Crossover scaling in Scheidegger's river-network model

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A crossover behavior is investigated in Scheidegger's river-network model [Bull. Int. Acco. Sci. Hydrol. **12**, 1 (1967); **12**, 15 (1967)] where a river meanders left with probability p and right with probability $1-p$. Near $p=1$ (or $p=0$), the crossover phenomenon occurs from linear rivers at smaller length scales than the crossover length t_c to the river network of a self-affine fractal at larger length scales than t_c. For $0<p<1$, the river network always crosses over the self-organized critical state. The mean river size $\langle S \rangle$ scales as $\langle S \rangle \approx t$ for $t<t_c$ and $\langle S \rangle \approx t^{d_b}$ ($d_b=1.50$) for $t>t_c$ where d_b is the scaling exponent of the drainage basin area. The crossover length t_c scales as $t_c \approx (\Delta p)^{-1/\phi}$ ($1/\phi=1.033\pm0.050$) where $\Delta p=1-p$ near $p=1$ (or $\Delta p=p$ near $p=0$). The mean river size is described by the scaling form $\langle S \rangle = tf(t/t_c)$ where $f(x)\approx 1$ for $x \ll 1$ and $f(x)\approx x^{d_b-1}$ for $x \gg 1$. For a sufficiently small Δp, the mean river size $\langle S \rangle$ also scales as $\langle S \rangle \approx \Delta p^{\gamma}$ ($\gamma=0.484\pm0.020$). The cumulative river size distribution N_S scales as $N_S \approx (\Delta p)^{-2\gamma/3} S^{-1/3}$.

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Recently, there has been increasing interest in scaling structures of growth processes such as river networks, cluster-cluster aggregations, rough surfaces and diffusion-limited aggregations [1–10]. Branched river networks are among nature's most common patterns, spontaneously producing fractal structures. Rivers have been studied extensively by a wide variety of researchers with a variety of techniques and goals. Some investigators have constructed models for the evolution of an entire drainage network [11–14]. The Scheidegger's river-network model is the simplest model that reveals the essential features of river formation. It has been known that the cumulative size distribution of rivers in Scheidegger's model satisfies the power law

$$N_S = P(\geq S) \approx S^{-1/3} \tag{1}$$

where S indicates the area of the drainage basin of a river [15,16].

Very recently we proposed the extended Scheidegger river-network model [17]. The extended river network shows the scaling behavior depending on the exponent of the flow-dependent meandering. The river model can describe the river network with a variety of exponents of the drainage basin. In the extended river model, we found that the river-size distribution $n_s(t)$ satisfies the dynamic scaling law

$$n_S(t) \approx S^{-\tau} f(S/t^z) , \tag{2}$$

where the dynamic exponent z is given by the exponent of the drainage basin area. The scaling relationship $(2-\tau)z=1$ was found. Also, we found that the flow distribution shows a typical multifractal character.

Doering and ben-Avraham investigated the simplest model of a diffusion-limited reaction [18]. This model also corresponds to the irreversible one-species aggregation process "with no injection" [19]. They showed the preasymptotic behavior corresponding to various initial conditions. The model can be related with the columnar growth [20]. Meakin and Krug [20] presented a theoretical analysis of the columnar growth by mapping to the coalescing particles on the line. Scheidegger's river model is equivalent to the irreversible one-species coagulation "with injection" [19]. The scaling behavior of the aggregation process with injection is different from that of the aggregation without injection [19]. However, in the aggregation process with injection, the preasymptotic behavior has not been investigated. It is interesting how the scaling behavior of river networks depends on the meandering probability p in real river networks. It will be expected that a crossover occurs from a nonfractal structure to the fractal structure, depending on the meandering probability p. In Scheidegger's river-network model, the crossover phenomena have been unknown until now. If the crossover occurs, there is an open question as to how the crossover phenomenon can be described by a crossover scaling.

In this paper, we study the crossover phenomenon in Scheidegger's river-network model where a river meanders left with probability p and right with probability $1-p$. We show that near $p=1$ (or $p=0$) the crossover phenomenon occurs from linear rivers at smaller length scales than the crossover length t_c to the river network of a self-affine fractal at larger length scales than t_c. We find that the mean river size $\langle S \rangle$ can be described by the scaling form $\langle S \rangle \approx tf(t/t_c)$ where $f(x)\approx 1$ for $x \ll 1$ and $f(x)\approx x^{d_b-1}$ for $x \gg 1$ (d_b: the exponent of the drainage basin area).

For later convenience, we introduce Scheidegger's river-network model [11,15]. Rains are assumed to be falling in a stationary and uniform manner on the sites of the oblique square lattice. One unit of water is injected into each site per unit of time. Then, the fallen raindrops go slowly down the slope. When two raindrops collide with each other, they join and make one drop which runs down just like before the collision. Any flow is prohibited from splitting. Flows are allowed to go left down with probability p or right down with probability $1-p$ only in preferred direction (downstream). As the result, the rivers do not contain any loops. All branches are directed upstream. The flow (channel discharge) I_i on the site i is defined as the amount of flowing water through the site i per unit of time. The flow on the site i is proportional to the area S of its drainage basin connecting upstream at the site i. Each site of the river network can be characterized by the flow of water. If the site i is labeled by the position (m,n), the flow (channel discharge) satisfies the equation

$$I(m+1,n)=w(m,n)I(m,n)+[1-w(m,n+1)]I(m,n+1)+1 , \qquad (3)$$

where m indicates the downstream direction, $w(m,n)$ denotes the realization of the flow direction at the site (m,n) which is equal to 1 when the flow at the site (m,n) goes right down and 0 when the flow goes left down, and $w(m,n)$ is given by

$$w(m,n)=\begin{cases}1 & \text{probability } 1-p\\ 0 & \text{probability } p\ .\end{cases} \qquad (4)$$

We perform the computer simulation of Eq. (3) for the square lattice $10\,000\times 10\,000$. The flow $I(m,n)$ on each site is calculated under a periodic lateral boundary condition. For illustration, Fig. 1 shows the typical patterns obtained by small-size simulations. The patterns (a) and (b) in Fig. 1 are obtained respectively under $p=1$ and $p=0.7$ for size 15×25. At the $p=1$ (or $p=0$) the river pattern shows the set of linear rivers. For $0<p<1$, the river network eventually approaches the self-affine fractal with the same scaling property as the original Scheidegger river model ($p=\frac{1}{2}$). We find the crossover phenomenon from the linear rivers to the self-affine river network. We define the mean river size as $\langle S(t)\rangle=\sum n_S(t)S^2/\sum n_S(t)S$ where $n_S(t)$ is the river-size distribution. Figure 2 shows the log-log plot of the mean river size $\langle S\rangle$ against the downstream length t near $p=1$. In the set of linear rivers ($p=1$), the mean river size scales as $\langle S\rangle\approx t$. In Scheidegger's river network ($p=\frac{1}{2}$) of the self-affine fractal the mean river size scales as $\langle S\rangle\approx t^{1.50}$. The value 1.50 of the exponent gives the size of the drainage basin area. Near $p=1$, the mean river size $\langle S\rangle$ crosses over from the linear rivers to the Scheidegger river network. We define the crossover length t_c as the point at which the tangential line of the slope 1.0 intersects with that of the slope 1.50. The mean river size scales as

$$\langle S\rangle\approx\begin{cases}t & \text{for } t<t_c,\\ t^{1.50} & \text{for } t>t_c\ .\end{cases} \qquad (5)$$

Figure 3 shows the log-log plot of the crossover length t_c against $\Delta p(=1-p)$. The crossover length t_c scales as

$$t_c\approx(\Delta p)^{-1/\phi} \quad \text{with } 1/\phi=1.033\pm 0.05\ . \qquad (6)$$

We propose the scaling ansatz

$$\langle S\rangle=tf(\Delta p t^{\phi}) \qquad (7)$$

where the scaling function is assumed as follows:

$$f(x)=\begin{cases}1 & \text{for } x\ll 1\\ x^{(d_b-1)/\phi} & \text{for } x\gg 1\ .\end{cases} \qquad (8)$$

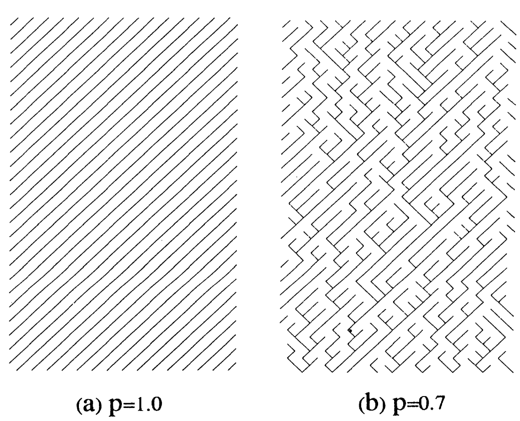

FIG. 1. The typical patterns of river networks generated by small-size simulation. The patterns (a) and (b) are obtained, respectively, under $p=1.0$ and 0.7 for size 15×25.

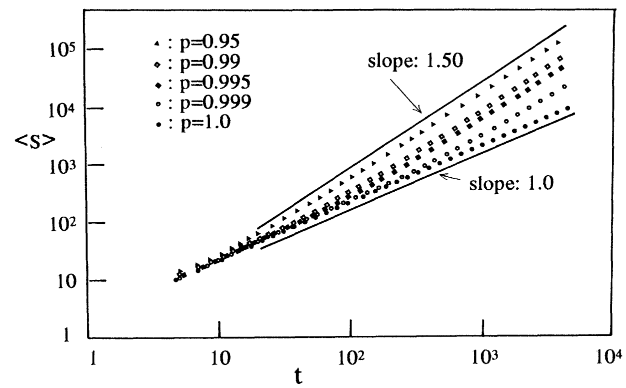

FIG. 2. The log-log plot of the mean river size $\langle S\rangle$ against the downstream length t near $p=1$.

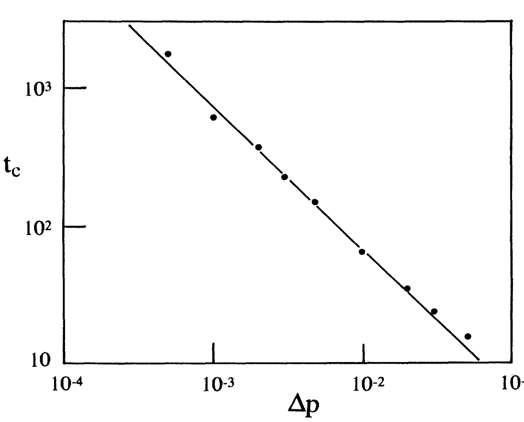

FIG. 3. The log-log plot of the crossover length t_c against $\Delta p(=1-p)$.

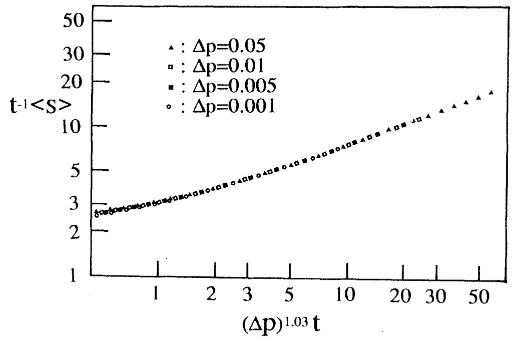

FIG. 4. The log-log plot of the rescaled mean river size $t^{-1}\langle S\rangle$ against the rescaled length $\Delta p^{1.03}t$ for various Δp.

Here $d_b(=1.50)$ is the exponent of the drainage basin area. The same crossover occurs near $p=0$. The crossover is also described by the scaling form (7). Figure 4 shows the log-log plot of $t^{-1}\langle S\rangle$ against $\Delta p^{1.03}t$ for various Δp ($\ll 1$). The data points collapse on a single curve. The scaling ansatz (7) is verified.

Figure 5 shows the log-log plot of the mean river size $\langle S\rangle$ against Δp at $t=2000$. For a sufficiently small Δp, the mean river size $\langle S\rangle$ also scales as

$$\langle S\rangle \approx \Delta p^{\gamma} \quad \text{with } \gamma=0.484\pm0.020 \,. \tag{9}$$

We study the scaling behavior of the cumulative river-size distribution. The cumulative river-size distribution N_S is defined as

$$N_S=P(\geq S)\equiv \sum_{S'=S}^{\infty} n_{S'} \,, \tag{10}$$

where n_S indicates the size distribution with size S and S is the area of the drainage basin. Figure 6 shows the log-log plot of the cumulative river size N_S against size S for various p at $t=2000$. With decreasing p, the cumulative river-size distribution N_S approaches that of Scheidegger's river network ($p=\frac{1}{2}$). In order to investigate the scaling behavior of the cumulative river-size distribution for Δp, we plot the rescaled cumulative size distribution against the rescaled size. Figure 7 shows the log-log plot of the rescaled cumulative size distribution $\Delta p^{\gamma}N_S$ against the rescaled size $\Delta p^{-\gamma}S$ for various p. The rescaled cumulative size distributions cross over from the flat line to the line of the slope $-\frac{1}{3}$. After the crossover occurs, all data points collapse on a single curve. On the scaling region, the cumulative river-size distribution scales as

$$N_S \approx (\Delta p)^{-2\gamma/3}S^{-1/3} \quad (\gamma=0.484\pm0.020) \,. \tag{11}$$

We find the dependence of the cumulative river-size distribution on Δp.

We try to interpret the found exponents ϕ and d_b in physical terms. The crossover exponent is very close to unity, showing that $\Delta p \ll 1$ introduces essentially only an anisotropic length unit, which is related to the turning probability. The turning probability is defined by the probability that a line coalesces with the nearest neighbors per unit length in the set of lines [Fig. 1(a)]. The turning probability is proportional to Δp. Therefore, the crossover length t_c is proportional to Δp^{-1}. We find that the crossover exponent $1/\phi$ equals 1. The scaling exponent d_b of the drainage basin area can be intuitively related to random walks. The basin's left and right boundaries are simple random walks. The drainage basin area is roughly given by the product $\langle S\rangle = wt$, where w and t indicate the basin's width and height. The width w is proportional to $t^{1/2}$ since the boundaries of the basin are

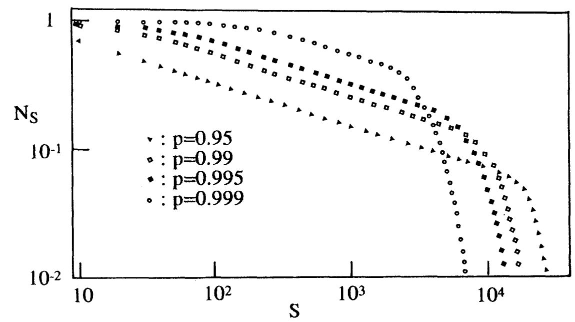

FIG. 6. The log-log plot of the cumulative river size N_S against size S for various p at $t=2000$.

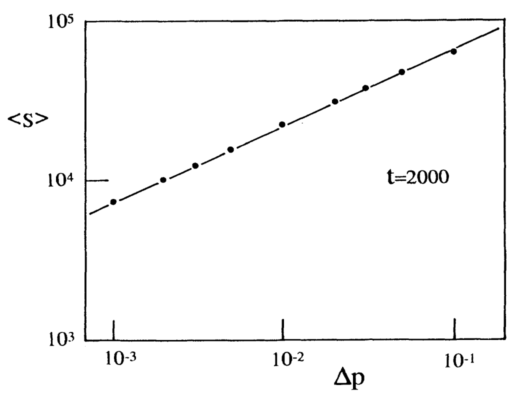

FIG. 5. The log-log plot of the mean river size $\langle S\rangle$ against Δp at $t=2000$.

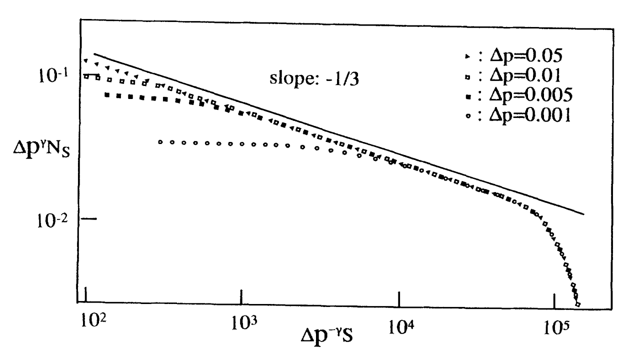

FIG. 7. The log-log plot of the rescaled cumulative size distribution $\Delta p^{\gamma}N_S$ against the rescaled size $\Delta p^{-\gamma}S$ for various p.

random walks. Therefore, the drainage basin area scales as $\langle S \rangle \approx t^{3/2}$. We find $d_b = \frac{3}{2}$.

In summary, we find the crossover phenomenon from the linear rivers to the self-affine river network. We show how the crossover can be described by the crossover scaling. We present the scaling form (7) of the mean river size $\langle S \rangle$. We find the scaling (11) of the cumulative river-size distribution for Δp.

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