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# Mechanism of self-organization in point vortex system

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Abstract. A mechanism of the self-organization in an unbounded two-dimensional (2D) point vortex system is discussed. A kinetic equation for the system with positive and negative vortices is derived using the Klimontovich formalism. Similar to the Fokker-Planck collision term, the obtained collision term consists of a diffusion term and a drift term. It is revealed that the mechanism for the self-organization in the 2D point vortex system at negative absolute temperature is mainly provided by the drift term. Positive and negative vortices are driven toward opposite directions respectively by the drift term. As a result, well-known, two isolated clumps with positive and negative vortices, respectively, are formed as an equilibrium distribution. Regardless of the number of species of the vortices, either single- or double-sign, it is found that the collision term has following physically good properties: (i) When the system reaches a quasi-stationary state near the thermal equilibrium state with negative absolute temperature, the sign of  $d\omega/d\psi$  is expected to be positive, where  $\omega$  is the vorticity and  $\psi$  is the stream function. In this case, the diffusion term decreases the mean field energy, while the drift term increases it. As a whole, the total mean field energy is conserved. (ii) Similarly, the diffusion term increases the Boltzmann entropy, while the drift term decreases it. As a whole, the total entropy production rate is positive or zero (H theorem), which ensures that the system relaxes to the global thermal equilibrium state characterized by the zero entropy production.

Keywords: Point vortex system, Negative absolute temperature, Self-organization

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# 1. Introduction

In this paper, we propose a general mechanism of the self-organization for the twodimensional (2D) point vortex system composed of double-sign vortices through a newly obtained kinetic equation. The kinetic equation clearly elucidates the mechanism of the self-organization, in other words, a condensation of the same-sign vortices and a separation of the different-sign vortices.

At first, let us briefly introduce a hierarchy of the plasma kinetic equations. There are several equations with different scales. The most microscopic equation is the Klimontovich equation which describes a time evolution of a microscopic phase density  $\hat{f}_{\alpha}(\boldsymbol{r}, \boldsymbol{v}, t)$  for the  $\alpha$ -th plasma species with the charge  $q_{\alpha}$  and the mass  $m_{\alpha}$  in a six-dimensional phase space

$$\frac{\partial \hat{f}_{\alpha}(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v} \cdot \nabla \hat{f}_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) + \frac{q_{\alpha}}{m_{\alpha}} (\hat{\boldsymbol{E}}(\boldsymbol{r},t) + \boldsymbol{v} \times \hat{\boldsymbol{B}}(\boldsymbol{r},t)) \cdot \frac{\partial \hat{f}_{\alpha}(\boldsymbol{r},\boldsymbol{v},t)}{\partial \boldsymbol{v}} = 0(1)$$

where  $\hat{E}(\mathbf{r},t)$  and  $\hat{B}(\mathbf{r},t)$  are the microscopic electric and magnetic fields. Equation (1) has a formal discretized solution

$$\hat{f}_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) = \sum_{i}^{N_{\alpha}} \delta(\boldsymbol{r} - \boldsymbol{r}_{\alpha,i}(t)) \delta(\boldsymbol{v} - \boldsymbol{v}_{\alpha,i}(t))$$
(2)

where the position vector and the velocity of the *i*-th particle of the  $\alpha$ -th component are given by  $\mathbf{r}_{\alpha,i}$  and  $\mathbf{v}_{\alpha,i}$ , respectively. Number of particles of the  $\alpha$ -th component is given by  $N_{\alpha}$ .

It is difficult to make direct use of the microscopic equation (1) because of its complexity. We shall therefore proceed to the ensemble-average. It is assumed that the microscopic phase density  $\hat{f}_{\alpha}(\boldsymbol{r}, \boldsymbol{v}, t)$  is composed of a macroscopic phase density  $f_{\alpha}(\boldsymbol{r}, \boldsymbol{v}, t)$  and a fluctuation  $\delta f_{\alpha}(\boldsymbol{r}, \boldsymbol{v}, t)$ 

$$\hat{f}_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) = f_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) + \delta f_{\alpha}(\boldsymbol{r},\boldsymbol{v},t), \qquad (3)$$

$$f_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) \equiv \langle \hat{f}_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) \rangle \tag{4}$$

where the operator  $\langle \cdot \rangle$  means the ensemble average. In the same manner, the other physical quantities are rewritten into the averaged value and the fluctuation. Inserting the above expressions into the microscopic equation (1) and averaging the equation, we obtain the following macroscopic equation with a collisional effect in the right hand side:

$$\frac{\partial f_{\alpha}}{\partial t} + \boldsymbol{v} \cdot \nabla f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_{\alpha}}{\partial \boldsymbol{v}} = \frac{q_{\alpha}}{m_{\alpha}} \left\langle (\delta \boldsymbol{E} + \boldsymbol{v} \times \delta \boldsymbol{B}) \cdot \frac{\partial \delta f_{\alpha}}{\partial \boldsymbol{v}} \right\rangle.(5)$$

This equation describes a time evolution of a system in terms of the continuous probability density function  $f_{\alpha}$  instead of the discretized microscopic phase density  $\hat{f}_{\alpha}$ . Expressing the collision term in a form of the perturbation expansion and gathering terms of the appropriate order, the Fokker-Planck type equation for a plasma is obtained

$$\frac{\partial f_{\alpha}}{\partial t} + \boldsymbol{v} \cdot \nabla f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_{\alpha}}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \left( \mathsf{D} \cdot \frac{\partial f_{\alpha}}{\partial \boldsymbol{v}} + \boldsymbol{A} f_{\alpha} \right)$$
(6)

where D is a diffusion tensor and A is a friction. The above procedure is called the Klimontovich formalism (Klimontovich 1967). In plasmas, long-range Coulomb interactions rather than collisions govern a whole dynamics of a system. For these systems, the Vlasov equation is appropriate, which is obtained by dropping the collision term in (6):

$$\frac{\partial f_{\alpha}}{\partial t} + \boldsymbol{v} \cdot \nabla f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_{\alpha}}{\partial t} = 0.$$
(7)

Namely, (7) is a collisionless equation by approximation.

We have noticed that the same hierarchy exist in the 2D Euler equation. The point vortex solution is a counterpart of (2) and we assume that this solution is a microscopic one. Therefore, we regard the 2D Euler equation which has the point vortex solution as a microscopic equation. Applying the Klimontovich formalism to the microscopic Euler equation, we will obtain a corresponding macroscopic equation to (6) with a collisional effect. By dropping the collision term from the obtained equation, we will obtain the inviscid 2D Euler equation in the usual sense.

The 2D point vortex system has been successfully applied to understand the various phenomena including 2D turbulence (Kida 1985, Eyink & Sreenivasan 2006, Tabeling 2002, Kraichnan & Montgomery 1980), neutral (Taylor & McNamara 1971) and nonneutral (Dubin & Jin 2001, Yatsuyanagi et al. 2003) plasmas. These phenomena share a common keyword, "self-organization". In the context of the self-organization, possibility of the negative temperature state in the 2D point vortex system was first pointed out by Onsager (1949). The concept of the negative temperature state is convenient to explain how a large scale structure, such as Jupiter's Great Red Spot and typhoons, is formed before stored energy is exhausted by a dissipative process. If the temperature is negative, no spatially homogeneous thermal equilibrium distribution exists. Such states have been discussed in several ways. Joyce & Montgomery (1973) derived the sinh-Poisson equation which determines the thermal equilibrium distribution of double-sign point vortices. Kida (1975) discussed the axisymmetric equilibrium distribution the well-known maximizing entropy techniquie.

To understand the relaxation process toward such thermal equilibrium states, it is necessary to develop a kinetic theory. A kinetic equation with a collisional effect describes how a system relaxes to an equilibrium state. Kinetic theory of the point vortex system has attracted a lot of attention. A general kinetic equation for the point vortex system has been obtained by Chavanis with several methods including projection operator, the BBGKY hierarchy and the Klimontovich formalism (Chavanis 2001, Chavanis 2008). The kinetic equations have a Fokker-Planck type collision term that is composed of a diffusion term and a drift term. The drift term was first evidenced in Robert & Sommeria (1992) and Chavanis (1998). A kinetic theory for multi-species point vortex system was discussed by Dubin & O'Neil (1988) and Dubin (2003) in the context of magnetized plasmas with the Klimontovich formalism. Chavanis & Lemou (2007) discussed the axisymmetric case with the Fokker-Planck type collision term. The result has an issue that a relaxation process stops before the system reaches a Boltzmann-type thermal equilibrium state, if a profile of an angular velocity is a monotonic decay function.

In the previous paper (Yatsuyanagi et al. 2015), we have derived a kinetic equation having a Fokker-Planck type collision term for a single-species point vortex system with a weak mean flow. We have paid a special attention to treat a weak mean flow case correctly, which is a complementary case to many works by Chavanis (1998), Chavanis (2001), Chavanis (2008), Dubin & O'Neil (1988) and Dubin (2003). The phrase "weak mean flow" means that the number of the point vortices N has a lower and an upper limits,

$$1 < \pi \left(\frac{R}{L}\right)^2 < N < \frac{\pi}{16} \left(\frac{R}{L}\right)^4 \tag{8}$$

where R is a characteristic system size and L is a characteristic microscopic size. See Appendix in Yatsuyanagi et al. (2015) for detail and we will present a refined estimation in section 5.2. With this limit, the approximation that a mean trajectory is linear in the microscopic time scale is validated. The obtained collision term has the Fokker-Planck form, namely, it is composed of the diffusion term and the drift term. It was revealed that the diffusion term dissipate the mean field energy, while the drift term increases it. As a whole, the total men field energy is conserved. In other words, the drift term accumulates the vortices in the same place, while the diffusion term disperses the accumulated vortices. In addition, the collision term exhibits several physically important properties: (a) it includes a nonlocal effect; (b) it satisfies the H theorem; (c) its effect vanishes in the thermal equilibrium state. This means that in contrast to Chavanis & Lemou (2007), the kinetic equation ensures that a system relaxes to a Boltzmann thermal equilibrium state even if the profile is a monotonically-decaying symmetric one.

The most remarkable feature of the self-organization in a 2D system is a large-scale vortex formation with the same-sign vorticity, which is expected to be connected with the inverse-cascade in the 2D turbulence. In such systems, it is quite common that positive and negative vortices coexist. However, the above-mentioned single-species model cannot handle a system with positive and negative vortices. Thus, to understand the self-organization process in a system in which vortices with the clockwise direction and with the counterclockwise direction coexist, we need to extend the previous singlespecies model to a double-species model. So, in this paper, we present a new model for a double-species point vortex system with a weak mean flow.

In a single-species point vortex system, a remarkable feature of the self-organization is a condensation of the same-sign vortices, although no attractive forces act between them. The previous single-species model can describe the condensation of the singlesign vortices correctly. In a double-species point vortex system, an additional remarkable feature of the self-organization appears. That is a "charge-separation" of positive and negative vortices. Namely, positive vortices isolate themselves from negative vortices and are condensed into a clump which is exclusively composed of positive vortices, and vice versa. We emphasize that a new finding in the double-species model is that the model can explain the feature of charge-separation brought by the drift term in addition to the clumping feature which is also provided by the drift term regardless of the single- or double-species system. Thus, the current model bears discussions for the 2D turbulence as the important features of the 2D turbulence, the condensation and the charge-separation, are incorporated in it. We will also demonstrate the obtained collision term has physically important properties similar to the single-species model.

The organization of this paper is as follows. In section 2, the point vortex system and a kinetic equation are briefly introduced. We demonstrate explicit formulae for the diffusion term and the drift term. In section 3, physical properties of the collision term are examined. The important role of the drift term in the 2D self-organization with negative absolute temperature will be discussed. In section 4, we present a numerical result of the self-organization of the point vortices. Finally in sections 4 and 5, we give a discussion and a conclusion.

#### 2. Kinetic equation for 2D point vortex system

Let us consider a 2D point vortex system consisting of  $N_+$  positive and  $N_-$  negative vortices (Newton 2001),

$$\hat{\omega}(\boldsymbol{r},t) \equiv \hat{\omega}_{+}(\boldsymbol{r},t) + \hat{\omega}_{-}(\boldsymbol{r},t), \qquad (9)$$

$$\hat{\omega}_{+}(\boldsymbol{r},t) = \sum_{i=1}^{\prime} \Omega \delta(\boldsymbol{r} - \boldsymbol{r}_{i}(t)), \qquad (10)$$

$$\hat{\omega}_{-}(\boldsymbol{r},t) = -\sum_{i=N_{+}+1}^{N_{+}+N_{-}} \Omega \delta(\boldsymbol{r}-\boldsymbol{r}_{i}(t)), \qquad (11)$$

where  $\mathbf{r} = (x, y)$  is the position vector on x - y plane,  $\hat{\omega}(\mathbf{r}, t)$  is the z-component of the vorticity, and  $\delta(\mathbf{r})$  is the Dirac delta function in two dimensions. The values of  $N_+$  and  $N_-$  are not necessarily the same. The circulation of each point vortex is given by either  $\Omega$  or  $-\Omega$  where  $\Omega$  is a positive constant. Magnitudes of  $N_+\Omega$  and  $N_-\Omega$  are finite. The position vector of the *i*-th point vortex is given by  $\mathbf{r}_i = \mathbf{r}_i(t)$ . The discretized vorticities (10) and (11) are formal solutions of the 2D Euler equations (12) and (13)

$$\frac{\partial}{\partial t}\hat{\omega}_{+}(\boldsymbol{r},t) + \nabla \cdot (\hat{\boldsymbol{u}}(\boldsymbol{r},t)\hat{\omega}_{+}(\boldsymbol{r},t)) = 0, \qquad (12)$$

$$\frac{\partial}{\partial t}\hat{\omega}_{-}(\boldsymbol{r},t) + \nabla \cdot (\hat{\boldsymbol{u}}(\boldsymbol{r},t)\hat{\omega}_{-}(\boldsymbol{r},t)) = 0$$
(13)

where  $\hat{\boldsymbol{u}}(\boldsymbol{r},t)$  is the velocity field which is determined by the stream function  $\psi(\boldsymbol{r},t)$ :

$$\hat{\boldsymbol{u}}(\boldsymbol{r},t) = -\hat{\boldsymbol{z}} \times \nabla \hat{\psi},\tag{14}$$

$$\hat{\psi}(\boldsymbol{r},t) = \int d\boldsymbol{r}' G(\boldsymbol{r}-\boldsymbol{r}') \hat{\omega}(\boldsymbol{r}',t)$$
  
=  $\sum_{i} \Omega_{i} G(\boldsymbol{r}-\boldsymbol{r}_{i}),$  (15)

$$G(\mathbf{r}) = -\frac{1}{2\pi} \ln |\mathbf{r}|.$$
(16)

Here,  $\hat{z}$  is the unit vector in the z-direction, and  $G(\mathbf{r})$  is the 2D Green function for the Laplacian operator with an infinite domain. From now on, for brevity, we shall omit the dependences on t and  $\mathbf{r}$  and denote the two equations for the positive and the negative vortices into the single formula with double-sign, if there is no ambiguity. For example (12) and (13) are combined into the following form:

$$\frac{\partial}{\partial t}\hat{\omega}_{\pm} + \nabla \cdot (\hat{\boldsymbol{u}}\hat{\omega}_{\pm}) = 0.$$
(17)

As is mentioned in section 1, we regard (17) as the microscopic equations because they have the discretized point vortex solutions (10) and (11). Applying the Klimontovich formalism to (17) (Klimontovich 1967), we obtain an intermediate result which corresponds to (5)

$$\frac{\partial}{\partial t}\omega_{\pm} + \nabla \cdot (\boldsymbol{u}\omega_{\pm}) = -\nabla \cdot \langle \delta \boldsymbol{u} \delta \omega_{\pm} \rangle \tag{18}$$

where the microscopic vorticity and the microscopic velocity field are defined by

$$\hat{\omega}_{\pm} = \langle \hat{\omega}_{\pm} \rangle + \delta \omega_{\pm} = \omega_{\pm} + \delta \omega_{\pm}, \tag{19}$$

$$\hat{\boldsymbol{u}} = \langle \hat{\boldsymbol{u}} \rangle + \delta \boldsymbol{u} = \boldsymbol{u} + \delta \boldsymbol{u}.$$
<sup>(20)</sup>

The term  $\langle \delta u \delta \omega_{\pm} \rangle$  is a diffusion flux and will be denoted by  $\Gamma_{\pm}$ 

$$\Gamma_{\pm} \equiv \langle \delta \boldsymbol{u} \delta \omega_{\pm} \rangle = -\int d\boldsymbol{r}' \boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}') \langle \delta \omega' \delta \omega_{\pm} \rangle, \qquad (21)$$

$$\boldsymbol{F}(\boldsymbol{r}) = \hat{\boldsymbol{z}} \times \nabla G(\boldsymbol{r}), \tag{22}$$

$$\delta \boldsymbol{u} = -\int d\boldsymbol{r}' \boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}') \delta \omega'. \qquad (23)$$

We note  $\delta \omega'$  for  $\delta \omega(\mathbf{r}', t)$ . Similarly, we note  $\omega'$  for  $\omega(\mathbf{r}', t)$ .

To evaluate the diffusion fluxes  $\Gamma_{\pm}$  explicitly, we introduce a small parameter  $\epsilon$ . Orders are given by:

$$\begin{split} \omega_{\pm} &\approx \nabla^2 \psi \approx O(\epsilon^0), \quad \boldsymbol{u} \approx \nabla \psi \approx O(\epsilon^0), \quad \nabla \boldsymbol{u} \approx \nabla^2 \psi \approx O(\epsilon^0), \\ \nabla \omega_{\pm} &\approx O(\epsilon^{1/2}), \quad \delta \omega_{\pm} \approx O(\epsilon^{1/2}), \quad \delta \boldsymbol{u} \approx O(\epsilon^{1/2}), \\ \frac{\partial \boldsymbol{u}}{\partial t} &\approx O(\epsilon^{1/2}), \quad \frac{\partial \omega_{\pm}}{\partial t} \approx O(\epsilon^{1/2}), \quad \nabla \nabla \boldsymbol{u} \approx O(\epsilon^{1/2}), \\ \boldsymbol{\Gamma}_{\pm} &\approx O(\epsilon). \end{split}$$
(24)

The expansion parameter  $\epsilon$  is similar to the one introduced by Chavanis in Chavanis (2001), Chavanis (2008), Chavanis (2012) and the references therein. In addition, we assume that the gradient of the vorticity profile is weak. Magnitude of  $\Omega$  scales as either  $1/N_+$  or  $1/N_-$  and there is an upper limit of  $N_{\pm}$ . These assumptions are due to the situation of the weak mean flow and are necessary for the validity that a mean trajectory is linear (A.2). We will discuss the limitation on the number of vortices in section 5.2. With these scalings, the left hand side of (18) is  $O(\epsilon^{1/2})$ , while the right hand side is  $O(\epsilon^{3/2})$ . Expressing  $\Gamma_{\pm}$  in the form of the perturbation expansion and gathering the terms of the appropriate order, an analytical formula for the diffusion fluxes will be obtained.

Although the detailed calculation process is not the same as the single-species case presented in Yatsuyanagi et al. (2015), there appears many similar techniques in the double species case. Thus, the detailed process for deriving an explicit formula of the diffusion fluxes  $\Gamma_{\pm}$  is given in appendix.

The final result is as follows. As the obtained  $\Gamma_{\pm}$  contain oscillatory terms, we perform a space-average for  $\Gamma_{\pm}$  to reveal the characteristics of the collision term. The space-averaged diffusion flux  $\Gamma_s \equiv \langle \Gamma \rangle_s$  for the kinetic equation

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\boldsymbol{u}\omega) = -\nabla \cdot \boldsymbol{\Gamma}_s(\boldsymbol{r}), \qquad (25)$$

with

$$\omega = \omega_+ + \omega_- \tag{26}$$

is given by

$$\Gamma_{s}(\mathbf{r}) = \Gamma_{s+}(\mathbf{r}) + \Gamma_{s-}(\mathbf{r})$$

$$= -K \int d\mathbf{r}' \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}') \cdot \nabla' \omega'}{|\mathbf{u} - \mathbf{u}'|^{3}}$$

$$\cdot \left[ (\omega'_{+} - \omega'_{-}) \nabla \omega - (\omega_{+} - \omega_{-}) \nabla' \omega' \right], \qquad (27)$$

$$\Gamma_{s\pm}(\mathbf{r}) \equiv -\mathsf{D}_s \cdot \nabla \omega_{\pm} \mp \mathbf{V}_s \omega_{\pm}, \tag{28}$$

$$\mathsf{D}_{s} = K \int d\mathbf{r}' \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}')(\omega'_{+} - \omega'_{-})}{|\mathbf{u} - \mathbf{u}'|^{3}}, \tag{29}$$

$$\mathbf{V}_{s} = K \int d\mathbf{r}' \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}') \cdot \nabla' \omega'}{|\mathbf{u} - \mathbf{u}'|^{3}}, \qquad (30)$$

$$K = \frac{\Omega}{(2\pi)^3} \left(\frac{\pi}{L}\right)^2 \frac{1}{k_{\min}},\tag{31}$$

where K is a constant depending on  $\Omega$ , a system size R and a coarse-graining scale L. Parameter  $k_{\min}$  is introduced to regularize a singularity, and is determined by the largest wave length that does not exceed the system size. It is worth stressing that the term  $V_s\omega_+$  has the opposite sign to the term  $V_s\omega_-$ . It provides a mechanism for the "charge separation" which is usually seen in the equilibrium distribution for systems with positive and negative vortices (Joyce & Montgomery 1973, Yatsuyanagi et al. 2005).

## 3. Physical properties of the diffusion flux

In this section, we examine several properties of the diffusion flux (27).

## 3.1. Diffusion flux in local and global equilibrium states

At first, let us examine if the diffusion flux (27) locally disappears in a local equilibrium state. We rewrite (27) into a symbolic form

$$\boldsymbol{\Gamma}_{s}(\boldsymbol{r}) = -K \int d\boldsymbol{r}' \boldsymbol{\gamma}[\omega, \psi; \omega', \psi']$$
(32)

where  $\gamma$  is a functional of  $\omega$ ,  $\psi$ ,  $\omega'$  and  $\psi'$ . Consider a state where temperature is locally uniform in each small region in the system, namely there are small regions with different  $\beta$ . In each small region, a local equilibrium condition is satisfied

$$\omega_{\rm leq\pm} = \omega_{0\pm} \exp(\mp \beta_{\rm leq} \Omega \psi_{\rm leq}). \tag{33}$$

Inserting (33) into  $\gamma$  in (32) and assuming that  $\boldsymbol{r}$  and  $\boldsymbol{r'}$  belong to the same subsystem, we find that

$$\gamma[\omega_{\mathrm{leq}}, \psi_{\mathrm{leq}}; \omega'_{\mathrm{leq}}, \psi'_{\mathrm{leq}}]$$

$$= \frac{(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})}{|\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}}|^{3}}$$

$$\cdot [(\omega'_{\mathrm{leq+}} - \omega'_{\mathrm{leq-}})\nabla(\omega_{\mathrm{leq+}} + \omega_{\mathrm{leq-}}) - (\omega_{\mathrm{leq+}} - \omega_{\mathrm{leq-}})\nabla'(\omega'_{\mathrm{leq+}} + \omega'_{\mathrm{leq-}})]$$

$$= -\beta_{\mathrm{leq}}\Omega(\omega'_{\mathrm{leq+}} - \omega'_{\mathrm{leq-}})(\omega_{\mathrm{leq+}} - \omega_{\mathrm{leq-}})$$

$$\times \frac{(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})}{|\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}}|^{3}} \cdot (\nabla\psi_{\mathrm{leq}} - \nabla'\psi'_{\mathrm{leq}})$$

$$= 0 \qquad (34)$$

where  $\boldsymbol{u}_{\text{leq}} = -\hat{\boldsymbol{z}} \times \nabla \psi_{\text{leq}}$  is used. As  $\boldsymbol{u}_{\text{leq}} - \boldsymbol{u}'_{\text{leq}}$  is perpendicular to  $\nabla \psi_{\text{leq}} - \nabla' \psi'_{\text{leq}}$ ,  $\gamma$  is equal to zero and this result indicates that the detailed balance is achieved.

When the system reaches a global thermal equilibrium state characterized by

$$\omega_{\rm eq\pm} = \omega_{0\pm} \exp(\mp \beta \Omega \psi_{\rm eq}), \tag{35}$$

with uniform  $\beta$  (Joyce & Montgomery 1973), we obtain

$$\nabla' \omega'_{\rm eq} = \nabla' (\omega'_{\rm eq+} + \omega'_{\rm eq+})$$
  
=  $-\beta \Omega (\omega'_{\rm eq+} - \omega'_{\rm eq-}) (\nabla' \psi'_{\rm eq} - \nabla \psi_{\rm eq} + \nabla \psi_{\rm eq}).$  (36)

As  $(\boldsymbol{u}_{eq} - \boldsymbol{u}'_{eq}) \cdot (\nabla' \psi'_{eq} - \nabla \psi_{eq}) = 0$ , the drift term in (30) is rewritten as

$$\boldsymbol{V}_{s,\mathrm{eq}} = -\beta \Omega \mathsf{D}_{s,\mathrm{eq}} \cdot \nabla \psi_{\mathrm{eq}} \tag{37}$$

and the total diffusion flux vanishes. Equation (37) is a counterpart of the Einstein relation (Chavanis 2001, Chavanis 2008).

#### 3.2. Sign of $d\omega/d\psi$ near thermal equilibrium states

If the sign of the inverse temperature  $\beta$  is negative, we obtain

$$\frac{d\omega_{\rm eq}}{d\psi_{\rm eq}} = -\beta\Omega(\omega_{\rm eq+} - \omega_{\rm eq-}) \ge 0$$
(38)

and

$$\nabla \cdot (\boldsymbol{u}_{\rm eq}\omega_{\rm eq}) = 0 \tag{39}$$

where equations (35) and

$$\omega_{\rm eq} = \omega_{\rm eq+} + \omega_{\rm eq-} \tag{40}$$

are used. The point vortex system easily approaches a quasi-stationary state near the thermal equilibrium state by a violent relaxation which is purely collisionless and driven by the mean field effects. The local equilibrium state is also categorized in the above state. In the local equilibrium state near the thermal equilibrium one, the following relation is expected to be satisfied (Yatsuyanagi et al. 2014)

$$\nabla \cdot (\boldsymbol{u}_{\text{leq}}\omega_{\text{leq}}) \approx 0 \tag{41}$$

or equivalently

$$\omega_{\rm leq} = \omega_{\rm leq}(\psi_{\rm leq}). \tag{42}$$

In this state, we may expect that

$$\frac{d\omega_{\rm leq}}{d\psi_{\rm leq}} \ge 0 \tag{43}$$

almost everywhere in the system. We will use this relation later.

# 3.3. Energy-conservative property of diffusion flux

It is shown that the obtained kinetic equation (25) conserves the total mean field energy E:

$$E \equiv \frac{1}{2} \int d\mathbf{r} \psi \omega$$
  
=  $\frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \omega' \omega.$  (44)

Time derivative of the total mean field energy E is given by

$$\frac{dE}{dt} = \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \left( \frac{\partial \omega'}{\partial t} \omega + \omega' \frac{\partial \omega}{\partial t} \right) 
= \int d\mathbf{r} \psi \frac{\partial \omega}{\partial t},$$
(45)

Inserting the space-averaged equation of motion (25) into (45), we obtain

$$\frac{dE}{dt} = \int d\mathbf{r}\psi \left(-\nabla \cdot (\mathbf{u}\omega) - \nabla \cdot \mathbf{\Gamma}_{s}\right) 
= \int d\mathbf{r}\nabla\psi \cdot \mathbf{u}\omega + \int d\mathbf{r}\nabla\psi \cdot \mathbf{\Gamma}_{s} 
= \int d\mathbf{r}\nabla\psi \cdot \mathbf{\Gamma}_{s} 
= -K \int d\mathbf{r} \int d\mathbf{r}'\nabla\psi \cdot \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}')}{|\mathbf{u} - \mathbf{u}'|^{3}} 
\cdot \left[(\omega'_{+} - \omega'_{-})\nabla\omega - (\omega_{+} - \omega_{-})\nabla'\omega'\right].$$
(46)

By permuting the dummy variables r and r' in (46) and taking the half-sum of the resulting expressions, we obtain

$$\frac{dE}{dt} = -\frac{K}{2} \int d\mathbf{r} \int d\mathbf{r}' (\nabla \psi - \nabla' \psi') \cdot \frac{\mathbf{u} - \mathbf{u}'}{|\mathbf{u} - \mathbf{u}'|^3} \times (\mathbf{u} - \mathbf{u}') \cdot \left[ (\omega'_+ - \omega'_-) \nabla \omega - (\omega_+ - \omega_-) \nabla' \omega' \right] = 0.$$
(47)

We conclude that the obtained diffusion flux conserves the total mean field energy.

It is also revealed that the energy conservation is achieved by ballancing the energy dissipation process due to the diffusion term and the energy production process due to the drift term. We divide the expression (46) into two parts, namely the term which corresponds to the diffusion term and the one to the drift term.

$$\frac{dE}{dt} = \frac{dE}{dt}\Big|_{\mathbf{D}} + \frac{dE}{dt}\Big|_{\mathbf{V}} = 0,$$
(48)

$$\left. \frac{dE}{dt} \right|_{\mathsf{D}} = -K \int d\mathbf{r} \int d\mathbf{r}' \nabla \psi \cdot \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}')}{|\mathbf{u} - \mathbf{u}'|^3} \cdot (\omega'_+ - \omega'_-) \nabla \omega, \quad (49)$$

$$\frac{dE}{dt}\Big|_{\boldsymbol{V}} = K \int d\boldsymbol{r} \int d\boldsymbol{r}' \nabla \psi \cdot \frac{(\boldsymbol{u} - \boldsymbol{u}')(\boldsymbol{u} - \boldsymbol{u}')}{|\boldsymbol{u} - \boldsymbol{u}'|^3} \cdot (\omega_+ - \omega_-) \nabla' \omega'.$$
(50)

If the vorticity  $\omega$  is a function of the stream function  $\psi$  (see (42)), equations (49) and (50) are rewritten as

$$\frac{dE}{dt}\Big|_{\mathsf{D}} = -K \int d\mathbf{r} \int d\mathbf{r}' \frac{|\nabla \psi \cdot (\mathbf{u} - \mathbf{u}')|^2}{|\mathbf{u} - \mathbf{u}'|^3} (\omega'_+ - \omega'_-) \frac{d\omega}{d\psi},\tag{51}$$

$$\frac{dE}{dt}\Big|_{\boldsymbol{V}} = K \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{|\nabla\psi\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{|\boldsymbol{u}-\boldsymbol{u}'|^3} (\omega_+ - \omega_-) \frac{d\omega'}{d\psi'}.$$
(52)

Thus, if  $d\omega/d\psi \ge 0$  with  $\beta < 0$ , it is concluded that

$$\left. \frac{dE}{dt} \right|_{V} = -\left. \frac{dE}{dt} \right|_{\mathsf{D}} \ge 0,\tag{53}$$

namely,

$$\left. \frac{dE}{dt} \right|_{\mathbf{V}} + \left. \frac{dE}{dt} \right|_{\mathbf{D}} = 0.$$
(54)

#### 3.4. H theorem

It is shown that the obtained kinetic equation (25) satisfies an H theorem. The entropy function S is defined by using the H function:

$$S = -k_B H,$$

$$H = \int d\mathbf{r} \frac{\omega_+}{\Omega} \ln \frac{\omega_+}{\Omega} + \frac{\omega_-}{-\Omega} \ln \frac{\omega_-}{-\Omega} + \text{const.}$$

$$= \frac{1}{\Omega} \int d\mathbf{r} \omega_+ \ln \omega_+ - \omega_- \ln |\omega_-| - 2N \ln \Omega + \text{const.}$$
(56)

The time derivative of the H function is given by

$$\frac{dH}{dt} = \frac{1}{\Omega} \int d\boldsymbol{r} \frac{\partial \omega_{+}}{\partial t} (\ln \omega_{+} + 1) - \frac{\partial \omega_{-}}{\partial t} (\ln |\omega_{-}| + 1)$$

$$= \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{u} \omega_{+} \cdot \nabla \ln \omega_{+} - \boldsymbol{u} \omega_{-} \cdot \nabla \ln |\omega_{-}|$$

$$+ \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{\Gamma}_{s+} \cdot \nabla \ln \omega_{+} - \boldsymbol{\Gamma}_{s-} \cdot \nabla \ln |\omega_{-}|$$

$$= \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{u} \cdot \nabla \omega_{+} - \boldsymbol{u} \cdot \nabla \omega_{-}$$

$$+ \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{\Gamma}_{s+} \cdot \nabla \ln \omega_{+} - \boldsymbol{\Gamma}_{s-} \cdot \nabla \ln |\omega_{-}|$$

$$= \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{\Gamma}_{s+} \cdot \nabla \ln \omega_{+} - \boldsymbol{\Gamma}_{s-} \cdot \nabla \ln |\omega_{-}|$$
(57)

For simplicity, we introduce the following notations.

$$\nabla \ln \omega_{+} = \boldsymbol{v}_{+}, \quad \nabla \ln |\omega_{-}| = \boldsymbol{v}_{-},$$
$$\boldsymbol{u} - \boldsymbol{u}' = \boldsymbol{U}.$$
(58)

Inserting (27) into (57), we obtain

$$\frac{dH}{dt} = -\frac{K}{\Omega} \int d\mathbf{r} \int d\mathbf{r}' \mathbf{v}_{+} \cdot \frac{\mathbf{U}\mathbf{U}}{|\mathbf{U}|^{3}} \cdot \left[(\omega_{+}' - \omega_{-}')\nabla\omega_{+} - \omega_{+}\nabla'\omega'\right] 
- \frac{K}{\Omega} \int d\mathbf{r} \int d\mathbf{r}' \mathbf{v}_{-} \cdot \frac{\mathbf{U}\mathbf{U}}{|\mathbf{U}|^{3}} \cdot \left[(\omega_{+}' - \omega_{-}')\nabla\omega_{-} + \omega_{-}\nabla'\omega'\right] 
= -\frac{K}{\Omega} \int d\mathbf{r} \int d\mathbf{r}' \frac{1}{|\mathbf{U}^{3}|} 
\times \left[\omega_{+}\omega_{+}'(\mathbf{v}_{+} \cdot \mathbf{U}\mathbf{U} \cdot (\mathbf{v}_{+} - \mathbf{v}_{+}') + \omega_{-}\omega_{-}'(\mathbf{v}_{-} \cdot \mathbf{U}\mathbf{U} \cdot (\mathbf{v}_{-} - \mathbf{v}_{-}') - \omega_{+}\omega_{-}'(\mathbf{v}_{+} \cdot \mathbf{U}\mathbf{U} \cdot (\mathbf{v}_{+} + \mathbf{v}_{-}') - \omega_{+}\omega_{-}'(\mathbf{v}_{-} \cdot \mathbf{U}\mathbf{U} \cdot (\mathbf{v}_{-} + \mathbf{v}_{+}')) \right].$$
(59)

By permuting the dummy variables r and r' in (59) and taking the half-sum of the resulting expressions, we obtain

$$\frac{dH}{dt} = -\frac{K}{2\Omega} \int d\mathbf{r} \int d\mathbf{r}' \frac{1}{|\mathbf{U}|^3} \times [\omega_+ \omega'_+ |(\mathbf{v}_+ - \mathbf{v}'_+) \cdot \mathbf{U}|^2 + \omega_- \omega'_- |(\mathbf{v}_- - \mathbf{v}'_-) \cdot \mathbf{U}|^2 - \omega_+ \omega'_- |(\mathbf{v}_+ - \mathbf{v}'_-) \cdot \mathbf{U}|^2 - \omega_- \omega'_+ |(\mathbf{v}_- - \mathbf{v}'_+) \cdot \mathbf{U}|^2] \le 0.$$
(60)

As the integrand of (60) is positive or equal to zero, dH/dt is negative or equal to zero. It is concluded that the entropy function (55) is the monotonically increasing function.

It is also revealed that the diffusion term increases the entropy, while the drift term decreases it. We divide the expression (57) into two parts (the entropy function S is

used instead of the H function).

$$\frac{dS}{dt} = \frac{dS}{dt}\Big|_{\mathsf{D}} + \frac{dS}{dt}\Big|_{\mathsf{V}} \ge 0,$$

$$\frac{dS}{dt}\Big|_{\mathsf{D}} = \frac{k_B}{\Omega} \int d\mathbf{r} \frac{\nabla\omega_+ \cdot \mathsf{D}_s \cdot \nabla\omega_+}{\omega_+} - \frac{\nabla\omega_- \cdot \mathsf{D}_s \cdot \nabla\omega_-}{\omega_-}$$
(61)

$$= k_B \frac{K}{\Omega} \int d\mathbf{r} \int d\mathbf{r}' \left( \frac{|\nabla \omega_+ \cdot (\mathbf{u} - \mathbf{u}')|^2}{\omega_+} - \frac{|\nabla \omega_- \cdot (\mathbf{u} - \mathbf{u}')|^2}{\omega_-} \right) \times \frac{\omega'_+ - \omega'_-}{|\mathbf{u} - \mathbf{u}'|^3},$$
(62)

$$\frac{dS}{dt}\Big|_{\boldsymbol{V}} = -\frac{k_B}{\Omega} \int d\boldsymbol{r} \boldsymbol{V}_s \cdot \nabla \omega$$
$$= -k_B \frac{K}{\Omega} \int d\boldsymbol{r} \int d\boldsymbol{r}' \nabla \omega \cdot \frac{(\boldsymbol{u} - \boldsymbol{u}')(\boldsymbol{u} - \boldsymbol{u}')}{|\boldsymbol{u} - \boldsymbol{u}'|^3} \cdot \nabla' \omega'.$$
(63)

Equation (62) indicates that

$$\left. \frac{dS}{dt} \right|_{\mathsf{D}} \ge 0 \tag{64}$$

regardless of the sign of  $d\omega/d\psi$ . On the other hand, if the system reaches a local equilibrium state, we expect that (43) is valid. In this case, equation (63) is rewritten as

$$\frac{dS}{dt}\Big|_{\boldsymbol{V}} = -k_B \frac{K}{\Omega} \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{|\nabla\psi\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{|\boldsymbol{u}-\boldsymbol{u}'|^3} \frac{d\omega}{d\psi} \frac{d\omega'}{d\psi'}$$
(65)

and we obtain

$$\left. \frac{dS}{dt} \right|_{\boldsymbol{V}} \le 0 \tag{66}$$

in the local equilibrium state. When the system reaches a thermal equilibrium state, the relation

$$\left. \frac{dS}{dt} \right|_{V} = -\left. \frac{dS}{dt} \right|_{\mathsf{D}} \le 0 \tag{67}$$

namely,

$$\left. \frac{dS}{dt} \right|_{V} + \left. \frac{dS}{dt} \right|_{\mathsf{D}} = 0 \tag{68}$$

is satisfied, and the entropy production stops.

This clearly indicates the crucial role of the drift term in the self-organization of the point vortex system. The diffusion term increases the entropy, while the drift term decreases it. Namely, in the self-organization of the 2D point vortex system with negative  $\beta$ , a background distribution outside the clumps is necessary to dump the entropy. The clump formation is driven by the drift term and the background distribution outside the clumps is made by the diffusion term. This conclusion is supported by the nonneutral plasma experiments (Sanpei et al. 2003, Soga et al. 2003, Jin & Dubin 1998, Fine et al. 1995) and the numerical simulation (Yatsuyanagi et al. 2005).



Figure 1. Time evolution of the vortices up to 40 turnover time of an initial small clump is shown. Characteristic turnover time is given by  $T \approx 5$ . Initial profile is quickly destroyed and two relatively large clumps with same-sign vortices are formed by the self-organizing feature of the system.

#### 4. Numerical result

We will demonstrate an example of the self-organization in the 2D point vortex system by numerical simulations. Twenty positive (red) and 20 negative (blue) clumps are initially arranged uniformly in a circular wall with radius R. Each clump is composed of the same-sign 283 vortices. Total number of the vortices is  $283 \times 40 = 11320$ . Characteristic time scale is given by a self-rotation time of a small clump  $T \approx 5$ . A motion of the vortices are traced by the following equation of motion

$$\frac{\partial \boldsymbol{r}_i}{\partial t} = \frac{1}{2\pi} \sum_{j \neq i} \Omega_i \frac{\hat{\boldsymbol{z}} \times (\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2} - \frac{1}{2\pi} \sum_j \Omega_i \frac{\hat{\boldsymbol{z}} \times (\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j)}{|\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j|^2}$$
(69)

where the circular wall effect is introduced by image vortices located at

$$\bar{\boldsymbol{r}}_i = \frac{R^2}{|\boldsymbol{r}_i|^2} \boldsymbol{r}_i. \tag{70}$$

Time evolution of the system is given in figure 1. The system finally settles down into a well-known self-organized thermal equilibrium state which is described by the sinh-Poisson equation (Joyce & Montgomery 1973, Pointin & Lundgren 1976). Remarkable features of this result are the charge-separation of the vortices and the condensation of the same-sign vortices. Energy belonging to the *i*-th point vortex  $H_i$  is defined by

$$H = \sum_{i} H_{i},$$

$$H_{i} = \frac{1}{2} \Omega_{i} \psi_{i},$$

$$\psi_{i} = -\frac{1}{2\pi} \sum_{j \neq i} \Omega_{j} \ln |\mathbf{r}_{i} - \mathbf{r}_{j}| + \frac{1}{2\pi} \sum_{j} \Omega_{j} \ln |\mathbf{r}_{i} - \bar{\mathbf{r}}_{j}|$$

$$-\frac{1}{2\pi} \sum_{j} \Omega_{j} \ln \left|\frac{R}{|\mathbf{r}_{j}|}\right|$$
(71)

where H is a total energy of the system. Energy of the vortices inside a clump is positive and large as the same-sign vortices are confined in a small region. Such configuration is enabled by the drift term as is shown in (53). In addition, the drift term plays an important role of separation of the positive and negative vortices. The positive vortices and the negative vortices are driven in the opposite directions by the drift term as the signs of the drift terms for the positive and the negative vortices in (28) are opposite.

On the other hand, the system is an energy-conserving one. Thus, if there are vortices that gains energy by the clumping, there must be vortices that loses energy. These vortices that loses energy go outside a clump and form a background distribution. This feature to lower the energy of the vortices is provided by the diffusion term as is shown in (53).

Another evidence for the charge-separation is given in figure 2. It can be seen that the center of gravity of the positive vortices goes downward and the center of gravity of the negative vortices goes upward. This figure clearly indicates the charge separation.

Following these observations, we conclude that Fokker-Planck type collision term for double-species 2D point vortex system provides an essential and crucial role for the self-organization in the system at negative absolute temperature.

#### 5. Discussion

#### 5.1. Confinment of positive and negative vortices in an infinite domain

In our kinetic theory, the vortices are located in an infinite domain without any boundary. A positive and a negative vortices with the same strength move in parallel straight lines. Many pairs of the positive and negative vortices escape from the initial center of vortices without violating the conservation of the inertia I,

$$I = \sum_{i} \Omega_i |\boldsymbol{r}_i|^2.$$
(72)

However, if the system temperature is negative, clumps consisting of the same sign vortices are expected to be formed by the rapid violent relaxation and two clumps with the different signs travel along the perpendicular bisector of the line joining the two centers of the clumps with nearly constant speed. So even if there is no boundary, our



Figure 2. Trajectories of centers of gravity of the positive (red line) and negative (blue line) vortices are plotted. Length scale is normalized by the radius of the circular wall R. Initial positions of both centers of gravity locate at the center of the wall. The distance between the both centers of gravity at the end of the simulation is approximately 0.20R.

kinetic theory describes the relaxation process in the reference flame moving with the center of the two clumps.

# 5.2. Magnitude of the expansion parameter $\epsilon$

We have estimated the magnitude of the expansion parameter  $\epsilon$  in the previous paper (Yatsuyanagi et al. 2015). We find a new estimation and will present the result.

Let us reexamine the order of  $\epsilon$ , which is given by the ratio of the drift velocity to the macroscopic fluid velocity,

$$\frac{|\mathbf{V}|}{|\mathbf{u}|} = \frac{\left| K \int d\mathbf{r}' \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}') \cdot \nabla' \omega'}{|\mathbf{u} - \mathbf{u}'|^3} \right|}{|\mathbf{u}|} = O\left(\frac{1}{16} \frac{1}{N_{\pm}} \left(\frac{R}{L}\right)^2\right).$$
(73)

Here, we have used the relations  $k_{\min} = 2\pi/R$ ,  $|\boldsymbol{u}| = R\omega$  and  $\omega = N_{\pm}\Omega/(\pi R^2)$ , which are introduced in Yatsuyanagi et al. (2015). The notation R is the characteristic length of the system, L(< R) the space-averaging size introduced in Appendix. As the order of the obtained diffusion flux (27) is  $O(\epsilon)$ , the following scaling is obtained,

$$\epsilon \approx \frac{1}{16} \frac{1}{N_{\pm}} \left(\frac{R}{L}\right)^2. \tag{74}$$

Let us introduce a notation  $N_L$  as

$$N_L \equiv N_{\pm} \left(\frac{L}{R}\right)^2. \tag{75}$$

This notation represents the number of vortices inside the space-averaging (coarsegraining) area with sides  $L \times L$ . Finally, the smallness parameter  $\epsilon$  is characterized by

$$\epsilon \approx \frac{1}{N_L} > \frac{1}{N_{\pm}}.\tag{76}$$

where  $N_{\pm}$  is given by a counterpart of (8)

$$1 < \pi \left(\frac{R}{L}\right)^2 < N_{\pm} < \frac{\pi}{16} \left(\frac{R}{L}\right)^4.$$
(77)

This quantity directly corresponds to the discreteness of matter for our kinetic theory.

# 5.3. Comparison of the directions of the diffusion and the drift with an ordinary Fokker-Planck system

In an ordinary Fokker-Planck system with positive  $\beta$ , particles are populated in a low energy state and the diffusion occurs toward a high energy state. Low energy particles diffuse toward the high energy state by the diffusion, while high energy particles lose their energy by the friction and go to the low energy state. On the other hand, in a point vortex system with negative  $\beta$ , vortices are populated in a high energy state and the diffusion occurs toward a low energy state. High energy vortices diffuse toward the low energy state by the diffusion, while low energy vortices gain their energy by the drift and go to the high energy state. Namely, regardless of the sign of  $\beta$ , the diffusion term works to lower the population of the particles (vortices). On the other hand, when  $\beta > 0$ , the friction term decreases the speed of the diffusion. When  $\beta < 0$ , the drift term works to accumulate the vortices against the diffusion. This effect of the drift term may be called "negative friction". Note that the directions of the diffusive effect and the drift effect are always opposite regardless the sign of  $\beta$ .

#### 6. Conclusion

We have demonstrated the simple and explicit formula (27) of the Fokker-Planck type collision term for double-species point vortex system without the collective effect. We have also demonstrated the strong evidence of the important role of the drift term in the self-organization of the 2D point vortex system at negative absolute temperature.

The previous model for the single-species point vortex system (Yatsuyanagi et al. 2015) corresponds to the guiding-center nonneutral plasmas. However, as the



Figure 3. Populations in the energy space are illustrated. In an ordinary Fokker-Planck system with positive  $\beta$ , particles are populated in a low energy state and the diffusion occurs toward a high energy state. On the other hand in a point vortex system with negative  $\beta$ , vortices are populated in a high energy state and the diffusion occurs toward a low energy state.

nonneutral plasma is not a perfect tool to understand all the phenomena in the 2D turbulence, we have motivated to extend the previous result to the double-species system allowing the different numbers of the positive and the negative vortices. The current model can be applied to researches on the 2D turbulence, including plasmas with the same magnitudes of the charges and the masses, e.g., electron-positron plasmas and hydrogen-antihydrogen plasmas.

The obtained diffusion flux  $\Gamma_s$  conserves the mean field energy. The *H* theorem ensures that a point vortex system of any type of the flow, including an axisymmetric one, relaxes to the Boltzmann thermal equilibrium state (35). The positive and the negative vortices independently relax to the thermal equilibrium states even if the numbers of the positive and the negative vortices are different.

It should be noted again that the drift term plays the important role in the selforganization in the 2D point vortex system. In (28), the sign of the drift term changes in accordance with the sign of the vorticity, while the sign of the diffusion term is always negative. This implies that the drift term provides the "charge separation" of the vortices, which is commonly observed in equilibrium states at negative temperature.

During a relaxation process in a closed system, (Boltzmann) entropy should increase even if the system energy is conserved. It is reasonable that a distribution of particles broaden and finally reaches a flat distribution. Thus, it is difficult to understand the clump formation usually seen in the 2D self-organization as the entropy seems to decrease. As is discussed in section 3.4, it is found that the drift term decreases the entropy, while the total entropy increases. The effect of the drift term to decrease the entropy, say negative entropy production, hide behind the effect of the diffusion term to increase the entropy. It has also been stressed the common and essential role of the background vortices in supporting the vortex condensation experimentally (Sanpei et al. 2003, Soga et al. 2003, Jin & Dubin 1998, Fine et al. 1995) and numerically (Yatsuyanagi et al. 2005). This role is provided by the drift term. Note that the negative entropy production and the negative friction exist also in a single-species point vortex system.

It was reported that a sinh-Poisson equilibrium state is observed in a 2D Navier-Stokes system with finite Reynolds number (Montgomery et al. 1992, Li & Montgomery 1996, Li et al. 1997), although the sinh-Poisson equation is derived not in a continuous fluid system but in a discretized point vortex system (Joyce & Montgomery 1973). It is conjectured that the collision term in the Navier-Stokes equation implicitly involves a turbulent drift-like effect at high Reynolds number in addition to the turbulent diffusion.

There are several outstanding issues remaining. First, the final formulae (29) and (30) include unknown parameters  $k_{min}$  and L. Second, the integrals in (29) for  $\mathsf{D}_s$  and (30) for  $V_s$  contain the divergent integrand, although combined terms  $\Gamma_s = \Gamma_{s+} + \Gamma_{s-}$ are regularized. A method to resolve this problem may be to introduce a corrective effect, some kind of screening. Third, the obtained diffusion flux does not conserve the inertia (72) for an arbitrary flow. If the flow is axisymmetric, the inertia conserves. A more rigorous justification will be needed for fixing the above issues.

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#### Appendix A. Outline of the calculation

In the following, we show an outline for deriving an explicit formula for the diffusion fluxes  $\Gamma_{\pm}$ .

To rewrite the diffusion fluxes (18), we introduce linearized equations obtained by inserting (19) and (20) into (17) and assembling the terms of the order up to  $\epsilon$ (Yatsuyanagi et al. 2015):

$$\frac{\partial}{\partial t}\delta\omega_{\pm} + \nabla \cdot (\boldsymbol{u}\delta\omega_{\pm}) = -\delta\boldsymbol{u} \cdot \nabla\omega_{\pm}.$$
(A.1)

As the macroscopic quantities  $\boldsymbol{u}$  appearing in the second term in the left-hand side and  $\nabla \omega_{\pm}$  in the right-hand side are supposed to be constant in the time scale of the microscopic fluctuation, equations (A.1) can be integrated

$$\delta\omega_{\pm} = -\int_{t_0}^t d\tau \delta \boldsymbol{u}(\boldsymbol{r} - \boldsymbol{u}(t - \tau), \tau) \cdot \nabla \omega_{\pm} + \delta\omega_{\pm}(\boldsymbol{r} - \boldsymbol{u}(t - t_0), t_0).$$
(A.2)

This approximation means that the mean trajectory is linear (straight) (Chavanis 2008, Yatsuyanagi et al. 2015). The value of  $t_0$  is chosen to satisfy  $t - t_0 \gg t_c$  where  $t_c$  is a correlation time of the fluctuation.

Substituting (A.2) into the correlation terms in (21), we obtain

$$\boldsymbol{\Gamma}_{\pm} = -\int_{t_0}^t d\tau \int d\boldsymbol{r}' \int d\boldsymbol{r}'' \boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}') \boldsymbol{F}(\boldsymbol{r} - \boldsymbol{u}(t - \tau) - \boldsymbol{r}'') \cdot \nabla \omega_{\pm} \\
\times \langle \delta \omega(\boldsymbol{r}'', \tau) \delta \omega' \rangle \\
- \int_{t_0}^t d\tau \int d\boldsymbol{r}' \int d\boldsymbol{r}'' \boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}') \boldsymbol{F}(\boldsymbol{r}' - \boldsymbol{u}'(t - \tau) - \boldsymbol{r}'') \cdot \nabla' \omega' \\
\times \langle \delta \omega(\boldsymbol{r}'', \tau) \delta \omega_{\pm} \rangle$$
(A.3)

where  $\nabla' \omega' = \nabla_{\mathbf{r}'} \omega(\mathbf{r}')$ . There are three correlation terms  $\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega' \rangle$ ,  $\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_+ \rangle$ and  $\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_- \rangle$  in (A.3). At first, we handle the term  $\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_+ \rangle$ .

$$\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_{+} \rangle$$

$$= \left\langle \sum_{i=1}^{N_{+}} \Omega_{i}^{2} \delta(\mathbf{r}'' - \mathbf{r}_{i}(\tau)) \delta(\mathbf{r} - \mathbf{r}_{i}) \right\rangle$$

$$+ \left\langle \sum_{i=1}^{N_{+}+N_{-}} \sum_{j \neq i}^{N_{+}} \Omega_{i} \Omega_{j} \delta(\mathbf{r}'' - \mathbf{r}_{i}(\tau)) \delta(\mathbf{r} - \mathbf{r}_{j}) \right\rangle$$

$$- \omega(\mathbf{r}'', \tau) \omega_{+}.$$
(A.4)

The first term in the right-hand side in (A.4) corresponds to the case of i = j, and the second term corresponds to the case of  $i \neq j$ .

For the i = j case, the formula is rewritten as

$$\left\langle \sum_{i=1}^{N_{+}} \Omega_{i}^{2} \delta(\boldsymbol{r}'' - \boldsymbol{r}_{i}(\tau) - \boldsymbol{r} + \boldsymbol{r}_{i}) \delta(\boldsymbol{r} - \boldsymbol{r}_{i}) \right\rangle$$

$$= \sum_{i=1}^{N_{+}} \Omega_{i}^{2} \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r} + \boldsymbol{u}_{i}(t - \tau) + \boldsymbol{\xi}_{i}) \right\rangle_{\boldsymbol{\xi}} \left\langle \delta(\boldsymbol{r} - \boldsymbol{r}_{i}) \right\rangle$$

$$= \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r} + \boldsymbol{u}(t - \tau) + \boldsymbol{\xi}) \right\rangle_{\boldsymbol{\xi}} \Omega \omega_{+}$$
(A.5)

Here we introduce a stochastic process to evaluate  $\boldsymbol{r}_i - \boldsymbol{r}_i(\tau)$ 

$$\boldsymbol{r}_{i} - \boldsymbol{r}_{i}(\tau) = \int_{\tau}^{t} \boldsymbol{u}(\boldsymbol{r}_{i}(\tau'), \tau') d\tau' + \boldsymbol{\xi}_{i}$$
  

$$\approx \boldsymbol{u}_{i}(t - \tau) + \boldsymbol{\xi}_{i}.$$
(A.6)

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The first term in (A.6) represents the approximation that the mean trajectory is linear and the second term represents a Brownian motion. The stochastic process represented by  $\langle \cdot \rangle_{\xi}$  includes all the possible motions to reach position  $\mathbf{r}_i$  at time t.

For the  $i \neq j$  case, we introduce an approximation that the correlation between the particles can be neglected and the term is rewritten as

$$\left\langle \sum_{i=1}^{N_{+}+N_{-}} \sum_{j\neq i}^{N_{+}} \Omega_{i} \Omega_{j} \delta(\boldsymbol{r}''-\boldsymbol{r}_{i}(\tau)) \delta(\boldsymbol{r}-\boldsymbol{r}_{j}) \right\rangle$$
  
=  $\left(\omega_{+}(\boldsymbol{r}'',\tau) + \omega_{-}(\boldsymbol{r}'',\tau)\right) \omega_{+}$   
 $- \frac{1}{N_{+}} \omega_{+}(\boldsymbol{r}'',\tau) \omega_{+},$  (A.7)

assuming

$$\sum_{i=1}^{N_{+}+N_{-}} \Omega_{i} \left\langle \delta(\boldsymbol{r}''-\boldsymbol{r}_{i}(\tau)) \right\rangle = \omega_{+}(\boldsymbol{r}'',\tau) + \omega_{-}(\boldsymbol{r}'',\tau), \qquad (A.8)$$

$$\omega_{+}(\boldsymbol{r}'',\tau) = N_{+}\Omega\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}_{i}(\tau))\right\rangle, \qquad (A.9)$$

$$\omega_{-}(\boldsymbol{r}'',\tau) = -N_{-}\Omega\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}_{i}(\tau))\right\rangle.$$
(A.10)

Combining the results of the i = j and the  $i \neq j$  cases, we rewrite (A.4) as

$$\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_{+} \rangle$$
  
=  $\Omega \langle \delta(\mathbf{r}'' - \mathbf{r} + \mathbf{u}(t - \tau) + \xi) \rangle_{\xi} \omega_{+}$   
-  $\frac{1}{N_{+}} \omega_{+}(\mathbf{r}'', \tau) \omega_{+}.$  (A.11)

Similarly, we obtain

$$\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_{-} \rangle$$

$$= -\Omega \langle \delta(\mathbf{r}'' - \mathbf{r} + \mathbf{u}(t - \tau) + \xi) \rangle_{\xi} \omega_{-}$$

$$- \frac{1}{N_{-}} \omega_{-}(\mathbf{r}'', \tau) \omega_{-}.$$
(A.12)

$$\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega' \rangle$$
  
=  $\Omega \langle \delta(\mathbf{r}'' - \mathbf{r}' + \mathbf{u}'(t - \tau) + \xi) \rangle_{\xi} (\omega'_{+} - \omega'_{-})$   
-  $\frac{1}{N_{+}} \omega_{+}(\mathbf{r}'', \tau) \omega'_{+} - \frac{1}{N_{-}} \omega_{-}(\mathbf{r}'', \tau) \omega'_{-}).$  (A.13)

To proceed with the evaluation of (A.11), (A.12) and (A.13), conservation laws are introduced

$$\int d\mathbf{r}' \left\langle \delta \omega(\mathbf{r}'', \tau) \delta \omega' \right\rangle = 0, \qquad (A.14)$$

$$\int d\boldsymbol{r} \left\langle \delta\omega(\boldsymbol{r}'',\tau)\delta\omega_{\pm}(\boldsymbol{r},t) \right\rangle = 0.$$
(A.15)

Using these formulae, each term in (A.3) is evaluated.

$$-\int_{t_0}^t d\tau \int d\mathbf{r}' \int d\mathbf{r}'' \mathbf{F}(\mathbf{r}-\mathbf{r}') \mathbf{F}(\mathbf{r}-\mathbf{u}(t-\tau)-\mathbf{r}'') \cdot \nabla \omega_{\pm}$$

$$\times \langle \delta \omega(\mathbf{r}'', \tau) \delta \omega' \rangle$$

$$= -\Omega \int d\mathbf{r}' \mathbf{F}(\mathbf{r} - \mathbf{r}') \int \frac{d\mathbf{k}}{(2\pi)^2}$$

$$\times \exp(i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')) \frac{\hat{\mathbf{z}} \times i\mathbf{k}}{|\mathbf{k}|^2} \cdot \left(\omega'_+ - \omega'_-\right) \nabla \omega_{\pm}$$

$$\times \left[ \pi \delta(\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')) - \frac{i\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')}{|\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')|^2 + \nu^2} \right], \qquad (A.16)$$

$$- \int_{t_0}^t d\tau \int d\mathbf{r}' \int d\mathbf{r}'' \mathbf{F}(\mathbf{r} - \mathbf{r}') \mathbf{F}(\mathbf{r}' - \mathbf{u}'(t - \tau) - \mathbf{r}'') \cdot \nabla' \omega'$$

$$\times \langle \delta \omega(\mathbf{r}'', \tau) \delta \omega_{\pm} \rangle$$

$$= \pm \Omega \int d\mathbf{r}' \mathbf{F}(\mathbf{r} - \mathbf{r}') \int \frac{d\mathbf{k}}{(2\pi)^2} \exp(i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')) \frac{\hat{\mathbf{z}} \times i\mathbf{k}}{|\mathbf{k}|^2} \cdot \omega_{\pm} \nabla' \omega'$$

$$\times \left[ \pi \delta(\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')) - \frac{i\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')}{|\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')|^2 + \nu^2} \right]. \qquad (A.17)$$

The whole results are given by

$$\Gamma_{\pm} = -\Omega \int d\mathbf{r}' \int \frac{d\mathbf{k}}{(2\pi)^2} \int \frac{d\mathbf{k}'}{(2\pi)^2} \exp(i(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{r} - \mathbf{r}'))$$

$$\times \left[ \pi \delta(\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')) - \frac{i\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')}{|\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')|^2 + \nu^2} \right]$$

$$\times \frac{\hat{\mathbf{z}} \times i\mathbf{k}'}{|\mathbf{k}'|^2} \frac{\hat{\mathbf{z}} \times i\mathbf{k}}{|\mathbf{k}|^2} \cdot \left[ (\omega'_+ - \omega'_-)\nabla\omega_{\pm} \mp \omega_{\pm}\nabla'\omega' \right]$$
(A.18)

where we have used the relation

$$F(r - q' - (u - u(q'))(t - \tau))$$
  
=  $\frac{1}{(2\pi)^2} \int d\mathbf{k} \frac{\hat{\mathbf{z}} \times i\mathbf{k}}{|\mathbf{k}|^2} \exp(i\mathbf{k} \cdot (r - q' - (u - u(q'))(t - \tau))).$  (A.19)

It should be noted that the obtained diffusion fluxes (A.18) can be divided into two parts, namely the diffusion tensor  $\mathsf{D}(\mathbf{r},t)$  proportional to  $\nabla \omega_{\pm}$  and the drift velocity  $\mathbf{V}(\mathbf{r},t)$  proportional to  $\omega_{\pm}$ .

$$\begin{split} \mathbf{\Gamma}_{\pm} &= -\mathsf{D} \cdot \nabla \omega_{\pm} \pm \mathbf{V} \omega_{\pm}, \quad (A.20) \\ \mathsf{D} &= \Omega \int d\mathbf{r}' \int \frac{d\mathbf{k}}{(2\pi)^2} \int \frac{d\mathbf{k}'}{(2\pi)^2} \exp(i(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{r} - \mathbf{r}')) \\ &\times \left[ \pi \delta(\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')) - \frac{i\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')}{|\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')|^2 + \nu^2} \right] \\ &\times \frac{\hat{\mathbf{z}} \times i\mathbf{k}'}{|\mathbf{k}'|^2} \frac{\hat{\mathbf{z}} \times i\mathbf{k}}{|\mathbf{k}|^2} (\omega'_+ - \omega'_-) \quad (A.21) \\ \mathbf{V} &= \Omega \int d\mathbf{r}' \int \frac{d\mathbf{k}}{(2\pi)^2} \int \frac{d\mathbf{k}'}{(2\pi)^2} \exp(i(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{r} - \mathbf{r}')) \\ &\times \left[ \pi \delta(\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')) - \frac{i\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')}{|\mathbf{k} \cdot (\mathbf{u} - \mathbf{u}')|^2 + \nu^2} \right] \\ &\times \frac{\hat{\mathbf{z}} \times i\mathbf{k}'}{|\mathbf{k}'|^2} \frac{\hat{\mathbf{z}} \times i\mathbf{k}}{|\mathbf{k}|^2} \cdot \nabla' \omega' \quad (A.22) \end{split}$$

Equations (A.21) and (A.22) include the oscillatory term  $\exp(i(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{r} - \mathbf{r}'))$ . To reveal the characteristics of the obtained collision term, we need to calculate the space average of the diffusion fluxes to drop the high-frequency component. Space average is calculated over the small rectangular area  $\Lambda(\mathbf{r})$  with sides both 2L located at  $\mathbf{r}$ . The space average of the diffusion fluxes  $\Gamma_{\pm}$  is defined by

$$\langle \Gamma_{\pm} \rangle_s \equiv \Gamma_{s\pm}(\boldsymbol{r}) = \frac{1}{|\Lambda(\boldsymbol{r})|} \int_{\Lambda(\boldsymbol{r})} d\boldsymbol{r}'' \Gamma_{\pm}(\boldsymbol{r}'').$$
 (A.23)

We assume that the macroscopic variables such as  $\boldsymbol{u}$  and  $\boldsymbol{\omega}$  may be constant inside  $\Lambda(\boldsymbol{r})$ and only the term  $\exp(i(\boldsymbol{k} + \boldsymbol{k}') \cdot (\boldsymbol{r} - \boldsymbol{r}'))$  should be space-averaged.

Finally, we obtain the following formulae for the diffusion and the drift terms.

$$\Gamma_{s\pm}(\mathbf{r}) \equiv -\mathsf{D}_s \cdot \nabla \omega_{\pm} \pm \mathbf{V}_s \omega_{\pm}, \tag{A.24}$$

$$\mathsf{D}_{s} = K \int d\mathbf{r}' \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}')(\omega'_{+} - \omega'_{-})}{|\mathbf{u} - \mathbf{u}'|^{3}}, \qquad (A.25)$$

$$\mathbf{V}_{s} = K \int d\mathbf{r}' \frac{(\mathbf{u} - \mathbf{u}')(\mathbf{u} - \mathbf{u}') \cdot \nabla' \omega'}{|\mathbf{u} - \mathbf{u}'|^{3}}, \qquad (A.26)$$

$$K = \frac{\Omega}{(2\pi)^3} \left(\frac{\pi}{L}\right)^2 \frac{1}{k_{\min}},\tag{A.27}$$

where the parameter  $k_{\min}$  is introduced to regularize a singularity. It is determined by the largest wave length that does not exceed the system size, namely  $k_{\min} = 2\pi/R$  where R is a characteristic system size determined by an initial distribution of the vortices. Note that in (A.25) and (A.26), two unknown parameters L and  $k_{\min}$  remain.

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