &+ A^2\big(\rho^2_+ B_-(B_- + A) + \rho^2_- B_+(B_+ + A)\big) \\ &+ A^3\big(\mu_-\mu^{-1}_+\rho^2_+(3B_- - A) + \mu_+\mu^{-1}_-\rho^2_-(3B_+ - A)\big) \end{aligned}$$

$$+ (\mu_+ B_+ + \mu_- B_-) A\big(\mu_+^{-1}\rho_+^2 B_-(B_- + A) + \mu_-^{-1}\rho_-^2 B_+(B_+ + A)\big)\}, \tag{4.28}$$

we have

$$\hat{h}(s,\xi') = f(B_+, B_-, A) L(B_+, B_-, A)^{-1}(\hat{g}_h + [\![\rho\hat{v}_n]\!]/[\![\rho]\!]). \tag{4.29}$$

Lemma 4.2. *Let $l = 0, 1$. For every $s \in \Sigma_{\epsilon,\gamma_0}$, we have*

$$\begin{aligned}|L(B_+, B_-, A)| &\geq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^3(|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2)\\ &\geq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^6,\end{aligned} \tag{4.30}$$

$$\begin{aligned}&|D_{\xi'}^{\alpha'}(\tau D_\tau)^l L(B_+, B_-, A)^{-1}|\\ &\quad\leq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-3}(|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2)^{-1} A^{-|\alpha'|}\\ &\quad\leq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-6} A^{-|\alpha'|}\end{aligned} \tag{4.31}$$

hold.

Proof. We use symbols that are used in Lemma 6.1 in [23]. Let δ and $\mathcal{O}(\delta)$ be a small number determined later and a symbol satisfying $|\mathcal{O}(\delta)| \leq C\delta$, respectively. Suppose $\delta \leq \min(\rho_+/\mu_+, \rho_-/\mu_-)$.

First we prove (4.30) in the case where

$$|\rho_\pm \mu_\pm^{-1} s A^{-2}| \leq \delta.$$

If we write

$$B_\pm = A(1 + \mathcal{O}(\delta)),$$

then we obtain from (4.29) and (4.21)

$$\begin{aligned}L(B_+, B_-, A) = s(\mu_+ + \mu_-)&A^5(9 + 16\mathcal{O}(\delta))\\ &+[\![\rho]\!]^{-2}\sigma A^3\{A^3(\rho_+^2 + \rho_-^2)(2 + 3\mathcal{O}(\delta))\\ &\qquad + A^3(\mu_-\mu_+^{-1}\rho_+^2 + \mu_+\mu_-^{-1}\rho_-^2)(2 + 3\mathcal{O}(\delta))\\ &\qquad + A^3(\mu_+ + \mu_-)(\mu_+^{-1}\rho_+^2 + \mu_-^{-1}\rho_-^2)(2 + 3\mathcal{O}(\delta))\}.\end{aligned}$$

Since $A \geq 2^{-1}(|s|^{1/2} + A)$, if we choose a δ properly, in the same way as the proof of Lemma 6.1 in [23] we obtain

$$\begin{aligned}|L(B_+, B_-, A)| &\geq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^3(|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2)\\ &\geq C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^6.\end{aligned}$$

Secondly, we prove (4.30) in the case where

$$|\rho_\pm \mu_\pm^{-1} s A^{-2}| \geq \delta.$$

By Lemma 4.6, Lemma 4.8 in [23] and Lemma 4.1,

$$\begin{aligned}|L(B_+, B_-, A)| &\geq |s||f(B_+, B_-, A)| - C\sigma[\![\rho]\!]^{-2}A^3(|s|^{1/2} + A)^3\\ &\geq C_1(|s|^{1/2} + A)^3(|s|(|s|^{1/2} + A)^2 - \sigma[\![\rho]\!]^{-2}A^3).\end{aligned}$$

Because

$$A \leq (\min(\rho_+/\mu_+, \rho_-/\mu_-))^{1/2}\delta^{-1/2}|s|^{1/2}, \ |s|^{-1} \leq \gamma_0^{-1},$$

there exists $C_2 > 0$ such that

$$|s|(|s|^{1/2} + A)^2 - [\![\rho]\!]^{-2} A^3 \geq |s|^2(1 - C_2 \gamma_0^{-1/2}).$$

Combining the inequality above and

$$\begin{aligned}|s|^2 = (|s|^2 + |s|^2)/2 &\geq C(|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2) \\ &\geq C_{\epsilon,\gamma_0} \gamma_0^{1/2} (|s|^{1/2} + |s|^{1/2})(|s|^{1/2} + A)^2 \\ &\geq C_{\epsilon,\gamma_0} (|s|^{1/2} + A)^3,\end{aligned}$$

we obtain

$$\begin{aligned}|L(B_+, B_-, A)| &\geq C_{\epsilon,\gamma_0} (|s|^{1/2} + A)^3 (|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2) \\ &\geq C_{\epsilon,\gamma_0} (|s|^{1/2} + A)^6.\end{aligned}$$

By the Bell formula, we obtain (4.31) as the similar manner in the proof of Lemma 6.1 in [23]. □

By using an extension

$$\hat{h}(s, \xi', x_n) = f(B_+, B_-, A))L(B_+, B_-, A)^{-1} e^{-Ax_n} (\hat{g}_h + [\![\rho \hat{v}_n]\!]/[\![\rho]\!]),$$

from Lemma 3.4 , Lemma 4.1 and Lemma 4.2, we have

$$e^{-\gamma t} \nabla h,\ e^{-\gamma t} \partial_t \nabla h,\ e^{-\gamma t} \nabla^2 h,\ e^{-\gamma t} \nabla^3 h,\ e^{-\gamma t} \partial_t \nabla^2 h \in L_p(\mathbb{R}, L_q(\mathbb{R}^n_+)).$$

Since the problem (4.3) is the case where we set $f_u = f_d = g_{u,k} = 0$ and $g_{u,n} = \sigma \Delta' h$ in the problem (4.2) and add two equations from below in (4.3), estimates (4.23)-(4.27) hold for $w_\pm$ and $\kappa_\pm$, too. In order to prove $e^{-\gamma t} h \in L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))$, we use an extension;

$$\hat{h}(s, \xi', x_n) = f(B_+, B_-, A)L(B_+, B_-, A)^{-1} e^{-B_+ x_n} (\hat{g}_h + [\![\rho \hat{v}_n]\!]/[\![\rho]\!])$$

and the identity;

$$\frac{f(B_+, B_-, A)}{L(B_+, B_-, A)} = \frac{1}{s} - \frac{L(B_+, B_- A) - s f(B_+, B_-, A)}{s L(B_+, B_-, A)}.$$

It is clear that

$$|D^{\alpha'}_{\xi'} (\tau D_\tau)^l s^{-1}| \leq |s|^{-1} (|s|^{1/2} + A)^{-|\alpha'|} \leq \gamma_0^{-1} (|s|^{1/2} + A)^{-|\alpha'|}$$

for $l = 0, 1$. By (4.28),

$$L(B_+, B_-, A) - s f(B_+, B_-, A) = \sigma [\![\rho]\!]^{-2} A M(s, \xi')$$

where $M(s, \xi')$ is a function satisfying

$$|D^{\alpha'}_{\xi'} (\tau D_\tau)^l M(s, \xi')| \leq C_{\epsilon,\gamma_0} (|s|^{1/2} + A)^5 A^{-|\alpha'|}.$$

In view of Lemma 3.4 , we see

$$e^{-\gamma t} h,\ e^{-\gamma t} \partial_t h,\ e^{-\gamma t} \partial_t^2 h,\ \Lambda^{3/2}_\gamma h,\ \nabla \Lambda^{3/2}_\gamma h \in L_p(\mathbb{R}, L_q(\mathbb{R}^n_+)).$$

In fact, by using the Volevich trick: if $f(x,y_n)\to 0\,(y_n\to\infty)$,

$$f(x,0)=-\int_0^\infty \frac{\partial f}{\partial y_n}(x,y_n)\,dy_n$$

and $B_+=\{(\rho_+/\mu_+)s+A^2\}/B_+$, we have

$$\begin{aligned}
&\mathcal{F}_{x'}\mathcal{L}_t[\partial_t^2 h](s,\xi',x_n)=s^2\hat{h}(s,\xi',x_n)\\
=&\Big(s-\frac{\sigma[\![\rho]\!]^{-2}sM(s,\xi')}{L(B_+,B_-,A)}A\Big)e^{-B_+x_n}\Big(\hat{g}_h(s,\xi',x_n)+\frac{[\![\rho\hat{v}_n(s,\xi',x_n)]\!]}{[\![\rho]\!]}\Big)\\
=&-\int_0^\infty \partial_y\Big\{\Big(s-\frac{\sigma[\![\rho]\!]^{-2}sM(s,\xi')}{L(B_+,B_-,A)}A\Big)e^{-B_+(x_n+y_n)}\Big(\hat{g}_h(s,\xi',y_n)+\frac{[\![\rho\hat{v}_n(s,\xi',y_n)]\!]}{[\![\rho]\!]}\Big)\Big\}dy\\
=&\int_0^\infty s^{\frac12}e^{-B_+(x_n+y_n)}\Big\{\frac{\rho_+}{\mu_+}\frac{s^{\frac12}}{B_+}\Big(s\hat{g}_h(s,\xi',x_n)+\frac{[\![\rho s\hat{v}_n(s,\xi',x_n)]\!]}{[\![\rho]\!]}\Big)\\
&\qquad -s^{\frac12}\partial_n\hat{g}_h(s,\xi',x_n)-s^{\frac12}\frac{[\![\rho\partial_n\hat{v}_n(s,\xi',x_n)]\!]}{[\![\rho]\!]}\Big\}\,dy\\
+&\int_0^\infty Ae^{-B_+(x_n+y_n)}\Big(\frac{A}{B_+}s\hat{g}_h(s,\xi',x_n)+\frac{A}{B_+}\frac{[\![\rho s\hat{v}_n(s,\xi',x_n)]\!]}{[\![\rho]\!]}\Big)\,dy\\
-&\int_0^\infty Ae^{-B_+(x_n+y_n)}\Big\{\frac{\sigma[\![\rho]\!]^{-2}B_+M(s,\xi')}{L(B_+,B_-,A)}\Big(s\hat{g}_h(s,\xi',x_n)+\frac{[\![\rho s\hat{v}_n(s,\xi',x_n)]\!]}{[\![\rho]\!]}\Big)\\
&\qquad -\frac{\sigma[\![\rho]\!]^{-2}s^{\frac12}M(s,\xi')}{L(B_+,B_-,A)}\Big(s^{\frac12}\partial_n\hat{g}_h(s,\xi',x_n)+\frac{[\![\rho s^{\frac12}\partial_n\hat{v}_n(s,\xi',x_n)]\!]}{[\![\rho]\!]}\Big)\Big\}\,dy.
\end{aligned}\tag{4.32}$$

Since g_h, $v\in\mathbb{G}_h$ and

$$\left|D_{\xi'}^{\alpha'}(\tau D_\tau)^l\frac{\sigma[\![\rho]\!]^{-2}\beta M(s,\xi')}{L(B_+,B_-,A)}\right|\le C_{\epsilon,\gamma_0}A^{-|\alpha'|},\quad \beta=s^{\frac12}\text{ or }B_+,$$

we could apply Lemma 3.4 to the solution formula (4.32). Here we regard $\nabla\Lambda_\gamma^{1/2}v_n$ in (4.34) as a given function because we know (4.25), (4.26), (4.27) and the relation

$$W_p^1(\mathbb{R};L_q(\mathbb{R}^n_\pm))\cap L_p(\mathbb{R};W_q^2(\mathbb{R}^n_\pm))\hookrightarrow H_p^{1/2}(\mathbb{R};W_q^1(\mathbb{R}^n_\pm))\tag{4.33}$$

(cf. Proposition 2.9 in [22]). After all, it holds that

$$\begin{aligned}
&\|e^{-\gamma t}(h,\partial_t h,\nabla h,\partial_t^2 h,\partial_t\nabla h,\nabla^2 h,\nabla^3 h,\partial_t\nabla^2 h,\Lambda_\gamma^{3/2}h,\nabla\Lambda_\gamma^{3/2}h)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))}\\
&\qquad\le C(\|e^{-\gamma t}(g_h,\partial_t g_h,\nabla g_h,\nabla^2 g_h,\partial_t v_n,\nabla\Lambda_\gamma^{1/2}v_n)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))}.
\end{aligned}\tag{4.34}$$

In the end, we refer that we could prove that v, w, τ, κ vanish for $t<0$ in the same way as Section 3 in [24]. We have thus proved Theorem 2.5.

5. Local L_p-L_q well-posedness; Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The nonlinear problem (1.1)-(1.3) can be transformed to a problem on $\dot{\mathbb{R}}^n := \mathbb{R}^n \setminus [\mathbb{R}^{n-1} \times \{0\}]$ by means of the transformations

$$\begin{aligned} u(t,x',x_n) &:= (u',u_n)^{\mathsf{T}}(t,x',x_n+h(t,x')),\\ \bar\theta(t,x',x_n) &:= \theta(t,x',x_n+h(t,x'))-\theta_\infty,\\ \bar\pi(t,x',x_n) &:= \pi(t,x',x_n+h(t,x'))-\pi_\infty, \end{aligned}$$

where $t \in J = [0,T]$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, $x_n \neq 0$. Here $\theta_\infty > 0$ denotes the (equilibrium) temperature at infinity and π_∞ the corresponding (equilibrium) pressure at infinity defined by the relations

$$[\![\psi(\theta_\infty)]\!] + [\![\pi_\infty/\rho]\!] = 0, \quad [\![\pi_\infty]\!] = 0.$$

With a slight abuse of notation we will denote in the sequel the transformed velocity again by u, the transformed temperature by θ, and the transformed pressure by π. For given initial data $u_0(x)$ and $\theta_0(x)$, we set again $u_0(x',x_n) := u_0(x',x_n+h_0(x'))$ and $\theta_0(x',x_n) := \theta_0(x',x_n+h_0(x'))-\theta_\infty$, and define

$$\mu_0 = \mu(\theta_\infty), \quad \kappa_0 = \kappa(\theta_\infty), \quad d_0 = d(\theta_\infty), \quad l_0 = l(\theta_\infty).$$

We remark that μ_0, κ_0, d_0 and l_0 are constants. With this notation we have the transformed problem

$$\begin{aligned} \rho\partial_t u - \mu_0\Delta u + \nabla\pi &= F_u(u,\pi,\theta,h) && \text{in } \dot{\mathbb{R}}^n, && t>0,\\ \operatorname{div} u &= F_d(u,h) && \text{in } \dot{\mathbb{R}}^n, && t>0,\\ -[\![\mu_0(\partial_n u' + \nabla' u_n)]\!] &= G_{u'}(u,\theta,h) && \text{on } \mathbb{R}^{n-1}, && t>0,\\ -2[\![\mu_0\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h &= G_{u_n}(u,\theta,h) && \text{on } \mathbb{R}^{n-1}, && t>0,\\ [\![u']\!] &= G(u,h) && \text{on } \mathbb{R}^{n-1}, && t>0,\\ \rho\kappa_0\partial_t\theta - d_0\Delta\theta &= F_\theta(u,\theta,h) && \text{in } \dot{\mathbb{R}}^n, && t>0,\\ [\![\theta]\!] &= 0 && \text{on } \mathbb{R}^{n-1}, && t>0,\\ -[\![d_0\partial_n\theta]\!] &= G_\theta(u,\theta,h) && \text{on } \mathbb{R}^{n-1}, && t>0,\\ -2[\![(\mu_0/\rho)\partial_n u_n]\!] + [\![\pi/\rho]\!] &= G_\pi(u,\theta,h) && \text{on } \mathbb{R}^{n-1}, && t>0,\\ \partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] &= G_h(u,h) && \text{on } \mathbb{R}^{n-1}, && t>0,\\ u(0) = u_0, \quad \theta(0) &= \theta_0 && \text{in } \dot{\mathbb{R}}^n, &&\\ h(0) &= h_0 && \text{on } \mathbb{R}^{n-1}, && \end{aligned} \tag{5.1}$$

where the phase flux j already has been eliminated, according to Section 1. Here it reads

$$j = \frac{[\![u_n]\!] - [\![u']\!]\cdot\nabla' h}{\sqrt{1+|\nabla' h|^2}[\![1/\rho]\!]} = [\![u_n]\!]\frac{\sqrt{1+|\nabla' h|^2}}{[\![1/\rho]\!]}.$$

The nonlinear right hand sides are defined by

$$\begin{aligned}
F_u(u,\pi,\theta,h) &= (F_{u'}(u,\pi,\theta,h), F_{u_n}(u,\theta,h))^{\mathsf{T}},\\
F_{u'}(u,\pi,\theta,h) &= (\mu(\theta)-\mu_0)\Delta u'\\
&+ \mu(\theta)(-\Delta' h\partial_n u' - 2\nabla' h\cdot\nabla'\partial_n u' + |\nabla' h|^2\partial_n^2 u')\\
&- \rho(u'\cdot\nabla' u' + u_n\partial_n u' - u'\cdot\nabla' h\partial_n u') + \rho\partial_t h\partial_n u' + \nabla' h\partial_n\pi\\
&+ \{(\nabla' u' + [\nabla' u']^{\mathsf{T}}) - (\nabla' h\otimes\partial_n u' + \partial_n u'\otimes\nabla' h)\}\mu'(\theta)\nabla'\theta\\
&+ (\partial_n u' + \nabla' u_n - \nabla' h\partial_n u_n)\mu'(\theta)\partial_n\theta,\\
F_{u_n}(u,\theta,h) &= (\mu(\theta)-\mu_0)\Delta u_n\\
&+ \mu(\theta)(-\Delta' h\partial_n u_n - 2\nabla' h\cdot\nabla'\partial_n u_n + |\nabla' h|^2\partial_n^2 u_n)\\
&- \rho(u'\cdot\nabla' u_n + u_n\partial_n u_n - u'\cdot\nabla' h\partial_n u_n) + \rho\partial_t h\partial_n u_n\\
&+ ([\partial_n u']^{\mathsf{T}} + [\nabla' u_n]^{\mathsf{T}} - \partial_n u_n[\nabla' h]^{\mathsf{T}})\mu'(\theta)\nabla'\theta + 2\partial_n u_n\mu'(\theta)\partial_n\theta,\\
F_d(u,h) &= \nabla' h\cdot\partial_n u' = \partial_n(\nabla' h\cdot u'),\\
G_{u'}(u,\theta,h) &= [\![(\mu(\theta)-\mu_0)(\partial_n u' + \nabla' u_n)]\!] - [\![\mu(\theta)(\nabla u' + [\nabla u']^{\mathsf{T}})]\!]\nabla' h\\
&+ [\![\mu(\theta)\{\nabla' h(\partial_n u'\cdot\nabla' h) + \partial_n u'|\nabla' h|^2 - \nabla' h\partial_n u_n\}]\!]\\
&+ [\![\mu(\theta)\{-(\partial_n u' + \nabla' u_n)\cdot\nabla' h + 2\partial_n u_n + \partial_n u_n|\nabla' h|^2\}]\!]\nabla' h\\
&+ [\![\rho^{-1}]\!](1+|\nabla' h|^2)[\![u_n]\!]^2\nabla' h,\\
G_{u_n}(u,\theta,h) &= [\![(\mu(\theta)-\mu_0)2\partial_n u_n]\!] - [\![\mu(\theta)(\partial_n u' + \nabla' u_n)\cdot\nabla' h]\!]\\
&+ [\![\mu(\theta)\partial_n u_n]\!]|\nabla' h|^2 - [\![\rho^{-1}]\!](1+|\nabla' h|^2)[\![u_n]\!]^2 - \sigma J(h),\\
G(u,h) &= -[\![u_n]\!]\nabla' h,\\
F_\theta(u,\theta,h) &= \rho(\kappa_0 - \kappa(\theta))\partial_t\theta + (d(\theta)-d_0)\Delta\theta\\
&+ \rho\kappa(\theta)\{\partial_t h\partial_n\theta - u'\cdot\nabla\theta + (u'\cdot\nabla' h)\partial_n\theta - u_n\partial_n\theta\}\\
&+ d'(\theta)\{|\nabla'\theta - \nabla' h\partial_n\theta|^2 + (\partial_n\theta)^2\}\\
&+ (\mu(\theta)/2)|\nabla' u' + [\nabla' u']^{\mathsf{T}} - \nabla' h\otimes\partial_n u' - \partial_n u'\otimes\nabla' h|^2\\
&+ \mu(\theta)[|\partial_n u' + \nabla' w - \partial_n u_n\nabla' h|^2 + 2|\partial_n u_n|^2],\\
G_\theta(u,\theta,h) &= [\![(d(\theta)-d_0)\partial_n\theta]\!] - [\![d(\theta)\nabla'\theta\cdot\nabla' h]\!]\\
&+ (l(\theta)/[\![1/\rho]\!])(1+|\nabla' h|^2)[\![u_n]\!],\\
G_\pi(u,\theta,h) &= -[\![\psi(\theta+\theta_\infty) - \psi(\theta_\infty)]\!] + 2[\![(\mu(\theta)-\mu_0)\partial_n u_n/\rho]\!]\\
&- [\![\frac{1}{2\rho^2}]\!](1+|\nabla' h|^2)[\![\frac{1}{\rho}]\!]^{-2}[\![u_n]\!]^2 - 2[\![\frac{\mu(\theta)}{\rho}\partial_n u'\cdot\nabla' h]\!]\\
&+ \frac{2}{1+|\nabla' h|^2}[\![\frac{\mu(\theta)}{\rho}\{(\nabla u'\nabla' h)\cdot\nabla' h - \nabla' u_n\cdot\nabla' h\}]\!],\\
G_h(u,h) &= -\frac{[\![\rho u'\cdot\nabla' h]\!]}{[\![\rho]\!]}.
\end{aligned}$$

The curvature of $\Gamma(t)$ is given by

$$H(\Gamma(t)) = \operatorname{div}_{x'}\left(\frac{\nabla' h(t,x')}{\sqrt{1+|\nabla' h(t,x')|^2}}\right) = \Delta' h - J(h),$$

with

$$J(h) = \frac{|\nabla' h|^2 \Delta' h}{(1+\sqrt{1+|\nabla' h|^2})\sqrt{1+|\nabla' h|^2}} + \frac{\nabla' h\cdot(\nabla'^2 h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}},$$

where $\nabla'^2 h$ denotes the Hessian of h.

Concerning the boundary condition

$$[\![\rho^{-1}]\!] j^2 \nu_\Gamma - [\![\mu(\theta)(\nabla u + [\nabla u]^\mathsf{T})]\!]\nu_\Gamma = (\sigma H_\Gamma - [\![\pi]\!])\nu_\Gamma \tag{5.2}$$

in (1.1), multiplying (5.2) by $\sqrt{1+|\nabla' h|^2}\nu$, $\nu = e_n$, we obtain

$$\sigma H_\Gamma - [\![\pi]\!] = -[\![\mu(\theta)(\nabla u + [\nabla u]^\mathsf{T})]\!]\sqrt{1+|\nabla' h|^2}\nu_\Gamma\cdot\nu + [\![\rho^{-1}]\!] j^2.$$

Inserting this relation into (5.2), we obtain the nonlinear term $G_v(u,\theta,h)$ which neither contains the curvature nor the pressure jump $[\![\pi]\!]$ (cf. [14, Section 4]).

Given $h_0 \in B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$ we define

$$\Theta_{h_0}(x) := (x', x_n + h_0(x')) \quad (x', x_n) \in \mathbb{R}^n \times \mathbb{R}.$$

Letting $\Omega_{h_0,\pm} := \{(x',x_n) \in \mathbb{R}^n \times \mathbb{R} : \pm(x_n - h_0(x')) > 0\}$ and $\Omega_{h_0} := \Omega_{h_0,+} \cup \Omega_{h_0,-}$. By the assumption $2 < p < \infty$, $n < q < \infty$ and $2/p + n/q < 1$, we obtain from Sobolev's embedding theorem that Θ_{h_0} yields a C^2-diffeomorphism between $\dot{\mathbb{R}}^n$ and Ω_{h_0}, $\mathbb{R}^n_+$ and $\Omega_{h_0,+}$, and $\mathbb{R}^n_-$ and $\Omega_{h_0,-}$. The inverse transform is given by $\Theta^{-1}_{h_0}(x',x_n) = (x', x_n - h_0(x'))$. It then follows from the chain rule and transformation rule for integrals that

$$\Theta^*_{h_0} \in \operatorname{Isom}(W^k_p(\dot{\mathbb{R}}^n), W^k_p(\Omega_{h_0})), \quad [\Theta^*_{h_0}]^{-1} = \Theta^{h_0}_* \quad k = 0,1,2,$$

where we use the notation

$$\Theta^*_{h_0} f = f \circ \Theta_{h_0} \quad f : \Omega_{h_0} \to \mathbb{R}^m,$$
$$\Theta^{h_0}_* g = g \circ \Theta^{-1}_{h_0} \quad g : \dot{\mathbb{R}}^n \to \mathbb{R}^m,$$

for the pull-back and push-forward operators, where m is non-negative integer.

Therefore it is enough to prove the following theorem instead of Theorem 1.1.

Theorem 5.1. *Let $2 < p < \infty$, $n < q < \infty$ and $2/p+n/q < 1$. Let $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ be such that*

$$\kappa_\pm(s) = -s\psi''_\pm(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0 \quad s \in (0,\infty),$$

and

$$(u_0, \theta_0, h_0) \in B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

be given. Assume that the compatibility conditions:

$$\operatorname{div}(\Theta^{h_0}_* u_0) = 0 \qquad \text{in } \Omega_0,$$

$$[\![\mu P_{\Gamma_0} E(\Theta^{h_0}_* u_0)\nu_0]\!] = 0, \quad [\![P_{\Gamma_0}\Theta^{h_0}_* u_0]\!] = 0 \qquad \text{on } \Gamma_0,$$

$$[\![\Theta_*^{h_0}\theta_0]\!] = 0, \quad [\![d\partial_{\nu_0}\Theta_*^{h_0}\theta_0]\!] + \ell(\Theta_*^{h_0}(\theta_0+\theta_\infty))[\![\rho^{-1}]\!]^{-1}[\![\Theta_*^{h_0}u_0\cdot\nu_0]\!] = 0 \quad \text{on } \Gamma_0. \tag{5.3}$$

Then there exists a constant ε_0 depending only on Ω_0, p, q, n such that if h_0 and u_0 satisfy $\|\nabla' h_0\|_{L_\infty(\dot{\mathbb{R}}^n)} + \|\Theta_*^{h_0}u_0\|_{L_\infty(\dot{\mathbb{R}}^n)} \le \varepsilon_0$, then there exist

$$T = T(\|\theta_0-\theta_\infty\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)}, \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}, \varepsilon_0) > 0$$

and a unique L_p-L_q solution (u,π,θ,h) of the nonlinear problem (5.1) *on* $[0,T]$ *of* $\mathbb{E}(J)$ *which is defined by* (2.8)*.*

Now we prove Theorem 5.1. There is an extension $F_d^* \in \mathbb{F}_d(J)$ which satisfies $F_d^*(0) = \operatorname{div} u_0$ (cf. [17, Theorem 6.3]). We define F_d^*, G_u^*, G^*, G_θ^* and G_π^* as

$$\begin{aligned}
&G_u^*(u_0,\theta_0,h_0) = e^{t\Delta'}G_u(u_0,\theta_0,h_0),\\
&G^*(u_0,h_0) = e^{t\Delta'}G(u_0,h_0), \quad G_\theta^*(u_0,\theta_0,h_0) = e^{t\Delta'}G_\theta(u_0,\theta_0,h_0),\\
&G_\pi^*(u_0,\theta_0,h_0) = e^{t\Delta'}G_\pi(u_0,\theta_0,h_0), \quad G_h^*(u_0,h_0) = e^{t\Delta'}G_h(u_0,h_0),
\end{aligned}$$

where $e^{t\Delta'}$ is a semigroup generated by Δ'. Let u^*, π^* and h^* be solutions of the next problem:

$$\begin{aligned}
\rho\partial_t u^* - \mu_0\Delta u^* + \nabla\pi^* &= 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
\text{div } u^* &= F_d^*(u_0,h_0) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
-2[\![\mu_0 D(u^*)\nu]\!] + [\![\pi^*]\!]\nu - \sigma\Delta' h^*\nu &= G_u^*(u_0,\theta_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
[\![u^{*\prime}]\!] &= G^*(u_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
\rho\kappa_0\partial_t\theta^* - d\Delta\theta^* &= 0 && \text{in } \mathbb{R}^n_\pm\ ,\ t>0,\\
[\![\theta^*]\!] &= 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
-[\![d_0\partial_n\theta^*]\!] &= G_\theta^*(u_0,\theta_0,h_0) && \text{on } \mathbb{R}^n_0\ ,\ t>0,\\
-2[\![\mu_0 D(u^*)\nu\cdot\nu/\rho]\!] + [\![\pi^*/\rho]\!] &= G_\pi^*(u_0,\theta_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
\partial_t h^* - [\![\rho u_n^*]\!]/[\![\rho]\!] &= G_h^*(u_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
u^*(0) = u_0\ ,\ \theta^*(0) &= \theta_0 && \text{in } \dot{\mathbb{R}}^n,\\
h^*(0) &= h_0 && \text{on } \mathbb{R}^n_0.
\end{aligned} \tag{5.4}$$

With these extensions, we may apply Theorem 2.6 in order to solve (5.4), because the right members of (5.4) satisfy the required regularity conditions and the required compatibility conditions. By Theorem 2.6, a unique solution of (5.4) satisfies

$$\begin{aligned}
&z^* = (u^*,\pi^*,\pi^*_\pm,\theta^*,h^*) \in \mathbb{E}(J),\\
&\|z^*\|_{\mathbb{E}(J)} \le C(\|u_0\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)} + \|\theta_0-\theta_\infty\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}).
\end{aligned} \tag{5.5}$$

We seek a solution of (5.1) of the form: $u = \bar{u} + u^*$, $\pi = \bar{\pi} + \pi^*$, $\theta = \bar{\theta} + \theta^*$, $h = \bar{h} + h^*$. Namely, $\bar{u}$, $\bar{\pi}$, $\bar{\theta}$ and $\bar{h}$ are solutions of the following equations whose initial values are 0:

$$\begin{aligned}
\rho\partial_t\bar{u} - \mu_0\Delta\bar{u} + \nabla\bar{\pi} &= F_u(\bar{u}+u^*, \bar{\pi}+\pi^*, \bar{\theta}+\theta^*, \bar{h}+h^*) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
\operatorname{div}\bar{u} &= F_d(\bar{u}+u^*, \bar{h}+h^*) - F_d^*(u_0,h_0) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
-2[\![\mu_0 D(\bar{u})\nu]\!] + [\![\bar{\pi}]\!]\nu - \sigma\Delta'\bar{h}\nu &= G_u(\bar{u}+u^*, \bar{\theta}+\theta^*, \bar{h}+h^*)\\
&\quad - G_u^*(u_0,\theta_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
[\![\bar{u}']\!] &= G(\bar{u}+u^*, \bar{h}+h^*) - G^*(u_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
\rho\kappa_0\partial_t\bar{\theta} - d\Delta\bar{\theta} &= F_\theta(\bar{u}+u^*, \bar{\theta}+\theta^*, \bar{h}+h^*) && \text{in } \mathbb{R}^n_\pm,\ t>0,\\
[\![\bar{\theta}]\!] &= 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
-[\![d_0\partial_n\bar{\theta}]\!] &= G_\theta(\bar{u}+u^*, \bar{\theta}+\theta^*, \bar{h}+h^*)\\
&\quad - G_\theta^*(u_0,\theta_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
-2[\![\mu_0 D(\bar{u})\nu\cdot\nu/\rho]\!] + [\![\bar{\pi}/\rho]\!] &= G_\pi(\bar{u}+u^*, \bar{\theta}+\theta^*, \bar{h}+h^*)\\
&\quad - G_\pi^*(u_0,\theta_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
\partial_t\bar{h} - [\![\rho\bar{u}_n]\!]/[\![\rho]\!] &= G_h(\bar{u}+u^*, \bar{h}+h^*) - G_h^*(u_0,h_0) && \text{on } \mathbb{R}^n_0,\ t>0,\\
\bar{u}(0) = 0\ ,\ \bar{\theta}(0) &= 0 && \text{in } \dot{\mathbb{R}}^n,\\
\bar{h}(0) &= 0 && \text{on } \mathbb{R}^n_0.
\end{aligned}\tag{5.6}$$

In what follows, we shall solve (5.6) by contraction mapping principle. We define the underlying space $X_{R,T}$ by

$$X_{R,T} = \{\bar{z} = (\bar{u}, \bar{\pi}, \bar{\pi}_\pm, \bar{\theta}, \bar{h}) \in \mathbb{E}_0(J), \quad J=[0,T], \quad \|\bar{z}\|_{\mathbb{E}(J)} \le R\},$$

where $\mathbb{E}_0(J) = \{\mathbb{E}(J) \mid \bar{z}(0) = 0\}$. Here T is a positive number determined later and R is a large number which satisfies

$$C(\|u_0\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)} + \|\theta_0 - \theta_\infty\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}) \le R,$$

where the left-hand side is the same as the right-hand side of (5.5). We set the equation (5.6) $L(\bar{z}) = N(\bar{z}+z^*), \bar{z}(0) = 0$. Given $\tilde{z} = (\tilde{u}, \tilde{\pi}, \tilde{\pi}_\pm, \tilde{\theta}, \tilde{h}) \in X_{R,T}$, let $\bar{z} = (\bar{u}, \bar{\pi}, \bar{\pi}_\pm, \bar{\theta}, \bar{h})$ be a solution to the equation $L(\bar{z}) = N(\tilde{z}+z^*), \bar{z}(0) = 0$. Our task is to show that if we define the map $\Phi(\tilde{z}) = \bar{z}$, then Φ is a contraction map from $X_{R,T}$ into itself. The following lemmas avail to estimate of the nonlinear terms and to prove contraction of Φ.

Lemma 5.2 (Embeddings)**.** *Set $J=[0,T]$ with $0<T<\infty$. We use the following embedding relations.*
(1) (Sobolev Embeddings in Bessel Potential Space) For $2<p<\infty$, it holds that

$$H^{1/2}_p(J) \hookrightarrow C^{1/2-1/p}(J),$$

where $C^{1/2-1/p}$ denotes Hölder space.
(2) For $n<q<\infty$, it holds that

$$W^1_q(\mathbb{R}^n_\pm) \hookrightarrow L_\infty(\mathbb{R}^n_\pm).$$

(3) For $2 < p < \infty$, it holds that

$$B^{2-2/p}_{q,p}(\mathbb{R}^n_\pm) \hookrightarrow W^1_q(\mathbb{R}^n_\pm).$$

(4) For $1 < p,\, q < \infty$, it holds that

$$W^1_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^2_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; B^{2-2/p}_{q,p}(\mathbb{R}^n_\pm)).$$

(5) For $n < q < \infty$ and $2/p + n/q < 1$, it holds that

$$H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; BUC(\mathbb{R}^n_\pm)).$$

Proof. Since the relations (1) to (3) are basic embeddings for Sobolev space and Besov space, we prove (4) and (5).
(4) We may utilize the embedding relation:

$$W^1_p(J; E_0) \cap L_p(J; E_1) \subset BUC(J; (E_0, E_1)_{1-1/p,p})$$

for any two Banach spaces E_0 and E_1 such that E_1 is dense in E_0, $1 < p < \infty$. Here, $(\cdot,\cdot)_{\theta,p}$ denotes the real interpolation with exponent $0 < \theta < 1$ and

$$(L_q(\mathbb{R}^n_\pm), W^2_q(\mathbb{R}^n_\pm))_{1-1/p,p} \simeq B^{2-2/p}_{q,p}(\mathbb{R}^n_\pm)$$

where we make use of

$$(F^{s_0}_{qp_0}, F^{s_1}_{qp_1})_{\theta,p} \simeq B^s_{qp},$$

where $0 < p, q, p_0, p_1 \le \infty$, $s_0 \ne s_1$ and $s = (1-\theta)s_0 + \theta s_1$ $(0 < \theta < 1)$.
(5) We have

$$\begin{aligned} H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)) &\hookrightarrow H^{\theta/2}_p(J; [L_q(\mathbb{R}^n_\pm), W^1_q(\mathbb{R}^n_\pm)]_\theta) \\ &= H^{\theta/2}_p(J; H^{1-\theta}_q(\mathbb{R}^n_\pm)), \end{aligned}$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation with exponent $0 < \theta < 1$. If $\theta/2 - 1/p > 0$ and $1 - \theta - n/q > 0$, namely

$$2/p < 1 - n/q,$$

then the embedding relation

$$H^{\theta/2}_p(J; H^{1-\theta}_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; BUC(\mathbb{R}^n_\pm))$$

holds. We may think as follows:

$$\begin{aligned} H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)) &\subset W^{\theta/2}_p(J; W^{1-\theta}_q(\mathbb{R}^n_\pm)) \\ &\simeq F^{\theta/2}_{pp}(J; F^{1-\theta}_{qq}(\mathbb{R}^n_\pm)) \end{aligned}$$

if $0 < \theta < 1$. There exist θ, φ and ϕ such that $2/p < \theta < 1 - n/q$, $1/p < \varphi < \theta/2$ and $n/q < \phi < 1 - \theta$ by density of $\mathbb{R}$, hence the next embeddings hold,

$$\begin{aligned} F^{\frac{1}{2}\theta}_{pp}(J) \hookrightarrow F^{\varphi}_{p2}(J) \simeq H^{\varphi}_p(J) \hookrightarrow C^{\varphi - \frac{1}{p}}(J), \\ F^{1-\theta}_{qq}(\mathbb{R}^n_\pm) \hookrightarrow F^{\phi}_{q2}(\mathbb{R}^n_\pm) \simeq H^{\phi}_q(\mathbb{R}^n_\pm) \hookrightarrow C^{\phi - \frac{n}{q}}(\mathbb{R}^n_\pm). \end{aligned}$$

Thus, we obtain

$$H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)) \hookrightarrow C^{\varphi - \frac{1}{p}}(J; C^{\phi - \frac{n}{q}}(\mathbb{R}^n_\pm)),$$

so we see boundedness and continuity by the definition of Hölder space.

□

Remark 5.3. We remark that if $n < q < \infty$ and $2/p + n/q < 1$, then it holds that $p > 2$. Indeed, it is clear that by $n < q < \infty$

$$\frac{2}{p} + \frac{n}{q} < 1 \Leftrightarrow \frac{2}{p} < \frac{n-q}{q}$$
$$\Leftrightarrow \frac{2q}{q-n} < p$$

and $0 < q - n < q$, therefore we have $2 < 2q/(q - n)$.

Lemma 5.4. *Suppose that* $1 < p, q < \infty$. *Set* $J = [0, T]$, $0 < T < \infty$.

(1) *For* $f \in \{f \mid \nabla^{i+1} f,\ \nabla^{i-1} f \in L_q\}$, *we have*

$$\|\nabla^i f\|_{L_q(\mathbb{R}^n_\pm)} \leq C \|\nabla^{i-1} f\|^{1/2}_{L_q(\mathbb{R}^n_\pm)} \|\nabla^{i+1} f\|^{1/2}_{L_q(\mathbb{R}^n_\pm)}. \tag{5.7}$$

(2) *For* $f \in L_p(J; X)$, $1 < r < \infty$, *we have*

$$\left(\int_0^T \|f\|_X^{p/r}\, dt \right)^{1/p} \leq T^{(r-1)/(rp)} \|f\|^{1/r}_{L_p(J;X)}. \tag{5.8}$$

(3) *For* $f \in W^1_{p,0}(J; X) = \{f \in W^1_p(J; X) \mid f|_{t=0} = 0\}$, *we have*

$$\|f\|_{L_\infty(J;X)} \leq T^{(p-1)/p} \|f\|_{W^1_p(J;X)}. \tag{5.9}$$

Here, $i \in \mathbb{N}$ *and we abbreviate* $L_q(\mathbb{R}^n_\pm)$ *and* $W^m_q(\mathbb{R}^n_\pm)$ *to* L_q *and* W^m_q *with non negative integer,* m, *respectively.* X *is* L_∞, L_q *or* W^m_q.

Proof. (5.7) is well known as the Gagliard-Nirenberg inequality and (5.8) is easily proved by the Hölder inequality, so we prove (5.9). Because $f|_{t=0} = 0$, we obtain

$$\|f\|_{L_\infty(J;X)} = \operatorname*{esssup}_{t\in[0,T]} \left\| \int_0^t \partial_s f\, ds \right\|_X \leq \int_0^T \|\partial_s f\|_X\, ds \leq T^{(p-1)/p} \|f\|_{W^1_p(J;X)}.$$

□

Lemma 5.5. *For* $f, g \in W^1_p(J; L_q) \cap L_p(J; W^2_q)$, *we have*

$$\|\nabla f \nabla g\|_{L_p(J;L_q)} \leq C T^{1/(2p)} \|f\|_{W^1_p(J;L_q)\cap L_p(J;W^2_q)} \|g\|_{W^1_p(J;L_q)\cap L_p(J;W^2_q)},$$

where we set $L_q = L_q(\mathbb{R}^n_\pm)$ *and* $W^2_q = W^2_q(\mathbb{R}^n_\pm)$.

Proof. By (5.7), (5.8) and Lemma 5.2 (5), we obtain

$$\begin{aligned}
\|\nabla f \nabla g\|_{L_p(J;L_q)} &\leq \|\nabla f\|_{L_\infty(J;L_\infty)} \Big(\int_0^T \|\nabla g(t)\|^p_{L_q} dt \Big)^{1/p} \\
&\leq C \|\nabla f\|_{L_\infty(J;L_\infty)} \Big(\int_0^T \|g(t)\|^{p/2}_{L_q} \|\nabla^2 g(t)\|^{p/2}_{L_q} dt \Big)^{1/p} \\
&\leq C T^{1/(2p)} \|f\|_{W^1_p(J;L_q)\cap L_p(J;W^2_q)} \|g\|_{W^1_p(J;L_q)\cap L_p(J;W^2_q)}.
\end{aligned}$$

□

Lemma 5.6 (Fractional order derivative)**.** *We propose that* $2 < p < \infty$, $1 < q < \infty$, $0 < T < \infty$ *and set* $J = [0,T]$ *and* $L_q = L_q(\mathbb{R}^n_\pm)$*. For* $f \in W^1_p(J; L_q)$ *and* $g \in H^{1/2}_p(J; L_q)$*, we have*

$$\|fg\|_{H^{1/2}_p(J;L_q)} \le CT^{1/(2p)} \|f\|^{1/2}_{W^1_p(J;L_q)} \|f\|^{1/2}_{H^{1/2}_p(J;L_q)} \|g\|_{H^{1/2}_p(J;L_q)}.$$

Proof. By the relation, $H^s_p \simeq F^s_{p2}$ for $s > 0$ and the Hölder inequality for Triebel-Lizorkin space, we have

$$\|fg\|_{H^{1/2}_p(J;L_q)} \le C\|f\|_{H^{1/2}_p(J;L_q)} \|g\|_{H^{1/2}_2(J;L_q)}.$$

$L_p \subset L_2$ holds from $|T| < \infty$ and $2 < p$, hence $\|g\|_{H^{1/2}_2(J;L_q)} \le C\|g\|_{H^{1/2}_p(J;L_q)}$. Making use of (1) in Lemma 5.2 and (5.8), we derive

$$\begin{aligned}\|f\|_{L_p(J;L_q)} = \Big(\int_0^T \|f\|^{p/2}_{L_q} \|f\|^{p/2}_{L_q}\, dt\Big)^{1/2} &\le C\|f\|^{1/2}_{H^{1/2}_p(J;L_q)} \Big(\int_0^T \|f\|^{p/2}_{L_q}\, dt\Big)^{1/2} \\ &\le CT^{1/(2p)} \|f\|^{1/2}_{H^{1/2}_p(J;L_q)} \|f\|^{1/2}_{L_p(J;L_q)}.\end{aligned}$$

In the same way as that,

$$\|\Lambda^{1/2}_\gamma f\|_{L_p(J;L_q)} \le T^{1/(2p)} \|f\|^{1/2}_{W^1_p(J;L_q)} \|f\|^{1/2}_{H^{1/2}_p(J;L_q)}.$$

Combining these estimates, we obtain the desired estimate. □

Lemma 5.7. *Set* $J = [0,T]$ *with* $0 < T < \infty$*. If* $1 < p < \infty$, $n < q < \infty$*, then we have for* $f \in L_p(J; W^3_q(\mathbb{R}^n_\pm))$ *and* $g \in W^1_{p,0}(J; L_q(\mathbb{R}^n_\pm)) \cap L_{p,0}(J; W^2_q(\mathbb{R}^n_\pm))$

$$\|(\nabla^3 f)g\|_{L_p(J;L_q(\mathbb{R}^n_\pm))} \le T^{1-1/p} C\|g\|_{W^1_p(J;L_q(\mathbb{R}^n_\pm)) \cap L_p(J;W^2_q(\mathbb{R}^n_\pm))} \|f\|_{L_p(J;W^3_q(\mathbb{R}^n_\pm))}.$$

Proof. Since $n < q < \infty$, by the Gagliard-Nirenberg inequality:

$$\|\nabla^j u\|_{L_p} \le C\|\nabla^m u\|^\alpha_{L_r} \|u\|^{1-\alpha}_{L_q},$$

where we suppose p, q, r $(1 \le p, q, r \le \infty)$, $m \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ satisfy

$$1/p = j/n + (1/r - m/n)\alpha + (1-\alpha)/q, \;\; j/m \le \alpha \le 1,$$

we have

$$\|g(t)\|_{L_\infty} \le C\|\nabla g(t)\|^{n/q}_{L_q} \|g(t)\|^{1-n/q}_{L_q}, \tag{5.10}$$

Combining (5.10), (5.9) and the Höder inequality, we have

$$\begin{aligned}\|(\nabla^3 f)g\|_{L_p(J;L_q(\mathbb{R}^n_\pm))} &\le C\left(\int_0^T \|\nabla^3 f\|^p_{L_q} \|\nabla g\|^{np/q}_{L_q} \|g\|^{p(1-n/q)}_{L_q}\, dt\right)^{1/p} \\ &\le C\|\nabla g\|^{n/q}_{L_\infty(J;L_q)} \|g\|^{1-n/q}_{L_\infty(J;L_q)} \|\nabla^3 f\|_{L_p(J;L_q)} \\ &\le C\|g\|_{L_\infty(J;W^1_q)} \|g\|_{L_\infty(J;L_q)} \|f\|_{L_p(J;W^3_q)} \\ &\le CT^{1-1/p} \|g\|_{W^1_p(J;L_q) \cap L_p(J;W^2_q)} \|f\|_{L_p(J;W^3_q)}.\end{aligned}$$

□

Remark 5.8. We remind that

$$F_d(u^*, h^*)|_{t=0} = F_d^*(u_0, h_0)|_{t=0}, \qquad G_u(u^*, \theta^*, h^*)|_{t=0} = G_u^*(u_0, \theta_0, h_0)|_{t=0},$$
$$G(u^*, h^*)|_{t=0} = G_u^*(u_0, h_0)|_{t=0}, \qquad G_\theta(u^*, \theta^*, h^*)|_{t=0} = G_\theta^*(u_0, \theta_0, h_0)|_{t=0},$$
$$G_\pi(u^*, \theta^*, h^*)|_{t=0} = G_\pi^*(u_0, \theta_0, h_0)|_{t=0}, \; G_h(u^*, h^*)|_{t=0} = G_h^*(u_0, h_0)|_{t=0}.$$

Now, we show the mapping Φ is contractive based on Theorem 2.6 and using Lemmas 5.2-5.7. We remind that we consider $L(\tilde{z}) = N(\bar{z} + z^*)$ with $\bar{z} = 0$, $z^*(0) = \bar{z}_0$. Nonlinear terms are classified into three types:

(I) highest order terms carry the difference of coefficient (e.g. $(\mu(\theta) - \mu_0)\partial_n u$ in $G_{u'}$),

(II) products between lower order terms (e.g. $\partial_t h \partial_n u$ in F_u),

(III) highest order terms carry $\nabla' h$ (e.g. $J(h)$ in G_{u_n}).

We consider a typical nonlinear term in each type.

(I): As a type of (I), we consider $(\mu(\theta) - \mu_0)\partial_n u$ in $G_{u'}(u, \theta, h)$. By $\mu \in C^2(0, \infty)$, Lemma 5.2 and Lemma 5.4, we obtain

$$\begin{aligned}
&\|(\mu(\theta) - \mu_0)\partial_n u\|_{L_p(J;W_q^1)} \\
&\le \|\mu'(\theta)\|_{L_\infty(J;L_\infty)} \|\nabla\theta \partial_n u\|_{L_p(J;L_q)} + 2\|\mu(\theta) - \mu_0\|_{L_\infty(J;L_\infty)} \|u\|_{L_p(J;W_q^2)} \\
&\le C T^{1/(4p)} R^2.
\end{aligned}$$

This estimate holds for both $u = \bar{u}$ and $u = u^*$. Combining the relation (4.33) and $\mu \in C^2(0, \infty)$, we estimate $H_p^{1/2}(J; L_q)$ norm of $(\mu(\theta) - \mu_0)\partial_n u$.

(II): As a type of (II), we consider $\partial_t h \partial_n u$ in $F_u(u, \pi, \theta, h)$. By Lemma 5.2 and Lemma 5.4

$$\begin{aligned}
\|\partial_t h \partial_n u\|_{L_p(J;L_q)} &\le C\|\partial_t h\|_{L_\infty(J;L_\infty)} \|\partial_n u\|_{L_p(J;L_q)} \\
&\le C\|h\|_{H_p^{3/2}(J;W_q^1)} \|u\|_{L_p(J;L_q)}^{1/2} \|\nabla^2 u\|_{L_p(J;L_q)}^{1/2} \\
&\le C\|h\|_{H_p^{3/2}(J;W_q^1)} T^{1/(4p)} \|u\|_{W_p^1(J;L_q)}^{1/4} \|u\|_{L_p(J;L_q)}^{1/4} \|\nabla^2 u\|_{L_p(J;L_q)}^{1/2} \\
&\le C T^{1/(4p)} R^2,
\end{aligned}$$

where

$$\begin{aligned}
\|u\|_{L_p(J;L_q)} &= \Big(\int_0^T \|u\|_{L_q}^p \, dt \Big)^{1/p} \\
&= \Big(\int_0^T \|u\|_{L_q}^{p/2} \|u\|_{L_q}^{p/2} \, dt \Big)^{1/p} \\
&\le \|u\|_{L_\infty(J;L_q)}^{1/2} \Big(\int_0^T \|u\|_{L_q}^{p/2} \, dt \Big)^{1/p} \\
&\le C\|u\|_{W_p^1(J;L_q)}^{1/2} T^{1/(2p)} \|u\|_{L_p(J;L_q)}^{1/2}.
\end{aligned}$$

This estimate holds for both $u = \bar{u}$ and $u = u^*$.

(III): As a type of (III), we consider $J(h)$ in $G_{u_n}(u,\theta,h)$. Lemma 5.6 could not be used because $J(h)$ is a fraction. Thus, in view of the relation,

$$W^1_p(J;L_q(\dot{\mathbb{R}}^n))\cap L_p(J;L_q(\dot{\mathbb{R}}^n))\subset H^{1/2}_p(J;L_q(\dot{\mathbb{R}}^n)),$$

we handle $L_p(J;L_q)$ norm of $\partial_t J(h)$. $J(h)$ has been defined as

$$J(h)=\frac{|\nabla' h|^2\Delta' h}{(1+\sqrt{1+|\nabla' h|^2})\sqrt{1+|\nabla' h|^2}}+\frac{\nabla' h\cdot(\nabla'^2 h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}}.$$

Let the first term and the second term of $J(h)$ be $J_1(h)$ and $J_2(h)$, respectively. We easily see that

$$\partial_t J_2(h)=\frac{\nabla'\partial_t h\cdot(\nabla'^2 h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}}+\frac{\nabla' h\cdot(\nabla'^2\partial_t h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}}+\frac{\nabla' h\cdot(\nabla'^2 h\cdot\nabla'\partial_t h)}{(1+|\nabla' h|^2)^{3/2}}$$
$$-3\nabla' h\cdot(\nabla'^2 h\cdot\nabla' h)(1+|\nabla' h|^2)^{-5/2}\nabla' f\cdot\partial_t\nabla' h,$$

therefore

$$|\partial_t J_2(h)|\le C(|\nabla'^2 h||\partial_t\nabla' h|+|\nabla' h||\nabla'^2\partial_t h|).$$

Combining the following relation and estimates derived by Lemma 5.2, (5.8), (5.9)

$$\partial_t\nabla'\bar{h},\partial_t\nabla' h^*\in H^{1/2}_p(J;L_q(\dot{\mathbb{R}}^n))\cap L_p(J;W^1_q(\dot{\mathbb{R}}^n))\subset BUC(J;BUC(\dot{\mathbb{R}}^n)),$$
$$\|\nabla'^2 h\|_{L_p(J;L_q)}\le CT^{1/(2p)}\|h\|^{1/2}_{W^1_p(J;W^2_q)}\|h\|^{1/2}_{L_p(J;W^2_q)},\quad h=h^*\text{ or }\bar{h}$$
$$\|\nabla'\bar{h}\|_{L_\infty(J;L_\infty)}\le CT^{(p-1)/p}\|\bar{h}\|_{W^1_p(J;W^2_q)}$$

and $\|\nabla' h^*\|_{L_\infty(J;L_\infty)}\le\varepsilon_0$ by the assumption, we estimate $\mathbb{G}_{u_n}$ norm of $\partial_t J_2(h)$. We could calculate $|\partial_t J_1(h)|$ similarly.

Finally we remark that the nonlinear terms $[\![u_n]\!]\nabla' h$ in $G(u,h)$ and $[\![\rho u'\cdot\nabla' h]\!]/[\![\rho]\!]$ in $G_h(u,h)$. By Theorem 2.6, it holds that

$$\|u_n^*\nabla'\bar{h}\|_{L_p(J;W^2_q)}\le\|u_n^*\|_{L_\infty(J;L_\infty)}\|\nabla'\bar{h}\|_{L_p(J;W^2_q)}\le\|u_n^*\|_{L_\infty(J;L_\infty)}R,$$

we have to impose the smallness assumption $\|u_0\|_{L_\infty(\dot{\mathbb{R}}^n)}\le\varepsilon_0$. Thus, ϵ_0 may be smaller than $T^{1/(4p)}$ and $T^{(p-1)/p}$. Indeed, there exists positive numbers that belong to a interval, $(0,\min\{T^{1/(4p)},T^{(p-1)/p}\}]$ by density of $\mathbb{R}$. Here, by $1/(4p)-(p-1)/p<0$ if $2<p$, $T^{1/4p}>T^{(p-1)/p}$ if $0<T<1$ and $T^{1/(4p)}<T^{(p-1)/p}$ in case $1<T$.

Combining the estimates above, we show the mapping Φ is contractive if we take time interval, T and ε_0 small. This completes the proof of Theorem 5.1.

6. Results for Large Initial Data

We remove the smallness condition $\|u_0\|_{L_\infty(\mathbb{R}^n)}\epsilon_0$.

Theorem 6.1. *Let $2<p<\infty$, $n<q<\infty$, $2/p+n/q<1$, $\rho_\pm>0$, $[\![\rho]\!]\ne 0$, and suppose $\psi_\pm\in C^3(0,\infty)$, $\mu_\pm$, $d_\pm\in C^2(0,\infty)$ are such that*

$$\kappa_\pm(s)=-s\psi''_\pm(s)>0,\quad \mu_\pm(s)>0,\quad d_\pm(s)>0\quad s\in(0,\infty).$$

Let the initial interface Γ_0 *be given by a graph* $x' \mapsto (x', h_0(x'))$. *Assume the regularity conditions:*

$$(u_0, \theta_0, h_0) \in B^{2-2/p}_{q,p}(\Omega_0)^n \times B^{2-2/p}_{q,p}(\Omega_0) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

and the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = 0 \quad & \text{in } \Omega_0, \\ P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] = 0, \quad P_{\Gamma_0}[\![u_0]\!] = 0 \quad & \text{on } \Gamma_0, \\ [\![\theta_0]\!] = 0, \quad (l(\theta_0)/[\![1/\rho]\!])[\![u_0 \cdot \nu_0]\!] + [\![d(\theta_0)\partial_{\nu_0}\theta_0]\!] = 0 \quad & \text{on } \Gamma_0, \end{aligned}$$

where $P_{\Gamma_0} = I - \nu_{\Gamma_0} \otimes \nu_{\Gamma_0}$ *denotes the projection onto the tangent bundle of* Γ_0. *Then there exists a constant* ε_0 *depending only on* Ω_0, p, q, n *such that if* h_0 *satisfies* $\|\nabla' h_0\|_{L_\infty(\dot{\mathbb{R}}^n)} \le \varepsilon_0$, *then there exist*

$$T = T(\|u_0\|_{B^{2-2/p}_{q,p}(\Omega_0)} + \|\theta_0\|_{B^{2-2/p}_{q,p}(\Omega_0)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}, \varepsilon_0) > 0$$

and a unique L_p-L_q *solution* (u, π, θ, h) *of* (1.1)-(1.3) *on* $[0, T]$ *in the class of* (7.6) *below.*

Remark 6.2.

(1) The notion of L_p-L_q-solution is explained in more detail below.
(2) We supposed $\|u_0\|_{L_\infty(\Omega_0)} + \|\nabla' h_0\|_{L_\infty(\dot{\mathbb{R}}^n)} \le \epsilon_0$ in [5]. In this paper we remove the smallness condition $\|u_0\|_{L_\infty(\Omega_0)} \le \epsilon_0$.
(3) In Prüss, Shimizu and Wilke [16], they considered the same problem in bounded domain when $p = q$ and proved local well-posedness in L_p-setting when $n + 2 < p < \infty$. Our result may treat the case when $2 < p < \infty$, $n < q < \infty$ and $2/p + n/q < 1$, which covers wider range than the results of [15]. Indeed, if $n + 2 < q < \infty$, then

$$2q/(q - n) < p \le n + 2$$

is permitted and if $n + 2 < p < \infty$, then $q = n + 2$ is permitted.
(4) The restriction of expornents of p, q comes from using the following embedding relations to treat nonlinear terms. When $n < q < \infty$, it holds that

$$W^1_q(\mathbb{R}^n_\pm) \hookrightarrow L_\infty(\mathbb{R}^n_\pm).$$

When $2 < p < \infty$, it holds that

$$B^{2-2/p}_{qp}(\mathbb{R}^n_\pm) \hookrightarrow W^1_q(\mathbb{R}^n_\pm).$$

Let $J = [0, T]$. When $2/p + n/q < 1$, it holds that

$$W^1_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^2_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; BUC(\mathbb{R}^n_\pm)),$$

and when $n < q < \infty$ and $2/p + n/q < 1$, it holds that

$$H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)) \hookrightarrow BUC(J; BUC(\mathbb{R}^n_\pm))$$

(cf. Lemma 4.2 in [5]).

7. Linearized Problem

Let $\mathbb{R}^n_0 = \mathbb{R}^{n-1} \times \{0\}$ and $\dot{\mathbb{R}}^n = \mathbb{R}^n \backslash \mathbb{R}^n_0$. In order to prove Theorem 6.1, we use maximal L_p-L_q-regularity of the modified principal linearized problem of (1.1),(1.2),(1.3).

$$\begin{aligned}
\rho\partial_t u - \mu(x)\Delta u + \nabla\pi &= \rho f_u && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\
\operatorname{div} u &= f_d && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\
[\![u']\!] + c(t,x)\nabla' h &= g_u && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
-[\![\mu(x)(\nabla' u_n + \partial_n u')]\!] &= g_\tau && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
-2[\![\mu(x)\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h &= g_n && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
u(0) &= u_0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{7.1}$$

$$\begin{aligned}
\rho\kappa(x)\partial_t\theta - d(x)\Delta\theta &= \rho\kappa(x) f_\theta && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\
-[\![d(x)\partial_n\theta]\!] &= g_\theta && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
[\![\theta]\!] &= 0 && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
\theta(0) &= \theta_0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
-2[\![(\mu(x)/\rho)\partial_n u_n]\!] + [\![\pi/\rho]\!] &= g_\pi && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
\partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] + b(t,x)\cdot\nabla' h/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
h(0) &= h_0 && \text{on } \mathbb{R}^n_0.
\end{aligned} \tag{7.3}$$

Here $c(t,x)$ and $b(t,x)$ are substitutions for u_n and u' close to initial velocity u_{0n} and $u_0{}'$.

Since (7.2) decouples from the remaining problem and it is well-known that this problem has maximal L_p-L_q-regularity (cf. Denk, Hieber and Prüss [4]), we concentrate on the remaining one. It reduces to the *modified asymmetric Stokes problem*:

$$\begin{aligned}
\rho\partial_t u - \mu(x)\Delta u + \nabla\pi &= f_u && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\
\operatorname{div} u &= f_d && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\
[\![u']\!] + c(t,x)\nabla' h &= g && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
-[\![\mu(x)(\nabla' u_n + \partial_n u')]\!] &= g_\tau && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
-2[\![\mu(x)\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h &= g_u && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
-2[\![(\mu(x)/\rho)\partial_n u_n]\!] + [\![\pi/\rho]\!] &= g_\pi && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
\partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] + b(t,x)\cdot\nabla' h/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^n_0, \quad t > 0,\\
u(0) &= 0 && \text{in } \dot{\mathbb{R}}^n,\\
h(0) &= 0 && \text{on } \mathbb{R}^n_0,
\end{aligned} \tag{7.4}$$

where $\omega \geq 0$. We add ωu for the first equation and ωh for the sixth equation in order to consider on time interval $\mathbb{R}_+$. The differences of the modified asymmetric Stokes problem (7.4) from the asymmetric Stokes problem (2.4) in [5] are in where the modified (7.4) contains $c(t,x)\nabla' h$ and $b(t,x)\cdot\nabla' h/[\![\rho]\!]$ in the 3rd and the 6th equation, respectively, and $\mu(x)$ is a function not a constant. We set

$$\begin{aligned}
L_{p,0,\gamma_0}(\mathbb{R};X) &= \{f:\mathbb{R}\to X \mid e^{-\gamma_0 t}f(t)\in L_p(\mathbb{R};X),\ f(t)=0 \text{ for } t<0\},\\
W^m_{p,0,\gamma_0}(\mathbb{R};X) &= \{f\in L_{p,0,\gamma_0}(\mathbb{R};X) \mid e^{-\gamma_0 t}D^j_t f(t)\in L_p(\mathbb{R};X),\ j=1,\cdots,m\},\\
\hat{W}^1_q(\mathbb{R}^n) &= \{\theta\in L_{q,loc}(\mathbb{R}^n) \mid \nabla\theta\in L_q(\mathbb{R}^n)^n\},\\
\hat{W}^{-1}_q(\mathbb{R}^n) &= \hat{W}^1_q(\mathbb{R}^n)^*,\quad 1/q+1/q'=1,\\
\|\theta\|_{\hat{W}^{-1}_q(\mathbb{R}^n)} &= \sup_{\varphi\in\hat{W}^1_q(\mathbb{R}^n),\ \|\nabla\varphi\|_{L_{q'}(\mathbb{R}^n)}=1} |\int_{\mathbb{R}^n}\theta\varphi\,dx|,
\end{aligned}$$

$$\begin{aligned}
< D_t >^\alpha f(t) &= \mathcal{F}^{-1}[(1+s^2)^{\frac{a}{2}}\mathcal{F}[f](s)](t)\quad \text{for } a\geq 0,\\
H^a_{p,0,\gamma_0}(\mathbb{R};X) &= \{f:\mathbb{R}\to X \mid e^{-\gamma t} < D_t >^a f(t)\in L_p(\mathbb{R};X)\\
&\qquad\qquad \text{for any } \gamma\geq\gamma_0,\ f(t)=0 \text{ for } t<0\},
\end{aligned}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and its inverse respectively, and set $\hat{W}^{-1}_q(\mathbb{R}^n)$ the dual space of $\hat{W}^1_{q'}(\mathbb{R}^n)$, where $1/q+1/q'=1$.

We set the following function spaces.

$$\begin{aligned}
C_\ell(\dot{\mathbb{R}}^n) = \{u\in C(\dot{\mathbb{R}}^n) \mid \exists C_+,C_->0, \text{s.t. } \forall\varepsilon>0, \exists r_0>0\\
|u_+(x)-C_+|<\varepsilon,\ |u_-(x)-C_-|<\varepsilon\ \text{ for } x\in\dot{\mathbb{R}}^n\setminus\overline{B_{r_0}(0)}\}.
\end{aligned}$$

We set the function spaces of the solution:

$$\begin{aligned}
\mathbb{E}_{u,\gamma_0}(\mathbb{R}) &:= [W^1_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n))\cap L_{p,0,\gamma_0}(\mathbb{R};W^2_q(\dot{\mathbb{R}}^n))]^n,\\
\mathbb{E}_{\pi,\gamma_0}(\mathbb{R}) &:= L_{p,0,\gamma_0}(\mathbb{R};\hat{W}^1_q(\dot{\mathbb{R}}^n)),\\
\mathbb{E}_{\pi\pm,\gamma_0}(\mathbb{R}) &:= H^{1/2}_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n_\pm))\cap L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\mathbb{R}^n_\pm)),\\
\mathbb{E}_{h,\gamma_0}(\mathbb{R}) &:= W^1_{p,0,\gamma_0}(\mathbb{R};W^2_q(\dot{\mathbb{R}}^n))\cap L_{p,0,\gamma_0}(\mathbb{R};W^3_q(\dot{\mathbb{R}}^n))\cap W^2_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n)),\\
\mathbb{E}_{\gamma_0}(\mathbb{R}) &:= \mathbb{E}_{u,\gamma_0}(\mathbb{R})\times\mathbb{E}_{\pi,\gamma_0}(\mathbb{R})\times\mathbb{E}_{\pi\pm,\gamma_0}(\mathbb{R})\times\mathbb{E}_{h,\gamma_0}(\mathbb{R}).
\end{aligned}$$

We set the function spaces of right members:

$$\begin{aligned}
&\mathbb{F}_{u,\gamma_0}(\mathbb{R}) := L_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n))^n,\\
&\mathbb{F}_{d,\gamma_0}(\mathbb{R}) := W^1_{p,0,\gamma_0}(\mathbb{R};\hat{W}^{-1}_q(\mathbb{R}^n))\cap L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\mathbb{R}}^n)),\\
&\mathbb{G}_{u,\gamma_0}(\mathbb{R})\times\mathbb{G}_{h,\gamma_0}(\mathbb{R}) := [W^1_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n))\cap L_{p,0,\gamma_0}(\mathbb{R};W^2_q(\dot{\mathbb{R}}^n))]^n,\\
&\mathbb{G}_{\tau,\gamma_0}(\mathbb{R})\times\mathbb{G}_{n,\gamma_0}(\mathbb{R})\times\mathbb{G}_{\pi,\gamma_0}(\mathbb{R}) := [H^{1/2}_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n))\cap L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\mathbb{R}}^n))]^{n+1},\\
&\quad\mathbb{F}_{\gamma_0}(\mathbb{R}) := \mathbb{F}_{u,\gamma_0}(\mathbb{R})\times\mathbb{F}_{d,\gamma_0}(\mathbb{R})\times\mathbb{G}_{u,\gamma_0}(\mathbb{R})\\
&\qquad\qquad\times\mathbb{G}_{\tau,\gamma_0}(\mathbb{R})\times\mathbb{G}_{n,\gamma_0}(\mathbb{R})\times\mathbb{G}_{\pi,\gamma_0}(\mathbb{R})\times\mathbb{G}_{h,\gamma_0}(\mathbb{R}).
\end{aligned}$$

For the modified (7.4), we have the following maximal L_p-L_q regularity result.

Theorem 7.1. *Let* $2 < p < \infty$, $n < q < \infty$, $2/p + n/q < 1$. *We assume that* $\mu(x) \in BUC^1(\dot{\mathbb{R}}^n) \cap C_\ell(\dot{\mathbb{R}}^n)$, $\mu > 0$, $(b(t,x), c(t,x)) \in [W^1_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W^2_q(\dot{\mathbb{R}}^n))]^n$ *and* $(b(t,x), c(t,x)) \in BUC(\mathbb{R}, BUC^1(\dot{\mathbb{R}}^n) \cap C_\ell(\dot{\mathbb{R}}^n))$ *If the data* $(f_u, f_d, g_u, g_\tau, g_n, , g_\pi, g_h) \in \mathbb{F}_{\gamma_0}(\mathbb{R})$ *satisfy the compatibility conditions:*

$$f_d(0) = 0,\ g_u(0) = g_\tau(0) = 0 \quad in\ \dot{\mathbb{R}}^n,$$

then the modified asymmetric Stokes problem (7.4) *admits a unique solution* $(u, \pi, \pi_\pm, h) \in \mathbb{E}_{\gamma_0}(\mathbb{R})$. *There exists* $C_{\gamma_0} > 0$ *such that the following estimate holds:*

$$\|(u, \pi, \pi_\pm, h)\|_{\mathbb{E}_{\gamma_0}(\mathbb{R})} \le C\|(f_u, f_d, g_u, g, g_\pi, g_h)\|_{\mathbb{F}_{\gamma_0}(\mathbb{R})}. \tag{7.5}$$

If $u \in L_{p,o,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n))$ for some $\gamma_0 > 1$, then for any T with $0 < T < \infty$, it holds that

$$\|u\|_{L_p(0,T;L_q(\mathbb{R}^n))} \le e^{\gamma_0 T}\|e^{-\gamma_0 t}u\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))}.$$

Hence, we may view the nonlinear problem in following spaces. Let $J = [0,T]$. We set the function spaces of the solution:

$$\begin{aligned}
\mathbb{E}_u(J) &:= [(W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n))]^n,\\
\mathbb{E}_\pi(J) &:= L_p(J; \hat{W}^1_q(\dot{\mathbb{R}}^n)),\\
\mathbb{E}_{\pi_\pm}(J) &:= H^{1/2}_p(J; L_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^1_q(\mathbb{R}^n_\pm)),\\
\mathbb{E}_\theta(J) &:= W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n)),\\
\mathbb{E}_h(J) &:= W^1_p(J; W^2_q(\dot{\mathbb{R}}^n)) \cap L_p(J; W^3_q(\dot{\mathbb{R}}^n)) \cap W^2_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n)),\\
\mathbb{E}(J) &:= \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\pi_\pm}(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_h(J).
\end{aligned} \tag{7.6}$$

We set the function spaces of right members:

$$\begin{aligned}
&\mathbb{F}_u(J) \times \mathbb{F}_\theta(J) := L_p(J; L_q(\mathbb{R}^n))^{n+1},\\
&\mathbb{F}_d(J) := W^1_p(J; \hat{W}^{-1}_q(\mathbb{R}^n)) \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n)),\\
&\mathbb{G}_u(J) = \mathbb{G}_h(J) := [W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n))]^n,\\
&\mathbb{G}_\tau(J) \times \mathbb{G}_n(J) \times \mathbb{G}_\theta(J) \times \mathbb{G}_\pi(J) := [H^{1/2}_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^1_q(\dot{\mathbb{R}}^n))]^{n+2},\\
&\mathbb{F}(J) := \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{F}_\theta(J) \times \mathbb{G}_u(J) \times \mathbb{G}_\tau(J) \times \mathbb{G}_n(J)\\
&\qquad\qquad \times \mathbb{G}_\theta(J) \times \mathbb{G}_\pi(J) \times \mathbb{G}_h(J).
\end{aligned}$$

We define the time trace space X_γ of $\mathbb{E}(J)$ as

$$X_\gamma = B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1}).$$

The main result which is maximal L_p-L_q regularity for linearized problem (7.1)-(7.3) is stated as follows.

Theorem 7.2. *Let $2 < p < \infty$, $n < q < \infty$, $2/p + n/q < 1$. We assume that $\mu(x), d(x) \in BUC^1(\dot{\mathbb{R}}^n) \cap C_\ell(\dot{\mathbb{R}}^n)$, $\mu_\pm > 0$, $d_\pm > 0$, $(b(t,x), c(t,x)) \in [W^1_p(J; L_q(\mathbb{R}^n)) \cap L_p(J; W^2_q(\dot{\mathbb{R}}^n))]^n$ with $J = [0,T]$ and $(b(t,x), c(t,x)) \in C(J, BUC^1(\dot{\mathbb{R}}^n) \cap C_\ell(\dot{\mathbb{R}}^n))$. If $(f_u, f_d, f_\theta, g_u, g_\tau, g_n, g_\theta, g_\pi, g_h) \in \mathbb{F}(J)$, and the initial data*

$$(u_0, \theta_0, h_0) \in X_\gamma = B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

satisfy the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = f_d(0) \quad &\text{in } \dot{\mathbb{R}}^n, \quad 2-2/p > 1+1/q,\\ -[\![\mu(x)(\nabla' u_{0n} + \partial_n u_0{}')]\!] = g_\tau(0) \quad &\text{on } \mathbb{R}^{n-1}, \quad 2-2/p > 1+1/q,\\ [\![u_0{}']\!] + c(0,x)\nabla' h_0 = g_u(0), \quad [\![\theta_0]\!] = 0 \quad &\text{on } \mathbb{R}^{n-1}, \quad 2-2/p > 1/q,\\ -[\![d(x)\partial_n \theta_0]\!] = g_\theta(0) \quad &\text{on } \mathbb{R}^{n-1}, \quad 2-2/p > 1+1/q, \end{aligned}$$

then the linearized problem (7.1)-(7.3) *admits a unique solution $(u, \pi, \pi_\pm, \theta, h) \in \mathbb{E}(J)$.*

Theorem 7.2 is proved by combining Theorem 7.1 and the results within [23] and [4]. Therefore it is key to prove Theorem 7.1.

The plan for this part is as follows. In Section 8, we prove maximal L_p-L_q regularity of (7.4) in the case where μ, b and c are constant. In section 9, we prove Theorem 7.1. Section 10 is devoted to prove local L_p-L_q well-posedness of the problem of (1.1) (1.2) (1.3).

8. Maximal L_p-L_q Regularity; for Constant Coefficients

Let $\mathbb{R}^n_0 = \mathbb{R}^{n-1} \times \{0\}$ and $\dot{\mathbb{R}}^n = \mathbb{R}^n \backslash \mathbb{R}^n_0$. We solve (7.4) in the case where μ, c and b are constants:

$$\begin{aligned} \rho\partial_t u - \mu_0 \Delta u + \nabla \pi = f_u \quad &\text{in} \quad \dot{\mathbb{R}}^n, \quad t > 0,\\ \operatorname{div} u = f_d \quad &\text{in} \quad \dot{\mathbb{R}}^n, \quad t > 0,\\ [\![u']\!] + c_0 \nabla' h = g_u \quad &\text{on} \quad \mathbb{R}^n_0, \quad t > 0,\\ -[\![\mu_0(\partial_n u' + \nabla' u_n)]\!] = g_\tau \quad &\text{on} \quad \mathbb{R}^n_0, \quad t > 0,\\ -2[\![\mu_0 \partial_n u_n]\!] + [\![\pi]\!] - \sigma \Delta' h = g_n \quad &\text{on} \quad \mathbb{R}^n_0, \quad t > 0,\\ -2[\![(\mu_0/\rho)\partial_n u_n]\!] + [\![\pi/\rho]\!] = g_\pi \quad &\text{on} \quad \mathbb{R}^n_0, \quad t > 0,\\ \partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] + b_0 \cdot \nabla' h/[\![\rho]\!] = g_h \quad &\text{on} \quad \mathbb{R}^n_0, \quad t > 0,\\ u(0) = 0 \quad &\text{in} \quad \dot{\mathbb{R}}^n,\\ h(0) = 0 \quad &\text{on} \quad \mathbb{R}^n_0, \end{aligned} \tag{8.1}$$

where $c_0 \in \mathbb{R}$, $b_0 \in \mathbb{R}^{n-1}$, $\mu_{0\pm} > 0$ are constants. We assume as always in this paper $[\![\rho]\!] = \rho_+ - \rho_- \neq 0$.

For problem (8.1), we have the following maximal L_p-L_q regularity result.

Theorem 8.1. *Let $1 < p, q < \infty$, and assume that $\sigma > 0$, $\mu_{0\pm} > 0$, $c_0 \in \mathbb{R}$ and $b_0 \in \mathbb{R}^{n-1}$ are constants. Suppose the data $(f_u, f_d, g_u, g_\tau, g_n, , g_\pi, g_h) \in \mathbb{F}_{\gamma_0}(\mathbb{R})$ satisfy the compatibility conditions:*

$$f_d(0) = 0,\ g_u(0) = g_\tau(0) = 0 \quad \text{in } \dot{\mathbb{R}}^n.$$

Then the asymmetric Stokes problem (8.1) admits a unique solution $(u, \pi, \pi_\pm, h) \in \mathbb{E}_{\gamma_0}(\mathbb{R})$. There exists $C > 0$ such that the following estimate holds:

$$\|(u, \pi, \pi_\pm, h)\|_{\mathbb{E}_{\gamma_0}(\mathbb{R})} \le C\|(f_u, f_d, g_u, g, g_\pi, g_h)\|_{\mathbb{F}_{\gamma_0}(\mathbb{R})}. \tag{8.2}$$

In the rest of the section, we prove Theorem 8.1. If we set $u = \upsilon + w$ and $\pi = \tau + \kappa$ for a solution (u, π) of (8.1), then (υ, τ) and (w, κ) satisfy the following problems:

$$\begin{aligned}
\rho_\pm\partial_t\upsilon_\pm - \mu_{0\pm}\Delta\upsilon_\pm + \nabla\tau_\pm &= f_u && \text{in } \mathbb{R}^n_\pm,\ t > 0,\\
\text{div}\upsilon_\pm &= f_d && \text{in } \mathbb{R}^n_\pm,\ t > 0,\\
[\![\upsilon']\!] &= g_u && \text{on } \mathbb{R}^n_0,\ t > 0,\\
[\![\mu_0(\partial_n\upsilon_k + \partial_k\upsilon_n)]\!] &= -g_\tau && \text{on } \mathbb{R}^n_0,\ t > 0,\\
[\![2\mu_0\partial_n\upsilon_n]\!] - [\![\tau]\!] &= -g_n && \text{on } \mathbb{R}^n_0,\ t > 0,\\
\upsilon_\pm(0) &= 0 && \text{in } \mathbb{R}^n_\pm,\\
[\![(2\mu_0/\rho)\partial_n\upsilon_n]\!] - [\![\tau/\rho]\!] &= -g_\pi && \text{on } \mathbb{R}^n_0,\ t > 0.
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\rho_\pm\partial_t w_\pm - \mu_{0\pm}\Delta w_\pm + \nabla\kappa_\pm &= 0 && \text{in } \mathbb{R}^n_\pm,\ t > 0,\\
\text{div}w_\pm &= 0 && \text{in } \mathbb{R}^n_\pm,\ t > 0,\\
[\![w']\!] &= -c_0\nabla' h && \text{on } \mathbb{R}^n_0,\ t > 0,\\
[\![\mu_0(\partial_n w_k + \partial_k w_n)]\!] &= 0 && \text{on } \mathbb{R}^n_0,\ t > 0,\\
[\![2\mu_0\partial_n w_n]\!] - [\![\kappa]\!] &= -\sigma\Delta' h && \text{on } \mathbb{R}^n_0,\ t > 0,\\
w_\pm(0) &= 0 && \text{in } \mathbb{R}^n_\pm,\\
[\![(2\mu_0/\rho)\partial_n w_n]\!] - [\![\kappa/\rho]\!] &= 0 && \text{on } \mathbb{R}^n_0,\ t > 0,\\
\partial_t h - [\![\rho w_n]\!]/[\![\rho]\!] + b_0 \cdot \nabla' h/[\![\rho]\!] &= g_h + [\![\rho\upsilon_n]\!]/[\![\rho]\!] && \text{on } \mathbb{R}^n_0,\ t > 0,\\
h_\pm(0) &= 0 && \text{on } \mathbb{R}^n_0.
\end{aligned} \tag{8.4}$$

Let $\mathcal{F}_{x'}$ and $\mathcal{F}^{-1}_{\xi'}$ denote the partial Fourier transform with respect to x' and its inversion transform

$$\begin{aligned}
\mathcal{F}_{x'}[u(\cdot, x_n)](\xi') &= \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n)\ dx',\\
\mathcal{F}^{-1}_{\xi'}[u(\cdot, \xi_n)](x') &= (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} u(\xi', \xi_n)\ d\xi',
\end{aligned}$$

and let $\mathcal{L}_t$ and $\mathcal{L}^{-1}_s$ denote the Laplace transform and its inversion transform

$$\mathcal{L}_t[u](s) = \int_{\mathbb{R}} e^{-st} u(t)\ dt, \quad \mathcal{L}^{-1}_s[u](t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{st} u(s)\ d\tau.$$

We use the symbol: $\hat{u} = \mathcal{F}_{x'}\mathcal{L}_t[u]$. Set

$$A = |\xi'| \ , \ B_\pm = (\rho_\pm s/\mu_\pm + A^2)^{\frac{1}{2}}.$$

First we solve (8.3). We could deduce the case where $f_u = f_d = 0$ in the problem (8.3) (e.g. Shibata and Shimizu [24, Section 3]). Using the Fourier transform with respect to x' and the Laplace transform with respect to t, we can convert the problem (8.3) into ordinary differential equations of x_n. It was solved in [5]. By Section 4 in [5], we know that there exists a unique solution of (8.3) and which satisfies the estimate:

$$\begin{aligned}
&\|e^{-\gamma t}(\gamma v,\, \partial_t v,\, \nabla^2 v,\, \nabla\tau)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))} + \|e^{-\gamma t}(< D_t >^{\frac{1}{2}} \tau_\pm,\, \nabla\tau_\pm)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))}\\
&\le C_{\gamma_0}(\|e^{-\gamma t}(\gamma g_u,\, \partial_t g_u,\, \nabla^2 g_u, \nabla(g_\tau, g_n, g_\pi),\, < D_t >^{\frac{1}{2}} (g_\tau, g_n, g_\pi)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))}.
\end{aligned} \tag{8.5}$$

Next, we solve the problem (8.4). In order to solve (8.4), we can use the solution formula of (8.3) in [5] with $f_u = f_d = g_\tau = g_\pi = 0$, $g_u = -\nabla' h$ and $g_n = \sigma\Delta' h$, we obtain the solution formula:

$$\begin{aligned}
\hat{\kappa}_+ &= \mu_+ R e^{-Ax_n}, \hat{w}_{+m} = P_m e^{-Ax_n} + Q_m e^{-B_+ x_n} \quad \text{for } x_n > 0\\
\hat{\kappa}_- &= \mu_- R' e^{Ax_n}, \hat{w}_{-m} = P'_m e^{Ax_n} + Q'_m e^{B_- x_n} \quad \text{for } x_n < 0.
\end{aligned}$$

where we set $\alpha_\pm = -\mu_\pm A^2(3B_\pm - A)/(2B_\pm(B_\pm + A))$, $\beta = (\mu_+ B_+ + \mu_- B_-)/2$,

$$\begin{aligned}
R = (\alpha_+ + \alpha_-\beta)^{-1}\Big[&+(\mu_-(1-\rho_+/\rho_-))^{-1}(\alpha_- + \mu_- A^2/(2B_-))(-\sigma A^2\hat{h})\\
&+ (\mu_+(1-\rho_-/\rho_+))^{-1}(\alpha_- - \beta - \mu_+ A^2/(2B_+))(-\sigma A^2\hat{h})\Big],\\
R' = (\alpha_+ + \alpha_-\beta)^{-1}\Big[&+(\mu_+(1-\rho_-/\rho_+))^{-1}(-\alpha_+ - \mu_+ A^2/(2B_+))(-\sigma A^2\hat{h}),\\
&+ (\mu_-(1-\rho_+/\rho_-))^{-1}(-\alpha_+ + \beta + \mu_- A^2/(2B_-))(-\sigma A^2\hat{h})\Big],
\end{aligned}$$

$$\begin{aligned}
&P_k = -i\mu_+\xi_k R/(\rho_+ s), \quad P_n = \mu_+ AR/(\rho_+ s),\\
&P'_k = -i\mu_-\xi_k R'/(\rho_- s), \quad P'_n = -\mu_- AR'/(\rho_- s),\\
&Q_n = (2\mu_+ B_+(1-\rho_-/\rho_+))^{-1}(-\sigma A^2\hat{h}) - (A/B_+)P_n - R/(2B_+),\\
&Q'_n = (2\mu_- B_-(1-\rho_+/\rho_-))^{-1}(-\sigma A^2\hat{h}) - (A/B_-)P'_n + R'/(2B_-).
\end{aligned}$$

Substituting

$$\begin{aligned}
[\![\rho\hat{w}_n]\!] = &-\rho_+(B_+ - A)(2B_+(B_+ + A))^{-1}R - \rho_-(B_- - A)(2B_-(B_- + A))^{-1}R'\\
&- \sigma A^2\hat{h}(\rho_+/(2\mu_+ B_+(1-\rho_-/\rho_+)) - \rho_-/(2\mu_- B_-(1-\rho_+/\rho_-)))
\end{aligned}$$

with $B_\pm^2 = \rho_\pm s/\mu_\pm + A^2$ for the second equation from below in (8.4), finally we obtain the description of $\hat{h}$

$$\hat{h} = f(B_+, B_-, A)L(B_+, B_-, A)^{-1}(\hat{g}_h + [\![\rho\hat{v}_n]\!]/[\![\rho]\!]), \tag{8.6}$$

where

$$\begin{aligned} f(B_+, B_-, A) &= \mu_+ A^2(3B_+ - A)B_-(B_- + A) + \mu_- A^2(3B_- - A)B_+(B_+ + A) \\ &\quad + (\mu_+ B_+ + \mu_- B_-)B_+ B_-(B_+ + A)(B_- + A), \end{aligned} \tag{8.7}$$

and

$$\begin{aligned} L(B_+, B_-, A) &= sf(B_+, B_-, A) + ib_0 \cdot \xi' [\![\rho]\!]^{-1} f(B_+, B_-, A) \\ &\quad + [\![\rho]\!]^{-2} \sigma A^2 \{2\rho_+\rho_- A^2 (B_+ - A)(B_- - A) \\ &\quad + A^2\big(\rho_+^2 B_-(B_- + A) + \rho_-^2 B_+(B_+ + A)\big) \\ &\quad + A^3\big(\mu_-\mu_+^{-1}\rho_+^2(3B_- - A) + \mu_+\mu_-^{-1}\rho_-^2(3B_+ - A)\big) \\ &\quad + (\mu_+ B_+ + \mu_- B_-)A\big(\mu_+^{-1}\rho_+^2 B_-(B_- + A) + \mu_-^{-1}\rho_-^2 B_+(B_+ + A)\big) \\ &\quad - [\![\rho]\!]\sigma^{-1} c_0 \big(\mu_-\rho_+(B_+ - A)((B_- - A)^3 + 4B_-^2 A) \\ &\qquad + \mu_+\rho_-(B_- - A)((B_+ - A)^3 + 4B_+^2 A)\big)\}, \end{aligned} \tag{8.8}$$

We remark that $f(B_+, B_-, A)$ is the same function as in the solution formula $\hat{h}$ of (3.3) in [5], however $L(B_+, B_-, A)$ is different. If we put $b_0 = 0$ and $c_0 = 0$ in (8.8), then it is the same formula for $\hat{h}$ of (3.3) in [5].

We consider $\mathcal{R}$-boundedness of solution operators defined in a sector $\Sigma_{\epsilon,\gamma_0} = \{s \in \mathbb{C}\backslash\{0\} \mid |\arg s| \le \pi - \epsilon,\ |s| \ge \gamma_0\}$ with $0 < \epsilon < \pi/2$ and $\gamma_0 \ge 0$.

Lemma 8.2 (Lemma 4.1 in [5])**.** *For $l = 0, 1$, $\gamma_0 \ge 1$ and $\epsilon \in (0, \pi/2)$, we have*

$$|f(B_+, B_-, A)| \ge C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^5,$$
$$|D_{\xi'}^{\alpha'}(\tau D_\tau)^l f(B_+, B_-, A)^{-1}| \le C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-5} A^{-|\alpha'|}.$$

Lemma 8.3. *For $l = 0, 1$, $\gamma_0 \ge 1$ and $\epsilon \in (0, \pi/2)$,*

$$\begin{aligned} |L(B_+, B_-, A)| &\ge C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^3(|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2) \\ &\ge C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^6, \end{aligned} \tag{8.9}$$

$$\begin{aligned} &|D_{\xi'}^{\alpha'}(\tau D_\tau)^l L(B_+, B_-, A)^{-1}| \\ &\quad \le C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-3}(|s|(|s|^{1/2} + A)^2 + \sigma A^3/[\![\rho]\!]^2)^{-1} A^{-|\alpha'|} \\ &\quad \le C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^{-6} A^{-|\alpha'|} \end{aligned} \tag{8.10}$$

hold.

Proof. We use symbols that are used in Lemma 6.1 in [23]. Let δ and $\mathcal{O}(\delta)$ be a small number determined later and a symbol satisfying $|\mathcal{O}(\delta)| \le C\delta$ respectively. Suppose $\delta \le \min(\rho_+/\mu_+, \rho_-/\mu_-)$. First we prove (8.9) in the case where $|\rho_\pm \mu_\pm^{-1} s A^{-2}| \le \delta$. If we write $B_\pm = A(1 + \mathcal{O}(\delta))$, then we obtain from (8.6) and (8.7)

$$\begin{aligned} L(B_+, B_-, A) &= s(\mu_+ + \mu_-)A^5(9 + 16\mathcal{O}(\delta)) \\ &\qquad + [\![\rho]\!]^{-2}\sigma A^6(i[\![\rho]\!]\sigma^{-1} b_0 \cdot \xi' A^{-1})(9 + 16\mathcal{O}(\delta)) \end{aligned}$$

$$
\begin{aligned}
+[\![\rho]\!]^{-2}\sigma A^3\big(&A^3(\rho_+^2+\rho_-^2)(2+3\mathcal{O}(\delta))\\
&+A^3((\mu_-\mu_+^{-1}\rho_+^2+\mu_+\mu_-^{-1}\rho_-^2)(2+3\mathcal{O}(\delta)))\\
&+A^3(\mu_++\mu_-)(\mu_+^{-1}\rho_+^2+\mu_-^{-1}\rho_-^2)(2+3\mathcal{O}(\delta))\\
&-A^3[\![\rho]\!]\sigma^{-1}c_0(\mu_-\rho_++\mu_+\rho_-)17\mathcal{O}(\delta)\big).
\end{aligned}
$$

Now, $A \geq 2^{-1}(|s|^{1/2}+A)$. Therefore, in the same way as the proof of Lemma 6.1 in [23], choosing a δ properly, we gain

$$
\begin{aligned}
|L(B_+,B_-,A)| &\geq C_{\epsilon,\gamma_0}(|s|^{1/2}+A)^3(|s|(|s|^{1/2}+A)^2+\sigma A^3/[\![\rho]\!]^2)\\
&\geq C_{\epsilon,\gamma_0}(|s|^{1/2}+A)^6.
\end{aligned}
$$

Secondly, we prove (8.9) in the case where $|\rho_\pm\mu_\pm^{-1}sA^{-2}| \geq \delta$. By Lemma 4.6, Lemma 4.8 in [23],

$$
\begin{aligned}
|L(B_+,B_-,A)| &\geq (|s|-C_1A)|f(B_+,B_-,A)|-C_2\sigma[\![\rho]\!]^{-2}A^3(|s|^{1/2}+A)^3\\
&\geq C_3(|s|^{1/2}+A)^3((|s|-C_1A)(|s|^{1/2}+A)^2-\sigma[\![\rho]\!]^{-2}A^3)
\end{aligned}
$$

Because

$$
A \leq (\min(\rho_+/\mu_+,\rho_-/\mu_-))^{1/2}\delta^{-1/2}|s|^{1/2}\ ,\ |s|^{-1}\leq\gamma_0^{-1},
$$

there exist $C_3, C_4 > 0$ such that

$$
\begin{gathered}
|s|-C_1A \geq (1-C_3\gamma_0^{-1/2})|s|,\\
|s|(|s|^{1/2}+A)^2-[\![\rho]\!]^{-2}A^3 \geq |s|^2(1-C_4\gamma_0^{-1/2}).
\end{gathered}
$$

Combining the inequality above and

$$
\begin{aligned}
|s|^2=(|s|^2+|s|^2)/2 &\geq C(|s|(|s|^{1/2}+A)^2+\sigma A^3/[\![\rho]\!]^2)\\
&\geq C_{\epsilon,\gamma_0}\gamma_0^{1/2}(|s|^{1/2}+|s|^{1/2})(|s|^{1/2}+A)^2\\
&\geq C_{\epsilon,\gamma_0}(|s|^{1/2}+A)^3,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|L(B_+,B_-,A)| &\geq C_{\epsilon,\gamma_0}(|s|^{1/2}+A)^3(|s|(|s|^{1/2}+A)^2+\sigma A^3/[\![\rho]\!]^2)\\
&\geq C_{\epsilon,\gamma_0}(|s|^{1/2}+A)^6
\end{aligned}
$$

from (8.8). By the Bell formula, we obtain (8.10) as the similar manner in the proof of Lemma 6.1 in [23]. □

In case we use an extension

$$
\hat{h}=f(B_+,B_-,A))L(B_+,B_-,A)^{-1}e^{-Ax_n}(\hat{g}_h+[\![\rho\hat{v}_n]\!]/[\![\rho]\!]),
$$

by Lemma 5.4 in [24] Lemma 2.1 and Lemma 2.2,

$$
e^{-\gamma t}\Delta h, e^{-\gamma t}\partial_t\Delta h, e^{-\gamma t}\nabla^3 h \in L_p(\mathbb{R}, L_q(\mathbb{R}^n_+)).
$$

Then, since the problem (8.4) is the case where we set $f_u = f_d = g_\tau = 0$ and $g_n = \sigma\Delta' h$ in the problem (8.3) and add two equations below in (8.4), estimates (8.5) hold for $\kappa_\pm$ and $w_\pm$, too. When we prove $e^{-\gamma t}h \in L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))$, we use an extension;

$$\hat{h} = f(B_+, B_-, A)L(B_+, B_-, A)^{-1}e^{-B_+x_n}(\hat{g}_h + [\![\rho\hat{v}_n]\!]/[\![\rho]\!])$$

and the identity;

$$\frac{f(B_+, B_-, A)}{L(B_+, B_-, A)} = \frac{1}{s} - \frac{L(B_+, B_- A) - sf(B_+, B_-, A)}{sL(B_+, B_-, A)}.$$

It is clear that

$$|D^{\alpha'}_{\xi'}(\tau D_\tau)^l s^{-1}| \le |s|^{-1}(|s|^{1/2} + A)^{-|\alpha'|} \le \gamma_0^{-1}(|s|^{1/2} + A)^{-|\alpha'|}$$

for $l = 0, 1$. By (8.8), $L(B_+, B_-, A) - sf(B_+, B_-, A) = \sigma[\![\rho]\!]^{-2}AM(s, \xi')$ where $M(s, \xi')$ is a function satisfying $|D^{\alpha'}_{\xi'}(\tau D_\tau)^l M(s, \xi')| \le C_{\epsilon,\gamma_0}(|s|^{1/2} + A)^5 A^{-|\alpha'|}$. In view of Lemma 5.4 in [24], we see

$$e^{-\gamma t}h, e^{-\gamma t}\partial_t h, e^{-\gamma t}\partial_t^2 h \in L_p(\mathbb{R}, L_q(\mathbb{R}^n_+)).$$

For example, in fact, we could obtain that

$$\mathcal{F}_{x'}\mathcal{L}_t[\partial_t^2 h] = s^2\hat{h} = \left(s^{\frac{1}{2}} - \frac{\sigma[\![\rho]\!]^{-2}s^{\frac{1}{2}}M(s, \xi')}{L(B_+, B_-, A)}A\right)e^{-B_+x_n}\left(s^{\frac{1}{2}}\hat{g}_h + s^{\frac{1}{2}}\frac{[\![\rho\hat{v}_n]\!]}{[\![\rho]\!]}\right),$$

$$|D^{\alpha'}_{\xi'}(\tau D_\tau)^l 1| \le C_{\epsilon,\gamma_0}(|s|^{\frac{1}{2}} + A)^{-|\alpha'|}$$

and

$$\left|D^{\alpha'}_{\xi'}(\tau D_\tau)^l \frac{\sigma[\![\rho]\!]^{-2}s^{\frac{1}{2}}M(s, \xi')}{L(B_+, B_-, A)}\right| \le C_{\epsilon,\gamma_0}A^{-|\alpha'|}.$$

After all, it holds that

$$\begin{aligned} &\|e^{-\gamma t}(\gamma h, \partial_t h, \nabla h)\|_{L_p(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\partial_t^2 h\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))} \\ &\qquad \le C(\|e^{-\gamma t}(\gamma g_h, \partial_t g_h, \nabla^2 g_h, \partial_t v_n, < D_t >^{\frac{1}{2}} v_n)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))}. \end{aligned} \tag{8.11}$$

In the same way as the section 3 in [24], we could prove that v, w, τ, κ vanish for $t < 0$. This completes the proof of Theorem 8.1.

9. Maximal L_p-L_q Regularity; for Variable Coefficients

In this section, we prove Theorem 7.1. We set

$$\tilde{\mu}_+(x) = \begin{cases} \mu_+(x) & x \in \mathbb{R}^n_+ \\ \mu_+(-x) & x \in \mathbb{R}^n_- \end{cases}, \quad \tilde{\mu}_-(x) = \begin{cases} \mu_-(-x) & x \in \mathbb{R}^n_+ \\ \mu_-(x) & x \in \mathbb{R}^n_-, \end{cases}$$

and $\tilde{f}_\pm$ and $\tilde{d}_\pm$are defined similarly by even extension. Cauchy problems of the Stokes equation with variable coefficients

$$\begin{aligned} \rho_+\partial_t\tilde{u}_+ - \tilde{\mu}_+(x)\Delta\tilde{u}_+ + \nabla\tilde{\pi}_+ = \rho_+\tilde{f}_{u+}, \quad &\operatorname{div}\tilde{u}_+ = \tilde{f}_{d+} \quad \text{in } \mathbb{R}^n, \quad t > 0, \\ &\tilde{u}_+(0) = 0 \qquad \text{in } \mathbb{R}^n. \end{aligned} \tag{9.1}$$

$$\begin{aligned} \rho_-\partial_t\tilde{u}_- - \tilde{\mu}_-(x)\Delta\tilde{u}_- + \nabla\tilde{\pi}_- = \rho_-\tilde{f}_{u-}, \quad \operatorname{div}\tilde{u}_- &= \tilde{f}_{d-} \quad \text{in } \mathbb{R}^n, \quad t>0,\\ \tilde{u}_-(0) &= 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{9.2}$$

are solved for $0 < \tilde{\mu}_\pm(x) \in BUC^1(\mathbb{R}^n) \cap C_\ell(\mathbb{R}^n)$ (cf. [?]). We set the solutions of (9.1) and (9.2) $(\tilde{u}_+, \tilde{\pi}_+)$ and $(\tilde{u}_-, \tilde{\pi}_-)$ respectively and $(\tilde{u}, \tilde{\pi}) = (\tilde{u}_+, \tilde{\pi}_+)$ for $x \in \mathbb{R}^n_+$ and $(\tilde{u}, \tilde{\pi}) = (\tilde{u}_-, \tilde{\pi}_-)$ for $x \in \mathbb{R}^n_-$. Then we reduce the problem (7.4) to the case $f_u = f_d = 0$ by setting $(u - \tilde{u}, \pi - \tilde{\pi})$.

9.1. $\mu(x)$**.** In this subsection, we consider the case where $\mu(x)$ is a variable coefficient and b and c are constants b_0 and c_0. By the assumption $\mu(x) \in C_\ell(\dot{\mathbb{R}}^n)$, there exists a large ball $B_{r_0}(0)$ and constants $C_\pm > 0$ such that for every $\varepsilon > 0$,

$$|\mu_+(x) - C_+| < \varepsilon, \ |\mu_-(x) - C_-| < \varepsilon \ \text{ for } x \in \dot{\mathbb{R}}^n \setminus \overline{B_{r_0}(0)}.$$

We set $U_0 = \dot{\mathbb{R}}^n \setminus \overline{B_{r_0}(0)}$. Since $\overline{B_{r_0}(0)}$ is compact and $\mu(x)$ is continuous, $\overline{B_{r_0}(0)}$ is covered by finite number of open balls $U_j = B_{r_j}(x_j)$ such that

$$|\mu(x) - \mu(x_j)| < \varepsilon \quad \text{if} \quad |x - x_j| < r_j \quad j = 1, \dots N.$$

Define coefficients $\mu^j(x)$ $(j = 0, 1, \dots, N)$ by reflection, i.e.

$$\mu^0(x) = \begin{cases} \mu(x) & x \notin \overline{B_{r_0}(0)} \\ \mu(r_0^2 \frac{x}{|x|^2}) & x \in \overline{B_{r_0}(0)}, \end{cases}$$

$$\mu^j(x) = \begin{cases} \mu(x) & x \in \overline{B_{r_j}(x_j)} \\ \mu(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}) & x \notin \overline{B_{r_j}(x_j)}, \end{cases}$$

for $j = 1, \dots N$ (cf. Section 5 in [?]). Then for each fixed j, $\mu^j(x)$ is uniformly continuous, i.e., it holds that

$$|\mu^j(x) - \mu(x_j)| < \varepsilon \quad \text{for} \quad \forall x \in \dot{\mathbb{R}}^n, \quad j = 0, 1 \dots N.$$

From Theorem 8.1, we obtain that the problem with coefficients $\mu^j(x)$ for fixed j:

$$\begin{aligned} \rho\partial_t u - \mu^j(x)\Delta u + \nabla\pi &= 0 && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\ \operatorname{div} u &= 0 && \text{in } \dot{\mathbb{R}}^n, \quad t > 0,\\ [\![u']\!] + c_0\nabla' h &= g && \text{on } \mathbb{R}^n_0, \quad t > 0,\\ -[\![\mu^j(x)(\nabla' u_n + \partial_n u')]\!] &= g_\tau && \text{on } \mathbb{R}^n_0, \quad t > 0,\\ -2[\![\mu^j(x)\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h &= g_u && \text{on } \mathbb{R}^n_0, \quad t > 0,\\ -2[\![(\mu^j(x)/\rho)\partial_n u_n]\!] + [\![\pi/\rho]\!] &= g_\pi && \text{on } \mathbb{R}^n_0, \quad t > 0,\\ \partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] + b_0 \cdot \nabla' h/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^n_0, \quad t > 0,\\ u(0) &= 0 && \text{in } \dot{\mathbb{R}}^n,\\ h(0) &= 0 && \text{on } \mathbb{R}^n_0. \end{aligned} \tag{9.3}$$

has maximal L_p-L_q regularity and the estimate (8.2).

We introduce cut-off functions

$$\varphi_j \in C^\infty(\mathbb{R}^n) \text{ s.t. } 0 \le \varphi_j(x) \le 1,\ \operatorname{supp}\varphi_j \in U_j,\ \sum_{j=1}^N \varphi_j(x) = 1,$$
$$\psi_j \in C^\infty(\mathbb{R}^n) \text{ s.t. } \psi_j(x) = 1 \text{ on } \operatorname{supp}\varphi_j,\ \operatorname{supp}\psi_j \subset U_j.$$

Multiplying (9.4) by φ_j we obtain

$$\begin{aligned}
&\rho\partial_t(\varphi_j u) - \mu^j(x)\Delta(\varphi_j u) + \nabla(\varphi_j \pi) && , \\
&\quad = -\mu^j(x)\{(\Delta\varphi_j)u + 2\nabla\varphi_j \cdot \nabla u)\} + (\nabla\varphi_j)\pi && \text{in } \dot{\mathbb{R}}^n, \quad t>0, \\
&\operatorname{div}(\varphi_j u) = (\nabla\varphi_j)u && \text{in } \dot{\mathbb{R}}^n, \quad t>0, \\
&[\![\varphi_j u']\!] + c_0\nabla'(\varphi_j h) = \varphi_j g_u + c_0(\nabla'\varphi_j)h && \text{on } \mathbb{R}^n_0, \quad t>0, \\
&-[\![\mu^j(x)\{\nabla'(\varphi_j u_n) + \partial_n(\varphi_j u')\}]\!] && \\
&= \varphi_j g_\tau - [\![\mu^j(x)\{(\nabla'\varphi_j)u_n + (\partial_n\varphi_j)u'\}]\!] && \text{on } \mathbb{R}^n_0, \quad t>0, \\
&-2[\![\mu^j(x)\partial_n(\varphi_j u_n)]\!] + [\![\varphi_j\pi]\!] - \sigma\Delta'(\varphi_j h) && \\
&\quad = \varphi_j g_n - 2[\![\mu^j(x)(\partial_n\varphi_j)u_n]\!] - \sigma(\Delta'\varphi_j)h - 2\sigma\nabla'\varphi_j\cdot\nabla' h && \text{on } \mathbb{R}^n_0, \quad t>0, \\
&-2[\![(\mu^j(x)/\rho)\partial_n(\varphi_j u_n)]\!] + [\![(\varphi_j\pi)/\rho]\!] && \\
&\quad = \varphi_j g_\pi - 2[\![(\mu^j(x)/\rho)(\partial_n\varphi_j)u_n]\!] && \text{on } \mathbb{R}^n_0, \quad t>0, \\
&\partial_t(\varphi_j h) - [\![\rho(\varphi_j u_n)]\!]/[\![\rho]\!] + b_0\cdot\nabla'(\varphi_j h)/[\![\rho]\!] && \\
&\quad = \varphi_j g_h + b_0\cdot(\nabla'\varphi_j)h/[\![\rho]\!] && \text{on } \mathbb{R}^n_0, \quad t>0, \\
&(\varphi_j u)(0) = 0 && \text{in } \dot{\mathbb{R}}^n, \\
&(\varphi_j h)(0) = 0 && \text{on } \mathbb{R}^n_0.
\end{aligned} \tag{9.4}$$

From (8.2) $\{(\varphi_j u, \varphi_j\pi, \varphi_j h)\}_{j=1}^N$ satisfy the estimate

$$\begin{aligned}
&\|(\varphi_j u, \varphi_j\pi, \varphi_j\pi_\pm, \varphi_j h)\|_{\mathbb{E}_{\gamma_0}(\mathbb{R})} \le C_{\gamma_0}\Big(\|\varphi_j(g_u, g_\tau, g_n, g_\pi, g_h)\|_{\mathbb{F}_{\gamma_0}(\mathbb{R})} \\
&\quad + \|e^{-\gamma t}\mu^j(x)\{(\Delta\varphi_j)u + 2\nabla\varphi_j\cdot\nabla u)\}\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} + \|e^{-\gamma t}(\nabla\varphi_j)\pi\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} \\
&\quad + \|(\nabla\varphi_j)u\|_{\mathbb{F}_{\gamma_0,d}(\mathbb{R})} + \|e^{-\gamma t}(\nabla'\varphi_j)h\|_{L_p(\mathbb{R};W^2_q(\dot{\mathbb{R}}^n)\cap H^1_p(\mathbb{R};L_q(\mathbb{R}^n))} \\
&\quad + \|e^{-\gamma t}\{\mu^j(x)(\nabla\varphi_j)u + (\Delta\varphi_j)h + \nabla'\varphi_j\cdot\nabla h\}\|_{L_p(\mathbb{R};W^1_q(\dot{\mathbb{R}}^n)\cap H^{\frac12}_p(\mathbb{R};L_q(\mathbb{R}^n))}\Big).
\end{aligned} \tag{9.5}$$

for $\gamma \ge \gamma_0$. For any $\epsilon > 0$, it holds that

$$\begin{aligned}
&\|e^{-\gamma t}\nabla u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} \le \epsilon\|e^{-\gamma t}\nabla^2 u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} + (4\epsilon\gamma)^{-1}\|e^{-\gamma t}\gamma u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))}, \\
&\|e^{-\gamma t}\nabla^2 h\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} \\
&\qquad\qquad \le \epsilon\|e^{-\gamma t}\nabla^3 h\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} + (4\epsilon\gamma)^{-1}\|e^{-\gamma t}\gamma\nabla h\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))}.
\end{aligned} \tag{9.6}$$

For any $R \in \mathbb{R}$ with $1 \le R < \infty$ and $\gamma \ge 1$, it holds that

$$\begin{aligned}
&\|e^{-\gamma t} < D_t >^{\frac{1}{2}} u\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))} \\
&\qquad \le CR^{-\frac{1}{2}} \|e^{-\gamma t} \partial_t u\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))} + CR^{\frac{1}{2}} \|e^{-\gamma t} \gamma u\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))}, \\
&\|e^{-\gamma t} < D_t >^{\frac{1}{2}} \nabla h\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))} \\
&\qquad \le CR^{-\frac{1}{2}} \|e^{-\gamma t} \partial_t \nabla h\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))} + CR^{\frac{1}{2}} \|e^{-\gamma t} \gamma \nabla h\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))} \qquad (9.7)
\end{aligned}$$

by Proposition 2.6 in [22]. Using (9.6), (9.7) and taking $\epsilon > 0$ sufficiently small and $R \ge 1$ sufficiently large, we absorb the right hand side norms except $\|e^{-\gamma t}(\nabla \varphi_j)\pi\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))}$ and $\|(\nabla \varphi_j) u\|_{\mathbb{F}_{\gamma_0, d}(\mathbb{R})}$ into the left hand side. We set the left hand side of the first equation of (9.4) F_j. In order to treat $\|e^{-\gamma t}(\nabla \varphi_j)\pi\|_{L_p(\mathbb{R}; L_q(\mathbb{R}^n))}$ and $\|(\nabla \varphi_j) u\|_{\mathbb{F}_{\gamma_0, d}(\mathbb{R})}$, we consider the following problem for φ_j and F_j $(j = 0, 1, \ldots, N)$ as in Subsection 7.2 in [16]

$$\begin{aligned}
\Delta \phi_j = u \cdot \nabla \varphi_j &= \operatorname{div}(u \varphi_j) && \text{in } \dot{\mathbb{R}}^n, \\
[\![\partial_n \phi_j]\!] &= [\![u_n \varphi_j]\!] && \text{on } \mathbb{R}^{n-1}, \\
[\![\phi_j]\!] &= \phi_j = 0 && \text{on } \mathbb{R}^{n-1}, \\
\Delta \psi_j &= \operatorname{div} F_j && \text{in } \dot{\mathbb{R}}^n, \\
[\![\psi_j]\!] &= \psi_j = 0 && \text{on } \mathbb{R}^{n-1} \qquad (9.8)
\end{aligned}$$

with $\omega \ge 0$. The system is uniquely solvable and

$$\nabla \phi_j \in H^1_p(\mathbb{R}_+; L_q(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; H^3_q(\dot{\mathbb{R}}^n)), \quad \nabla \psi_j \in L_p(\mathbb{R}; L_q(\mathbb{R}^n)).$$

Defining

$$\widetilde{\phi_j u} = \phi_j u - \nabla \phi_j, \quad \widetilde{\phi_j \pi} = \phi_j \pi - \psi_j + \rho \partial_t \phi_j - \mu(x) \Delta \phi_j.$$

Along Corollary 1 in [11] and [**?**], we see that there are situations where π has additional time regularity in $\dot{\mathbb{R}}^n$.

Corollary 9.1. *Assume in addition to the hypotheses of Theorem 7.1 that*

$$\begin{aligned}
&u_0 = 0, \quad h_0 = f_d = 0, \quad \mathit{div}\, f_u = 0 && \textit{in } \dot{\mathbb{R}}^n, \\
&[\![u_n]\!] = 0, \quad [\![f_{un}]\!] = 0 && \textit{on } \mathbb{R}^n_0.
\end{aligned}$$

Then $\pi \in H^\alpha_{\gamma_0, 0, p}(\mathbb{R}; L_q(\mathbb{R}^n))$ *for each* $\alpha \in (0, \frac{1}{2} - \frac{1}{2p})$.

Proof. Since $C^\infty_0(\mathbb{R}^n)$ is dense in $L_q(\mathbb{R}^n)$, we give $g \in C^\infty_0(\mathbb{R}^n)$ and consider the problem

$$\begin{aligned}
\Delta \psi &= g && \text{in } \dot{\mathbb{R}}^n, \\
[\![\rho \psi]\!] &= 0 && \text{on } \mathbb{R}^n_0, \\
[\![\partial_n \psi]\!] &= 0 && \text{on } \mathbb{R}^n_0.
\end{aligned}$$

This problem is uniquely solvable. We set $\phi = \rho \psi$. Then

$$(\pi, g)_{\mathbb{R}^n} = (\pi, \Delta \psi)_{\mathbb{R}^n} = (\frac{\pi}{\rho}, \Delta \phi)_{\mathbb{R}^n}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^{n-1}} [\![\frac{\pi}{\rho}\partial_n\phi]\!]\,dx' - (\frac{\nabla\pi}{\rho},\nabla\phi)_{\mathbb{R}^n}\\
&= -\int_{\mathbb{R}^{n-1}} [\![\frac{\pi}{\rho}\partial_n\phi]\!]\,dx' - (\frac{\mu}{\rho}\Delta u,\nabla\phi)_{\mathbb{R}^n}\\
&= \int_{\mathbb{R}^n} \frac{\mu}{\rho}\nabla u : \nabla^2\phi\,dx + \int_{\mathbb{R}^{n-1}} \left([\![\frac{\mu\partial_n u}{\rho}\nabla\phi]\!] - [\![\frac{\pi}{\rho}\partial_n\phi]\!]\right)dx'.
\end{aligned}
$$

We know $\nabla u \in H^{\frac12}_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n))$, and also by Section 7 in Prüss-Simonett [**?**] $e^{-\gamma t}[\![\pi]\!]$, $e^{-\gamma t}[\![\partial_n u]\!] \in F^{\frac12-\frac1{2p}}_{p,q}(\mathbb{R}; L_q(\mathbb{R}^{n-1}))$ for $\gamma \ge \gamma_0$, where $F^s_{p,q}$ is a Lizorkin-Triebel space. Applying ∂_t^α to this identity we obtain
$\|e^{-\gamma t}{<}D_t{>}^\alpha\partial_t^\alpha\pi\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))} \le C\left(\|e^{-\gamma t}({<}D_t{>}^\alpha(\nabla u,\ [\![\pi]\!],\ \partial_n u))\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))}\right)$
for each $\alpha \in (0, 1/2 - 1/2p)$, which completes the proof of the corollary. □

When $\{(\varphi_j u, \varphi_j\pi, \varphi_j h)\}_{j=1}^N$ change into $\{(\widetilde{\varphi_j u}, \widetilde{\varphi_j\pi}, \varphi_j h)\}_{j=1}^N$, the first and second equations of (9.4) are changed into

$$
\begin{aligned}
&\rho\partial_t(\widetilde{\varphi_j u}) - \mu^j(x)\Delta(\widetilde{\varphi_j u}) + \nabla(\widetilde{\varphi_j\pi}) = F_j - \nabla\psi_j && \text{in } \dot{\mathbb{R}}^n, \quad t>0,\\
&\operatorname{div}(\widetilde{\varphi_j u}) = 0 && \text{in } \dot{\mathbb{R}}^n, \quad t>0,
\end{aligned}
$$

with

$$
\begin{aligned}
&\operatorname{div}(F_j - \nabla\psi_j) = 0 && \text{in } \dot{\mathbb{R}}^n,\\
&[\![(F_j - \nabla\psi_j)_n]\!] = [\![(\widetilde{\varphi_j u})]\!] = 0 && \text{on } \mathbb{R}^n_0.
\end{aligned}
$$

Therefore $\|(\nabla\varphi_j)u\|_{\mathbb{F}_{\gamma_0,d}(\mathbb{R})}$ is absence and $\|e^{-\gamma t}(\nabla\varphi_j)\pi\|_{L_p(\mathbb{R};L_q(\mathbb{R}^n))}$ has time regularity.

Taking finite sum $j = 1$ to N for (9.5), we obtain the estimate (7.5). Setting the left hand side operator $L : \mathbb{E}_{\gamma_0}(\mathbb{R}) \to \mathbb{F}_{\gamma_0}(\mathbb{R})$, then we obtain L is injective and has closed range. Surjectivity of L is proved in similar way in [16] with base results are obtained in Section 7 in [**?**].

9.2. $b(t,x)$**,** $c(t,x)$**.** In this subsection, we consider the case where not only $\mu(x)$ but also $b(t,x)$ and $c(t,x)$ are variable coefficients. Since

$$b(t,x), c(t,x) \subset BUC(\mathbb{R}, BUC^1(\dot{\mathbb{R}}^n))$$

by the assumption, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t - t_j| < \delta \quad \Rightarrow \quad |c(t,\cdot) - c(t_j,\cdot)|_{BUC^1(\dot{\mathbb{R}}^n)} < \epsilon$$

We set $U_1 = [0,\delta)$, $U_j = ((j-1)\delta, (j+1)\delta)$, $j = 1, 2, \ldots$.. We choose a partition of unity $\chi_j \in C^\infty(\mathbb{R}_+)$ as

$$\sum_{j=0}^{\infty}\chi_j(t) = 1 \quad \text{on } \mathbb{R}_+, \quad 0 \le \chi_j(t) \le 1, \quad \operatorname{supp}\chi_j \subset U_j.$$

We set

$$c_j(x) = \frac{1}{\delta}\int_{j\delta}^{(j+1)\delta} c(t,x)\,dt.$$

$b_j(x)$ is defined similarly. Multiplying (7.4) by χ_j we obtain

$$\begin{aligned}
&\rho\partial_t(\chi_j u) - \mu(x)\Delta(\chi_j u) + \nabla(\chi_j \pi) = \rho(\partial_t \chi_j)u && \text{in } \dot{\mathbb{R}}^n, \quad t>0,\\
&\mathrm{div}\,(\chi_j u) = 0 && \text{in } \dot{\mathbb{R}}^n, \quad t>0,\\
&[\![\chi_j u']\!] + c_j(x)\nabla(\chi_j h) && \\
&\quad = \chi_j g_u + c_j(x)(\nabla\chi_j)h - \chi_j\big((c(t,x) - c_j(x)\big)\nabla h && \text{on } \mathbb{R}^n_0, \quad t>0,\\
&-[\![\mu(x)\{\nabla'(\chi_j u_n) + \partial_n(\chi_j u')\}]\!] = \chi_j g_\tau && \text{on } \mathbb{R}^n_0, \quad t>0,\\
&-2[\![\mu(x)\partial_n(\chi_j u_n)]\!] + [\![\chi_j\pi]\!] - \sigma\Delta'(\chi_j h) = \chi_j g_n && \text{on } \mathbb{R}^n_0, \quad t>0,\\
&-2[\![(\mu(x)/\rho)\partial_n(\chi_j u_n)]\!] + [\![(\chi_j\pi)/\rho]\!] = \chi_j g_\pi && \text{on } \mathbb{R}^n_0, \quad t>0,\\
&\partial_t(\chi_j h) - \frac{[\![\rho(\chi_j u_n)]\!]}{[\![\rho]\!]} + \frac{[\![b(x)\cdot\nabla'(\chi_j h)]\!]}{[\![\rho]\!]} = \chi_j g_h + (\partial_t\chi_j)h && \\
&\quad + \frac{[\![b(x)\cdot(\nabla'\chi_j)h]\!]}{[\![\rho]\!]} - \frac{[\![\chi_j(b(t,x) - b(x))\cdot\nabla' h]\!]}{[\![\rho]\!]} && \text{on } \mathbb{R}^n_0, \quad t>0,\\
&(\chi_j u)(0) = 0 && \text{in } \dot{\mathbb{R}}^n,\\
&(\chi_j h)(0) = 0 && \text{on } \mathbb{R}^n_0.
\end{aligned} \tag{9.9}$$

By using the result in Subsection 4.1, we obtain that $\{(\chi_j u, \chi_j\pi, \chi_j\pi_\pm, \chi_j h)\}_{j=1}^\infty$ satisfy the estimate

$$\begin{aligned}
&\|(\chi_j u, \chi_j\pi, \chi_j\pi_\pm, \chi_j h)\|_{\mathbb{E}_{\gamma_0}(\mathbb{R})} \le \|\chi_j(g_u, g_\tau, g_n, g_\pi, g_h)\|_{\mathbb{F}_{\gamma_0}(\mathbb{R})}\\
&+ \|\rho(\partial_t\chi_j)u\|_{\mathbb{F}_{u,\gamma_0}(\mathbb{R})} + \|(\partial_t\chi_j)h\|_{\mathbb{G}_{h,\gamma_0}(\mathbb{R})}\\
&+ |c_j(x)\nabla\chi_j|_\infty \|h\|_{\mathbb{G}_{u,\gamma_0}(\mathbb{R})(U_j)} + \epsilon\|\nabla h\|_{\mathbb{G}_{u,\gamma_0}(\mathbb{R})(U_j)}\\
&+ [\![\rho]\!]^{-1}|c_j(x)\nabla\chi_j|_\infty \|h\|_{\mathbb{G}_{h,\gamma_0}(\mathbb{R})(U_j)} + \epsilon[\![\rho]\!]^{-1}\|\nabla h\|_{\mathbb{G}_{h,\gamma_0}(\mathbb{R})(U_j)}
\end{aligned} \tag{9.10}$$

where we use for every $\epsilon > 0$

$$|c(t,x) - c_j(x)| \le \frac{1}{\delta}\int_{j\delta}^{(j+1)\delta} |c(t,x) - c(s,x)|\,ds < \epsilon.$$

in front of the highest order term $\|\nabla h\|_{G_{u,\gamma_0}(\mathbb{R})}$ and $\|\nabla h\|_{G_{h,\gamma_0}(\mathbb{R})}$. We take sum $j = 1$ to ∞ for (9.10). $\sum_{j=1}^\infty \|h\|_{G_{h,\gamma_0}(\mathbb{R})(U_j)} \le 2\|h\|_{G_{h,\gamma_0}(\mathbb{R})}$, $\|u\|_{\mathbb{F}_{u,\gamma_0}(\mathbb{R})}$, $\|h\|_{G_{u,\gamma_0}(\mathbb{R})}$ are lower order terms so we absorb the terms into the left hand side using (9.6)-(9.7) and choosing γ and R sufficiently large. Therefore we obtain required estimate (7.5). Setting the left hand side operator $L : \mathbb{E}_{\gamma_0}(\mathbb{R}) \to \mathbb{F}_{\gamma_0}(\mathbb{R})$, then we obtain L is injective and has closed range, i.e., L is semi-Fredholm operator. In order to show surjectivity of L, we employ the continuation method for semi-Fredholm operators [16]. We introduce the continuation parameter $\alpha \in [0,1]$ by replacing the 3rd equation and 7th equation of (7.4) into

$$\begin{aligned}
&\|u'\| + (1-\alpha)c_0\nabla' h + \alpha c(t,x)\nabla' h = f_d && \text{on } \mathbb{R}^n_0, \quad t>0\\
&\partial_t h - \frac{[\![\rho u_n]\!]}{[\![\rho]\!]} + (1-\alpha)\frac{[\![b_0\cdot\nabla' h]\!]}{[\![\rho]\!]} + \alpha\frac{[\![b(t,x)\cdot\nabla' h]\!]}{[\![\rho]\!]} = g_h && \text{on } \mathbb{R}^n_0, \quad t>0
\end{aligned}$$

The problem with $\alpha = 0$ is solved in Subsection 4.1, this shows that we have subjectivity in the case $\alpha = 0$, We can prove that the a priori estimates are uniform with respect to $\alpha \in [0,1]$. Hence by continuation method we have surjectivity for $\alpha = 1$. In this way we reduce time and space variable coefficients problem into space variable coefficients problem. The proof of Theorem 7.1 is now complete.

10. Local L_p-L_q well-posedness; Proof of Theorem 6.1

In this section, we prove Theorem 6.1. The nonlinear problem (1.1)-(1.3) can be transformed to a problem on $\dot{\mathbb{R}}^n := \mathbb{R}^n \setminus [\mathbb{R}^{n-1} \times \{0\}]$ by means of the transformations

$$\begin{aligned}
\bar{u}(t,x',x_n) &:= (u',u_n)^{\mathsf{T}}(t,x',x_n+h(t,x')),\\
\bar{\theta}(t,x',x_n) &:= \theta(t,x',x_n+h(t,x')),\\
\bar{\pi}(t,x',x_n) &:= \pi(t,x',x_n+h(t,x')),
\end{aligned}$$

where $t \in J = [0,T]$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, $x_n \neq 0$. With a slight abuse of notation we will denote in the sequel the transformed velocity again by u, the transformed temperature by θ, and the transformed pressure by π. For given initial data $\theta_0(x)$, we set $\mu(x) := \mu(\theta_0(x))$, $\kappa(x) = \kappa(\theta_0(x))$, $d(x) = d(\theta_0(x))$, $c(t,x) = e^{\Delta t}[\![u_{0n}]\!]$ and $b(t,x) = e^{\Delta t}[\![\rho u_0{}']\!]$, where $e^{\Delta t}$ is the heat semigroup. With this notation we have the transformed problem:

$$\begin{aligned}
\rho\partial_t u - \mu(x)\Delta u + \nabla\pi &= F_u(u,\pi,\theta,h) && \text{in} \quad \dot{\mathbb{R}}^n, && t>0,\\
\operatorname{div} u &= F_d(u,h) && \text{in} \quad \dot{\mathbb{R}}^n, && t>0,\\
[\![u']\!] + c(t,x)\nabla' h &= G_u(u,h) && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
-[\![\mu(x)(\partial_n u' + \nabla' u_n)]\!] &= G_\tau(u,\theta,h) && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
-2[\![\mu(x)\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h &= G_n(u,\theta,h) && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
\rho\kappa(x)\partial_t\theta - d(x)\Delta\theta &= F_\theta(u,\theta,h) && \text{in} \quad \dot{\mathbb{R}}^n, && t>0,\\
[\![\theta]\!] &= 0 && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
-[\![d(x)\partial_n\theta]\!] &= G_\theta(u,\theta,h) && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
-2[\![(\mu(x)/\rho)\partial_n u_n]\!] + [\![\pi/\rho]\!] &= G_\pi(u,\theta,h) && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
\partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] + b(t,x)\cdot\nabla' h/[\![\rho]\!] &= G_h(u,h) && \text{on} \quad \mathbb{R}^{n-1}, && t>0,\\
u(0) = u_0, \quad \theta(0) &= \theta_0 && \text{in} \quad \dot{\mathbb{R}}^n,\\
h(0) &= h_0 && \text{on} \quad \mathbb{R}^{n-1}.
\end{aligned} \tag{10.1}$$

This problem is slightly different from the problem (4.1) in [5] which is linearization around an equilibrium. The right hand sides of (10.1) are defined by

$$\begin{aligned}
&F_u(u,\pi,\theta,h) = (F_{u'}(u,\pi,\theta,h), F_{u_n}(u,\theta,h))^{\mathsf{T}},\\
&F_{u'}(u,\pi,\theta,h) = (\mu(\theta) - \mu(\theta_0))\Delta u'
\end{aligned}$$

$$
\begin{aligned}
&+ \mu(\theta)(-\Delta' h\partial_n u' - 2\nabla' h\cdot\nabla'\partial_n u' + |\nabla' h|^2\partial_n^2 u')\\
&- \rho(u'\cdot\nabla' u' + u_n\partial_n u' - u'\cdot\nabla' h\partial_n u') + \rho\partial_t h\partial_n u' + \nabla' h\partial_n\pi\\
&+ \{(\nabla' u' + [\nabla' u']^{\mathsf T}) - (\nabla' h\otimes\partial_n u' + \partial_n u'\otimes\nabla' h)\}\mu'(\theta)\nabla'\theta\\
&+ (\partial_n u' + \nabla' u_n - \nabla' h\partial_n u_n)\mu'(\theta)\partial_n\theta,\\
F_{u_n}(u,\theta,h) &= (\mu(\theta)-\mu(\theta_0))\Delta u_n\\
&+ \mu(\theta)(-\Delta' h\partial_n u_n - 2\nabla' h\cdot\nabla'\partial_n u_n + |\nabla' h|^2\partial_n^2 u_n)\\
&- \rho(u'\cdot\nabla' u_n + u_n\partial_n u_n - u'\cdot\nabla' h\partial_n u_n) + \rho\partial_t h\partial_n u_n\\
&+ ([\partial_n u']^{\mathsf T} + [\nabla' u_n]^{\mathsf T} - \partial_n u_n[\nabla' h]^{\mathsf T})\mu'(\theta)\nabla'\theta + 2\partial_n u_n\mu'(\theta)\partial_n\theta,\\
F_d(u,h) &= \nabla' h\cdot\partial_n u' = \partial_n(\nabla' h\cdot u'),\\
G_u(u,h) &= [\![(e^{\Delta t}u_{0n} - u_n)\nabla' h]\!],\\
G_\tau(u,\theta,h) &= [\![(\mu(\theta)-\mu(\theta_0))(\partial_n u' + \nabla' u_n)]\!] - [\![\mu(\theta)(\nabla u' + [\nabla u']^{\mathsf T})]\!]\nabla' h\\
&+ [\![\mu(\theta)\{\nabla' h(\partial_n u'\cdot\nabla' h) + \partial_n u'|\nabla' h|^2 - \nabla' h\partial_n u_n\}]\!]\\
&+ [\![\mu(\theta)\{-(\partial_n u' + \nabla' u_n)\cdot\nabla' h + 2\partial_n u_n + \partial_n u_n|\nabla' h|^2\}]\!]\nabla' h\\
&+ [\![\rho^{-1}]\!](1+|\nabla' h|^2)[\![u_n]\!]^2\nabla' h,\\
G_n(u,\theta,h) &= [\![(\mu(\theta)-\mu(\theta_0))2\partial_n u_n]\!] - [\![\mu(\theta)(\partial_n u' + \nabla' u_n)\cdot\nabla' h]\!]\\
&+ [\![\mu(\theta)\partial_n u_n]\!]|\nabla' h|^2 - [\![\rho^{-1}]\!](1+|\nabla' h|^2)[\![u_n]\!]^2 - \sigma J(h),\\
F_\theta(u,\theta,h) &= \rho(\kappa(\theta_0)-\kappa(\theta))\partial_t\theta + (d(\theta)-d(\theta_0))\Delta\theta\\
&+ \rho\kappa(\theta)\{\partial_t h\partial_n\theta - u'\cdot\nabla\theta + (u'\cdot\nabla' h)\partial_n\theta - u_n\partial_n\theta\}\\
&+ d'(\theta)\{|\nabla'\theta - \nabla' h\partial_n\theta|^2 + (\partial_n\theta)^2\}\\
&+ (\mu(\theta)/2)|\nabla' u' + [\nabla' u']^{\mathsf T} - \nabla' h\otimes\partial_n u' - \partial_n u'\otimes\nabla' h|^2\\
&+ \mu(\theta)[|\partial_n u' + \nabla' w - \partial_n u_n\nabla' h|^2 + 2|\partial_n u_n|^2],\\
G_\theta(u,\theta,h) &= [\![(d(\theta)-d(\theta_0))\partial_n\theta]\!] - [\![d(\theta)\nabla'\theta\cdot\nabla' h]\!]\\
&+ (l(\theta)/[\![1/\rho]\!])(1+|\nabla' h|^2)[\![u_n]\!],\\
G_\pi(u,\theta,h) &= -[\![\psi(\theta)]\!] + 2[\![(\mu(\theta)-\mu(\theta_0))\partial_n u_n/\rho]\!]\\
&- [\![\frac{1}{2\rho^2}]\!](1+|\nabla' h|^2)[\![\frac{1}{\rho}]\!]^{-2}[\![u_n]\!]^2 - 2[\![\frac{\mu(\theta)}{\rho}\partial_n u'\cdot\nabla' h]\!]\\
&+ \frac{2}{1+|\nabla' h|^2}[\![\frac{\mu(\theta)}{\rho}\{(\nabla u'\nabla' h)\cdot\nabla' h - \nabla' u_n\cdot\nabla' h\}]\!],\\
G_h(u,h) &= \frac{[\![\rho(e^{\Delta t}u_0{}' - u')\cdot\nabla' h]\!]}{[\![\rho]\!]}.
\end{aligned}
$$

The curvature of $\Gamma(t)$ is given by

$$
H(\Gamma(t)) = \operatorname{div}_{x'}\left(\frac{\nabla' h(t,x')}{\sqrt{1+|\nabla' h(t,x')|^2}}\right) = \Delta' h - J(h),
$$

with

$$J(h) = \frac{|\nabla' h|^2 \Delta' h}{(1+\sqrt{1+|\nabla' h|^2})\sqrt{1+|\nabla' h|^2}} + \frac{\nabla' h \cdot (\nabla'^2 h \cdot \nabla' h)}{(1+|\nabla' h|^2)^{3/2}},$$

where $\nabla'^2 h$ denotes the Hessian of h.

Given $h_0 \in B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$ we define

$$\Theta_{h_0}(x) := (x', x_n + h_0(x')) \quad (x', x_n) \in \mathbb{R}^n \times \mathbb{R}.$$

Letting $\Omega_{h_0,\pm} := \{(x', x_n) \in \mathbb{R}^n \times \mathbb{R} : \pm(x_n - h_0(x')) > 0\}$ and $\Omega_{h_0} := \Omega_{h_0,+} \cup \Omega_{h_0,-}$. By the assumption $2 < p < \infty$, $n < q < \infty$ and $2/p + n/q < 1$, we obtain from Sobolev's embedding theorem that Θ_{h_0} yields a C^2-diffeomorphism between $\dot{\mathbb{R}}^n$ and Ω_{h_0}, $\mathbb{R}^n_+$ and $\Omega_{h_0,+}$, and $\mathbb{R}^n_-$ and $\Omega_{h_0,-}$. The inverse transform is given by $\Theta^{-1}_{h_0}(x', x_n) = (x', x_n - h_0(x'))$. It then follows from the chain rule and transformation rule for integrals that

$$\Theta^*_{h_0} \in \mathrm{Isom}(W^k_p(\dot{\mathbb{R}}^n), W^k_p(\Omega_{h_0})), \quad [\Theta^*_{h_0}]^{-1} = \Theta^{h_0}_* \quad k = 0, 1, 2,$$

where we use the notation

$$\begin{aligned} \Theta^*_{h_0} f &= f \circ \Theta_{h_0} \quad f : \Omega_{h_0} \to \mathbb{R}^m, \\ \Theta^{h_0}_* g &= g \circ \Theta^{-1}_{h_0} \quad g : \dot{\mathbb{R}}^n \to \mathbb{R}^m, \end{aligned}$$

for the pull-back and push-forward operators, where m is non-negative integer.

Therefore it is enough to prove the following theorem instead of Theorem 6.1.

Theorem 10.1. *Let $2 < p < \infty$, $n < q < \infty$ and $2/p + n/q < 1$. Let $\psi_\pm \in C^3(0, \infty)$, $\mu_\pm$, $d_\pm \in C^2(0, \infty)$ be such that*

$$\kappa_\pm(s) = -s\psi''_\pm(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0 \quad s \in (0, \infty),$$

and

$$(u_0, \theta_0, h_0) \in B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

be given. Assume that the compatibility conditions:

$$\begin{aligned} &\mathrm{div}\,(\Theta^{h_0}_* u_0) = 0 && \text{in } \Omega_0, \\ &[\![\mu P_{\Gamma_0} E(\Theta^{h_0}_* u_0)\nu_0]\!] = 0, \quad [\![P_{\Gamma_0} \Theta^{h_0}_* u_0]\!] = 0 && \text{on } \Gamma_0, \\ &[\![\Theta^{h_0}_* \theta_0]\!] = 0, \quad [\![d\partial_{\nu_0} \Theta^{h_0}_* \theta_0]\!] + \ell(\Theta^{h_0}_*(\theta_0 + \theta_\infty))[\![\rho^{-1}]\!]^{-1}[\![\Theta^{h_0}_* u_0 \cdot \nu_0]\!] = 0 && \text{on } \Gamma_0. \end{aligned} \tag{10.2}$$

Then there exists a constant $\varepsilon_0 > 0$ depending only on Ω_0, p, q, n such that if h_0 and u_0 satisfy $\|\nabla' h_0\|_{L_\infty(\dot{\mathbb{R}}^n)} \le \varepsilon_0$, then there exist

$$T = T(\|u_0\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)}, \|\theta_0\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)}, \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}, \varepsilon_0) > 0$$

and a unique L_p-L_q solution (u, π, θ, h) of the nonlinear problem (10.1) *on* $[0, T]$ *of* $\mathbb{E}(J)$ *which is defined by* (7.6)*.*

A proof of Theorem 10.1 is the same proof as for Theorem 4.1 in [5] instead of the nonlinear terms $G_u(u,h)$ and $G_h(u,h)$. It is possible to take $\epsilon > 0$

$$|[\![e^{\Delta t}u_{0n} - u_n]\!]| < \epsilon, \quad |[\![e^{\Delta t}u_0{}' - u']\!]| < \epsilon$$

small as we want if we take time T small. Therefore we have proved Theorem 10.1.

Part 2. **The Case of Equal Densities**

11. The Substance of The Case of Equal Densities

In (1.1)-(1.3), setting $\rho_+ = \rho_- = 1$, we have the following:

$$\begin{aligned} \partial_t u + u\cdot\nabla u - \operatorname{div} T(u,\pi,\theta) &= 0 && \text{in } \ \Omega(t),\ t>0,\\ \operatorname{div} u &= 0 && \text{in } \ \Omega(t),\ t>0,\\ [\![T(u,\pi,\theta)\nu_\Gamma]\!] + \sigma H_\Gamma \nu_\Gamma &= 0 && \text{on } \ \Gamma(t),\ t>0,\\ [\![u]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0,\\ u(0) &= u_0 && \text{in } \ \Omega(t), \end{aligned} \tag{11.1}$$

$$\begin{aligned} \kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) - 2\mu(\theta)|D(u)|_2^2 &= 0 && \text{in } \ \Omega(t),\ t>0,\\ l(\theta)j + [\![d(\theta)\partial_{\nu_\Gamma}\theta]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0,\\ [\![\theta]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0,\\ \theta(0) &= \theta_0 && \text{in } \ \mathbb{R}^n, \end{aligned} \tag{11.2}$$

$$\begin{aligned} [\![\psi(\theta)]\!] - [\![T(u,\pi,\theta)\nu_\Gamma\cdot\nu_\Gamma]\!] &= 0 && \text{on } \ \Gamma(t),\ t>0,\\ V_\Gamma - u\cdot\nu_\Gamma + j &= 0 && \text{on } \ \Gamma(t),\ t>0,\\ \Gamma(0) &= \Gamma_0, \end{aligned} \tag{11.3}$$

Changing variables of (11.1)-(11.3) with $y_n = x_n - h(x',t)$, we have the following quasilinear-problem:

$$\begin{aligned} \partial_t u - \mu_0\Delta u + \nabla\pi &= F_u(u,\pi,\theta,h) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\ \operatorname{div} u &= F_d(u,h) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\ -[\![\mu_0(\partial_n u' + \nabla' u_n)]\!] &= G_{u'}(u,\theta,h) && \text{on } \mathbb{R}^n_0,\ t>0,\\ -2[\![\mu_0\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h &= G_{u_n}(u,\theta,h) && \text{on } \mathbb{R}^n_0,\ t>0,\\ [\![u]\!] &= 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\ \kappa_0\partial_t\theta - d_0\Delta\theta &= F_\theta(u,\theta,h) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\ [\![\theta]\!] &= 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\ l_1\theta + \sigma\Delta' h &= G_{\theta,2}(u,\theta,h) && \text{on } \mathbb{R}^n_0,\ t>0, \end{aligned}$$

$$\begin{aligned} \partial_t h - [\![d_0\partial_n\theta]\!]/l_0 = G_{h,2}(u,\theta,h) \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ u(0)=u_0,\ \theta_0=\theta_0 \quad & \text{in } \dot{\mathbb{R}}^n, \\ h(0)=h_0 \quad & \text{on } \mathbb{R}^n_0, \end{aligned} \tag{11.4}$$

where μ_0, κ_0, d_0 and l_0 have been defined in Section 5. We suppose that l_1 is a constant defined by $l_1 = \lim_{t\to\infty} l_1(x,t)|_{\mathbb{R}^n_0}$ where $l_1(x,t) = [\![\psi'(e^{t\Delta'}\theta_0)]\!]$ (cf. Section 5, Section 7 in [14] and Section 5). $G_{\theta,2}$, $G_{h,2}$ are defined as

$$\begin{aligned} G_{\theta,2}(u,\theta,h) &= -(l_1(x,t)-l_1))\theta - [\![\psi(\theta)]\!] + \sigma J(h) \\ G_{h,2}(u,\theta,h) &= l(\theta)^{-1}[\![d(\theta)(-\Sigma_{k=1}^{n-1}(\partial_k h)\partial_k\theta + |\nabla' h|^2\partial_n\theta]\!] \\ &\quad + (l(\theta)^{-1} - l_0^{-1})[\![d(\theta)\partial_n\theta]\!] + l_0^{-1}[\![(d(\theta)-d_0)\partial_n\theta]\!] - (u'\cdot\nabla')h + u_n, \end{aligned}$$

respectively. Thus, we treat the following linearized problem to solve (11.1)-(11.3):

$$\begin{aligned} \partial_t u - \mu_0\Delta u + \nabla\pi = f_u \quad & \text{in } \dot{\mathbb{R}}^n,\ t>0, \\ \operatorname{div} u = f_d \quad & \text{in } \dot{\mathbb{R}}^n,\ t>0, \\ -[\![\mu_0(\partial_n u' + \nabla' u_n)]\!] = g_{u'} \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ -2[\![\mu_0\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h = g_{u_n} \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ [\![u]\!] = 0 \quad & \text{in } \dot{\mathbb{R}}^n, \end{aligned} \tag{11.5}$$

$$\begin{aligned} \kappa_0\partial_t\theta - d_0\Delta\theta = f_\theta \quad & \text{in } \dot{\mathbb{R}}^n,\ t>0, \\ l_1\theta = -\sigma\Delta' h + g_\theta \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ \partial_t h = [\![d_0\partial_n\theta]\!]/l_0 + g_h \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ \theta(0)=\theta_0 \quad & \text{in } \dot{\mathbb{R}}^n, \\ h(0)=h_0 \quad & \text{on } \dot{\mathbb{R}}^n_0. \end{aligned} \tag{11.6}$$

We may use results of [23] for (11.5). In order to prove maximal $L_p - L_q$ regularity of the problem (11.6), we set $\theta_0 = h_0 = 0$ once and solve (11.6). First dividing the problem, (11.6) into the nexts:

$$\begin{aligned} \kappa_{0\pm}\partial_t U_\pm - d_{0\pm}\Delta U_\pm = f^e_{\theta,\pm} \quad & \text{in } \mathbb{R}^n,\ t>0, \\ U_\pm(x,0) = 0 \quad & \text{in } \mathbb{R}^n, \end{aligned} \tag{11.7}$$

and

$$\begin{aligned} \kappa_{0\pm}\partial_t V_\pm - d_{0\pm}\Delta V_\pm = 0 \quad & \text{in } \mathbb{R}^n_\pm,\ t>0, \\ [\![V]\!] = -[\![U]\!] \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ l_1 V = -\sigma\Delta' h + g_\theta - l_1 U \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ l_0\partial_t h - [\![d_0\partial_n V]\!] = l_0 g_h + [\![d_0\partial_n U]\!] \quad & \text{on } \mathbb{R}^n_0,\ t>0, \\ V_0 = 0 \quad & \text{in } \mathbb{R}^n, \\ h_0 = 0 \quad & \text{on } \mathbb{R}^n, \end{aligned} \tag{11.8}$$

where $f^e_{\theta,\pm}$ are even extensions to $\dot{\mathbb{R}}^n$:

$$f^e_{\theta,+}(x,t) = \begin{cases} f_{\theta,+}(x',x_n,t) & \text{for } x_n > 0 \\ f_{\theta,+}(x',-x_n,t) & \text{for } x_n < 0, \end{cases}$$

$$f^e_{\theta,-}(x,t) = \begin{cases} f_{\theta,-}(x',-x_n,t) & \text{for } x_n > 0 \\ f_{\theta,-}(x',x_n,t) & \text{for } x_n < 0, \end{cases}$$

respectively, we could write $\theta = U + V$. Using Fourier transform for (11.7),

$$(\kappa_{0\pm} s + d_{0\pm}|\xi|^2)\mathcal{F}_\xi \mathcal{L}_t[U]_\pm = \mathcal{F}_\xi \mathcal{L}_t[f^e_{\theta,\pm}],$$

we gain

$$U_\pm = \mathcal{L}_s^{-1}\mathcal{F}_\xi^{-1}\left[\frac{1}{\kappa_{0\pm}s + d_{0\pm}|\xi|^2}\mathcal{F}_x\mathcal{L}_t[f^e_{\theta,\pm}]\right]. \tag{11.9}$$

Then, $\partial_n U_\pm|_{x_n=0} = 0$ holds. Indeed, by $B_+ = (\kappa_{0+}d_{0+}^{-1}s + |\xi'|^2)^{1/2}$, (11.9), the definition of Fourier transform and Laplace transform,

$$\begin{aligned}
\mathcal{F}_{x'}\mathcal{L}_t[\partial_n U_+|_{x_n=0}] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi_n}{\kappa_{0+}s + d_{0+}|\xi|^2}\mathcal{F}_x\mathcal{L}_t[f^e_{\theta,+}]\,d\xi_n \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi_n}{d_{0+}B_+^2 + d_{0+}\xi_n^2}\mathcal{F}_x\mathcal{L}_t[f^e_{\theta,+}]\,d\xi_n \\
&= \frac{1}{2\pi d_{0+}}\int_{-\infty}^{\infty} \frac{i\xi_n}{B_+^2 + \xi_n^2}\mathcal{F}_x\mathcal{L}_t[f^e_{\theta,+}]\,d\xi_n \\
&= \frac{1}{2\pi d_{0+}}\int_{-\infty}^{\infty} \frac{i\xi_n}{B_+^2 + \xi_n^2}\left(\int_0^\infty e^{-i\zeta_n\xi_n}\mathcal{F}_{x'}\mathcal{L}_t[f_{\theta,+}](\xi',\zeta_n)\,d\zeta_n\right. \\
&\qquad \left.\int_{-\infty}^0 e^{-i\zeta_n\xi_n}\mathcal{F}_{x'}\mathcal{L}_t[f_{\theta,+}](\xi',-\zeta_n)\,d\zeta_n\right)d\xi_n \\
&= \frac{1}{2\pi d_{0+}}\int_0^\infty \mathcal{F}_{x'}\mathcal{L}_t[f_{\theta,+}](\xi',\zeta_n)\Bigg(\\
&\qquad \int_{-\infty}^{\infty}\frac{i\xi_n e^{-i\zeta_n\xi_n}}{(\xi_n + iB_+)(\xi_n - iB_+)}\,d\xi_n \\
&\qquad + \int_{-\infty}^{\infty}\frac{i\xi_n e^{i\zeta_n\xi_n}}{(\xi_n + iB_+)(\xi_n - iB_+)}\,d\xi_n\Bigg)\,d\zeta_n \\
&= \frac{1}{2\pi d_{0+}}\int_0^\infty \mathcal{F}_{x'}\mathcal{L}_t[f_{\theta,+}](\xi',\zeta_n)\Big(\frac{1}{2}e^{-B_+\zeta_n} - \frac{1}{2}e^{-B_+\zeta_n}\Big)\,d\zeta_n \\
&= 0,
\end{aligned}$$

where we use next formulas:

$$\int_{-\infty}^{\infty}\frac{ixe^{\pm iax}}{x^2+b^2}\,dx = \mp\pi e^{-ab} \quad \text{for } a>0,\ b\in\mathbb{C}\,(\operatorname{Re} b > 0). \tag{11.10}$$

Hence, $\partial_n U_+|_{x_n=0} = 0$ and we have $\partial_n U_-|_{x_n=0} = 0$ in the same way, so $[\![d\partial_n U]\!] = 0$ holds. We prove (11.10) with complex integration. Defining C_1, C_2 and $f_\pm(z)$ as

$$\begin{aligned}
&C_1 = \{z \in \mathbb{C} \mid z = t, -R \le t \le R\} \cup \{z \in \mathbb{C} \mid z = Re^{it}, 0 \le t \le \pi\},\\
&C_2 = \{z \in \mathbb{C} \mid z = -t, -R \le t \le R\} \cup \{z \in \mathbb{C} \mid z = Re^{it}, -\pi \le t \le 0\},\\
&f_\pm(z) = \frac{ize^{\pm iaz}}{z^2 + b^2} \quad \text{for } a > 0,\ b \in \mathbb{C}\,(\mathrm{Re}\, b > 0),
\end{aligned}$$

we have

$$\begin{aligned}
\int_{C_1} f_+(z)\,dz &= \int_{-R}^{R} f_+(t)\,dt + \int_0^\pi f_+(Re^{it}) iR\,dt,\\
\int_{C_2} f_-(z)\,dz &= -\int_{-R}^{R} f_+(-t)\,dt + \int_{-\pi}^0 f_-(Re^{it}) iR\,dt.
\end{aligned}$$

By $\sin t \ge 0\,(0 \le t \le \pi)$ and $\sin t \le 0\,(-\pi \le t \le 0)$, it is clear that

$$\begin{aligned}
f_+(Re^{it})iR &= \frac{iRe^{it}e^{iaRe^{it}}}{R^2e^{2it} + b^2} iR\\
&= -\frac{R^2e^{it}e^{-aR\sin t}e^{iaR\cos t}}{R^2e^{2it} + b^2} \to 0 \quad (0 \le t \le \pi, R \to \infty),\\
f_-(Re^{it})iR &= \frac{iRe^{it}e^{-iaRe^{it}}}{R^2e^{2it} + b^2} iR\\
&= -\frac{R^2e^{it}e^{aR\sin t}e^{-iaR\cos t}}{R^2e^{2it} + b^2} \to 0 \quad (-\pi \le t \le 0, R \to \infty).
\end{aligned}$$

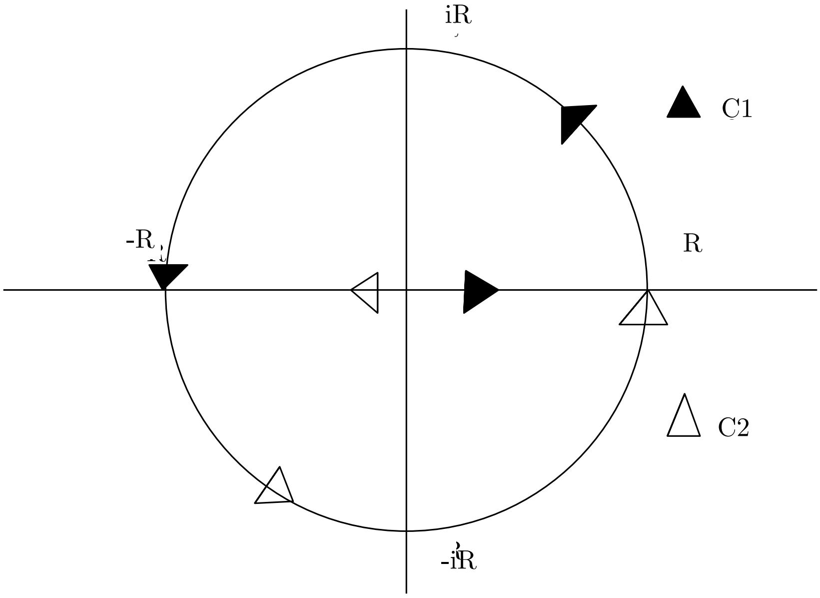
By the way, from theorem of residue, we see that

$$\begin{aligned}
\int_{C_1} f_+(z)\,dz &= 2\pi i \lim_{z \to ib} \frac{1}{0!}\frac{d^0}{dz^0}((z - ib)f_+(z))\\
&= 2\pi i \lim_{z \to ib} \frac{ize^{iaz}}{z + ib}\\
&= -\pi e^{-ab},\\
\int_{C_2} f_-(z)\,dz &= 2\pi i \lim_{z \to -ib} \frac{1}{0!}\frac{d^0}{dz^0}((z + ib)f_-(z))\\
&= 2\pi i \lim_{z \to -ib} \frac{ize^{iaz}}{z - ib}\\
&= -\pi e^{-ab}.
\end{aligned}$$

Thus, we obtain;

$$\int_{-\infty}^{\infty} f_+(t)\,dt = -\pi e^{-ab}, \quad \int_{-\infty}^{\infty} f_-(t)\,dt = -\lim_{R \to \infty} \int_{C_2} f_-(z)\,dz = \pi e^{-ab}$$

i.e. we prove (11.10).



From $[\![d\partial_n U]\!] = 0$ and (11.8), it follows that:

$$(B_\pm^2 - \partial_n^2)\hat{V}_\pm = 0 \qquad \text{in } \mathbb{R}^n_\pm, \tag{11.11}$$

$$[\![\hat{V}]\!] = -[\![\hat{U}]\!] \qquad \text{on } \mathbb{R}^n_0, \tag{11.12}$$

$$l_1\hat{V} = \sigma|\xi'|^2\hat{h} + \hat{g}_\theta - l_1\hat{U} \quad \text{on } \mathbb{R}^n_0, \tag{11.13}$$

$$l_0 s\hat{h} - [\![d_0\partial_n\hat{V}]\!] = l_0\hat{g}_h \qquad \text{on } \mathbb{R}^n_0, \tag{11.14}$$

where we set $B_\pm = (\kappa_{0\pm}d_{0\pm}^{-1}s + |\xi'|^2)^{1/2}$ ($\mathrm{Re}\, B_\pm > 0$). From (11.11), we look for solutions whose forms are;

$$\hat{V}_\pm = P_\pm e^{\mp B_\pm x_n} \quad \text{for } x_n \gtrless 0.$$

By (11.12) and (11.14), it holds that

$$\begin{cases} P_+ - P_- = -[\![\hat{U}]\!] \\ d_{0+}B_+P_+ + d_{0-}B_-P_- = \mathcal{F}_{\xi'}\mathcal{L}_t[l_0 g_h - l_0\partial_t h], \end{cases}$$

so we see the following:

$$\begin{cases} P_+ = (d_{0+}B_+ d_{0-}B_-)^{-1}(\mathcal{F}_{\xi'}\mathcal{L}_t[l_0 g_h - l_0\partial_t h] - d_{0-}B_-[\![\hat{U}]\!]) \\ P_- = (d_{0+}B_+ d_{0-}B_-)^{-1}(\mathcal{F}_{\xi'}\mathcal{L}_t[l_0 g_h - l_0\partial_t h] + d_{0+}B_+[\![\hat{U}]\!]) \end{cases}$$

Then, we have the description:

$$V_\pm = \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{e^{\mp B_\pm x_n}}{d_{0+}B_+ + d_{0-}B_-}\mathcal{F}_{x'}\mathcal{L}_t[l_0 g_h - l_0\partial_t h]\right]$$

$$\mp \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{d_{0\mp}B_{\mp}e^{\mp B_{\pm}x_n}}{d_{0+}B_+ + d_{0-}B_-}\mathcal{F}_{x'}\mathcal{L}_t[\,[\![U]\!]\,]\right]. \tag{11.15}$$

Making use of this formula of V and (11.13), we obtain the description of h:

$$\begin{aligned} h = {} & \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{1}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}\mathcal{F}_{x'}\mathcal{L}_t[g_h]\right] \\ & + \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}l_1^{-1}(d_{0+}B_+ + d_{0-}B_-)}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}\mathcal{F}_{x'}\mathcal{L}_t[g_\theta]\right] \\ & - \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}d_{0+}B_+}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}\mathcal{F}_{x'}\mathcal{L}_t[U_+]\right] \\ & - \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}d_{0-}B_-}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}\mathcal{F}_{x'}\mathcal{L}_t[U_-]\right]. \end{aligned} \tag{11.16}$$

Suppose that

$$\begin{aligned} & f_\theta \in L_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n)),\ g_\theta \in W^1_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R};W_q^2(\dot{\mathbb{R}}^n)), \\ & g_h \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R};W_q^1(\dot{\mathbb{R}}^n)), \end{aligned}$$

and $l_0 l_1 > 0$.

We could prove $U \in W^1_{p,0,\gamma_0}(\mathbb{R};L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R};W_q^2(\dot{\mathbb{R}}^n))$ in the same way as the proof of Theorem 3.1 in [24] from (11.9). Therefore, we analyze (11.15) and (11.16).

Lemma 11.1. *For $s \in \Sigma_{\epsilon,\gamma_0}$*

$$|s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2| \geq C(|s|+(|s|^{1/2}+|\xi'|)\sigma|\xi'|^2) \tag{11.17}$$

$$|s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2| \geq C(|s|^{1/2}+|\xi'|)^2 \tag{11.18}$$

Proof. We prove (11.17) like that

$$\begin{aligned} & |s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2| \\ & \qquad = |d_{0+}B_+ + d_{0-}B_-||s/(d_{0+}B_+ + d_{0-}B_-) + l_0^{-1}l_1^{-1}\sigma|\xi'|^2| \\ & \qquad \geq C(|s|^{1/2}+|\xi'|)(|s|/(|s|^{1/2}+|\xi'|)+\sigma|\xi'|^2) \\ & \qquad = C(|s|+(|s|^{1/2}+|\xi'|)\sigma|\xi'|^2) \end{aligned}$$

by $s/(d_{0+}B_+ + d_{0-}B_-) \in \Sigma_{\epsilon,\gamma_0}$. (11.18) is proved from (11.17) in the same way as Lemma 3.4 . □

From

$$\begin{aligned} & \frac{1}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2} \\ & = \frac{1}{s} - \frac{(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|}{s(s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2)}|\xi'|, \end{aligned}$$

it holds that

$$\frac{s}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2} = 1 - \frac{(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}|\xi'|,$$

so we could derive that h, $\partial_t h \in L_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n))$ with (11.18) and the extension:

$$\begin{aligned}
h &= \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{1}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp B_\pm x_n}\mathcal{F}_{x'}\mathcal{L}_t[g_h]\right] \\
&+ \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}l_1^{-1}(d_{0+}B_+ + d_{0-}B_-)}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp B_\pm x_n}\mathcal{F}_{x'}\mathcal{L}_t[g_\theta]\right] \\
&- \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}d_{0+}B_+}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp B_\pm x_n}\mathcal{F}_{x'}\mathcal{L}_t[U_+]\right] \\
&- \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}d_{0-}B_-}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp B_\pm x_n}\mathcal{F}_{x'}\mathcal{L}_t[U_-]\right] \quad \text{for } x_n \gtrless 0.
\end{aligned}$$

Making use of the following extension:

$$\begin{aligned}
h &= \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{1}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp|\xi'|x_n}\mathcal{F}_{x'}\mathcal{L}_t[g_h]\right] \\
&+ \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}l_1^{-1}(d_{0+}B_+ + d_{0-}B_-)}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp|\xi'|x_n}\mathcal{F}_{x'}\mathcal{L}_t[g_\theta]\right] \\
&- \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}d_{0+}B_+}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp|\xi'|x_n}\mathcal{F}_{x'}\mathcal{L}_t[U_+]\right] \\
&- \mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{l_0^{-1}d_{0-}B_-}{s+(d_{0+}B_+ + d_{0-}B_-)l_0^{-1}l_1^{-1}\sigma|\xi'|^2}e^{\mp|\xi'|x_n}\mathcal{F}_{x'}\mathcal{L}_t[U_-]\right] \quad \text{for } x_n \gtrless 0
\end{aligned}$$

and (11.17), $\nabla\partial_t h$, $\nabla^3\Lambda_\gamma^{1/2}h$, $\nabla^4 h \in L_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n))$ could be proved with Lemma 3.4 .

By Stefan condition:

$$\partial_t h = [\![\partial_n\theta]\!]/l_0 + g_h \quad \text{on } \mathbb{R}^n_0,\ t>0,$$

we extend h like that

$$\tilde{h} = -\mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}[s^{-1}e^{\mp B_\pm x_n}\mathcal{F}_{x'}\mathcal{L}_t[[\![\partial_n\theta]\!]/l_0 - g_h]] \quad \text{in } \mathbb{R}^n_\pm,\ t>0. \tag{11.19}$$

In view of (11.19), if $\theta \in L_{p,0,\gamma_0}(\mathbb{R}; W_q^2(\dot{\mathbb{R}}^n)) \cap H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; W_q^1(\dot{\mathbb{R}}^n))$, it hold that $\tilde{h} \in H^{3/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\dot{\mathbb{R}}^n))$. From the boundary condition:

$$l_1\theta = -\sigma\Delta' h + g \quad \text{on } \mathbb{R}^n_0,\ t>0,$$

defining an extension of θ, $\tilde{\theta}$ as

$$\tilde{\theta} = -\mathcal{L}_s^{-1}\mathcal{F}_{\xi'}^{-1}[l_0^{-1}e^{\mp|\xi'|x_n}\mathcal{F}_{x'}\mathcal{L}_t[\sigma\Delta' h - g]] \quad \text{in } \mathbb{R}^n_\pm,\ t>0,$$

we realize $\theta = \tilde{\theta}$ on $\mathbb{R}^n_0$ and $\tilde{\theta} \in L_{p,0,\gamma_0}(\mathbb{R}; W^2_q(\mathbb{R}^n_\pm)) \cap H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; W^1_q(\mathbb{R}^n_\pm))$. Thus, $\tilde{h}$ has the regularity, $H^{3/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n))$. Utilizing this facts, we could prove that θ belongs to $W^1_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n))$. So, we see an extension of h, $\tilde{\tilde{h}}$ satisfies that $\partial_t \Delta' \tilde{\tilde{h}} \in L_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n))$. We set $\theta_0 = h_0 = 0$ above but add conditions for initial values that are not 0 identically to gain the following theorem.

Theorem 11.2. *Let $1 < p,\, q < \infty$ and assume that κ, d, σ are positive constants and $l_0 l_1 > 0$. Suppose initial values $(u_0, \theta_0, h_0) \in B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)^n \times B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n) \times B^{4-2/p-1/q}_{q,p}(\dot{\mathbb{R}}^n)$ and the data $(f_u, f_d, g_u, f_\theta, g_\theta, g_h)$ satisfy the following conditions:*

$$
\begin{aligned}
&f_u \in L_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)),\\
&f_d \in W^1_{p,0,\gamma_0}(\mathbb{R}; W^{-1}_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W^2_q(\dot{\mathbb{R}}^n)),\\
&g_u \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n))^n \cap L_{p,0,\gamma_0}(\mathbb{R}; W^1_q(\dot{\mathbb{R}}^n))^n,\\
&f_\theta \in L_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)),\\
&g_\theta \in W^1_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W^2_q(\dot{\mathbb{R}}^n)),\\
&g_h \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W^1_q(\dot{\mathbb{R}}^n)),
\end{aligned}
$$

and the compatibility condition:

$$
\begin{aligned}
\operatorname{div} u_0 &= f_d(0) && \text{in } \dot{\mathbb{R}}^n, && 2 - 2/p > 1 + 1/q,\\
-[\![\mu P_{\mathbb{R}^{n-1}} D(u_0)]\!] &= P_{\mathbb{R}^{n-1}} g_u(0) && \text{on } \mathbb{R}^{n-1}, && 2 - 2/p > 1 + 1/q,\\
[\![u_0']\!] = g(0), \quad [\![\theta_0]\!] &= 0 && \text{on } \mathbb{R}^{n-1}, && 2 - 2/p > 1/q,\\
l_1 \theta_0 &= \sigma \Delta' h_0 + g_\theta(0) && \text{on } \mathbb{R}^{n-1}, && 2 - 2/p > 1/q.
\end{aligned}
$$

Then, the linearized Stefan problem (11.5) *and* (11.6) *admits a unique solution (u, π, θ, h) with regularity:*

$$
\begin{aligned}
u &\in W^1_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n))^n \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^2_q(\dot{\mathbb{R}}^n))^n,\\
\pi &\in L_{p,0,\gamma_0}(\mathbb{R}_+; \hat{W}^1_q(\dot{\mathbb{R}}^n)),\\
\pi_\pm &\in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}_+; L_q(\mathbb{R}^n_\pm)) \cap L_{p,0,\gamma_0}(\mathbb{R}_+; W^1_q(\mathbb{R}^n_\pm)),\\
\theta &\in W^1_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)),\\
\tilde{\theta} &\in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; W^1_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W^2_q(\dot{\mathbb{R}}^n)),\\
h &\in W^1_{p,0,\gamma_0}(\mathbb{R}; W^1_q(\dot{\mathbb{R}}^n)) \cap H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; W^3_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W^4_q(\dot{\mathbb{R}}^n)),\\
\tilde{h} &\in H^{3/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)),\\
\nabla'^2 \tilde{\tilde{h}} &\in W^1_{p,0,\gamma_0}(\mathbb{R}; L_q(\mathbb{R}^n)).
\end{aligned}
$$

We could solve (11.4) with the result of Theorem 11.2 in the same way as (5.1) but the height function, h is different from that of (5.1), so we need to notice difference between (11.4) and (5.1) (cf. Section 6 and Section 7 in [14]). In particular, we observe nonlinear terms $J(h)$ in $G_{\theta,2}$ and $|\nabla' h|^2\partial_n\theta$ in $G_{h,2}$. Recall $J(h)$ in Section 5:

$$J(h) = \frac{|\nabla' h|^2\Delta' h}{(1+\sqrt{1+|\nabla' h|^2})\sqrt{1+|\nabla' h|^2}} + \frac{\nabla' h\cdot(\nabla'^2 h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}}.$$

We have set the first term and the second term of $J(h)$ be $J_1(h)$ and $J_2(h)$, respectively. We have easily seen that

$$\begin{aligned}\partial_t J_2(h) &= \frac{\nabla'\partial_t h\cdot(\nabla'^2 h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}} + \frac{\nabla' h\cdot(\nabla'^2\partial_t h\cdot\nabla' h)}{(1+|\nabla' h|^2)^{3/2}} + \frac{\nabla' h\cdot(\nabla'^2 h\cdot\nabla'\partial_t h)}{(1+|\nabla' h|^2)^{3/2}}\\ &\quad - 3\nabla' h\cdot(\nabla'^2 h\cdot\nabla' h)(1+|\nabla' h|^2)^{-5/2}\nabla' h\cdot\partial_t\nabla' h,\end{aligned}$$

therefore

$$|\partial_t J_2(h)| \le C(|\nabla' h||\nabla'^2 h||\partial_t\nabla' h| + |\nabla' h|^2|\nabla'^2\partial_t h|).$$

Now, using $\partial_t\nabla' h, \partial_t\nabla'^2 h \in L_p(J; L_q(\mathbb{R}^n))$ where $J=(0,T]$ $(0<T<\infty)$, we may estimate $\nabla' h$ and $\nabla'^2 h$ like type **(III)** in Section 5. In case we couldn't use the smallness condition, as a result we derive a power of the time, T with Lemma 5.7. We could calculate $|\partial_t h_1(h)|$ similarly. By Lemma 5.2, Lemma 5.4 and the proof of Lemma 5.6, we gain

$$\begin{aligned}&\||\nabla' h|^2\partial_n\theta\|_{H_p^{1/2}(J;L_q)}\\ &\le C\|\nabla' h\|_{H_p^{1/2}(J;L_q)}\|\nabla' h\|_{H_p^{1/2}(J;L_q)}\|\partial_n\theta\|_{H_p^{1/2}(J;L_q)}\\ &= C\|\nabla'\Lambda_\gamma^{1/2} h\|_{L_p(J;L_q)}\|\nabla' h\|_{H_p^{1/2}(J;L_q)}\|\partial_n\theta\|_{H_p^{1/2}(J;L_q)}\\ &\le C\|\Lambda_\gamma^{1/2} h\|_{L_p(J;L_q)}^{1/2}\|\nabla'^2\Lambda_\gamma^{1/2} h\|_{L_p(J;L_q)}^{1/2}\|\nabla' h\|_{H_p^{1/2}(J;L_q)}\|\partial_n\theta\|_{H_p^{1/2}(J;L_q)}\\ &\le CT^{1/(4p)}\|h\|_{H_p^{3/2}(J;L_q)}^{1/4}\|h\|_{H_p^{1/2}(J;L_q)}^{1/4}\|h\|_{H_p^{1/2}(J;W_q^2)}^{1/2}\|h\|_{H_p^{1/2}(J;W_q^1)}\\ &\qquad\qquad\qquad\qquad\times\|\theta\|_{W_p^1(J;L_q)\cap L_p(J;W_q^2)}\\ &\le CT^{1/(4p)}\|h\|_{H_p^{3/2}(J;L_q)}^{1/2}\|h\|_{H_p^{1/2}(J;W_q^2)}^{3/2}\|\theta\|_{W_p^1(J;L_q)\cap L_p(J;W_q^2)}\end{aligned}$$

by

$$\begin{aligned}\left(\int_0^T \|\Lambda_\gamma^{1/2} h\|_{L_q}^p\,dt\right)^{1/p} &= \left(\int_0^T \|\Lambda_\gamma^{1/2} h\|_{L_q}^{p/2}\|\Lambda_\gamma^{1/2} h\|_{L_q}^{p/2}\,dt\right)^{1/p}\\ &\le \|\Lambda_\gamma^{1/2} h\|_{L_\infty(J;L_q)}^{1/2}\left(\int_0^T \|\Lambda_\gamma^{1/2} h\|_{L_q}^{p/2}\,dt\right)^{1/p}\\ &\le C\|\Lambda_\gamma^{1/2} h\|_{W_p^1(J;L_q)}^{1/2}T^{1/(2p)}\|\Lambda_\gamma^{1/2} h\|_{L_p(J;L_q)}^{1/2}\\ &\le CT^{1/(2p)}\|h\|_{H_p^{3/2}(J;L_q)}^{1/2}\|h\|_{H_p^{1/2}(J;L_q)}^{1/2}.\end{aligned}$$

In the same way as (2.8), define $\mathbb{E}_{\theta,2}(J)$, $\mathbb{E}_{\tilde{\theta}_\pm}(J)$, $\mathbb{E}_{h,2}(J)$, $\mathbb{E}_{\tilde{h}_\pm}(J)$ and $\mathbb{E}_{\tilde{\tilde{h}}_\pm}(J)$ as:

$$\begin{aligned}
&\mathbb{E}_{\theta,2}(J) := W^1_p(J; L_q(\mathbb{R}^n))\\
&\mathbb{E}_{\tilde{\theta}_\pm}(J) := H^{1/2}_p(J; W^1_q(\mathbb{R}^n_\pm)) \cap L_p(J; W^2_q(\mathbb{R}^n_\pm)),\\
&\mathbb{E}_{h,2}(J) := W^1_p(J; W^1_q(\dot{\mathbb{R}}^n)) \cap H^{1/2}_p(J; W^3_q(\dot{\mathbb{R}}^n)) \cap L_p(J; W^4_q(\dot{\mathbb{R}}^n))\\
&\mathbb{E}_{\tilde{h}_\pm}(J) := H^{3/2}_p(J; \mathbb{R}^n_\pm),\\
&\mathbb{E}_{\tilde{\tilde{h}}_\pm}(J) := \{h \in L_p(J; L_q(\mathbb{R}^n)) \mid \partial_t \Delta' h_\pm \in L_p(J; L_q(\mathbb{R}^n_\pm))\}
\end{aligned}$$

and set

$$\mathbb{E}_2(J) := \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\pi_\pm}(J) \times \mathbb{E}_{\theta,2}(J) \times \mathbb{E}_{\tilde{\theta}_\pm}(J) \times \mathbb{E}_{h,2}(J) \times \mathbb{E}_{\tilde{h}_\pm}(J) \times \mathbb{E}_{\tilde{\tilde{h}}_\pm}(J). \tag{11.20}$$

We state the result for (11.4):

Theorem 11.3. *Let $p < \infty$, $n < q < \infty$, $2/p + n/q < 1$ and suppose $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ are such that*

$$\kappa_\pm(s) = -s\psi''_\pm(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0 \quad s \in (0,\infty).$$

Let the initial interface Γ_0 be given by a graph $x' \mapsto (x', h_0(x'))$, $\theta_\infty > 0$ be the constant temperature at infinity. And let

$$(u_0, \theta_0, h_0) \in B^{2-2/p}_{q,p}(\Omega_0)^n \times B^{2-2/p}_{q,p}(\Omega_0) \times B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})$$

be given. Assume that the compatibility conditions:

$$\begin{aligned}
\operatorname{div} u_0 = 0 \quad &\text{in } \Omega_0,\\
P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] = 0, \quad P_{\Gamma_0}[\![u_0]\!] = 0 \quad &\text{on } \Gamma_0,\\
[\![\theta_0]\!] = 0, \quad [\![\psi(\theta_0)]\!] + \sigma H_{\Gamma_0} \quad &\text{on } \Gamma_0
\end{aligned}$$

and the **well-posedness condition:**

$$l(\theta_0) \neq 0 \quad \text{on } \Gamma_0 \qquad \text{and} \qquad \theta_0 > 0 \quad \text{in } \Omega_0.$$

Then there exists a constant ε_0 depending only on Ω_0, p, q, n such that if h_0 and u_0 satisfy $\|\nabla' h_0\|_{L_\infty(\dot{\mathbb{R}}^n)} + \|u_0\|_{L_\infty(\Omega_0)} \le \varepsilon_0$, then there exist

$$T = T(\|\theta_0 - \theta_\infty\|_{B^{2-2/p}_{q,p}(\dot{\mathbb{R}}^n)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbb{R}^{n-1})}, \varepsilon_0) > 0$$

and a unique L_p-L_q solution (u, π, θ, h) of (11.1)-(11.3) on $[0,T]$ in the class of (11.20).

Part 3. **Appendix**

12. Solution Formulas

In Appendix, we exhibit calculation of solution formulas in Section 4. From (4.4), (4.5), (4.11)-(4.14) we have

$$(B^2_+ - A^2)P_k + i\xi_k R = 0\ ,\ (B^2_+ - A^2)P_n - AR = 0.$$

Therefore we obtain P_k $(k = 1, \ldots, n)$ as (4.17), and the relation

$$\Sigma_{k=1}^{n-1} i\xi_k P_k - AP_n = 0. \tag{12.1}$$

In the same way, we obtain P'_k $(k = 1, \ldots, n)$ as (4.18), and the relation

$$\Sigma_{k=1}^{n-1} i\xi_k P'_k + AP'_n = 0. \tag{12.2}$$

From (4.6), (4.11)-(4.14), (12.1) and (12.2), we obtain

$$\Sigma_{k=1}^{n-1} i\xi_k Q_k - B_+ Q_n = 0, \tag{12.3}$$

$$\Sigma_{k=1}^{n-1} i\xi_k Q'_k + B_- Q'_n = 0. \tag{12.4}$$

By (4.13), (4.14), (4.11) and (4.12), we have

$$[\![2\mu\partial_n \hat{v}_n]\!] = 2\mu_+(-AP_n - B_+Q_n) - 2\mu_-(AP'_n +_- Q_n), [\![\hat{\tau}]\!] = \mu_+ R - \mu_- R'.$$

Inserting them into (4.9) and we have

$$\mu_+(-2AP_n - 2B_+Q_n - R) - \mu_-(2AP'_n + 2B_-Q'_n - R') = -\hat{g}_{u,n}. \tag{12.5}$$

Similarly we have

$$(\mu_+/\rho_+)(-2AP_n - 2B_+Q_n - R) - (\mu_-/\rho_-)(2AP'_n + 2B_-Q'_n - R') = -\hat{g}_\pi. \tag{12.6}$$

If we compute difference between (12.5) and the equation (12.6) multiplied by ρ_- and ρ_+, it is derived that

$$Q_n = (2\mu_+B_+(1 - \rho_-/\rho_+))^{-1}(-\rho_-\hat{g}_\pi + \hat{g}_{u,n}) - (A/B_+)P_n - R/(2B_+), \tag{12.7}$$

$$Q'_n = (2\mu_-B_-(1 - \rho_+/\rho_-))^{-1}(-\rho_+\hat{g}_\pi + \hat{g}_{u,n}) - (A/B_-)P'_n + R'/(2B_-). \tag{12.8}$$

From (4.13) and (4.14),

$$\partial_n \hat{v}_{+k} + i\xi_k \hat{v}_{+n} = -AP_k e^{-Ax_n} - B_+Q_k e^{-B_+x_n} + i\xi_k P_n e^{-Ax_n} + i\xi_k Q_n e^{-B_+x_n},$$

$$\partial_n \hat{v}_{-k} + i\xi_k \hat{v}_{-n} = AP'_k e^{Ax_n} + B_-Q'_k e^{B_-x_n} + i\xi_k P'_n e^{Ax_n} + i\xi_k Q'_n e^{B_-x_n}.$$

Substituting them into (4.8), we obtain

$$\begin{aligned}&\mu_+B_+Q_k + \mu_-B_-Q'_k \\ &= \mu_+ i\xi_k(P_n + Q_n) - \mu_+AP_k - \mu_- i\xi_k(P'_n + Q'_n) - \mu_-AP'_k + \hat{g}_{u,k}.\end{aligned} \tag{12.9}$$

Combining (4.7), (4.13) and (4.14), we have;

$$\begin{cases} \mu_-B_-Q_k - \mu_-B_-Q'_k = \mu - B_-(\hat{g}_k - P_k + P'_k) \\ -\mu_+B_+Q_k + \mu_+B_+Q'_k = -\mu_+B_+(\hat{g}_k - P_k + P'_k). \end{cases} \tag{12.10}$$

Combining (12.9) and (12.10);

$$\begin{aligned}(\mu_+B_+ + \mu_-B_-)Q_k = \mu_-B_-(\hat{g}_k - P_k + P'_k) + \mu_+ i\xi_k Q_n - \mu_+(AP_k - i\xi_k P_n) \\ -\mu_- i\xi_k Q'_n - \mu_-(AP'_k + i\xi_k P'_n) + \hat{g}_{u,k}.\end{aligned}$$

So, we pay attention to (12.1) and (12.2) and see the following;

$$(\mu_+B_+ + \mu_-B_-)\Sigma_{k=1}^{n-1} i\xi_k Q_k = \mu_-B_-\widehat{\mathrm{div}_{x'} g} + \widehat{\mathrm{div}_{x'} g_u} + P_n(-\mu_-AB_- - 2\mu_+A^2)$$

$$+ P_n'(-\mu_- AB_- + 2\mu_- A^2) - \mu_+ A^2 Q_n + \mu_- A^2 Q_n', \tag{12.11}$$

$$(\mu_+ B_+ + \mu_- B_-)(-B_+ Q_n) = (-\mu_+ B_+^2 - \mu_- B_- B_+)Q_n. \tag{12.12}$$

Substituting (12.11) and (12.12) into (12.3), we have

$$\begin{aligned}0 &= \mu_- B_- \hat{\mathrm{div}_{x'}} g + \hat{\mathrm{div}_{x'}} g_u \\ &+ (\mu_+ A/(\rho_+ s))R(-\mu_- AB_- - 2\mu_+ A^2) - (\mu_- A/(\rho_- s))R'(-\mu_- AB_- + 2\mu_- A^2) \\ &- (\mu_+ B_+^2 + \mu_+ A^2 + \mu_- B_- B_+)Q_n + \mu_- A^2 Q_n',\end{aligned}$$

where we use (4.17) and (4.18). Similarly, keeping in mind for (12.4), we obtain

$$\begin{aligned}0 &= -\mu_+ B_+ \hat{\mathrm{div}_{x'}} g + \hat{\mathrm{div}_{x'}} g_u \\ &+ (\mu_+ A/(\rho_+ s))R(\mu_+ AB_+ - 2\mu_+ A^2) - (\mu_- A/(\rho_- s))R'(\mu_+ AB_+ + 2\mu_- A^2) \\ &- \mu_+ A^2 Q_n + (\mu_- B_-^2 + \mu_- A^2 + \mu_+ B_+ B_-)Q_n'.\end{aligned}$$

By using $B_\pm^2 = \rho_\pm s/\mu_\pm + A^2$, we obtain

$$\begin{aligned}&R(\mu_+ A^2(3B_+ - A)/(2B_+(B_+ + A)) + (\mu_+ B_+ + \mu_- B_-)/2) \\ &+R'\mu_- A^2(3B_- - A)/(2B_-(B_- + A)) \\ &= -\mu_- B_- \hat{\mathrm{div}_{x'}} g - \hat{\mathrm{div}_{x'}} g_u \\ &-(2\mu_+ B_+(1 - \rho_-/\rho_+))^{-1}(\mu_+ B_+^2 + \mu_+ A^2 + \mu_- B_- B_+)(\rho_- \hat{g}_\pi - \hat{g}_{u,n}) \\ &-(2B_-(1 - \rho_+/\rho_-))^{-1} A^2(-\rho_+ \hat{g}_\pi + \hat{g}_{u,n})\end{aligned} \tag{12.13}$$

and

$$\begin{aligned}&R\mu_+ A^2(3B_+ - A)/(2B_+(B_+ + A)) \\ &+R'(\mu_- A^2(3B_- - A)/(2B_-(B_- + A)) + (\mu_+ B_+ + \mu_- B_-)/2) \\ &= \mu_+ B_+ \hat{\mathrm{div}_{x'}} g - \hat{\mathrm{div}_{x'}} g_u \\ &-(2\mu_- B_-(1 - \rho_+/\rho_-))^{-1}(\mu_- B_-^2 + \mu_- A^2 + \mu_+ B_+ B_-)(-\rho_+ \hat{g}_\pi + \hat{g}_{u,n}) \\ &-(2B_+(1 - \rho_-/\rho_+))^{-1} A^2(\rho_- \hat{g}_\pi - \hat{g}_{u,n}),\end{aligned} \tag{12.14}$$

respectively. Solving the simultaneous equations, (12.13) and (12.14), we can describe R and R' as (4.15) and (4.16). By (4.17), (4.18), (12.7) and (12.8), we obtain Q_k and Q_k' as (4.19) and (4.20) Thus, we obtain the description of solutions of (4.4)-(4.10).

Next, we solve the problem (4.3). In order to solve (4.3), we can use the solution formula of (4.2) with $f_u = f_d = g_{u,k} = g_\pi = g_k = 0$ and $g_{u,n} = \sigma\Delta' h$. And then we solve the last two equations in (4.3). Making use of

$$\begin{aligned}R = (\alpha_+ + \alpha_-\beta)^{-1}\Big[&+ (\mu_-(1 - \rho_+/\rho_-))^{-1}(\alpha_- + \mu_- A^2/(2B_-))(-\sigma A^2 \hat{h}) \\ &+ (\mu_+(1 - \rho_-/\rho_+))^{-1}(\alpha_- - \beta - \mu_+ A^2/(2B_+))(-\sigma A^2 \hat{h})\Big],\end{aligned} \tag{12.15}$$

$$\begin{aligned} R' = (\alpha_+ + \alpha_-\beta)^{-1}\Big[&+ (\mu_+(1-\rho_-/\rho_+))^{-1}(-\alpha_+ - \mu_+A^2/(2B_+))(-\sigma A^2\hat{h}) \\ &+ (\mu_-(1-\rho_+/\rho_-))^{-1}(-\alpha_+ + \beta + \mu_-A^2/(2B_-))(-\sigma A^2\hat{h})\Big], \end{aligned} \tag{12.16}$$

we can write

$$\begin{aligned} \hat{\kappa}_+ &= \mu_+ Re^{-Ax_n}, \hat{w}_{+m} = P_m e^{-Ax_n} + Q_m e^{-B_+x_n} \quad \text{for } x_n > 0 \\ \hat{\kappa}_- &= \mu_- R' e^{Ax_n}, \hat{w}_{-m} = P'_m e^{Ax_n} + Q'_m e^{B_-x_n} \quad \text{for } x_n < 0. \end{aligned}$$

Noting that

$$\begin{aligned} Q_n &= (2\mu_+B_+(1-\rho_-/\rho_+))^{-1}(-\sigma A^2\hat{h}) - (A/B_+)P_n - R/(2B_+), \\ Q'_n &= (2\mu_-B_-(1-\rho_+/\rho_-))^{-1}(-\sigma A^2\hat{h}) - (A/B_-)P'_n + R'/(2B_-), \end{aligned}$$

we obtain

$$\begin{aligned} [\![\rho\hat{w}_n]\!] = &-\rho_+(B_+-A)(2B_+(B_++A))^{-1}R - \rho_-(B_--A)(2B_-(B_-+A))^{-1}R' \\ &- \sigma A^2\hat{h}(\rho_+/(2\mu_+B_+(1-\rho_-/\rho_+)) - \rho_-/(2\mu_-B_-(1-\rho_+/\rho_-))) \end{aligned}$$

with $B_\pm^2 = \rho_\pm s/\mu_\pm + A^2$. By the second equation below in (4.3), (12.13), (12.15) and (12.16), finally we obtain the description of $\hat{h}$

$$\hat{h} = f(B_+, B_-, A)L(B_+, B_-, A)^{-1}(\hat{g}_h + [\![\rho\hat{v}_n]\!]/[\![\rho]\!]), \tag{4.29}$$

where we define $f(B_+, B_-, A)$ and $L(B_+, B_-, A)$ as (4.21) and (4.28), respectively. By the way, we don't use the formula like (4.24) with in order to obtain (4.29) since it is complicated. The formula like (4.24) avails to estimate w_n itself.

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