From ballistic deposition to the Kardar-Parisi-Zhang equation through a limiting procedure

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(Received 29 December 1997)

We show a direct connection between the ballistic deposition and the Kardar-Parisi-Zhang (KPZ) equation. We derive the KPZ equation from the ballistic deposition models, using an important limiting procedure. The cellular automaton rule is transformed into an integrable difference-difference equation through the limiting procedure. By applying the perturbation method to the difference-difference equation, the difference-difference equation is reduced to the KPZ equation through the Burgers equation. We apply the procedure to several types of ballistic deposition models. [S1063-651X(98)00307-9]

PACS number(s): 81.15.−z, 02.10.Jf, 68.35.−p

I. INTRODUCTION

Deposition of particles on surfaces is a phenomenon of scientific interest with a broad range of practical applications. The scaling properties of surface roughness have been investigated by means of both numerical and analytical tools [1–22]. The surface roughness problem belongs to the active field of nonequilibrium statistical physics and irreversible growth phenomena. The ballistic deposition model is the most basic and simplest one of irreversible growth models [1–3]. It is now generally believed that the ballistic deposition model is the discrete version of the continuous model introduced by Kardar, Parisi, and Zhang [22]. The continuous model is described by the nonlinear partial differential equation called the Kardar-Parisi-Zhang (KPZ) equation. However, one cannot formally derive the KPZ equation. One can develop a set of plausibility arguments using physical principles, which motivate the addition of a nonlinear term to the linear Edward-Wilkinson equation [21]. The nonlinear term represents the lateral growth when a new particle is added to the surface. Until now, there seemed to be no direct and formal derivation of the KPZ equation from the ballistic deposition model.

Recently, Tokihiro *et al.* [23] proposed such an ultradiscretization method in which the Korteweg–de Vries equation is transformed to a cellular automaton. They showed a direct connection between a cellular automaton and integrable nonlinear wave equations. In this paper we show a direct and formal derivation of the KPZ equation from the ballistic deposition models.

II. LIMITING PROCEDURE

The ballistic deposition model for a two-dimensional square lattice can be described as follows. At time t, the height of the interface of site i is $h(i,t)$. We choose a random position above the surface and allow a particle to fall vertically toward it. The particle sticks to the first site along its trajectory that has an occupied nearest neighbor. Figure 1 shows the schematic representation of the ballistic deposition model. At time $t+1$, the height $h(i,t+1)$ is given by

$$h(i,t+1)=\max[h(i-1,t),h(i,t)+1,h(i+1,t)], \quad (1)$$

where max[ ] is the maximum function. We consider the difference of heights between nearest neighbors:

$$\begin{aligned}h(i,t+1)-h(i-1,t+1)\\ &=\max[h(i-1,t),h(i,t)+1,h(i+1,t)]\\ &\quad-\max[h(i-2,t),h(i-1,t)+1,h(i,t)]. \quad (2)\end{aligned}$$

Next we take an important step using the limiting procedure, which is the key to making a transformation from a cellular automaton to a difference-difference equation. The identity for the limiting procedure is given by

$$\max[A,B,C]=\lim_{\varepsilon\to 0^+}\varepsilon\ln(e^{A/\varepsilon}+e^{B/\varepsilon}+e^{C/\varepsilon}), \quad (3)$$

where ε is a positive infinitesmal value. By applying the identity (3) to Eq. (2) we obtain

$$h(i,t+1)-h(i-1,t+1)=\lim_{\varepsilon\to 0^+}\varepsilon\ln\frac{[e^{h(i-1,t)/\varepsilon}+e^{[h(i,t)+1]/\varepsilon}+e^{h(i+1,t)/\varepsilon}]}{[e^{h(i-2,t)/\varepsilon}+e^{[h(i-1,t)+1]/\varepsilon}+e^{h(i,t)/\varepsilon}]}. \quad (4)$$

By replacing $e^{[h(i,t)-h(i-1,t)]/\varepsilon}$ by $c(i,t)$, we obtain the difference-difference equation

$$\begin{aligned}c(i,t+1)=[&\delta c(i-1,t)+c(i-1,t)c(i,t)\\ &+\delta c(i-1,t)c(i,t)c(i+1,t)][\delta+c(i-1,t)\\ &+\delta c(i-1,t)c(i,t)]^{-1}, \quad (5)\end{aligned}$$

where $\delta=e^{-1/\varepsilon}$.

III. PERTURBATION METHOD

We now consider the hydrodynamic mode in the rough surface on coarse-grained scales. The simplest way to describe the hydrodynamic mode is the long-wavelength expansion. We apply the perturbation method to Eq. (5) [24]. We wish to extract slow scales for the space variable i and time variable t. For $|\Delta x|\ll 1$, we therefore define the slow variables X and T,

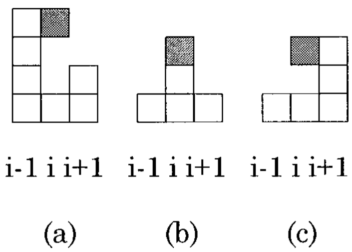

FIG. 1. Schematic representation of the ballistic deposition model. A particle falls vertically. The particle sticks to the first site along its trajectory that has an occupied nearest neighbor.

$$X=(\Delta x)i, \quad T=\delta(\Delta x)^2 t. \tag{6}$$

By setting ln $c(i,t)=(\Delta x)v(\Delta x i,\delta(\Delta x)^2 t)=(\Delta x)v(X,T)$, we expand $c(i,t)$ to order $(\Delta x)^3$. We obtain

$$c(i,t)=\exp[(\Delta x)v(X,T)]=1+(\Delta x)v+(\Delta x)^2 v^2/2 + (\Delta x)^3 v^3/6+\cdots , \tag{7}$$

where $v=v(X,T)$ in the second equality. We expand $c(i-1,t)$ to order $(\Delta x)^3$. We obtain

$$\begin{aligned} c(i-1,t) &= 1+(\Delta x)v(\Delta x(i-1),\delta(\Delta x)^2 t) \\ &\quad +(\Delta x)^2 v(\Delta x(i-1),\delta(\Delta x)^2 t)^2/2 \\ &\quad +(\Delta x)^3 v(\Delta x(i-1),\delta(\Delta x)^2 t)^3/6+\cdots \\ &= 1+(\Delta x)v+(\Delta x)^2(v^2/2-\partial_X v) \\ &\quad +(\Delta x)^3(v^3/6-v\,\partial_X v+\partial_X^2 v/2)+\cdots, \end{aligned} \tag{8}$$

where

$$\begin{aligned} v(\Delta x(i-1),\delta(\Delta x)^2 t) &= v(\Delta x i,\delta(\Delta x)^2 t) \\ &\quad -(\Delta x)\partial_X v(\Delta x i,\delta(\Delta x)^2 t) \\ &\quad +(\Delta x)^2 \partial_X^2 v(\Delta x i,\delta(\Delta x)^2 t)/2 \\ &= v(X,T)-(\Delta x)\partial_X v(X,T) \\ &\quad +(\Delta x)^2 \partial_X^2 v(X,T)/2, \end{aligned}$$

$$\partial_X=\partial/\partial X, \quad \partial_X^2=\partial^2/\partial X^2.$$

Similarly, we obtain

$$c(i+1,t)=1+(\Delta x)v+(\Delta x)^2(v^2/2+\partial_X v) +(\Delta x)^3(v^3/6+v\,\partial_X v+\partial_X^2 v/2)+\cdots , \tag{9}$$

$$c(i,t+1)=1+(\Delta x)v+(\Delta x)^2 v^2/2+(\Delta x)^3(v^3/6+\delta\partial_T v) +\cdots , \tag{10}$$

where $\partial_T=\partial/\partial T$. By substituting the long-wavelength expansions (7)–(10) into Eq. (5), the first- and second-order terms cancel. Only the third-order term remains. Thus we obtain the Burgers equation

$$\partial_T v=(2v\,\partial_X v+\partial_X^2 v)/(1+2\delta). \tag{11}$$

By setting $v(X,T)=(h(i,t)-h(i-1,t))/(\varepsilon\Delta x)=\partial_X h$, the KPZ equation is obtained

$$\partial_T h=[(\partial_X h)^2+\partial_X^2 h]/(1+2\delta). \tag{12}$$

Here constant δ is given by the ratio of the time increment Δt to the square of the space increment Δx. The ballistic deposition process on coarse-grained scales can be described by the KPZ equation.

IV. MODIFIED MODELS

Here we consider some modified versions of the ballistic deposition model. First we consider the modified version of the ballistic deposition with the next-nearest-neighbor (NNN) sticking rule [1], as shown in Fig. 2. At time $t+1$, the height $h(i,t+1)$ is given by

$$h(i,t+1)=\max[h(i-1,t)+1,h(i,t)+1,h(i+1,t)+1]. \tag{13}$$

By applying the limiting procedure similar to Eq. (1), the difference-difference equation is obtained

$$\begin{aligned} c(i,t+1) &= [c(i-1,t)+c(i-1,t)c(i,t) \\ &\quad +c(i-1,t)c(i,t)c(i+1,t)][1+c(i-1,t) \\ &\quad +c(i-1,t)c(i,t)]^{-1}. \end{aligned} \tag{14}$$

By applying the perturbation method to Eq. (14), we obtain the KPZ equation

$$\partial_T h=[(\partial_X h)^2+\partial_X^2 h]/3. \tag{15}$$

By comparing Eq. (15) with Eq. (12), only a coefficient changes from $1/(1+2\delta)$ to $\frac{1}{3}$. In the NNN ballistic model, the unknown coefficient δ does not appear.

We consider the one-sided deposition model shown in Fig. 3. The depositing particle sticks on position $h(i,t)+1$ or on position $h(i-1,t)+1$ on the left-hand side. The lateral growth occurs only on the left-hand side. The one-sided deposition model is described by the cellular automaton rule

$$h(i,t+1)=\max[h(i-1,t)+1,h(i,t)+1]. \tag{16}$$

By apply the limiting procedure to Eq. (16), we obtain the difference-difference equation

$$c(i,t+1)=[c(i-1,t)+c(i-1,t)c(i,t)]/[1+c(i-1,t)]. \tag{17}$$

We define the slow variables $X=\Delta x(i-t/2)$ and $T=(\Delta x)^2 t$. By setting

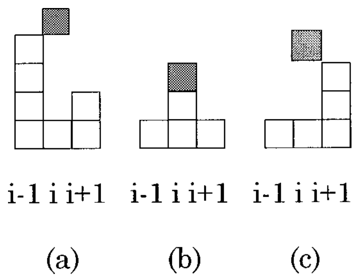

FIG. 2. Modified version of the ballistic deposition with the next-nearest-neighbor sticking rule. The lateral growth occurs at the next-nearest neighbors.

$$\ln c(i,t)=(\Delta x)v(\Delta x(i-t/2),\ (\Delta x)^2t)=(\Delta x)v(X,T),$$

the same long-wavelength expansions of $c(i,t)$, $c(i-1,t)$, and $c(i+1,t)$ as in Eqs. (7)–(9) are obtained. The long-wavelength expansion of $c(i,t+1)$ is obtained as

$$c(i,t+1)=1+(\Delta x)v+(\Delta x)^2(v^2/2-\partial_X v/2)+(\Delta x)^3(v^3/6-v\,\partial_X v/2+\partial_X^2 v/8+\partial_T v)+\cdots, \tag{18}$$

where $X=(\Delta x)(i-t/2)$, $T=(\Delta x)^2t$, $\partial_T=\partial/\partial T$, $\partial_X=\partial/\partial X$, and $\partial_X^2=\partial^2/\partial X^2$. By inserting the long-wavelength expansions (7)–(9) and (18) into Eq. (17), the KPZ equation is obtained through the Burgers equation

$$\partial_T h=[(\partial_X h)^2+\partial_X^2 h]/8, \tag{19}$$

where $X=(\Delta x)(i-t/2)$ and $T=(\Delta x)^2t$. By comparing the one-sided model with the NNN model, the coefficient $\frac{1}{8}$ of Eq. (19) is different from the coefficient $\frac{1}{3}$ of Eq. (15). In Eq. (19) the variable $X=(\Delta x)^2(i-t/2)$ is also different from $X=(\Delta x)^2 i$ in Eq. (15). In the one-sided deposition model, the deposition process can be described by the KPZ equation in terms of the moving frame with velocity $\frac{1}{2}$.

In the original ballistic deposition model, at each time, a particle is released from a chosen position above the surface, located at a distance larger than the maximum height of the interface. We extend one depositing particle to n particles. At time $t+1$, the height $h(i,t+1)$ is given by

$$h(i,t+1)=\max[h(i-1,t)+n-1,h(i,t)+n,h(i+1,t)+n-1]. \tag{20}$$

When we apply the same limiting procedure and the perturbation method to this model, the n-particle model reduces to Eq. (12).

We consider the NNN model with depositing n particles. The NNN model is described by

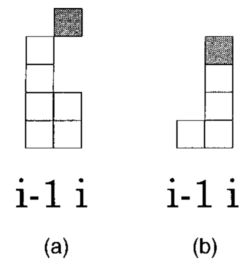

FIG. 3. One-sided ballistic deposition model. The lateral growth occurs only on the left-hand side.

$$h(i,t+1)=\max[h(i-1,t)+n,h(i,t)+n,h(i+1,t)+n]. \tag{21}$$

Similarly, we can show that this model reduces to Eq. (15). Therefore, the number of depositing particles at a unit of time does not affect the resulting KPZ equation.

We consider the ballistic deposition model for a three-dimensional cubic lattice described as follows. At time t, the height of the interface of site (i,j) is $h(i,j,t)$. We choose a random position above the surface and allow a particle to fall vertically toward it. The particle sticks to the first site along its trajectory that has an occupied nearest neighbor. At time $t+1$, the height $h(i,j,t+1)$ is given by

$$h(i,j,t+1)=\max[h(i,j-1,t),h(i-1,j,t),h(i,j,t)+1,h(i+1,j,t),h(i,j+1,t)]. \tag{22}$$

We apply the limiting procedure and the perturbation method to Eq. (22). Unfortunately, we have not been successful in obtaining the two-dimensional KPZ equation.

V. SUMMARY

In summary, we derived the KPZ equation from the ballistic deposition models using both the limiting procedure and perturbation method. We showed a direct connection between the ballistic deposition and the KPZ equation. In the one-sided ballistic deposition model, the growth process was also described in terms of the KPZ equation. We were successful in the derivation of the KPZ equation in $1+1$ dimensions, but could not derive the KPZ equation from the $(2+1)$-dimensional ballistic deposition.

Our method for deriving the nonlinear equation from the cellular automaton is general and can be applied to other cellular automata as well. Combining the limiting procedure and the perturbation method, we are able to obtain continuous analogs to discrete models in cases where the limiting procedure alone would not work. Thus we consider our method to be a powerful tool for the investigation of the analytical relationship between discrete and continuous models.

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