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メタデータ	言語: eng
	出版者:
	公開日: 2015-06-30
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	キーワード (En):
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URL	https://doi.org/10.14945/00008858

Reports of Faculty of Science, SHIZUOKA UNIVERSITY Vol. 10, pp. 9 to 15 (1975)

Free Algebras over All Fields and Pseudo-fields

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(*Received Oct. 11, 1975*)

Birkhoff [1] defined the notion of a free algebra over $\boldsymbol{K}$ which was a class of algebras. Hence, we can consider a free algebra over the class of all fields (simply we say, a free algebra over all fields). Since a free algebra over all groups is a group, a free algebra over all groups is called a free group. But it is easily seen that a free algebra over all fields is not a field. We will consider what kind of algebras the free algebras over all fields are, and show that any free algebra over all fields is a pseudo-field defined in §2. In contrast with the case of the free algebra over all fields, any free algebra over all pseudo-fields is a pseudo-field. Therefore, we can call a free algebra over all pseudo-fields a free pseudo-field. Further, we will show that any irreducible pseudo-field is a field and that a free pseudo-field is isomorphic to the free algebra over all fields with same cardinality of generators.

§ 1. Free algebras over all fields.

The first order language $\mathscr{L}$ that we consider has two nullary function symbols 0, 1, two unary function symbols $-$, $^{-1}$, two binary function symbols $+$, $\cdot$ and no relation symbols other than $=$.

We now give the axiomatic system of fields.

A1. $(x+y)+z=x+(y+z)$

A2. $x+y=y+x$

A3. $x+0=x$

A4. $x+(-x)=0$

A5. $(x\cdot y)\cdot z=x\cdot(y\cdot z)$

A6. $x\cdot 1=x$

A7. $1\cdot x=x$

A9. $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$

A9. $(y+z)\cdot x=(y\cdot x)+(z\cdot x)$

A10. $x\neq 0 \rightarrow x^{-1}\cdot x=1\wedge x\cdot x^{-1}=1.$

Note that the commutative law $x\cdot y=y\cdot x$ is not contained within this axiomatic system. The axioms A1–A9 constitute the axiomatic system of unitary rings. Our definition of fields permits a field in which $1=0$ holds. Such a field has only one element 0.

Let W be the set of all terms of the language $\mathscr{L}$. We define a relation $\sim$ on W as follows:

$X \sim Y \Leftrightarrow X=Y$ is derivable from the axiomatic system of fields.

Then, the relation $\sim$ is a congruence relation, and $W/\sim$ is naturally an algebra of same type as fields. We call this algebra a free algebra over all fields with ω generators. Let $[x]$ be the element of $W/\sim$ which contains x, then the subalgebra of $W/\sim$ generated by $[x_1]$, $[x_2]$, ..., $[x_n]$ is called a free algebra over all fields with n generators. We denote this by $FF_n(n \leqq \omega)$.

Theorem 1.1. $FF_n(n \leqq \omega)$ is not a field.

Proof. We abbreviate $1+1$ by 2. $[2] \neq 0$ but $[2] \cdot [2]^{-1} = [2 \cdot 2^{-1}] \neq [1]$. Because $2 \cdot 2^{-1} = 1$ is not derivable. Q.E.D.

§2. Pseudo-fields.

We begin with the presentation of the axiomatic system of pseudo-fields. We abbreviate $x+(-y)$ and $(x \cdot y) \cdot z$ by $x-y$ and $x \cdot y \cdot z$, respectively.

The axioms P1–P9 are same as the axioms A1–A9.

P10. $x^{-1}-y^{-1}=(x-y)\cdot(x-y)^{-1}\cdot(x^{-1}-y^{-1})$

P11. $x \cdot y = x \cdot y \cdot x^{-1} \cdot x$.

A pseudo-field is an algebra $\langle A; 0, 1, -, ^{-1}, +, \cdot \rangle$ which is a model of the axiomatic system of pseudo-fields. We say simply that A is a pseudo-field, when $\langle A; 0, 1, -, ^{-1}, +, \cdot \rangle$ is a pseudo-field.

Prof. P.M. Cohn has informed me that the axiomatic system (P1–P9)+P11 is equivalent to the axiomatic system (P1−P9)+(*), where (*) $x = x^{-1} \cdot x^2$.

Every axioms of pseudo-fields are derivable from the axiomatic system of fields. Therefore, any field is a pseudo-field. The following theorem gives another axiomatic system of fields.

Theorem 2.1. The axiomatic system of fields is equivalent to the axiomatic system obtained by adding the following axiom to the axiomatic system of pseudo-fields:

(2.1) $x \cdot y = 0 \rightarrow x = 0 \vee y = 0$.

Proof. Clearly, (2.1) is derivable from the axiomatic system of fields. Conversely, in the axiom P11, we substitute 1 for y and transpose x. Then, we have

(2.2) $(x \cdot x^{-1} - 1) \cdot x = 0, \ x \cdot (x^{-1} \cdot x - 1) = 0$.

By (2.1), $x \neq 0 \rightarrow x \cdot x^{-1} = 1 \wedge x^{-1} \cdot x = 1$. Q.E.D.

Definition 2.2. Let A be a pseudo-field. A non-empty subset J of A is *an ideal* of A if it satisfies the following two conditions:

1) $x \in J$ and $y \in J \Rightarrow x - y \in J$

2) $x\in J$ and $y\in A \Rightarrow x\cdot y\in J$ and $y\cdot x\in J$.

Definition 2.3. Let A be a pseudo-field and J be an ideal of A. We define a relation $\sim_J$ on A as follows:

$$x\sim_J y \Leftrightarrow x-y\in J.$$

The relation $\sim_J$ is compatible with 0, 1, $-$, $+$ and $\cdot$ similarly to the case of rings. Further, $\sim_J$ is compatible with $^{-1}$ by the axiom P10. Thus we obtain the following theorem.

Theorem 2.4. For any pseudo-field A and any ideal J of A, the relation $\sim_J$ is a congruence relation, and $A/\sim_J$ is naturally a pseudo-field. ($A/\sim_J$ is denoted by A/J.)

Homomorphism Theorem is easily shown.

Theorem 2.5. (Homomorphism Theorem) Let A and B be pseudo-fields, and φ: $A\to B$ a homomorphism of A onto B. Then $J=\varphi^{-1}(0_B)$ is an ideal of A, and A/J is isomorphic to B, where the isomorphism is given by $[a]\mapsto\varphi(a)$ $(a\in A)$.

We now define the term 'irreducible' which is called 'subdirectly irreducible' in Birkhoff [2]. Since there exists a pseudo-field in which $1=0$ holds, the exceptional case occurs. Such a pseudo-field has only one element 0.

Definition 2.6. Let A be a pseudo-field, x be a non-zero element of A. A is *irreducible with respect to* x if x is contained within any ideal of A which contains at least an element other than 0. A is *irreducible*, if there exists a non-zero element such that A is irreducibie with respect to the element, or A has only one element 0.

It is valid in fields that $x=0$ or $y=0$ if $x\cdot y=0$. But it is not always valid in pseudo-fields. The following lemma says that it is valid in irreducible pseudo-fields.

Lemma 2.7. Let A be an irreducible pseudo-field, and x, $y\in A$.

Then, $$x\cdot y=0 \rightarrow x=0\vee y=0.$$

Proof. Clearly, the lemma holds when A has only one element. Suppose that $x\neq 0$, $y\neq 0$ and $x\cdot y=0$. Let $z(\neq 0)$ be an element which A is irreducible with respect to. Because z is contained within the ideal generated by x, there exist elements a_1, a_2, ..., a_n, b_1, b_2, ..., b_n of A such that $z=\sum_{i=1}^{n} a_i\cdot x\cdot b_i$.

Similarly, there exist elements c_1, c_2, ..., c_m, d_1, d_2, ..., d_m of A such that $z=\sum_{i=1}^{m} c_i\cdot y\cdot d_i$.

$$\begin{aligned}\text{Hence, } z\cdot z^{-1}\cdot z &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\cdot x\cdot b_i\cdot z^{-1}\cdot c_j\cdot y\cdot d_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\cdot x\cdot b_i\cdot z^{-1}\cdot c_j\cdot x^{-1}\cdot x\cdot y\cdot d_j \quad \text{(by P11)} \\ &= 0 \qquad \text{(by } x\cdot y=0).\end{aligned}$$

In the axiom P11, we substitute z and 1 for x and y respectively. Then we have $z=z\cdot z^{-1}\cdot z=0$. This is a contradiction. Q.E.D.

The following theorem is very important. It completely characterizes irreducible pseudo-fields. The results in §3 can be regarded as its corollaries.

Theorem 2.8. A pseudo-field A is irreducible if and only if A is a field.

Proof. Suppose that A is a field. When A has only one element 0, it holds obviously. Let a be a non-zero element of A. If an ideal J contains a, J contains 1 because of $a\cdot a^{-1}=1$. Hence, A is irreducible with respect to 1. Q.E.D.

Conversely, suppose that a pseudo-field A is irreducible. Obviously, the axioms of fields other than $A10$ are valid in A. By Lemma 2.7 and (2.2), for any element x of A, if $x\neq 0$, then $x\cdot x^{-1}=1$ and $x^{-1}\cdot x=1$. Hence, the axiom $A10$ is valid in A. Therefore, A is a field. Q.E.D.

§3. Identities and Free algebras.

An identity is an atomic formula of the language $\mathscr{L}$, that is, the form is $t=s$ where t and s are terms of $\mathscr{L}$. Let A be a pseudo-field and let $Id(A)$ denote the set of all identities valid in A. For a class $\mathbf{K}$ of pseudo-fields, $Id(\mathbf{K})$ denotes the set of identities $\bigcap_{A\in\mathbf{K}} Id(A)$ as in Grätzer [3]. The following decomposition theorem is very useful for the study of $Id(\mathbf{K})$.

Theorem 3.1. Let $\mathbf{K}$ be a class of pseudo-fields. Then, there exists a class of irreducible pseudo-fields $\mathbf{K}_1$ such that $Id(\mathbf{K})=Id(\mathbf{K}_1)$.

Proof. Let $A\in\mathbf{K}$ and $s=t\notin Id(A)$. Then there exists an assignment f of A such that $f(s)\neq f(t)$. Let J be an ideal of A which is maximal with respect to $f(s)-f(t)\neq 0$, that is, a maximal element of the set $\{J \mid J \text{ is an ideal of } A \text{ and } (f(s)-f(t))\notin J\}$. The existance of J is obtained by Zorn's lemma. We can show that the pseudo-field A/J is irreducible, $s=t\notin Id(A/J)$, and $Id(A)\subseteq Id(A/J)$. Hence, if we put $\mathbf{K}_2=\{A/J \mid A\in\mathbf{K} \text{ and } J \text{ is an ideal of } A\}$ and $\mathbf{K}_1=\{A \mid A\in\mathbf{K}_2 \text{ and } A \text{ is irreducible}\}$, we have $Id(\mathbf{K})=Id(\mathbf{K}_1)$. Q.E.D.

The above theorem is easily obtained for the general algebras by the subdirect reduction theorem in Birkhoff [2].

We denote the class of all fields and the class of all pseudo-fields by $\mathbf{F}$ and $\mathbf{PF}$, respectively. Clearly, $\mathbf{F}\subseteq\mathbf{PF}$ and $A\times A\notin\mathbf{F}$ and $A\times A\in\mathbf{PF}$ where A is any field having at least two element and $A\times A$ is the direct product of A and A. Hence, we have $\mathbf{F}\subsetneqq\mathbf{PF}$.

By Theorem 2.8, $\mathbf{F}$ is a class of all irreducible pseudo-fields. Hence, the following theorem is obvious by Theorem 3.1.

Tneorem 3.2. $Id(\mathbf{PF})=Id(\mathbf{F})$.

Since $\mathbf{PF}$ is a variety (or equational class), we have the following corollary.

Corollary 3.3. $\mathbf{PF}$ is the minimum variety including $\mathbf{F}$.

We denote the free algebra over all pseudo-fields with n generators by $FPF_n(n\leqq\omega)$.

Because **PF** is a variety, $FPF_n \in \mathbf{PF}$. Hence, we can say that FPF_n is a free pseudo-field.

$Id(\mathbf{K})$ determines the structure of the free algebra over **K**, so we have the following theorem.

Theorem 3.4. $FF_n \cong FPF_n (n \leqq \omega)$.

§4. Some identities derivable from the axiomatic system of pseudo-fields.

It can be indirectly (model theoretically) shown by Theorem 3.2 that the identites in the next theorem are derivable from the axiomatic system of pseudo-fields. But, we give the direct (proof theoretical) proof of it.

Theorem 4.1. The following identities are derivable from the axiomatic system of pseudo-fields:

(4.1) $\quad x \cdot x^{-1} = x^{-1} \cdot x$

(4.2) $\quad y \cdot x = x \cdot x^{-1} \cdot y \cdot x.$

Proof. (4.1) $x \cdot x^{-1} - x^{-1} \cdot x = x \cdot x^{-1} \cdot x^{-1} \cdot x - x^{-1} \cdot x$ (by P11)

$= x \cdot x^{-1} - 1) \cdot x^{-1} \cdot x$ (by P9)

$= (x \cdot x^{-1} - 1) \cdot x^{-1} \cdot (x \cdot x^{-1} - 1)^{-1} \cdot (x \cdot x^{-1} - 1) \cdot x$ (by P11)

$= (x \cdot x^{-1} - 1) \cdot x^{-1} \cdot (x \cdot x^{-1} - 1)^{-1} \cdot (x \cdot x^{-1} \cdot x - x)$ (by P9)

$= 0$ (In P11, substitute 1 for y.).

(4.2) $y \cdot x - x \cdot x^{-1} \cdot y \cdot x = (1 - x \cdot x^{-1}) \cdot y \cdot x$ (by P9)

$= (1 - x \cdot x^{-1}) \cdot y \cdot (1 - x \cdot x^{-1})^{-1} \cdot (1 - x \cdot x^{-1}) \cdot x$ (by P11)

$= (1 - x \cdot x^{-1}) \cdot y \cdot (1 - x \cdot x^{-1})^{-1} \cdot (x - x \cdot x^{-1} \cdot x)$ (by P9)

$= 0$ (In P11, substitute 1 for y.). Q.E.D.

At first, we regarded above two identities as the axioms of pseudo-fields. But, after we noticed that they were not used in the proofs of Theorems in §2, they have been deleted from the axiomatic system of pseudo-fields and the direct proof of them has been invented.

We define the functions 0, 1, $-$, $+$, and $\cdot$ on $Z_3 = \{0, 1, 2\}$ as same as usual functions on Z_3 (mod 3). We define the function $^{-1}$ on Z_3 as follows: $0^{-1} = 1$, $1^{-1} = 1$, $2^{-1} = 2$. Then, Z_3 is a field but $(x^{-1})^{-1} = x$ and $(x \cdot y) = y^{-1} \cdot x^{-1}$ are not valid in Z_3. Hence, these identities are not derivable from the axiomatic system of pseudo-field. But, we add the next axiom to the axiomatic system of pseudo-fields, then these identities are derivable.

P12. $\quad 0^{-1} = 0.$

We say that a pseudo-field (or field) is desirable when the pseudo-field (or field) satisfies P12.

Theorem 4.2. The following identities are derivable from the axiomatic system of desirable pseudo-field :

(4.3) $(x^{-1})^{-1}=x$

(4.4) $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$.

Proof. (4.3) We substitute 0 for y in P10 and use P12, then we have

(4.5) $x^{-1}=x\cdot x^{-1}\cdot x^{-1}$.

$$
\begin{aligned}
(x^{-1})^{-1} &= x^{-1}\cdot(x^{-1})^{-1}\cdot(x^{-1})^{-1} && \text{(by (4.5))}\\
&= x\cdot x^{-1}\cdot x^{-1}\cdot(x^{-1})^{-1}\cdot(x^{-1})^{-1} && \text{(by (4.5))}\\
&= x\cdot x^{-1}\cdot(x^{-1})^{-1} && \text{(by P11 and (4.1))}\\
&= x\cdot x\cdot x^{-1}\cdot x^{-1}\cdot(x^{-1})^{-1} && \text{(by (4.5))}\\
&= x\cdot x\cdot x^{-1} && \text{(by P11 and (4.1))}\\
&= x && \text{(by P11 and (4.1)).}
\end{aligned}
$$

(4.4)

$$
\begin{aligned}
(x\cdot y)^{-1} &= x\cdot y\cdot(x\cdot y)^{-1}\cdot(x\cdot y)^{-1} && \text{(by (4.5))}\\
&= y^{-1}\cdot x^{-1}\cdot x\cdot y\cdot x\cdot y\cdot(x\cdot y)^{-1}\cdot(x\cdot y)^{-1} && \text{(by (4.1) and (4.2))}\\
&= y^{-1}\cdot x^{-1}\cdot x\cdot y\cdot(x\cdot y)^{-1} && \text{(by (4.5))}\\
&= y^{-1}\cdot x^{-1}\cdot y^{-1}\cdot x^{-1}\cdot(x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)^{-1} && \text{(by (4.1) and (4.2))}\\
&= y^{-1}\cdot x^{-1}\cdot y^{-1}\cdot x^{-1}\cdot x\cdot y && \text{(by P11 and (4.1))}\\
&= y^{-1}\cdot x^{-1}\cdot y^{-1}\cdot x^{-1}\cdot(x^{-1})^{-1}\cdot(y^{-1})^{-1} && \text{(by (4.3))}\\
&= y^{-1}\cdot x^{-1} && \text{(by P11 and (4.1)).}
\end{aligned}
$$

Q.E.D.

The identity (4.3) declares that the function $^{-1}$ on a desirable pseudo-field is bijective and that the inverse function of it is itself. This character of the function $^{-1}$ is desirable. We denote the class of all desirable fields and the class of all desirable pseudo-fields by **DF** and **DPF**, respectively. It is easily known that Theorem 3.2, Corollary 3.3 and Theorem 3.4 hold, even if we replace **PF** and **F** by **DPF** and **DF**, respectively.

§5. The independences of the axioms P10 and P11.

In §4, we have shown that P12 is independent of the axioms P1–P11. In this section, we prove that the axion P10 is not derivoble from the axiom P1–P9 and P11–P12, and the axiom P11 is not derivable from the axioms P1–P10 and P12.

Theorem 5.1. The axiom P11 is independent of the axioms P1–P10 and P12.

Proof. We define the functions 0, 1, $-$, $+$, and $\cdot$ on $Z_2=\{0, 1\}$ as same as usual functions on Z_2 (mod 2). We define the function $^{-1}$ on Z_2 as follows: $0^{-1}=0$, $1^{-1}=0$. Then, it is easily shown that the axioms P1–P10 and P12 are valid in Z_2. In the axiom P11, we substitute 1 for x and y. We have that the left side$=1$ and the right side$=0$. Hence, the axiom P11 is not valid in Z_2. Q.E.D.

Theorem 5.2. The axiom P10 is independent of the axioms P1–P9 and P11–P12.

Proof. We define the functions 0, 1, $-$, $+$, and $\cdot$ on $Z_6=\{0, 1, 2, 3, 4, 5\}$ as same as usual functions on Z_6 (mod 6). We define the function $^{-1}$ on Z_6 as follows:

$0^{-1}=0$, $1^{-1}=1$, $2^{-1}=2$, $3^{-1}=1$, $4^{-1}=1$, $5^{-1}=5$. Then, it is easily shown that the axioms P1–P9 and P11–P12 are valid in Z_6. In the axiom P10, we substitute 3 and 0 for x and y respectively. We have that the left side=1 and the the right side=3. Hence, the axiom P10 is not valid in Z_6. Q.E.D.

§6. The finite model property.

In the study of the intermediate logics, it has been known that many varieties of pseudo-boolean algebras have the finite model property (*fmp*) and some varieties of them have not *fmp*. (cf. [4], [5], [6].)

We will show that the varieties **PF** and **DPF** have not *fmp*.

Definition 6.1. A class **K** of algebras has *fmp* if for any identity $u \notin Id(\mathbf{K})$ there exists a finite algebra $A \notin \mathbf{K}$ such that $u \notin Id(A)$.

If **K** is a class of all infinite boolean algebras, then clearly **K** has not *fmp*. But, in this case, the above definition is uninteresting. In the above definition, one of the interesting cases is that **K** is a variety.

Theorem 6.2. The varieties **PF** and **DPF** have not *fmp*.

Proof. Suppose that **PF** has *fmp*. Since $x \cdot y = y \cdot x \notin Id(\mathbf{PF})$, there exists a finite pseudo-field A such that $x \cdot y = y \cdot x \notin Id(A)$. Then, there exists an assignment f of A such that $f(x \cdot y) \neq f(y \cdot x)$. Let J be an ideal of A which is maximal with resprct to $f(x \cdot y) - f(y \cdot x) \neq 0$. Then, A/J is an irreducible pseudo-field, that is, a field, and finite and $x \cdot y = y \cdot x \notin Id(A/J)$. Therefore, there exists a finite field in which $x \cdot y = y \cdot x$ is not valid. This is contrary to the well-known theorem that any finite field is commutative. The proof about **DPF** is entirely similar to the above. Q.E.D.

They are complicated, the examples of varieties without *fmp* of pseudo-boolean algebras. The variety **PF** (or **DPE**) is one of the simplest examples without *fmp*. Another simple example will be given in a subsequent paper.

References

1) G. Birkhoff, On the structure of abstract algebras, Proc. Cambridge Phil. Soc. **31** (1935), 433–454.

2) G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloq. Publ. vol. **25**, revised edition, Amer. Math. Soc., New York, N.Y., 1948.

3) G. Grätzer, Universal Algebra, D. Van Nostrand Company, Inc., Princeton etc., 1968.

4) T. Hosoi and H. Ono, Intermediate propositional logics (A survey), J. of Tsuda College, **5** (1973), 67–82.

5) V.A. Jankov, Constructing a sequence of strongly independent superintuitionistic propositional calculi, Soviet Math. Dokl., **9** (1968), 806–807.

6) Y. Komori, The finite model property of the intermediate propositional logics on finite slices, J. Fac. Sci., Univ. Tokyo, Sec. IA, **22** (1975), 117–120.