Calculation Note for N = (4,0) Super-Liouville Theory

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Abstract

We give proofs for the important formulae required in the articles [1] and [2].

Contents

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1 Introduction

The formulae which we would like to explictly show in this note are

$$\int_{\mathcal{O}_1}\int dxd^4\theta\ y=0, \qquad \int_{\mathcal{O}_1}\int dxd^4\theta\ yf=0,$$
$$\int_{\mathcal{O}_1}\int dxd^4\theta\ y\varphi_c=0, \qquad \int_{\mathcal{O}_1}\int dxd^4\theta\ y\varphi^c=0,$$
$$\int dxd^4\theta\ y\partial y=d[\cdots].$$

We use the same notation here as in [1] and [2] , except for $\partial\equiv\partial_x$. The $N=(4,0)$ superspace with supercoordinates (x,θ_a,θ^a,t) is extended to a fictitious space

$$(\ \underbrace{x,\ \theta_a,\ \theta^a}_{\substack{\text{left}-\text{moving}\\ \text{sector}}},\ \underbrace{t_1,\ t_2,\cdots,\ t_n}_{\substack{\text{extended}\\ \text{right}-\text{moving sector}\\ \mathcal{O}_n}}\ ).$$

y is a 1-form in the extended right-moving sector, while f,φ_a,φ^a are 0-forms. All of them are functions in the left-moving sector as well. $\int_{\mathcal{O}_1}$ implies integraton along an orbit in the extended right-moving sector $\mathcal{O}_n$.

To show all of the above formulae we have expanded the integrands in the fermionic cordinates θ_a and θ^a, and have been involved in calculating their top component, i.e., the terms proportional to $(\theta\cdot\theta)^2$. As for y the expansion of the form

$$y=[y]_{\theta^0}+[y]_{\theta^1}+[y]_{\theta^2}+[y]_{\theta^3}+[y]_{\theta^4}$$

is explicitly given in Sec. 2. For the work in [1] it is particularly important that the top component $[y]_{\theta^4}$ is merely a boundary term of integration as

$$y=\cdots+(\theta\cdot\theta)^2\Big(d[\cdots]+\partial[\cdots]\Big).$$

Hence we give detailed calculations for this in Sec. 3.

2 Various formulae

2.1 Superfields in components

$$f=h+\theta\cdot\psi+\psi\cdot\theta+\theta\cdot\theta l+(\theta\sigma^i\theta)t^i+\frac{1}{2}\epsilon_{ab}\theta^a\theta^b m+\frac{1}{2}\epsilon^{ab}\theta_a\theta_b n$$
$$+(\theta\cdot\theta)(\theta\cdot\omega)+(\theta\cdot\theta)(\omega\cdot\theta)+(\theta\cdot\theta)^2 g,$$
$$\varphi^c=\xi(x-\theta\cdot\theta)\Big[\theta^c+\eta^c(x-\theta\cdot\theta)+\frac{1}{2}\epsilon_{ab}\theta^a\theta^b\alpha^c(x-\theta\cdot\theta)\Big],$$
$$\varphi_c=\rho(x+\theta\cdot\theta)\Big[\theta_c+\eta_c(x+\theta\cdot\theta)+\frac{1}{2}\epsilon^{ab}\theta_a\theta_b\alpha_c(x+\theta\cdot\theta)\Big].$$

Here the inner product implies that

$$\theta\cdot\psi\equiv\theta_a\psi^a, \qquad (\theta\sigma^i\theta)=\theta_a(\sigma\theta)^a, \qquad etc.$$

The components of φ^c and φ_c have arguments shifted as $x\to x-\theta\cdot\theta$, so that φ^c and φ_c satisfy the chirality conditions $D^a\varphi^c=0$ and $D_a\varphi_c=0$. Expanding them in

$\theta \cdot \theta$ we calculate the superconformal conditions $D_a f = \varphi_c D_a \varphi^c$ and $D^a f = \varphi^c D^a \varphi_c$ in components. Then we find them to be

$$\begin{aligned}
&\partial h + \Big(\rho\eta \cdot \partial(\xi\eta)\Big) - \Big(\partial(\rho\eta)\cdot\xi\eta\Big) = \rho\xi,\\
&\psi^a = \rho\xi\eta^a, \qquad \psi_a = -\rho\xi\eta_a,\\
&m = -\rho\eta\cdot\xi\alpha, \qquad n = \rho\alpha\cdot\xi\eta,\\
&t^i = 0,\\
&l = \partial\Big(\rho\eta\cdot\xi\eta\Big),\\
&\omega^a = \partial(\rho\xi\eta^a), \qquad \omega = \partial(\rho\xi\eta_a),\\
&g = -\frac{1}{2}\Big(\rho\eta\cdot\partial^2(\xi\eta)\Big) + \frac{1}{2}\Big(\partial^2(\rho\eta)\cdot\xi\eta\Big) + \frac{1}{2}\xi\partial\rho + \frac{1}{2}\rho\partial\xi,\\
&\xi\partial\rho = \rho\partial\xi,\\
&\xi\alpha_a = 2\epsilon_{ab}\partial(\xi\eta^b), \qquad \rho\alpha^a = 2\epsilon^{ab}\partial(\rho\eta_b).
\end{aligned}$$

Here use was made of the formulae for the fermionic coordinates θ_c and θ^c

$$\begin{aligned}
&\frac{1}{2}\epsilon_{ab}\theta^a\theta^b\frac{1}{2}\epsilon^{cd}\theta_c\theta_d = -\frac{1}{2}(\theta\cdot\theta)^2,\\
&(\theta\cdot\theta)\theta_a\theta^b = \frac{1}{2}(\theta\cdot\theta)^2\delta_a^b,\\
&\frac{1}{2}\epsilon_{ab}\theta^a\theta^b\theta_c = \theta\cdot\theta\epsilon_{cd}\theta^d, \qquad \frac{1}{2}\epsilon^{ab}\theta_a\theta_b\theta^c = -\theta\cdot\theta\epsilon^{cd}\theta_d,\\
&\frac{1}{2}\epsilon_{ab}\theta^a\theta^b\theta_c\theta_d = -\frac{1}{2}\epsilon_{cd}(\theta\cdot\theta)^2, \qquad \frac{1}{2}\epsilon^{ab}\theta_a\theta_b\theta^c\theta^d = -\frac{1}{2}\epsilon^{cd}(\theta\cdot\theta)^2,\\
&\frac{1}{2}\epsilon_{ab}\theta^a\theta^b\frac{1}{2}\epsilon^{cd}\psi_c\psi_d = -\frac{1}{2}(\psi\cdot\theta)^2.
\end{aligned}$$

Putting these constraints in the above superfields we find

$$\begin{aligned}
f = {} & h + \rho\xi\Big[(\theta\cdot\eta) - (\eta\cdot\theta)\Big]\\
&+\theta\cdot\theta\partial\Big(\rho\eta\cdot\xi\eta\Big) + 2\frac{\xi}{\rho}\Big(\rho\eta\cdot\theta\Big)\Big(\partial(\rho\eta)\cdot\theta\Big) + 2\frac{\rho}{\xi}\Big(\theta\cdot\xi\eta\Big)\Big(\theta\cdot\partial(\xi\eta)\Big)\\
&+\theta\cdot\theta\Big[\Big(\theta\cdot\partial(\rho\xi\eta)\Big) + \Big(\partial(\xi\rho\eta)\cdot\theta\Big)\Big]\\
&+\frac{1}{2}(\theta\cdot\theta)^2\Big[-\Big(\rho\eta\cdot\partial^2(\xi\eta)\Big) + \Big(\partial^2(\rho\eta)\cdot\xi\eta\Big) + \xi\partial\rho + \rho\partial\xi\Big],\\
\varphi^c = {} & \xi\eta^c + \xi\theta^c - \theta\cdot\theta\partial(\xi\eta^c) + \frac{1}{2}\epsilon_{ab}\theta^a\theta^b\xi\alpha^c - \theta\cdot\theta\theta^c\partial\xi + \frac{1}{2}(\theta\cdot\theta)^2\partial^2(\xi\eta^c),\\
\varphi_c = {} & \rho\eta_c + \rho\theta_c + \theta\cdot\theta\partial(\rho\eta_c) + \frac{1}{2}\epsilon^{ab}\theta_a\theta_b\rho\alpha_c + \theta\cdot\theta\theta_c\partial\rho + \frac{1}{2}(\theta\cdot\theta)^2\partial^2(\rho\eta_c).
\end{aligned}$$

It is worth noting that the components are given in terms of $h, \rho\eta_a, \rho\eta^a, \xi\eta_a, \xi\eta^a$, which are still constrained by

$$\begin{aligned}
&\partial h + \Big(\rho\eta\cdot\partial(\xi\eta)\Big) - \Big(\partial(\rho\eta)\cdot\xi\eta\Big) = \rho\xi,\\
&\xi\partial\rho = \rho\partial\xi.
\end{aligned}$$

The last constraints are used frequently in the following calculations. Using these superfield expansions we calculate the following quantities.

$$\begin{aligned}
\Delta &= \partial f + \varphi_a\partial\varphi^a + \varphi^a\partial\varphi_a\\
&= \rho\xi\Big[1 + 2\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big) - 2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\\
&\qquad +4\theta\cdot\theta\frac{1}{\rho\xi}\partial(\rho\eta)\cdot\partial(\xi\eta) - \epsilon_{ab}\theta^a\theta^b\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\alpha\Big) + \epsilon^{ab}\theta_a\theta_b\Big(\alpha\cdot\frac{\partial(\xi\eta)}{\xi}\Big)
\end{aligned}$$

$$+4\theta\cdot\theta\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}+4\theta\cdot\theta\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\frac{\partial\xi}{\xi}+\frac{1}{2}\epsilon_{ab}\theta^a\theta^b\Big(\theta\cdot\partial\alpha\Big)-\frac{1}{2}\epsilon^{ab}\theta_a\theta_b\Big(\partial\alpha\cdot\theta\Big)$$
$$+\frac{1}{2}\Big(\theta\cdot\theta\Big)^2\Big\{-\frac{\partial^2\xi}{\xi}-\frac{\partial^2\rho}{\rho}+6\frac{\partial\xi}{\xi}\frac{\partial\rho}{\rho}+\frac{4}{\rho\xi}\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\frac{4}{\rho\xi}\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)$$
$$+\partial\alpha\cdot\alpha-\alpha\cdot\partial\alpha\Big\}\Big],$$

$$\frac{1}{\Delta}=\frac{1}{\rho\xi}\Big[1-2\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)$$
$$-4\theta\cdot\theta\frac{1}{\rho\xi}\partial(\rho\eta)\cdot\partial\xi\eta)-8\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)$$
$$-6\theta\cdot\theta\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}-6\theta\cdot\theta\Big(\frac{\partial(\rho\eta}{\rho}\cdot\theta\Big)\frac{\partial\xi}{\xi}$$
$$+2\theta\cdot\theta\Big(\theta\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\theta\cdot\theta\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\theta\Big)$$
$$+16\theta\cdot\theta\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{1}{\rho\xi}\partial(\rho\eta)\cdot\partial(\xi\eta)-16\theta\cdot\theta\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\frac{1}{\rho\xi}\partial(\rho\eta)\cdot\partial(\xi\eta)$$
$$+8\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)-8\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)^2$$
$$+\frac{1}{2}\Big(\theta\cdot\theta\Big)^2\Big\{\frac{\partial^2\xi}{\xi}+\frac{\partial^2\rho}{\rho}-6\frac{\partial\xi}{\xi}\frac{\partial\rho}{\rho}\Big\}\Big],$$

and

$$df+\varphi_a d\phi^a+\varphi^a d\varphi_a$$
$$=dh+\Big(\rho\eta\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\xi\eta\Big)$$
$$+2\Big(\theta\cdot d(\xi\eta)\Big)\rho-2\Big(d(\rho\eta)\cdot\theta\Big)\xi$$
$$+2(\theta\cdot\theta)\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)+2(\theta\cdot\theta)\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)+(\theta\cdot\theta)(\rho d\xi-\xi d\rho)$$
$$+4\frac{\xi}{\rho}\Big(d(\rho\eta)\cdot\theta\Big)\Big(\partial(\rho\eta)\cdot\theta\Big)+4\frac{\rho}{\xi}\Big(\theta\cdot\partial(\xi\eta)\Big)\Big(\theta\cdot d(\xi\eta)\Big)$$
$$+2(\theta\cdot\theta)\Big(\theta\cdot d(\xi\eta)\Big)\partial\rho+2(\theta\cdot\theta)\Big(d(\rho\eta)\cdot\theta\Big)\partial\xi$$
$$-2(\theta\cdot\theta)\Big[\xi\Big(d\partial(\rho\eta)\cdot\theta\Big)-2\frac{\xi}{\rho}\Big(\partial(\rho\eta)\cdot\theta\Big)d\rho\Big]$$
$$+2(\theta\cdot\theta)\Big[2\frac{\rho}{\xi}\Big(\theta\cdot\partial(\xi\eta)\Big)d\xi-\Big(\theta\cdot d\partial(\xi\eta)\Big)\Big]\rho$$
$$+\frac{1}{2}(\theta\cdot\theta)^2\Big\{-2\Big(d(\rho\eta)\cdot\partial^2(\xi\eta)\Big)+2\Big(\partial^2(\rho\eta)\cdot d(\xi\eta)\Big)$$
$$+3\partial\rho d\xi+3d\rho\partial\xi-\rho d\partial\xi-\xi d\partial\rho$$
$$-6\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)+6\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)$$
$$+8(\frac{d\xi}{\xi}-\frac{d\rho}{\rho})\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big\}.$$

Finally we calculate the product of the two quntities.

$$y=[df+\varphi_a d\varphi^a+\varphi^a d\varphi_a]\frac{1}{\Delta}$$
$$=\frac{dh}{\rho\xi}+\frac{P}{\rho\xi}$$
$$+2\Big(\theta\cdot\frac{d(\xi\eta)}{\xi}\Big)-2\Big(\frac{d(\rho\eta)}{\rho}\cdot\theta\Big)+(\frac{dh}{\rho\xi}+\frac{P}{\rho\xi})\Big[-2\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\Big]$$

$$
\begin{aligned}
&-4\theta\cdot\theta(\frac{dh}{\rho\xi}+\frac{P}{\rho\xi})\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-8(\frac{dh}{\rho\xi}+\frac{P}{\rho\xi})\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\\
&\quad+2\theta\cdot\theta\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]+\theta\cdot\theta(\frac{d\xi}{\xi}-\frac{d\rho}{\rho})\\
&\quad+4\Big(\theta\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)+4\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\theta\Big)\\
&+\theta\cdot\theta(\frac{dh}{\rho\xi}+\frac{P}{\rho\xi})\Big[-6\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}-6\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\frac{\partial\xi}{\xi}\\
&\qquad\qquad+2\Big(\theta\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\theta\Big)\\
&\qquad\qquad+16\Big(\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\quad+(\frac{dh}{\rho\xi}+\frac{P}{\rho\xi})\Big[8\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\Big)-8\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\Big)^2\Big]\\
&\quad+\theta\cdot\theta\Big[-8\Big(\theta\cdot\frac{d(\xi\eta)}{\xi}\Big)+8\Big(\frac{d(\rho\eta)}{\rho}\cdot\theta\Big)\Big)\Big]\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\\
&\quad+\Big[-8\Big(\theta\cdot\frac{d(\xi\eta)}{\xi}\Big)+8\Big(\frac{d(\rho\eta)}{\rho}\cdot\theta\Big)\Big)\Big]\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\\
&\quad+\theta\cdot\theta\Big[-2\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\Big)\Big](\frac{d\xi}{\xi}-\frac{d\rho}{\rho})\\
&\quad+4\theta\cdot\theta\Big[-\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\Big)\Big]\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\quad+\theta\cdot\theta\Big[2\Big(\theta\cdot\frac{d(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}+2\Big(\frac{d(\rho\eta)}{\rho}\cdot\theta\Big)\frac{\partial\xi}{\xi}\\
&\qquad-2\Big(\theta\cdot\frac{d\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}-2\Big(\frac{d\partial(\rho\eta)}{\rho}\cdot\theta\Big)\\
&\qquad+4\Big(\theta\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{d\xi}{\xi}+4\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\theta\Big)\frac{\partial\rho}{\rho}\Big]\\
&+(\theta\cdot\theta)^2\Big(d\Big[\frac{1}{2}h\partial\Big\{\frac{\partial\xi}{\rho\xi^2}+\frac{\partial\rho}{\rho^2\xi}\Big\}\Big]-\partial\Big[\frac{1}{2}hd\Big\{\frac{\partial\xi}{\rho\xi^2}+\frac{\partial\rho}{\rho^2\xi}\Big\}\Big]\\
&\qquad+d\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big]+\partial\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big]\\
&\qquad+\partial\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big).
\end{aligned}
$$

The calculations to show these formulae are tedious, but can be done straightforwardly. It is worth noting that the coefficient of $(\theta\cdot\theta)^2$ in the expansion of y is of the form $d(\cdots)+\partial(\cdots)$. For this we give the detailed calculations in Section 3.

2.2 Notations special for this note

P and Q defined as follows are frequently used for simplifying the presentation.

$$
\begin{aligned}
&P\equiv\Big(\rho\eta\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\xi\eta\Big),\qquad R\equiv-\Big(\rho\eta\cdot\partial(\xi\eta)\Big)+\Big(\partial(\rho\eta)\cdot\xi\eta\Big),\\
&dR+\partial P=2\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-2\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big),\\
&dP=2\Big(d(\rho\eta)\cdot d(\xi\eta)\Big).
\end{aligned}
$$

3 Proof of $[y]_{\theta^4} = (\theta\cdot\theta)^2(d[\cdots] + \partial[\cdots])$

$[y]_{\theta^4}$ indicates the top component in the expansion of the superfield y

$$y = \cdots\cdots + (\theta\cdot\theta)^2(\cdots).$$

In this section we obtain an explicit form of this top component. It is calcucuculated by epanding as

$$\begin{aligned}
[y]_{\theta^4} &= [df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^0} \times [\frac{1}{\Delta}]_{\theta^4} + [df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^1} \times [\frac{1}{\Delta}]_{\theta^3} \\
&+ [df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^2} \times [\frac{1}{\Delta}]_{\theta^2} + [df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^3} \times [\frac{1}{\Delta}]_{\theta^1} \\
&+ [df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^4} \times [\frac{1}{\Delta}]_{\theta^0}.
\end{aligned}$$

Below we calculate the r.h.s. piece by piece and give the coefficient alone, i.e., without $(\theta\cdot\theta)^2$.

3.1 $[df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^0} \times [\frac{1}{\Delta}]_{\theta^4}$

$O(\eta^0)$:

$$\begin{aligned}
&\frac{1}{2}dh\Big\{\frac{\partial^2\xi}{\rho\xi^2} + \frac{\partial^2\rho}{\rho^2\xi} - 6\frac{\partial\xi\partial\rho}{(\rho\xi)^2}\Big\} = \frac{1}{2}dh\partial\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\} \\
&= \frac{1}{2}\partial hd\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\} + d\Big[\frac{1}{2}h\partial\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}\Big] - \partial\Big[\frac{1}{2}hd\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}\Big] \\
&= \frac{1}{2}\rho\xi d\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\} \qquad (1)\\
&+ d\Big[\frac{1}{2}h\partial\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}\Big] - \partial\Big[\frac{1}{2}hd\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}\Big] \\
&+ \underbrace{\frac{1}{2}Rd\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}}_{O(\eta^2)}. \qquad (2)
\end{aligned}$$

$O(\eta^2)$:

$$\frac{1}{2}P\partial\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}. \qquad (3)$$

3.2 $[df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^1} \times [\frac{1}{\Delta}]_{\theta^3}$

$O(\eta^0)$: none.
$O(\eta^2)$:

$$6\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\frac{\partial\xi}{\xi} - 6\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho} \qquad (4)$$

$$-2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big) + 2\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big). \qquad (5)$$

$O(\eta^4)$: vanishing

3.3 $[df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^2} \times [\frac{1}{\Delta}]_{\theta^2}$

$O(\eta^0)$: none.
$O(\eta^2)$: vanishing.
$O(\eta^4)$: vanishing.

3.4 $[df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^3} \times [\frac{1}{\Delta}]_{\theta^1}$

$O(\eta^0)$: none.
$O(\eta^2)$:

$$-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}+2\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\xi}{\xi} \tag{6}$$

$$-2\Big(\frac{d\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d\partial(\xi\eta)}{\xi}\Big) \tag{7}$$

$$-4\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)(\frac{d\rho}{\rho}-\frac{d\xi}{\xi}). \tag{8}$$

$O(\eta^4)$: none.

3.5 $[df + \varphi_a d\varphi^a + \varphi^a d\varphi_a]_{\theta^4} \times [\frac{1}{\Delta}]_{\theta^0}$

$O(\eta^0)$:

$$\frac{1}{2}\frac{1}{\rho\xi}\Big\{3\partial\rho d\xi+3d\rho\partial\xi-\rho d\partial\xi-\xi d\partial\rho\Big\}. \tag{9}$$

$O(\eta^2)$:

$$-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big) \tag{10}$$

$$-3\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d\partial(\xi\eta)}{\xi}\Big)+3\Big(\frac{d\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big) \tag{11}$$

$$+4(\frac{d\xi}{\xi}-\frac{d\rho}{\rho})\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big). \tag{12}$$

$O(\eta^4)$: none.

We calculate $(1)+(2)+\cdots+(10)$ to show that there remain ∂- and d-boundaries alone as follows.

$$(1)+(9)=0,$$

$$(2)+(3)=-\frac{1}{2}[dR+\partial P]\partial(-\frac{1}{\rho\xi})$$
$$+d\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big]+\partial\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big], \tag{a}$$

$$(5)+(7)+(10)+(11)$$
$$=-\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)$$
$$-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{d\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)$$
$$=\partial\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$
$$-(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi})\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big], \tag{b}$$

$$(8)+(12)=0,$$

(a) $in((2)+(3))+(b)$ $in((5)+(7)+(10)+(11))=0.$

Summary of Sec. 3

$$[y]_{\theta^4} = (\theta\cdot\theta)^2\Big(d\Big[\frac{1}{2}h\partial\Big\{\frac{\partial\xi}{\rho\xi^2}+\frac{\partial\rho}{\rho^2\xi}\Big\}\Big] - \partial\Big[\frac{1}{2}hd\Big\{\frac{\partial\xi}{\rho\xi^2}+\frac{\partial\rho}{\rho^2\xi}\Big\}\Big]$$
$$+d\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big] + \partial\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big]$$
$$+\partial\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big)$$

4 Proof of $\int_{\mathcal{O}_1}\int dxd\theta^4[yf]_{\theta^4} = 0$

$$\int d\theta^4[yf]_{\theta^4}$$
$$= \int d\theta^4\Big([y]_{\theta^0}\times[f]_{\theta^4} + [y]_{\theta^1}\times[f]_{\theta^3} + [y]_{\theta^2}\times[f]_{\theta^2} + [y]_{\theta^3}\times[f]_{\theta^1} + +[y]_{\theta^4}\times[f]_{\theta^0}\Big).$$

Below we calculate the r.h.s. piece by piece to each order of η. We are interested in a result after the integration. Therefore either ∂- or d-boundary term is discareded once it happens to appear, by notifying as "*boundary*" without giving explicit forms.

4.1 $O(\eta^0)$

$$\int d^4\theta[y]_{\theta^0}\times[f]_{\theta^4}$$
$$= \frac{1}{2}\frac{dh}{\rho\xi}(\xi\partial\rho+\rho\partial\xi) = \frac{1}{2}dh\partial(\log\rho+\log\xi)$$
$$= \frac{1}{2}\partial hd(\log\rho+\log\xi) + \textit{boundary term}$$
$$= \underbrace{\frac{1}{2}Rd(\log\rho+\log\xi)}_{O(\eta^2)}$$
$$+\textit{boundary term},$$

$$\int d^4\theta[y]_{\theta^2}\times[f]_{\theta^2} = \frac{d\xi}{\xi} - \frac{d\rho}{\rho} = \textit{boundary term}.$$

$$\int d^4\theta[y]_{\theta^4}\times[f]_{\theta^0} = -d(\frac{1}{4}h^2)\partial\Big\{\frac{\partial\xi}{\rho\xi^2}+\frac{\partial\rho}{\rho^2\xi}\Big\} + \partial(\frac{1}{4}h^2)d\Big\{\frac{\partial\xi}{\rho\xi^2}+\frac{\partial\rho}{\rho^2\xi}\Big\} = \textit{boundary term}.$$

Summary for $O(\eta^0)$

$$\int d^4\theta[yf]_{\theta^4} = \underbrace{\frac{1}{2}Rd(\log\rho+\log\xi)}_{O(\eta^2)} + \textit{boundary term}$$

This will be cancelled in the calculation to order $O(\eta^2)$ in the next subsection.

4.2 $O(\eta^2)$

$$
\begin{aligned}
&\int d^4\theta[y]_{\theta^0}\times[f]_{\theta^4}\\
&=\frac{1}{2}\frac{dh}{\rho\xi}\Big\{-\Big(\rho\eta\cdot\partial^2(\xi\eta)\Big)+\Big(\partial^2(\rho\eta)\cdot\xi\eta\Big)\Big\} &(1)\\
&\quad+\frac{1}{2}\frac{P}{\rho\xi}(\xi\partial\rho+\rho\partial\xi), &(2)
\end{aligned}
$$

$$
\begin{aligned}
&\int d^4\theta[y]_{\theta^1}\times[f]_{\theta^3}\\
&=\frac{1}{2}\frac{dh}{\rho\xi}\Big\{-2\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\frac{\partial\rho}{\rho}+2\Big(\rho\eta\cdot\partial(\xi\eta)\Big)\frac{\partial\xi}{\xi}\Big\} &(3)\\
&\quad+\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\rho\xi\eta)\Big)-\Big(\partial(\xi\rho\eta)\cdot\frac{d(\xi\eta)}{\xi}\Big), &(4)
\end{aligned}
$$

$$
\int d^4\theta[y]_{\theta^2}\times[f]_{\theta^2}=\partial\Big(\rho\eta\cdot\xi\eta\Big)(\frac{d\xi}{\xi}-\frac{d\rho}{\rho}), \qquad (5)
$$

$$
\begin{aligned}
&\int d^4\theta[y]_{\theta^3}\times[f]_{\theta^1}\\
&=\frac{1}{2}\frac{dh}{\rho\xi}\Big\{6\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\frac{\partial\xi}{\xi}-6\Big(\rho\eta\cdot\partial(\xi\eta)\Big)\frac{\partial\rho}{\rho} &(6)\\
&\qquad-2\Big(\partial^2(\rho\eta)\cdot\xi\eta\Big)+2\Big(\rho\eta\cdot\partial^2(\xi\eta)\Big)\Big\} &(7)\\
&\quad-(\frac{d\xi}{\xi}-\frac{d\rho}{\rho})\Big[\Big(\rho\eta\cdot\partial(\xi\eta)\Big)+\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\Big] &(8)\\
&\quad+\Big[\Big(\rho\eta\cdot d(\xi\eta)\Big)\frac{\partial\rho}{\rho}-\Big(d(\rho\eta)\cdot\xi\eta\Big)\frac{\partial\xi}{\xi}\Big] &(9)\\
&\quad+\Big[\Big(d\partial(\rho\eta)\cdot\xi\eta\Big)-\Big(\rho\eta\cdot\ d\partial(\xi\eta)\Big)\Big] &(10)\\
&\quad+\Big[-2\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\frac{d\rho}{\rho}+2\Big(\rho\eta\cdot\ \partial(\xi\eta)\frac{d\xi}{\xi}\Big)\Big], &(11)
\end{aligned}
$$

$$
\begin{aligned}
&\int d^4\theta[y]_{\theta^4}\times[f]_{\theta^0}\\
&=-dh\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big]-\partial h\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big]\\
&\quad-\partial h\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&=-dh\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big]-\rho\xi\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big] &(12)\\
&\quad-\rho\xi\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] &(13)\\
&\quad-\underbrace{R\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^4)}\\
&\quad-\underbrace{R\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big]}_{O(\eta^4)}.
\end{aligned}
$$

We calculate (1)+(2)+$\cdots$+(13) to order $O(\eta^2)$ *plus* a contribution from the calculation to order $O(\eta^0)$

$$\int d^4\theta < [y]_{\theta^0} \times [f]_{\theta^4} >_{from\ O(\eta^0)} = \underbrace{\frac{1}{2}Rd(\log\rho + \log\xi)}_{O(\eta^2)} + boundary\ term, \qquad < a >$$

To order $O(\eta^2)$ there remain boundary terms alone. In the following we show how cancellations occur in the sum.

$$\begin{aligned}
&(1) + (3) + (6) + (7) \\
&\quad = \frac{1}{2}\frac{dh}{\rho\xi}\Big\{4\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\frac{\partial\xi}{\xi} - 4\Big(\rho\eta\cdot\partial(\xi\eta)\Big)\frac{\partial\rho}{\rho} \\
&\qquad\qquad -\Big(\partial^2(\rho\eta)\cdot\xi\eta\Big) + \Big(\rho\eta\cdot\partial^2(\xi\eta)\Big)\Big\} \\
&\quad = \frac{1}{2}dh\partial[R(-\frac{1}{\rho\xi})] + \frac{1}{2}dhR\partial(-\frac{1}{\rho\xi}), \\
&\{(1) + (3) + (6) + (7)\} + (2) + (12) = \frac{1}{2}dh\partial[R(-\frac{1}{\rho\xi})] \\
&\qquad = -\frac{1}{2}d(\rho\xi)R(-\frac{1}{\rho\xi}) - \underbrace{\frac{1}{2}dRR(-\frac{1}{\rho\xi})}_{O(\eta^4)}, \qquad < b > \\
&\{(4) + (10)\} + (9) = \partial\Big[\Big(d(\rho\xi)\cdot\xi\eta\Big) - \Big(\rho\eta\cdot d(\xi\eta)\Big)\Big] = boundary\ term, \\
&(5) + (8) = 0, \\
&(11) = -R(\frac{d\rho}{\rho} + \frac{d\xi}{\xi}) + \partial\Big(\rho\eta\cdot\xi\eta\Big)(\frac{d\rho}{\rho} - \frac{d\xi}{\xi}), \\
&(11) = -R(\frac{d\rho}{\rho} + \frac{d\xi}{\xi}) + boundary\ term, \qquad < c > \\
&(13) = \frac{1}{2}[dR + \partial P] = boundary\ term,
\end{aligned}$$

$$< a > + < b > + < c > = -\underbrace{\frac{1}{2}dRR(-\frac{1}{\rho\xi})}_{O(\eta^4)}.$$

Summary for $O(\eta^2)$

$$\int d^4\theta[yf]_{\theta^4} = -\underbrace{\frac{1}{2}R\partial P(-\frac{1}{\rho\xi})}_{O(\eta^4)} - \underbrace{R\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big]}_{O(\eta^4)} - \underbrace{dRR(-\frac{1}{\rho\xi})}_{O(\eta^4)} + boundary\ term,$$

The terms of order $O(\eta^4)$ will be canceled in the calculation to order $O(\eta^4)$ in the next subsection.

4.3 $O(\eta^4)$

$$\int d^4\theta[y]_{\theta^0} \times [f]_{\theta^4} = \frac{1}{2}\frac{P}{\rho\xi}\Big[-\Big(\rho\eta\cdot\partial^2(\xi\eta)\Big) + \Big(\partial^2(\rho\eta)\cdot\xi\eta\Big)\Big], \qquad (1)$$

$$\int d^4\theta [y]_{\theta^1} \times [f]_{\theta^3} = \frac{P}{\rho\xi}\Big[\Big(\partial(\xi\rho\eta)\cdot\frac{\partial(\xi\eta)}{\xi}\Big) - \Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\rho\xi\eta)\Big)\Big], \tag{2}$$

$$\int d^4\theta [y]_{\theta^2} \times [f]_{\theta^2} = 0,$$

$$\int d^4\theta [y]_{\theta^3} \times [f]_{\theta^1}$$

$$= -\frac{dR+\partial P}{\rho\xi}\Big[\Big(\partial(\rho\eta)\cdot\xi\eta\Big) - \Big(\rho\eta\cdot\partial(\xi\eta)\Big)\Big] \tag{3}$$

$$+\frac{1}{2}\frac{P}{\rho\xi}\Big\{6\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\frac{\partial\xi}{\xi} - 6\Big(\rho\eta\cdot\partial(\xi\eta)\Big)\frac{\partial\rho}{\rho} \tag{4}$$

$$-2\Big(\partial^2(\rho\eta)\cdot\xi\eta\Big) + 2\Big(\xi\eta\cdot\partial^2(\xi\eta)\Big)\Big\}. \tag{5}$$

We calculate (1) + (2) + $\cdots$ + (5) *plus* a contribution from the calculation to order $O(\eta^2)$

$$\int d^4\theta < [yf]_{\theta^4} >_{from\ O(\eta^2}>= -\underbrace{\frac{1}{2}R\partial P(-\frac{1}{\rho\xi})}_{O(\eta^4)} - \underbrace{R\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big]}_{O(\eta^4)} \tag{6}$$

$$-\underbrace{dRR(-\frac{1}{\rho\xi})}_{O(\eta^4)} + boundary\ term. \tag{7}$$

It is shown that there remains only boundary terms as follows.

$$(1) + (6) = 0,$$

$$(2) + (4) = P\Big\{\Big(\partial(\rho\eta)\cdot\xi\eta\Big)\partial(-\frac{1}{\rho\xi}) - \Big(\rho\eta\cdot\partial(\xi\eta)\Big)\partial(-\frac{1}{\rho\xi})\Big\}$$

$$= \frac{\partial P}{\rho\xi}\Big\{\Big(\partial(\rho\eta)\cdot\xi\eta\Big) - \Big(\rho\eta\cdot\partial(\xi\eta)\Big)\Big\}$$

$$+\frac{P}{\rho\xi}\Big\{\Big(\partial^2(\rho\eta)\cdot\xi\eta\Big) - \Big(\rho\eta\cdot\partial^2(\xi\eta)\Big)\Big\},$$

$$\{(2) + (4)\} + (3) + (5) + (7) = 0.$$

5 Proof of $\int_{\mathcal{O}_1}\int dx d\theta^4 [y\varphi_c]_{\theta^4} = 0$

$$[y\varphi_c]_{\theta^4} = [y]_{\theta^0}\times[\varphi_c]_{\theta^4} + [y]_{\theta^1}\times[\varphi_c]_{\theta^3} + [y]_{\theta^2}\times[\varphi_c]_{\theta^2} + [y]_{\theta^3}\times[\varphi_c]_{\theta^1} + +[y]_{\theta^4}\times[\varphi_c]_{\theta^0}.$$

Below we calculate piece by piece the r.h.s. to each order of η. Here also d- and ∂-boundary terms are discarded once they happen to appear, by notifying as "*boundary*" without giving explicit forms.

5.1 $O(\eta^1)$

$$\int d^4\theta [y]_{\theta^0} \times [\varphi_c]_{\theta^4} = \frac{1}{2}\frac{dh}{\rho\xi}\partial^2(\rho\eta_c), \tag{1}$$

$$\int d^4[y]_{\theta^1} \times [\varphi_c]_{\theta^3} = \partial\rho[-\frac{dh}{\rho\xi}\frac{\partial(\rho\eta_c)}{\rho} + \frac{d(\rho\eta_c)}{\rho}] = -\partial\rho\frac{dh}{\rho\xi}\frac{\partial(\rho\eta_c)}{\rho} + boundary\ term, \tag{2}$$

$$\int d^4\theta[y]_{\theta^2} \times [\varphi_c]_{\theta^2} = \partial(\rho\eta_c)(\frac{d\xi}{\xi} - \frac{d\rho}{\rho}) = boundary\ term,$$

$$\int d^4\theta[y]_{\theta^3} \times [\varphi_c]_{\theta^1}$$

$$= \frac{1}{2}\frac{dh}{\rho\xi}\{6\partial(\rho\eta_c)\frac{\partial\xi}{\xi} - 2\partial^2(\rho\eta_c)\} - d(\rho\eta_c)\frac{\partial\xi}{\xi} \tag{3}$$

$$+d\partial(\rho\eta_c) - 2\partial((\rho\eta_c)\frac{d\rho}{\rho}, \tag{4}$$

$$\int d^4\theta[y]_{\theta^4} \times [\varphi_c]_{\theta^0} = \rho\eta_c d\Big[\frac{1}{2}h\partial\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}\Big] - \rho\eta_c\partial\Big[\frac{1}{2}hd\Big\{\frac{\partial\xi}{\rho\xi^2} + \frac{\partial\rho}{\rho^2\xi}\Big\}\Big]$$

$$= \frac{1}{2}\rho\eta_c dh\partial^2(-\frac{1}{\rho\xi}) - \frac{1}{2}\rho\eta_c\partial h d\partial(-\frac{1}{\rho\xi})$$

$$= \frac{1}{2}\rho\eta_c dh\partial^2(-\frac{1}{\rho\xi}) \tag{5}$$

$$-\frac{1}{2}\rho\eta_c\rho\xi d\partial(-\frac{1}{\rho\xi}) \tag{6}$$

$$\underbrace{-\frac{1}{2}\rho\eta_c R d\partial(-\frac{1}{\rho\xi})}_{O(\eta^3)},$$

In the following we calculate (1) + (2) + $\cdots$ + (6) to show that there remain only boundary terms to order $\mathcal{O}(\eta^1)$.

$$(1) + (2) + (3) = \frac{1}{2}dh\Big\{2\partial(\rho\eta_c)\partial(-\frac{1}{\rho\xi}) - \frac{1}{\rho\xi}\partial^2(\rho\eta_c)\Big\},$$

$$(1) + (2) + (3) + (5) = \frac{1}{2}dh\partial^2\Big[\rho\eta_c(-\frac{1}{\rho\xi})\Big] = \frac{1}{2}d\partial^2 h\Big[\rho\eta_c(-\frac{1}{\rho\xi})\Big]$$

$$= -\frac{1}{2}\partial^2 h\Big[d(\rho\eta_c)(-\frac{1}{\rho\xi}) + \rho\eta_c d(-\frac{1}{\rho\xi})\Big]$$

$$= -\frac{1}{2}\partial(\rho\xi)\Big[d(\rho\eta_c)(-\frac{1}{\rho\xi}) + \rho\eta_c d(-\frac{1}{\rho\xi})\Big] + boundary\ term$$

$$= \underbrace{-\frac{1}{2}\partial R\Big[d(\rho\eta_c)(-\frac{1}{\rho\xi}) + \rho\eta_c d(-\frac{1}{\rho\xi})\Big]}_{O(\eta^3)},$$

$$(4) = -2d(\rho\eta_c)\frac{\partial\rho}{\rho} + boundary\ term$$

$$= -2d(\rho\eta_c)\frac{\partial\rho}{\rho} + boundary\ term, = -d(\rho\eta_c)\frac{\partial(\rho\xi)}{\rho\xi} + boundary\ term,$$

$$(6) = \frac{1}{2}d(\rho\eta_c)\rho\xi\partial(-\frac{1}{\rho\xi}) + \frac{1}{2}\rho\eta_c d(\rho\xi)\partial(-\frac{1}{\rho\xi}) + boundary\ term,$$

$$(4) + (6) + \{(1) + (2) + (3) + (5)\} = boundary\ term.$$

Summary for $O(\eta^1)$

$$\int d^4\theta [y\varphi_c]_{\theta^4} = -\underbrace{\frac{1}{2}\rho\eta_c R d\partial(-\frac{1}{\rho\xi})}_{O(\eta^3)} - \underbrace{\frac{1}{2}\partial R\Big[d(\rho\eta_c)(-\frac{1}{\rho\xi}) + \rho\eta_c d(-\frac{1}{\rho\xi})\Big]}_{O(\eta^3)} + boundary\ term$$

The terms of order $O(\eta^3)$ will be cancelled in the calculation to order $O(\eta^3)$ in the next subsection.

5.2 $O(\eta^3)$

$$\int d^4\theta [y]_{\theta^0} \times [\varphi_c]_{\theta^4} = \frac{1}{2}\frac{P}{\rho\xi}\partial^2(\rho\eta_c), \tag{7}$$

$$\int d^4\theta [y]_{\theta^1} \times [\varphi_c]_{\theta^3} = \frac{1}{2}\frac{P}{\rho\xi}(-2\partial\rho\frac{\partial\rho}{\rho}), \tag{8}$$

$$\int d^4\theta [y]_{\theta^2} \times [\varphi_c]_{\theta^2} = boundary\ term,$$

$$\int d^4\theta [y]_{\theta^3} \times [\varphi_c]_{\theta^1} = \frac{1}{2}\frac{P}{\rho\xi}\Big\{6\partial(\rho\eta_c)\frac{\partial\xi}{\xi} - 2\partial^2(\rho\eta_c)\Big\} - \frac{dR + \partial P}{\rho\xi}\partial(\rho\eta_c), \tag{9}$$

$$\int d^4\theta [y]_{\theta^4} \times [\varphi_c]_{\theta^0} = \rho\eta_c d\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big] + \rho\eta_c\partial\Big[\frac{1}{2}P\partial(-\frac{1}{\rho\xi})\Big] \tag{10}$$

$$+\rho\eta_c\partial\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big) + \Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]. \tag{11}$$

We calculate (7) + (8) + ⋯ + (11) *plus* a contribution from the calculation to order $O(\eta^1)$

$$\int d^4\theta < [y\varphi_c]_{\theta^4} >_{from\ O(\eta^1)}$$

$$= -\underbrace{\frac{1}{2}\rho\eta_c R d\partial(-\frac{1}{\rho\xi})}_{O(\eta^3)} - \underbrace{\frac{1}{2}\partial R\Big[d(\rho\eta_c)(-\frac{1}{\rho\xi}) + \rho\eta_c d(-\frac{1}{\rho\xi})\Big]}_{O(\eta^3)} + boundary\ term, \tag{12}$$

$$(7) + (10) = -\frac{1}{2}\frac{\partial P}{\rho\xi}\partial(\rho\eta_c) - d(\rho\eta_c)\Big[\frac{1}{2}R\partial(-\frac{1}{\rho\xi})\Big],$$

$$(8) + (9) = \frac{1}{2}\frac{P}{\rho\xi}\Big\{4\partial(\rho\eta_c)\frac{\partial\xi}{\xi} - 2\partial^2(\rho\eta_c)\Big\} - \frac{dR + \partial P}{\rho\xi}\partial(\rho\eta_c)$$

$$= \frac{1}{2}P\Big\{-2\partial\Big[\partial(\rho\eta_c)\frac{1}{\rho\xi}\Big]\Big\} - \frac{dR + \partial P}{\rho\xi}\partial(\rho\eta_c) = -\frac{dR}{\rho\xi}\partial(\rho\eta_c),$$

$$(12) = \Big[\frac{1}{2}d(\rho\eta_c)R\partial(-\frac{1}{\rho\xi}) + \frac{1}{2}\rho\eta_c dR\partial(-\frac{1}{\rho\xi})\Big] - \frac{1}{2}dR\partial\Big[\rho\eta_c(-\frac{1}{\rho\xi})\Big] + boundary\ term,$$

$$= \frac{1}{2}d(\rho\eta_c)R\partial(-\frac{1}{\rho\xi}) - \frac{1}{2}dR\partial(\rho\eta_c)(-\frac{1}{\rho\xi}) + boundary\ term,$$

$$\{(7) + (10)\} + \{(8) + (10)\} + (12) = -\frac{1}{2}\frac{dR + \partial P}{\rho\xi}\partial(\rho\eta_c). \tag{13}$$

Summing this and (11) we conclude that $\int d^4\theta y\varphi_c$ vanishes modulo boundary terms. There is no contribution to higher orders than $O(\eta^3)$.

6 Proof of $\int dx d^4\theta y\partial y = d[\cdots] = 0$

We show

$$\int dx d\theta^4 \; y\partial y = d[.....].$$

To this end we use the expansion formula of y given by in Sec. 2, of which components have been denoted as

$$y = [y]_{\theta^0} + [y]_{\theta^1} + [y]_{\theta^2} + [y]_{\theta^3} + [y]_{\theta^4}.$$

We expand here furthermore each component in the fermionic components η_a and η^a.

$$[y]_{\theta^n} = [y]_{\theta^n,\eta^0} + [y]_{\theta^n,\eta^1} + [y]_{\theta^n,\eta^2} + [y]_{\theta^n,\eta^3} + [y]_{\theta^n,\eta^4}, \qquad n = 1,2,3,4.$$

We are interested only in $[y]_{\theta^4}$. The integration over the left-moving sector is calculated as

$$\int dx d^4\theta \; [y\partial y]_{\theta^4} = \int dx d^4\theta \; 2\Big([y]_{\theta^0} \times \partial[y]_{\theta^4} + [y]_{\theta^1} \times \partial[y]_{\theta^3} + \frac{1}{2}[y]_{\theta^2} \times \partial[y]_{\theta^2}\Big).$$

In the following we calculate $\int d^4\theta [y\partial y]_{\theta^4}$ in this integration with the convention

$$\int d^4\theta (\theta\cdot\theta)^2 = 1.$$

Then d-boundary terms are given with concrete expressions. But for ∂-boundary terms their exsistence is merely notified, because it disappears anyway after integration over x.

6.1 $O(\eta^0)$

$$\int d^4\theta [y]_{\theta^1,\eta^0} \times \partial[y]_{\theta^3,\eta^0} = \int d^4\theta [y]_{\theta^2,\eta^0} \times \partial[y]_{\theta^2,\eta^0} = 0,$$

$$\begin{aligned}
&\int d^4\theta [y]_{\theta^0,\eta^0} \times \partial[y]_{\theta^4,\eta^0} \\
&= \frac{dh}{\rho\xi}\partial\Big[d\Big(\frac{1}{2}h\partial\Big\{\frac{\partial\rho}{\rho^2\xi} + \frac{\partial\xi}{\rho\xi^2}\Big\} - \partial\Big(\frac{1}{2}hd\Big\{\frac{\partial\rho}{\rho^2\xi} + \frac{\partial\xi}{\rho\xi^2}\Big\}\Big]\\
&= \frac{1}{2}\frac{dh}{\rho\xi}\partial\Big[d\Big(h\partial^2(-\frac{1}{\rho\xi})\Big) - \partial\Big(hd\partial(-\frac{1}{\rho\xi})\Big)\Big]\\
&= \frac{1}{2}\frac{dh}{\rho\xi}\partial\Big[dh\partial^2(-\frac{1}{\rho\xi}) - \partial hd\partial(-\frac{1}{\rho\xi})\Big]\\
&= \frac{1}{2}\Big\{\frac{dh}{\rho\xi}(d\partial h)\partial^2(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial^2 hd\partial(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial hd\partial^2(-\frac{1}{\rho\xi})\Big\}\\
&= \frac{1}{2}\Big\{dh\Big(d\log(\rho\xi)\Big)\partial^2(-\frac{1}{\rho\xi}) - dh\Big(\partial\log(\rho\xi)\Big)d\partial(-\frac{1}{\rho\xi})\\
&+d\Big[dh\partial^2(-\frac{1}{\rho\xi})\Big]\\
&+ \underbrace{\frac{dh}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^2)}\Big\}\\
&= \frac{1}{2}\Big\{dh\Big(d\log(\rho\xi)\Big)\partial^2(-\frac{1}{\rho\xi}) + dh\Big(d\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})
\end{aligned}$$

$$+d\Big[dh\partial^2(-\frac{1}{\rho\xi}) + dh\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]$$
$$+\underbrace{\frac{dh}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^2)}\Big\}$$
$$= \frac{1}{2}\Big\{dh\partial\Big[\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]$$
$$+d\Big[dh\partial^2(-\frac{1}{\rho\xi}) + dh\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]$$
$$+\underbrace{\frac{dh}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^2)}\Big\}$$
$$= \frac{1}{2}\Big\{d\Big[dh\partial^2(-\frac{1}{\rho\xi}) + dh\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]$$
$$+\underbrace{\frac{dh}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi}) - dR\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})}_{O(\eta^2)}\Big\}$$
$$+\partial\text{–}boundary\ term.$$

Summary for $O(\eta^0)$

$$\int d^4\theta[y]_{\theta^1}\times\partial[y]_{\theta^3} = \int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2} = 0$$
$$\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}$$
$$= \frac{1}{2}d\Big[dh\partial^2(-\frac{1}{\rho\xi}) + dh\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big] \quad \cdots \quad d\text{–}boundary\ term$$
$$+\underbrace{\frac{1}{2}\Big\{\frac{dh}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi}) - \frac{dh}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi}) - dR\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big\}}_{O(\eta^2)}$$
$$+\partial\text{–}boundary\ term$$

The terms of order $\mathcal{O}(\eta^2)$ will be canceled in the calculation to order $\mathcal{O}(\eta^2)$ in the next subsection.

6.2 $O(\eta^2)$

$$\int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2} = 0,$$

$$\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta}$$
$$= \Big\{\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big) - 2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big] \tag{1}$$
$$+\Big[2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big) - 2\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big\}\partial(\frac{dh}{\rho\xi}) \tag{2}$$
$$+\Big\{2(\frac{d\xi}{\xi} + \frac{d\rho}{\rho})\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big) + \Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \tag{3}$$
$$+\Big[2\Big(\partial(\frac{\partial\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big) - 2\Big(\partial(\frac{d\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\Big\}\frac{dh}{\rho\xi} \tag{4}$$

$$+\frac{dh}{\rho\xi}\Big\{6\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\frac{\partial\xi}{\xi}-6\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho} \tag{5}$$

$$-2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)+2\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big\} \tag{6}$$

$$+2(\frac{d\xi}{\xi}+\frac{d\rho}{\rho})\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)+\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \tag{7}$$

$$+4\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big), \tag{8}$$

In the following we calculate (1) + (2) + $\cdots$ + (8) to show that there remain only boundary terms to order $\mathcal{O}(\eta^2)$.

$$\begin{aligned}
(1)+(3) &= -4\frac{dh}{\rho\xi}(\frac{d\xi}{\xi}+\frac{d\rho}{\rho})\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \\
&\quad+\underbrace{\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^4)} \\
&= -4dhd(-\frac{1}{\rho\xi})\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \\
&\quad+\underbrace{\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^4)} \\
&= -4\frac{dh}{\rho\xi}d\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \\
&\quad+d\Big\{-4\frac{dh}{\rho\xi}\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\Big\} \\
&\quad+\underbrace{\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^4)} \\
&= 4\frac{dh}{\rho\xi}\Big\{\frac{1}{\rho\xi}\Big[\Big(d\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial^2(\xi\eta)\Big)\Big] \\
&\qquad+[\frac{1}{\xi}d(\frac{1}{\rho})+\frac{1}{\rho}d(\frac{1}{\xi})]\Big[\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big] \\
&\qquad-\frac{1}{\rho\xi}\Big[-\Big(\partial^2(\rho\eta)\cdot d\partial(\xi\eta)\Big)+\Big(d\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big]\Big\} \\
&\quad+d\Big\{-4\frac{dh}{\rho\xi}\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\Big\} \\
&\quad+\underbrace{\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^4)},
\end{aligned}$$

$$\begin{aligned}
(4)+(5)+(6) &= -4\frac{dh}{\rho\xi}\Big[\Big(\partial(\frac{\partial\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{d\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \\
&\quad+4\frac{dh}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\frac{\partial\xi}{\xi}-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}\Big],
\end{aligned}$$

(2) + (4) + (5) + (6) + (7)

$$\begin{aligned}
&= -4\frac{dh}{\rho\xi}\Big[\Big(\partial(\frac{\partial\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{d\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \\
&\quad+4dh\partial(-\frac{1}{\rho\xi})\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+4(\frac{d\xi}{\xi}+\frac{d\rho}{\rho})\Big[-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)+\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\qquad+\underbrace{\Big[2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-2\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big\}\frac{dR}{\rho\xi}}_{O(\eta^4)}\\
&=-4\frac{dh}{\rho\xi}\Big[\Big(\partial(\frac{\partial\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{d\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\\
&\quad+4\frac{dh}{\rho\xi}\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\quad\underbrace{+2\frac{dR}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^4)}\\
&\quad+\partial\text{–}boundary\ term\\
&=4\frac{dh}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial^2(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial^2(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\quad\underbrace{+2\frac{dR}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^4)}\\
&\quad+\partial\text{–}boundary\ term\\
&=4\frac{dh}{\rho\xi}\Big\{\frac{1}{\rho\xi}\Big[\Big(\partial(\rho\eta)\cdot d\partial^2(\xi\eta)\Big)-\Big(d\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\\
&\qquad+2\frac{1}{\rho}\partial(\frac{1}{\xi})\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)-2\frac{1}{\xi}\partial(\frac{1}{\rho})\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\\
&\qquad+\frac{1}{\rho}\partial^2\frac{1}{\xi}\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\frac{1}{\xi}\partial^2\frac{1}{\rho}\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big\}\\
&\quad\underbrace{+2\frac{dR}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^4)}\\
&\quad+\partial\text{–}boundary\ term,
\end{aligned}
$$

$$
\begin{aligned}
&\{(1)+(3)\}+\{(2)+(4)+(5)+(6)+(7)\}\\
&=4\frac{dh}{\rho\xi}\Big\{[\frac{1}{\xi}d(\frac{1}{\rho})+\frac{1}{\rho}d(\frac{1}{\xi})]\Big[\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big] && <1>\\
&\qquad-\frac{1}{\rho\xi}\Big[\Big(d\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)-\Big(\partial^2(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big] && <2>\\
&\qquad+2\frac{1}{\rho}\partial(\frac{1}{\xi})\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)-2\frac{1}{\xi}\partial(\frac{1}{\rho})\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big) && <3>\\
&\qquad+\frac{1}{\rho}\partial^2\frac{1}{\xi}\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\frac{1}{\xi}\partial^2\frac{1}{\rho}\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big\} && <4>\\
&\quad+d\Big\{-4\frac{dh}{\rho\xi}\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\Big\}\cdots d\text{–}boundary\ term\\
&\quad\underbrace{+\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^4)}\\
&\quad\underbrace{+2\frac{dR}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^4)}
\end{aligned}
$$

$+\partial-boundary\ term,$

The 1st line in the above result can be calculated as

$$
\begin{aligned}
<1>= & 4dhd[\frac{1}{2}\frac{1}{(\rho\xi)^2}]\Big[\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big] \\
& = -2dh\frac{1}{(\rho\xi)^2}\Big[\partial\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\partial\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big) \\
& \qquad -2\Big(d\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)+2\Big(\partial^2(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big] \\
& \quad +d\Big\{-2dh\frac{1}{(\rho\xi)^2}\Big[\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big]\Big\} \\
& = 2d(-\frac{1}{\rho\xi})\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big] && <1a> \\
& \quad -4dh\frac{\partial(\rho\xi)}{(\rho\xi)^3}\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big] && <1b> \\
& \quad +4dh\frac{1}{(\rho\xi)^2}\Big[\Big(d\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)-\Big(\partial^2(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big] && <1c> \\
& \quad +\underbrace{2dR\frac{1}{(\rho\xi)^2}\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big]}_{O(\eta^4)} \\
& \quad +d\Big\{-2dh\frac{1}{(\rho\xi)^2}\Big[\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big]\Big\} \\
& \quad +\partial-boundary\ term,
\end{aligned}
$$

$$
\begin{aligned}
<1a>= & -4\frac{1}{\rho\xi}\Big(d\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big) \\
& +d\Big\{-\frac{2}{\rho\xi}\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big]\Big\}, \\
<1b> & + <3>=0, \\
<1c> & + <2>=0,
\end{aligned}
$$

$$
\begin{aligned}
(8) = & 4\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big) \\
= & 4\Big(\frac{d\partial(\rho\xi)}{\rho}\cdot\frac{d\partial(\xi\eta)}{\xi}\Big)+4\Big(d(\rho\eta)\cdot d(\xi\eta)\Big)\partial(\frac{1}{\rho})\partial(\frac{1}{\xi})+2\partial\Big(d(\rho\eta)\cdot d(\xi\eta)\Big)\partial(\frac{1}{\rho\xi}).
\end{aligned}
$$

Summary 1 for $O(\eta^2)$

$$\int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2}=0,$$

$$\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta}$$
$$=4\frac{dh}{\rho\xi}\Big\{\frac{1}{\rho}\partial^2\frac{1}{\xi}\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\frac{1}{\xi}\partial^2\frac{1}{\rho}\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big\}$$
$$+4\Big(d(\rho\eta)\cdot d(\xi\eta)\Big)\partial(\frac{1}{\rho})\partial(\frac{1}{\xi})$$
$$+2\partial\Big(d(\rho\eta)\cdot d(\xi\eta)\Big)\partial(\frac{1}{\rho\xi})$$
$$+d\Big\{-4\frac{dh}{\rho\xi}\Big[-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\Big\}\cdots d\text{–}boudary\ term$$
$$+d\Big\{-\frac{2}{\rho\xi}\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big]\Big\}\qquad\cdots\cdots\cdots d\text{–}boudary\ term$$
$$+\underbrace{\frac{dh}{\rho\xi}\Big[2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^4)}$$
$$+\underbrace{2\frac{dR}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^4)}$$
$$+\underbrace{2dR\frac{1}{(\rho\xi)^2}\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big]}_{O(\eta^4)}$$
$$+\partial\text{–}boundary\ term$$

$$\int d^4\theta[y]_{\theta^0,\eta^0}\times\partial[y]_{\theta^4,\eta^2}$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}\partial\Big\{d\Big(R(\frac{\partial\rho}{\rho^2\xi}+\frac{\partial\xi}{\rho\xi^2})+\partial\Big(P(\frac{\partial\rho}{\rho^2\xi}+\frac{\partial\xi}{\rho\xi^2})\Big)$$
$$+2\partial\Big[\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big]\Big\}$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}\partial\Big\{[dR+\partial P]\partial(-\frac{1}{\rho\xi})+Rd\partial(-\frac{1}{\rho\xi})+P\partial^2(-\frac{1}{\rho\xi})$$
$$-\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]\Big\}$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}\Big\{\partial\Big\{[dR+\partial P]\partial(-\frac{1}{\rho\xi})\Big\}+\partial Rd\partial(-\frac{1}{\rho\xi})+Rd\partial^2(-\frac{1}{\rho\xi})+\partial P\partial^2(-\frac{1}{\rho\xi})+P\partial^3(-\frac{1}{\rho\xi})\Big\}$$
$$+\frac{1}{2}\partial(\frac{dh}{\rho\xi})\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]+\partial\text{–}boundary\ term,$$

$$\int d^4\theta[y]_{\theta^0,\eta^2}\times\partial[y]_{\theta^4,\eta^0}$$
$$=\frac{1}{2}\Big\{\frac{P}{\rho\xi}(d\partial h)\partial^2(-\frac{1}{\rho\xi})+\frac{P}{\rho\xi}dh\partial^3(-\frac{1}{\rho\xi})-\frac{P}{\rho\xi}\partial^2hd\partial(-\frac{1}{\rho\xi})-\frac{P}{\rho\xi}\partial hd\partial^2(-\frac{1}{\rho\xi})\Big\}$$
$$=\frac{1}{2}\Big\{\frac{P}{\rho\xi}d(\rho\xi)\partial^2(-\frac{1}{\rho\xi})+\frac{P}{\rho\xi}dh\partial^3(-\frac{1}{\rho\xi})-\frac{P}{\rho\xi}\partial(\rho\xi)d\partial(-\frac{1}{\rho\xi})-Pd\partial^2(-\frac{1}{\rho\xi})\Big\}$$

$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}$$

$$=\frac{1}{2}\Big\{-\partial P\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})-Pd\Big[\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]$$
$$+\frac{P}{\rho\xi}dh\partial^3(-\frac{1}{\rho\xi})-Pd\partial^2(-\frac{1}{\rho\xi})\Big\}$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}+\partial\text{−}boundary\ term,$$

Contribution from the calculation to order $O(\eta^0)$

$$\int d^4\theta < [y]_{\theta^0}\times\partial[y]_{\theta^4} >_{from\ O(\eta^0)}$$
$$=\frac{1}{2}\Big\{\frac{dh}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{dh}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{dh}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})-dR\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big\},$$

$$\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}$$
$$=[y]_{\theta^0,\eta^0}\times\partial[y]_{\theta^4,\eta^2}+[y]_{\theta^0,\eta^2}\times\partial[y]_{\theta^4,\eta^0}+<[y]_{\theta^0}\times\partial[y]_{\theta^4}>_{from\ O(\eta^0)}$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}\partial\Big\{[dR+\partial P]\partial(-\frac{1}{\rho\xi})\Big\}+\frac{1}{2}\frac{dh}{\rho\xi}[dR+\partial P]\partial^2(-\frac{1}{\rho\xi})+\frac{1}{2}\partial(\frac{dh}{\rho\xi})\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]$$
$$-\frac{1}{2}[dR+\partial P]\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})-\frac{1}{2}Pd\Big[\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]-\frac{1}{2}Pd\partial^2(-\frac{1}{\rho\xi})$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}$$
$$+\partial\text{−}boundary\ term,$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}[dR+\partial P]\Big\{2\partial^2(-\frac{1}{\rho\xi})+\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}+\frac{1}{2}\Big(d\log(\rho\xi)\Big)\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]$$
$$-\frac{1}{2}[dR+\partial P]\Big(d\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})-\frac{1}{2}Pd\Big[\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]-\frac{1}{2}Pd\partial^2(-\frac{1}{\rho\xi})$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}$$
$$+\underbrace{\frac{1}{2}\frac{dR}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]}_{O(\eta^4)}+\partial\text{−}boundary\ term,$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}[dR+\partial P]\Big\{2\partial^2(-\frac{1}{\rho\xi})+\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}+\frac{1}{2}\Big(d\log(\rho\xi)\Big)\frac{\partial[dR+\partial P]}{\rho\xi}$$
$$-\frac{1}{2}Pd\Big[\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]-\frac{1}{2}Pd\partial^2(-\frac{1}{\rho\xi})$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}$$
$$+\underbrace{\frac{1}{2}\frac{dR}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]}_{O(\eta^4)}+\partial\text{−}boundary\ term,$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}[dR+\partial P]\Big\{2\partial^2(-\frac{1}{\rho\xi})+\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}+\frac{1}{2}d(-\frac{1}{\rho\xi})\partial[dR+\partial P]$$

$$-\frac{1}{2}Pd\Big[\Big(\partial\log(\rho\xi)\Big)\partial(-\frac{1}{\rho\xi})\Big]-\frac{1}{2}Pd\partial^2(-\frac{1}{\rho\xi})$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}$$
$$+\underbrace{\frac{1}{2}\frac{dR}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]}_{O(\eta^4)}+\partial\text{–}\textit{boundary term},$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}[dR+\partial P]\Big\{2\partial^2(-\frac{1}{\rho\xi})+\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}+\frac{1}{2}\Big\{2\partial^2(\frac{1}{\rho\xi})-\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}dP$$
$$+d\Big[-\frac{1}{2}\frac{1}{\rho\xi}\partial[dR+\partial P]+\frac{1}{2}P\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2+\frac{1}{2}P\partial^2(-\frac{1}{\rho\xi})\Big]$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}+\underbrace{\frac{1}{2}\frac{dR}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]}_{O(\eta^4)}$$
$$+\partial\text{–}\textit{boundary term}.$$

Summary 2 for $O(\eta^2)$

$$\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}$$
$$=\frac{1}{2}\frac{dh}{\rho\xi}[dR+\partial P]\Big\{2\partial^2(-\frac{1}{\rho\xi})+\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}+\frac{1}{2}\Big\{2\partial^2(\frac{1}{\rho\xi})-\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2\Big\}dP$$
$$+d\Big[-\frac{1}{2}\frac{1}{\rho\xi}\partial[dR+\partial P]+\frac{1}{2}P\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2+\frac{1}{2}P\partial^2(-\frac{1}{\rho\xi})\Big]$$
$$+\underbrace{\frac{1}{2}\frac{P}{\rho\xi}dR\partial^2(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}\partial Rd\partial(-\frac{1}{\rho\xi})-\frac{1}{2}\frac{P}{\rho\xi}Rd\partial^2(-\frac{1}{\rho\xi})}_{O(\eta^4)}+\underbrace{\frac{1}{2}\frac{dR}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]}_{O(\eta^4)}$$
$$+\partial\text{–}\textit{boundary term}$$

By the constraint $\frac{\partial\rho}{\rho}=\frac{\partial\xi}{\xi}$ we can easily show that

$$-\partial(\frac{1}{\rho})\partial(\frac{1}{\xi})+\frac{1}{2}\partial^2(\frac{1}{\rho\xi})=\frac{1}{4}\frac{1}{\rho\xi}\Big(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi}\Big)^2-\frac{1}{2}\frac{1}{\rho\xi}\partial\Big(\frac{\partial\xi}{\xi}+\frac{\partial\rho}{\rho}\Big),$$
$$\frac{1}{\rho}\partial^2\frac{1}{\xi}-\frac{1}{\rho}\partial^2\frac{1}{\xi}=0,$$
$$\frac{1}{\rho}\partial^2\frac{1}{\xi}+\frac{1}{\rho}\partial^2\frac{1}{\xi}=\frac{1}{2}\frac{1}{\rho\xi}\Big(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi}\Big)^2-\frac{1}{\rho\xi}\partial\Big(\frac{\partial\xi}{\xi}+\frac{\partial\rho}{\rho}\Big),$$
$$2\partial^2(-\frac{1}{\rho\xi})+\rho\xi\Big(\partial(\frac{1}{\rho\xi})\Big)^2=-\frac{1}{\rho\xi}\Big(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi}\Big)^2+2\frac{1}{\rho\xi}\partial\Big(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi}\Big).$$

By means of these formulae We sum over the results of *Summaries* 1, 2 for $O(\eta^2)$ to find only d- and ∂-boundary terms remaining. The terms of order $\mathcal{O}(\eta^4)$ will be cancelled in the calculation to order $O(\eta^4)$ in the next subsection.

6.3 $O(\eta^4)$

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2,\eta^4}\times\partial[y]_{\theta^2,\eta^0}=0,$$

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2,\eta^2}\times\partial[y]_{\theta^2,\eta^2}$$

$$= 24\frac{dh}{\rho\xi}\frac{d\partial h}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2 \tag{1}$$

$$-8\frac{dh}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \tag{2}$$

$$-16\frac{dh}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big) \tag{3}$$

$$+\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big) \tag{4}$$

$$+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big) \tag{5}$$

$$+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \tag{6}$$

$$+2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \tag{7}$$

$$+4\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{d(\rho\xi)}{\rho}\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big] \tag{8}$$

$$+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \tag{9}$$

$$+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big] \tag{10}$$

$$+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big]\Big\}. \tag{11}$$

We have

$$(1)=24\frac{dh}{\rho\xi}\frac{d(\rho\xi)}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2+\underbrace{24\frac{dh}{\rho\xi}\frac{dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2}_{O(\eta^6)},$$

$$\begin{aligned}(3)+(6) &= -16dh\Big\{\frac{1}{(\rho\xi)^3}\Big(\partial(\rho\xi)\cdot\partial(\xi\eta)\Big)d\Big(\partial(\rho\xi)\cdot\partial(\xi\eta)\Big)\\
&\quad-\frac{1}{2}\frac{1}{\rho\xi}(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi})\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\\
&= -24\frac{dh}{\rho\xi}d\log(\rho\xi)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\\
&\quad-8dh\partial(\frac{1}{\rho\xi})\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\\
&\quad+d\Big[8\frac{dh}{(\rho\eta)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2\Big],\end{aligned}$$

$$\begin{aligned}(4)+(5) &= -8\frac{dh}{\rho\xi}\Big\{\partial\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\quad+\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{\partial P+dR}{2\rho\xi}\Big\},\end{aligned}$$

$$\begin{aligned}&(2)+(3)+(4)+(5)+(6)\\
&\quad= -24\frac{dh}{\rho\xi}d\log(\rho\xi)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\\
&\quad+8\frac{d\partial h}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&-8\frac{dh}{\rho\xi}\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{\partial P+dR}{2\rho\xi}\\
&+d\Big[8\frac{dh}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2\Big]+\partial\text{−}boundary\ terms\\
&=-24\frac{dh}{\rho\xi}d\log(\rho\xi)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\\
&+8\frac{d(\rho\xi)}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \qquad (a)\\
&-8\frac{dh}{\rho\xi}\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{\partial P+dR}{2\rho\xi}\\
&+d\Big[8\frac{dh}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2\Big]\\
&+\underbrace{8\frac{dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)}_{O(\eta^6)}\\
&+\partial\text{−}boundary\ terms,
\end{aligned}
$$

$$
\begin{aligned}
&(10)+(11)\\
&=4\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\\
&-4\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big\}\\
&=\Big\{2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)+\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&+2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big\}\\
&-\Big\{2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big]\\
&+2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big]\Big\}\\
&=4\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)+\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&-2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&+2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&-4\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big]\\
&+\partial\text{−}boundary\ terms\\
&=\Big\{4d(\frac{1}{(\rho\xi)^2})\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)+\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big) \qquad (b)\\
&+4(\frac{1}{(\rho\xi)^2})\Big[\Big(d\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big]\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big) \qquad (c)\\
&+d\Big\{-4(\frac{1}{(\rho\xi)^2})\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)+\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big\}\\
&-2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \qquad (d)
\end{aligned}
$$

$$+2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$
$$-4\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\Big[\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big],$$

$(a)\ in\ ((2)+(3)+(4)+(5)+(6))+(b)\ in\ ((10)+(11))=0,$
$(7)+(d)\ in\ ((10)+(11))=0,$
$(8)+(c)\ in\ ((10)+(11))=0.$

Summary 1 for $O(\eta^4)$

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2}=\int d^4\theta[y]_{\theta^2,\eta^0}\times\partial[y]_{\theta^2,\eta^4}+\frac{1}{2}\int d^4\theta[y]_{\theta^2,\eta^2}\times\partial[y]_{\theta^2,\eta^2}$$
$$-8\frac{dh}{\rho\xi}\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{\partial P+dR}{2\rho\xi}\qquad <1>$$
$$+4\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\qquad <2>$$
$$+2\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$
$$\times\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\qquad <3>$$
$$-4\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$
$$\times\Big[\Big((\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\partial(\frac{\partial\xi\eta}{\xi})\Big)\Big]\qquad <4>$$
$$+d\Big[8\frac{dh}{(\rho\eta)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2\Big]\cdots\cdots\cdots\ d\text{–}boundary\ terms$$
$$+d\Big[-4(\frac{1}{(\rho\xi)^2})\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)+\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]$$
$$\cdots\cdots\cdots\ d\text{–}boundary\ terms$$
$$\underbrace{+24\frac{dh}{\rho\xi}\frac{dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2}_{O(\eta^6)}$$
$$\underbrace{+8\frac{dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]}_{O(\eta^6)}\Big]$$
$$+\partial\text{–}boundary\ terms$$

$$\int d^4\theta[y]_{\theta^3,\eta^3}\times\partial[y]_{\theta^1,\eta^1}$$
$$=\frac{P}{\rho\xi}\Big\{-6\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}+6\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\frac{\partial\xi}{\xi}\qquad(12)$$
$$+2\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)-2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)\Big\}\qquad(13)$$
$$-8\Big[\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)$$
$$+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\qquad(14)$$

$$-4\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$
$$\times\Big[\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\Big] \tag{15}$$

$$+\frac{P}{\rho\xi}\Big\{-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\partial(\frac{dh}{\rho\xi}) \tag{16}$$

$$+\frac{P}{\rho\xi}\Big\{6\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}-6\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\frac{\partial\xi}{\xi} \tag{17}$$

$$-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big\}\frac{dh}{\rho\xi} \tag{18}$$

$$-8\Big\{\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)$$
$$+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\frac{dh}{\rho\xi} \tag{19}$$

$$+4\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]$$
$$\times\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\frac{dh}{\rho\xi}, \tag{20}$$

$$\int d^4\theta[y]_{\theta^3,\eta}\times\partial[y]_{\theta^1,\eta^3}$$

$$=\frac{dh}{\rho\xi}\Big\{-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\partial(\frac{P}{\rho\xi}) \tag{21}$$

$$+\Big\{2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-2\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\partial(\frac{P}{\rho\xi}) \tag{22}$$

$$+\frac{dh}{\rho\xi}\Big\{6\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}-6\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\frac{\partial\xi}{\xi}\Big\}\frac{P}{\rho\xi} \tag{23}$$

$$+\frac{dh}{\rho\xi}\Big\{-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho\xi})\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big((\frac{\partial^2(\rho\eta)}{\rho\xi})\cdot\partial\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}\frac{P}{\rho\xi} \tag{24}$$

$$+\Big[2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big](\frac{d\xi}{\xi}-\frac{d\rho}{\rho})\frac{P}{\rho\xi} \tag{25}$$

$$+\Big[2(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)-2(\partial(\frac{d(\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{P}{\rho\xi} \tag{26}$$

$$+\Big[-4\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{d\xi}{\xi}+4\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\frac{d\rho}{\rho}\Big]\frac{P}{\rho\xi}, \tag{27}$$

Contribution from the calculation to order $O(\eta^2)$

$$\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta^1}$$

$$=-2\frac{dR}{\rho\xi}\frac{dh}{\rho\xi}\Big[\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)\Big] \tag{28}$$

$$+2\frac{dR}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big], \tag{29}$$

$$+2\frac{dR}{(\rho\xi)^2}\Big[\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)-\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big] \tag{30}$$

$(17) + (18) + (23) + (24) = 0,$

$$
\begin{aligned}
&(16) + (21) + (25) + (27) + (28) \\
&= \frac{dh}{\rho\xi}\frac{dR+\partial P}{\rho\xi}\Big\{-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&-\frac{P}{\rho\xi}\Big[-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big](\frac{d\partial h}{\rho\xi}+\frac{d(\rho\xi)}{\rho\xi}) \\
&= \frac{dh}{\rho\xi}\frac{dR+\partial P}{\rho\xi}\Big\{-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&-2\frac{P}{\rho\xi}\Big[-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{d(\rho\xi)}{\rho\xi} \\
&\underbrace{-\frac{P}{\rho\xi}\Big[-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^6)},
\end{aligned}
$$

$$
\begin{aligned}
&(12) + (13) + (22) + (26) \\
&= -\partial(\frac{P}{\rho\xi})\Big\{2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-2\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&+\frac{P}{\rho\xi}\Big\{-4\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{\partial\rho}{\rho}+4\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\frac{\partial\xi}{\xi}\Big\} \\
&+4\frac{P}{\rho\xi}\Big\{\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)\Big\} \\
&= -2\frac{\partial P}{\rho\xi}\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&-4P\partial(\frac{1}{\rho\xi})\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&+4\frac{P}{\rho\xi}\Big\{\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\partial(\frac{d(\xi\eta)}{\xi})\Big)\Big\} \\
&= 2\frac{\partial P}{\rho\xi}\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&+4\frac{P}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial^2(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial^2\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \\
&+\partial\text{–}\mathit{boundary\ term},
\end{aligned}
$$

$$
\begin{aligned}
&(12) + (13) + (22) + (26) + (29) \\
&= 2\frac{dR+\partial P}{\rho\xi}\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
&+4\frac{P}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial^2(\frac{d(\xi\eta)}{\xi})\Big)-\Big(\partial^2\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi})\Big)\Big] \qquad [A] \\
&+\partial\text{–}\mathit{boundary\ term},
\end{aligned}
$$

$$
\begin{aligned}
[A] = & 4\frac{P}{\rho\xi}\Big[\Big\{\frac{1}{\rho\xi}\Big(\partial(\rho\eta)\cdot d\partial^2(\xi\eta)\Big)+\frac{1}{\rho}\partial^2(\frac{1}{\xi})\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)+2\frac{1}{\rho}\partial(\frac{1}{\xi})\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)\Big\} \\
& -4\frac{P}{\rho\xi}\Big\{\frac{1}{\rho\xi}\Big(d\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)+\frac{1}{\xi}\partial^2(\frac{1}{\rho})\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)+2\frac{1}{\xi}\partial(\frac{1}{\rho})\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big\}\Big] \\
& = 4P\Big\{\frac{1}{(\rho\xi)^2}\Big[\Big(\partial(\rho\eta)\cdot d\partial^2(\xi\eta)\Big)-\Big(d\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \qquad +\frac{1}{(\rho\xi)^2}\Big((\frac{\partial\rho}{\rho})^2-\partial(\frac{\partial\rho}{\rho})\Big)\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \qquad +\frac{1}{2}\partial(\frac{1}{(\rho\xi)^2})\Big[\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)-\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\Big\} \\
& = -2\frac{\partial P}{(\rho\xi)^2}\Big[\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)-\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad +2\frac{P}{(\rho\xi)^2}\Big[\Big(\partial(\rho\eta)\cdot d\partial^2(\xi\eta)\Big)-\Big(d\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad -2\frac{P}{(\rho\xi)^2}\Big[\Big(\partial^2(\rho\eta)\cdot d\partial(\xi\eta)\Big)-\Big(d\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)\Big] \\
& \quad +4\frac{P}{(\rho\xi)^2}\Big((\frac{\partial\rho}{\rho})^2-\partial(\frac{\partial\rho}{\rho})\Big)\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad +\partial\text{–}bounday\ term \\
& = -2\frac{\partial P}{(\rho\xi)^2}\Big[\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)-\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad +2\frac{P}{(\rho\xi)^2}d\Big[\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)-\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad -4P\Big(\frac{1}{4}[\partial(\frac{1}{\rho\xi})]^2+\frac{1}{2}\frac{1}{\rho\xi}\partial^2(-\frac{1}{\rho\xi})\Big)\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad +\partial\text{–}bounday\ term.
\end{aligned}
$$

Here use was made of

$$
\begin{aligned}
& \frac{1}{\xi}\partial\frac{1}{\rho}=\frac{1}{\rho}\partial\frac{1}{\xi}=\frac{1}{2}\partial(\frac{1}{\rho\xi}), \\
& \frac{1}{\xi}\partial^2\frac{1}{\rho}=\frac{1}{\rho}\partial^2\frac{1}{\xi} \\
& \qquad =\frac{1}{\rho\xi}(\frac{\partial\rho}{\rho})^2-\frac{1}{\rho\xi}\partial(\frac{\partial\rho}{\rho})=\frac{1}{2}\Big(\frac{1}{\rho\xi}[(\frac{\partial\rho}{\rho})^2+(\frac{\partial\xi}{\xi})^2]-\frac{1}{\rho\xi}\partial(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi})\Big) \\
& \qquad =-\frac{1}{4}\frac{1}{\rho\xi}(\frac{\partial\rho}{\rho}+\frac{\partial\xi}{\xi})^2+\frac{1}{2}\partial^2(\frac{1}{\rho\xi})=-\frac{1}{4}[\partial(\frac{1}{\rho\xi})]^2+\frac{1}{2}\partial^2(\frac{1}{\rho\xi}),
\end{aligned}
$$

which can be proved by $\frac{\partial\rho}{\rho}=\frac{\partial\xi}{\xi}$. Using this [A] we find

$$
\begin{aligned}
& (12)+(13)+(22)+(26)+(29) \\
& \ = 2\frac{dR+\partial P}{\rho\xi}\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)-\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \\
& \quad -2\frac{\partial P}{(\rho\xi)^2}\Big[\Big(\partial(\rho\eta)\cdot d\partial(\xi\eta)\Big)-\Big(d\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad +2\frac{P}{(\rho\xi)^2}d\Big[\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)-\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad -4P\Big(\frac{1}{4}[\partial(\frac{1}{\rho\xi})]^2+\frac{1}{2}\frac{1}{\rho\xi}\partial^2(-\frac{1}{\rho\xi})\Big)\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \\
& \quad +\partial\text{–}bounday\ term,
\end{aligned}
$$

$$
\begin{aligned}
&(12)+(13)+(22)+(26)+(29)+(30)\\
&\quad = 2\frac{P}{(\rho\xi)^2}d\Big[\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)-\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\\
&\quad -4P\Big(\frac{1}{4}[\partial(\frac{1}{\rho\xi})]^2+\frac{1}{2}\frac{1}{\rho\xi}\partial^2(-\frac{1}{\rho\xi})\Big)\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\\
&\quad +\partial\text{–}\mathit{bounday\ term},
\end{aligned}
$$

$$
\begin{aligned}
&(19)+(20)\\
&= -4\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\frac{dh}{\rho\xi}.
\end{aligned}
$$

Summary 2 for $O(\eta^4)$

$$
\begin{aligned}
&\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta^1}\\
&= \int d^4\theta[y]_{\theta^3,\eta^3}\times\partial[y]_{\theta^1,\eta^1}+\int d^4\theta[y]_{\theta^3,\eta^1}\times\partial[y]_{\theta^1,\eta^3}\\
&= -8\Big[\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \quad <5>\\
&\quad -4\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\\
&\qquad\qquad\times\Big[\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)\Big] \quad <6>\\
&\quad +\frac{dh}{\rho\xi}\frac{dR+\partial P}{\rho\xi}\Big\{-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial^2(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial^2(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\} \quad <7>\\
&\quad -2\frac{P}{\rho\xi}\Big[-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{d(\rho\xi)}{\rho\xi} \quad <8>\\
&\quad +2\frac{P}{(\rho\xi)^2}d\Big[\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big)-\Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \quad <9>\\
&\quad -4P\Big(\frac{1}{4}[\partial(\frac{1}{\rho\xi})]^2+\frac{1}{2}\frac{1}{\rho\xi}\partial^2(-\frac{1}{\rho\xi})\Big)\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big] \quad <10>\\
&\quad -4\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{dR+\partial P}{2\rho\xi}\frac{dh}{\rho\xi} \quad <11>\\
&\quad \underbrace{-\frac{P}{\rho\xi}\Big[-2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)+2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]\frac{dR}{\rho\xi}}_{O(\eta^6)}\\
&\quad +\partial\text{–}\mathit{bounday\ term}
\end{aligned}
$$

$$\int d^4\theta [y]_{\theta^0} \times \partial [y]_{\theta^4}$$

$$= \frac{1}{2}\frac{P}{\rho\xi}\partial\Big\{ d\Big(R(\frac{\partial\rho}{\rho^2\xi} + \frac{\partial\xi}{\rho\xi^2}) + \partial\Big(P(\frac{\partial\rho}{\rho^2\xi} + \frac{\partial\xi}{\rho\xi^2}) + 2\partial\Big[\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big) - \Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big]\Big\}$$

$$= \frac{1}{2}\frac{P}{\rho\xi}\partial\Big\{ [dR + \partial P]\partial(-\frac{1}{\rho\xi}) + Rd\partial(-\frac{1}{\rho\xi}) + P\partial^2(-\frac{1}{\rho\xi}) - \partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]\Big\}$$

$$= \frac{1}{2}\frac{P}{\rho\xi}\Big\{ \partial[dR + \partial P]\partial(-\frac{1}{\rho\xi}) + [dR + \partial P]\partial^2(-\frac{1}{\rho\xi}) + \partial R d\partial(-\frac{1}{\rho\xi}) + Rd\partial^2(-\frac{1}{\rho\xi}) + \partial P\partial^2(-\frac{1}{\rho\xi})\Big\} + \frac{1}{2}\partial(\frac{P}{\rho\xi})\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big] + \partial\text{–}\mathit{boundary\ term} \qquad [B]$$

Contribution from the calculation to order $O(\eta^2)$

$$\int d^4\theta [y]_{\theta^0} \times \partial [y]_{\theta^4}$$

$$= \frac{1}{2}\Big\{ \frac{P}{\rho\xi} dR\partial^2(-\frac{1}{\rho\xi}) - \frac{P}{\rho\xi} Rd\partial^2(-\frac{1}{\rho\xi}) - \frac{P}{\rho\xi}\partial R d\partial(-\frac{1}{\rho\xi})\Big\} + \frac{1}{2}\frac{dR}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big]. \qquad [C]$$

$$[B] + [C] = \frac{P}{\rho\xi}[dR + \partial P]\partial^2\Big(-\frac{1}{\rho\xi}\Big) + \frac{1}{2}P[dR + \partial P]\Big(\partial(\frac{1}{\rho\xi})\Big)^2 + \frac{1}{2}\frac{\partial P}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big] + \partial\text{–}\mathit{boundary\ term},$$

Summary 3 for $O(\eta^4)$

$$\int d^4\theta [y]_{\theta^0} \times \partial [y]_{\theta^4}$$

$$= \frac{P}{\rho\xi}[dR + \partial P]\partial^2\Big(-\frac{1}{\rho\xi}\Big) + \frac{1}{2}P[dR + \partial P]\Big(\partial(\frac{1}{\rho\xi})\Big)^2 \qquad \text{<} 12 \text{>}$$

$$+\frac{1}{2}\frac{dR+\partial P}{\rho\xi}\partial\Big[\frac{dR+\partial P}{\rho\xi}\Big] \qquad \text{<} 13 \text{>}$$

$+\partial$–*boundary term*

We sum over the results of *Summaries* 1, 2, 3 for $O(\eta^4)$ to find only a d-boundary term such as

$$< 1 > + < 7 > + < 11 >= 0,$$

$$< 2 > + < 8 > + < 9 >= d\Big\{ -2\frac{P}{(\rho\xi)^2}\Big[\Big(\partial(\rho\eta)\cdot\partial^2(\xi\eta)\Big) - \Big(\partial^2(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\Big\}$$

$\cdots\cdots\cdots$ d–*boundary term*

$$\{< 3 > + < 13 >\} + \{< 4 > + < 5 > + < 6 >\} = 0,$$

$$< 10 > + < 12 >= 0.$$

The terms of order $\mathcal{O}(\eta^6)$ will be cancelled in the calculation to order $O(\eta^6)$ in the next subsection.

6.4 $O(\eta^6)$

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2,\eta^4}\times\partial[y]_{\theta^2,\eta^2}$$

$$= 48\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\partial\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\frac{dh}{\rho\xi} + 48\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\partial(\frac{dh}{\rho\xi}) \quad (1)$$

$$-8\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\partial\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \quad (2)$$

$$-16\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{d(\xi\eta))}{\xi})\Big)+\Big(\partial(\frac{d(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \quad (3)$$

$$-16\frac{P}{\rho\xi}\Big\{\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big\}. \quad (4)$$

We calculate each line to find

$$(1) = 24\frac{dh}{\rho\xi}\frac{\partial P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2 + 24\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\frac{d\partial h}{\rho\xi}$$

$$= 24\frac{dh}{\rho\xi}\frac{\partial P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2 \quad <1>$$

$$+24\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\frac{d(\rho\xi)}{\rho\xi} \quad <2>$$

$$+\underbrace{24\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\frac{dR}{\rho\xi}}_{O(\eta^8)}$$

$$+\partial\text{–}boundary\ term,$$

$$(3) = -16\frac{P}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)d\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)$$

$$+8\frac{P}{\rho\xi}\frac{\partial(\rho\xi)}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$

$$= -8\frac{dP}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta\Big)^2 - 24P\frac{d(\rho\xi)}{(\rho\xi)^4}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta\Big)^2 \quad <3>$$

$$-8P\partial(\frac{1}{\rho\xi})\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big] \quad <4>$$

$$+d\Big[8\frac{P}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta\Big)^2\Big],$$

$$(4) = -8\frac{P}{\rho\xi}\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big) \quad <5>$$

$$-4\frac{P}{\rho\xi}\Big[\frac{\partial P+dR}{\rho\xi}\Big]\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)-\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big], \quad <6>$$

$$\int d^4\theta[y]_{\theta^3,\eta^5}\times\partial[y]_{\theta^1,\eta^1} = 0,$$

$$\int d^4\theta[y]_{\theta^3,\eta^3}\times\partial[y]_{\theta^1,\eta^3}$$

$$= \frac{P}{\rho\xi}\Big[-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)+2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\partial(\frac{P}{\rho\xi}) \qquad <7>$$

$$-4\Big[\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)-\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)\Big]$$

$$\times\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)-\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\frac{P}{\rho\xi}, \qquad <8>$$

$$\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}=0,$$

Contribution from the calculation to order η^4.

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2}$$

$$=24\frac{dh}{\rho\xi}\frac{dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2 \qquad <9>$$

$$+8\frac{dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big], \qquad <10>$$

$$\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta^1}$$

$$=\frac{P}{\rho\xi}\Big[-2\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\partial(\frac{\partial(\xi\eta)}{\xi})\Big)+2\Big(\partial(\frac{\partial(\rho\eta)}{\rho})\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]\frac{dR}{\rho\xi}. \qquad <11>$$

We have

$$(2)+<4>+<5>+<10>$$

$$=8\frac{\partial P+dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big[\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)+\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big]$$

$$+\partial\text{−}boundary\ term,$$

$$<1>+<9>=24\frac{dh}{\rho\xi}\frac{\partial P+dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2,$$

$$<2>+<3>=-8\frac{dP}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2,$$

$$<6>+<7>+<8>+<11>=0.$$

Then we find

$$(2)+<4>+<5>+<10>=16\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)$$

$$+\partial\text{−}boundary\ term,$$

$$<1>+<9>=0.$$

owing to the fermion properties $[\partial(\rho\eta_a)]^3=0$ and $[\partial(\xi\eta^a)]^3=0$ with no sum over a. Using this fermion property and explicitly writing down the inner product in spinor components, we have

$$\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{d(\xi\eta)}{\xi}\Big)\Big(\frac{d(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)$$

$$=\Big(\frac{\partial(\rho\eta)}{\rho}\Big)_1\Big(\frac{\partial(\xi\eta)}{\rho}\Big)^1\cdot\Big(\frac{\partial(\rho\eta)}{\rho}\Big)_2\Big(\frac{d(\xi\eta)}{\rho}\Big)^2\cdot\Big(\frac{d(\rho\eta)}{\rho}\Big)_2\Big(\frac{\partial(\xi\eta)}{\rho}\Big)^2$$

$$+\Big(\frac{\partial(\rho\eta)}{\rho}\Big)_2\Big(\frac{\partial(\xi\eta)}{\rho}\Big)^2\cdot\Big(\frac{\partial(\rho\eta)}{\rho}\Big)_1\Big(\frac{d(\xi\eta)}{\rho}\Big)^1\cdot\Big(\frac{d(\rho\eta)}{\rho}\Big)_1\Big(\frac{\partial(\xi\eta)}{\rho}\Big)^1$$

$$\frac{1}{2}dP\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2$$

$$=\Big[\Big(\frac{d(\rho\eta)}{\rho}\Big)_1\Big(\frac{d(\xi\eta)}{\rho}\Big)^1+\Big(\frac{d(\rho\eta)}{\rho}\Big)_2\Big(\frac{d(\xi\eta)}{\rho}\Big)^2\Big]$$

$$\times 2\Big(\frac{\partial(\rho\eta)}{\rho}\Big)_1\Big(\frac{\partial(\xi\eta)}{\rho}\Big)^1\cdot\Big(\frac{\partial(\rho\eta)}{\rho}\Big)_2\Big(\frac{\partial(\xi\eta)}{\rho}\Big)^2$$

Therefore

$\{(2)+ < 4 > + < 5 > + < 10 >\} + \{< 2 > + < 3 >\} = \partial\!-\!boundary\ term,$

Summary for $O(\eta^6)$

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2}+\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}+\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta^1}$$
$$= d\Big[8\frac{P}{(\rho\xi)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2\Big] + \underbrace{24\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\frac{dR}{\rho\xi}}_{O(\eta^8)}$$

The terms of order $\mathcal{O}(\eta^8)$ will be cancelled in the calculation to order $O(\eta^8)$ in the next subsection.

6.5 $O(\eta^8)$

$$\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}=0,$$

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2,\eta^4}\times\partial[y]_{\theta^2,\eta^4}=24\frac{P}{\rho\xi}\partial(\frac{P}{\rho\xi})\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2,$$
$$\int d^4\theta[y]_{\theta^2,\eta^5}\times\partial[y]_{\theta^2,\eta^3}=none,$$

$$\int d^4\theta[y]_{\theta^3,\eta^5}\times\partial[y]_{\theta^1,\eta^3}=0,$$
$$\int d^4\theta[y]_{\theta^3,\eta^3}\times\partial[y]_{\theta^1,\eta^5}=none.$$

We have also a contribution from the calculation to order η^6 such as

$$\frac{1}{2}\int d^4\theta[y]_{\theta^2,\eta^4}\times\partial[y]_{\theta^2,\eta^4}=24\frac{P}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2\frac{dR}{\rho\xi},$$
$$\int d^4\theta[y]_{\theta^2,\eta^6}\times\partial[y]_{\theta^2,\eta^2}=none.$$

Summary for $O(\eta^8)$

$$\int d^4\theta[y]_{\theta^0}\times\partial[y]_{\theta^4}=\int d^4\theta[y]_{\theta^3}\times\partial[y]_{\theta^1}=0,$$
$$\frac{1}{2}\int d^4\theta[y]_{\theta^2}\times\partial[y]_{\theta^2}=24\frac{P}{\rho\xi}\frac{\partial P+dR}{\rho\xi}\Big(\frac{\partial(\rho\eta)}{\rho}\cdot\frac{\partial(\xi\eta)}{\xi}\Big)^2=0$$

This is the end of the long calculations in Sec. 6. The formula which we have desired

$$\int dx d^4\theta\ y\partial y = d[\cdots]$$

has been shown. To be concrete it is given by

$$
\begin{aligned}
&\int dxd^4\theta\ y\partial y \\
&= 2d\int dx\Big\{\frac{1}{2}dh\Big[\partial_x^2(-\frac{1}{\rho\xi})+\partial_x\log(\rho\xi)\partial_x(-\frac{1}{\rho\xi})\Big] \\
&\qquad +4\frac{dh}{(\rho\xi)^2}\Big[\Big(\partial_x^2(\rho\eta)\cdot\partial_x(\xi\eta)\Big)-\Big(\partial_x(\rho\eta)\cdot\partial_x^2(\xi\eta)\Big)\Big] \\
&\qquad -\frac{2}{\rho\xi}\Big[\Big(d\partial_x(\rho\eta)\cdot\partial_x(\xi\eta)\Big)-\Big(\partial_x(\rho\eta)\cdot d\partial_x(\xi\eta)\Big)\Big] \\
&\qquad -\frac{1}{\rho\xi}\partial_x\Big[\Big(\partial_x(\rho\eta)\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\partial_x(\xi\eta)\Big)\Big] \\
&\qquad +\frac{1}{2}\Big[\Big(\rho\eta\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\xi\eta\Big)\Big]\Big[\rho\xi\Big(\partial_x(\frac{1}{\rho\xi})\Big)^2+\partial_x^2(-\frac{1}{\rho\xi})\Big] \\
&\qquad +8\frac{dh}{(\rho\eta)^3}\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big)^2 \\
&\qquad -4(\frac{1}{(\rho\xi)^2})\Big[\Big(\partial(\rho\eta)\cdot d(\xi\eta)\Big)+\Big(d(\rho\eta)\cdot\partial(\xi\eta)\Big)\Big]\Big(\partial(\rho\eta)\cdot\partial(\xi\eta)\Big) \\
&\qquad -\frac{2}{(\rho\xi)^2}\Big[\Big(\rho\eta\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\xi\eta\Big)\Big]\Big[\Big(\partial_x(\rho\eta)\cdot\partial_x^2(\xi\eta)\Big)-\Big(\partial_x^2(\rho\eta)\cdot\partial_x(\xi\eta)\Big)\Big] \\
&\qquad +\frac{8}{(\rho\xi)^3}\Big[\Big(\rho\eta\cdot d(\xi\eta)\Big)-\Big(d(\rho\eta)\cdot\xi\eta\Big)\Big]\Big(\partial_x(\rho\eta)\cdot\partial_x(\xi\eta)\Big)^2\Big\}.
\end{aligned}
$$

References

[1] S. Aoyama, "$N=(4,0)$ Super-Liouville Theory on the Coadjoint Orbit and PSU(1,1|2)", Phys. Lett. **B785**(2018)59, arXiv:1804.05179 [hep-th].

[2] S. Aoyama, Y. Honda, "$N=4$ Super-Schwarzian Theory on the Coadoint Orbit and PSU(1,1|2)", JHEP**06**(2018)070, arXiv:1801.06800 [hep-th].