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Multifractality of flow distribution in the river-network model of Scheidegger

Takashi Nagatani
College of Engineering, Shizuoka University, Hamamatsu 432, Japan

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The multifractal structure of flow distribution is investigated in the river-network model of Scheidegger [Bull. IASH **12**, 15 (1967)]. It is shown that the partition function $Z(q)\equiv\sum_i I_i^q$ scales as $Z(q)\approx L^{\zeta(q)}$ where I_i is the flow of water passing over the bond i within the river network, the summation ranges over all bonds, and L is the size of the river network. In the limit of a sufficiently large q, $\zeta(q)/q$ gives the exponent of the drainage basin of a river. The exponent also equals to the fractal dimension d_f of a single river. The $f-\alpha$ spectrum of the normalized flow distribution is calculated. It is found that the fractal dimension d_f of a river is exactly given by $d_f=2-\alpha(\infty)$. The flow distribution shows a characteristic multifractal structure for the river network. The river-width distribution also shows the multifractality if the width w of a river scales as $w\approx I^{\beta}$.

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Recently, there has been increasing interest in fractal structures of growth processes such as diffusion-limited aggregation (DLA), ballistic deposition, and river network. The DLA model presents a prototype of the pattern formation of diffusive systems including electrodeposition, crystal growth, viscous fingering, and bacterial colonies [1–9]. The ballistic-deposition model provides a basis for understanding deposition processes used to prepare a wide variety of thin-film devices [9]. Branched river networks are among nature's most common patterns, spontaneously producing fractal structure [10–13]. Rivers have been studied extensively by a wide variety of researchers with a variety of techniques and goals. Geomorphologists have found scaling relationships among various combinations of basin statistics from field data, such as drainage density and branching ratios. Hydrologists have likewise extracted power laws for channel parameters such as width, depth, and velocity as functions of total channel discharge. Some investigators have constructed models for the evolution of an entire drainage network [12–14]. The Scheidegger's model is the simplest model that reveals the essential features of river formation. It has been known that the river pattern of Scheidegger's model shows the fractal structure with the fractal dimension 1.5 and the size distribution of rivers also satisfies the power law [11]. However, the flow distribution in the river network has not been studied in the Scheidegger river-network model. There is an open question as to whether or not the flow distribution in the river network shows the multifractal structure.

Multifractal properties of the DLA and the random resistor network have recently attracted considerable attention [7,9,15]. It has become clear that the DLA aggregate cannot be fully characterized by its fractal dimensionality. In order to characterize the aggregate, furthermore, it is necessary to derive the multifractal structure of the growth probability distribution. From the multifractality, one can obtain detailed information on the capability of each perimeter site to grow and therefore, more information on the surface structure [16–20]. Also, de Arcangelis, Redner, and Coniglio [21] have found that the current distribution on the percolation cluster shows the multifractal structure. It has been shown that electrical properties of self-similar resistor networks should be characterized by an infinite set of exponents.

In this paper, we investigate the flow distribution of water flowing through the river network by using the Scheidegger-river network model. We address the open question as to whether or not the flow distribution shows the multifractal structure. Also, we discuss the multifractal structure of river-width distribution.

First, we introduce the river-network model of Scheidegger. Rains are assumed to be falling stationary and uniformly on the sites of the oblique square lattice. One unit of water is injected into each site per unit time. Then, fallen raindrops walk down the slope. When two raindrops collide with each other, they join and make one drop, which runs down just like before the collision. Any flow is prohibited to split. Flows are allowed to go right down or left down only with preferred direction (downstream). Figure 1 shows a simulation result of the Scheidegger river-network model. The rivers do not contain any loop. All branches are directed to the upstream. One of the most important quantities in this system is the distribution of flow rates on the river network. The rate of flow on the river network is proportional to the area of its drainage basin. The flow rate I_i on the bond i is defined as the amount of flowing water through the bond i per unit time. Each bond of the river network can be characterized by the flow rate of water flowing through it. If the bond (or site) is labeled by the position (m,n), where m indicates the downstream direction, the flow rate satisfies the equation

$$I(m+1,n)=w(m,n)I(m,n)+[1-w(m,n+1)]I(m,n+1)+1\ , \qquad (1)$$

where $w(m,n)$ denotes the realization of the water at the site (m,n) which is equal to 1 when the flow at the site (m,n) goes right down and 0 when the flow goes left

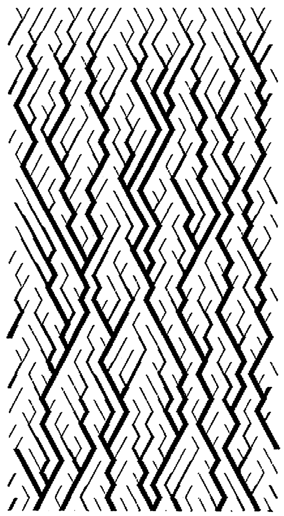

FIG. 1. Typical river-network pattern generated by the Scheidegger river-network model. This run was done on a 20×40 square lattice under a periodic lateral boundary condition for an illustration. The width of lines is proportional to the flow rate through each bond where larger flow rates than 8 are indicated by the same width.

down, and $w(m,n)$ is given by

$$w(m,n)=\begin{cases}1, & \text{probability } \frac{1}{2} \\ 0, & \text{probability } \frac{1}{2}\end{cases} . \tag{2}$$

For an illustration, Fig. 1 shows a typical river-network pattern generated by the Scheidegger river-network model. This run was done on a 20×40 square lattice under a periodic lateral boundary condition. The flow rate on each bond is indicated by the width of the river, where larger flow rates than 8 are represented by the same width. The partition function $Z(q)$ is defined as the moments of the flow rate

$$Z(q)\equiv \sum_i I_i^q , \tag{3}$$

where the summation ranges over all bonds on the river network.

We perform the computer simulation of the Scheidegger river-network model for the square lattice 200×300. By the use of Eq. (1), the flow rate on each bond is calculated under a periodic lateral boundary condition. We study the scaling behavior of the partition function (3). Figure 2 shows the log-log plot of the moments against the vertical size L. It is confirmed that for sufficiently large L the partition function scales with size L as

$$Z(q)\approx L^{\zeta(q)} . \tag{4}$$

Figure 3 shows the $\zeta(q)$ behavior against q. The positive exponents $\zeta(q)$ are a characteristic property of the coalescence of water drops with injection of water. In the resistor network of the percolation cluster, the exponents are negative values since the current distribution is described in terms of a fragmentation process of electric current.

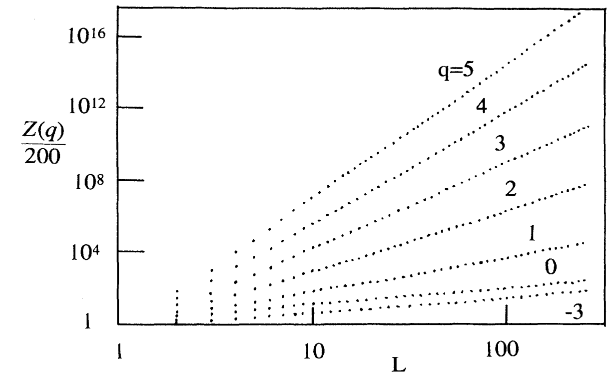

FIG. 2. Log-log plot of the moments (3) against size L showing scaling behavior.

The exponent $\zeta(0)$ equals to 1. The exponent $\zeta(1)$ gives the dimension $d=2$ of the river network. For a sufficiently large q, $\zeta(q)/q$ gives the scaling exponent 1.50 ± 0.02 of largest flow rate, which is consistent with the exponent of the drainage basin. Also, it equals to the fractal dimension $d_f=1.50$ of a single river.

In order to characterize the multifractality of flow distribution, it is convenient to normalize the flow rate. The normalized flow rate I_i' on the bond i is given by

$$I_i'=I_i/Z(1) . \tag{5}$$

We define the normalized partition function $Z'(q)$ as

$$Z'(q)=Z(q)/\{Z(1)\}^q . \tag{6}$$

For a sufficiently large L, the partition function scales as

$$Z'(q)\approx L^{-\tau(q)} . \tag{7}$$

Figure 4 shows the plot of exponent $\tau(q)$ against q. With the Legendre transformation of $\tau(q)$, we obtain the $f-\alpha$ spectrum

$$f(q)=q\alpha(q)-\tau(q) , \tag{8}$$

where $\alpha(q)=\partial\tau(q)/\partial q$ is the variable conjugate to q. Figure 5 indicates the $f-\alpha$ spectrum. The flow distribution shows a characteristic $f-\alpha$ spectrum for the river network. The maximum value $f(0)$ of $f(\alpha)$ is related to the

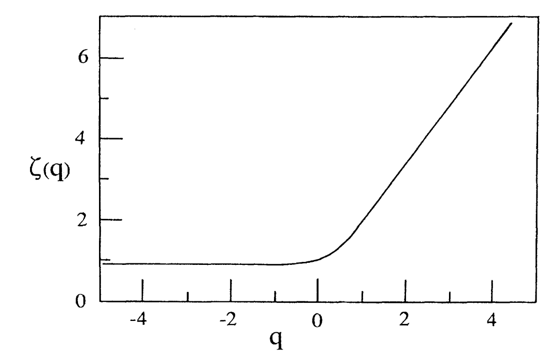

FIG. 3. Behavior of the scaling exponent $\zeta(q)$ against q.

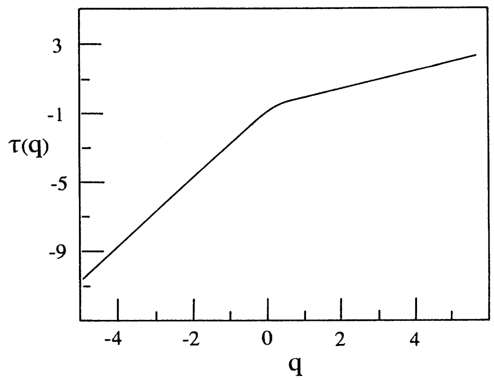

FIG. 4. Plot of the exponent $\tau(q)$ against q.

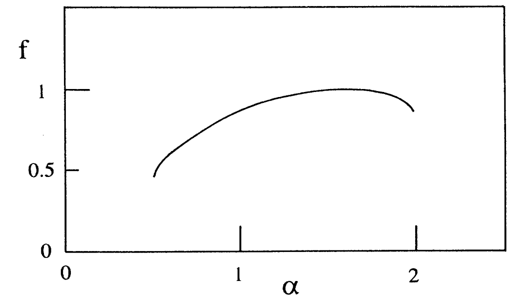

FIG. 5. $f-\alpha$ spectrum of flow distribution in the Scheidegger river network.

dimension $d=2$ of the river network

$$f(0)=d-1=1 . \tag{9}$$

The maximum value of α gives the minimum fraction of flow rate. The minimum value of α gives the maximum fraction of flow rate. The minimum value $\alpha(\infty)$ is exactly related to the fractal dimension d_f of a single river

$$\alpha(\infty)=\left[\frac{\partial\tau}{\partial q}\right]_{q=\infty}=-\left[\frac{\partial}{\partial q}\left\{\frac{\ln Z(q)}{\ln L}\right\}\right]_{q=\infty}+\frac{\ln Z(1)}{\ln L}$$
$$=2-d_f , \tag{10}$$

where $\ln Z(1)/\ln L=2$. The minimum value $\alpha(\infty)$ obtained from the simulation is given by 0.49 ± 0.02. We obtain the fractal dimension $d_f=1.51\pm0.02$ from Eq. (10). This value is consistent with 1.50 obtained by the direct simulation [11]. The properties of river networks should be characterized by the infinite set of exponents or the $f-\alpha$ spectrum.

We consider the scaling properties of the river-width distribution. From a variety of empirical investigations [22,23], it is found that the width w of a river scales as a function of channel discharge I:

$$w\approx I^{\beta} , \tag{11}$$

where the value of the exponent β ranges from 0.25 to 0.50 [13]. Here the channel discharge represents the flow rate. The partition function $Z_w(q)$ of the river-width distribution is defined as the moments of the river width

$$Z_w(q)\equiv\sum_i w_i^q , \tag{12}$$

where the summation ranges over all bonds on the river network. The partition function scales as

$$Z_w(q)\approx L^{\xi(q)} . \tag{13}$$

By using Eq. (11), the exponent $\xi(q)$ is related with the exponent $\zeta(q)$

$$\xi(q)=\beta\zeta(q) . \tag{14}$$

Therefore, the river-width distribution shows the multifractality. The scaling properties of the river width should be characterized by the infinite set of exponents. For a sufficiently large q, $\xi(q)/q$ gives the scaling exponent of the largest river width. Similar to the flow rate, the normalized partition function $Z'_w(q)$ is defined as

$$Z'_w(q)=Z_w(q)/\{Z_w(1)\}^q . \tag{15}$$

For a sufficiently large L, the partition function $Z'_w(q)$ scales as

$$Z'_w(q)\approx L^{-\tau_w(q)} . \tag{16}$$

The following relationships are found:

$$\tau_w(q)=\beta\tau(q) ,\quad \alpha_w(q)=\beta\alpha(q) ,\quad f_w(q)=\beta f(q) , \tag{17}$$

where $\tau_w(q)=q\alpha_w(q)-f_w(q)$. Therefore, if the width w of rivers scales as Eq. (11), the river-width distribution also shows the multifractality.

In summary, we found the multifractality of the flow distribution in the Scheidegger river-network model. We derived the $f-\alpha$ spectrum of the normalized flow distribution. We showed that the exponent of the drainage basin of a single river was exactly given by $d_f=2-\alpha(\infty)$. We also found that the river-width distribution showed the multifractality.

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